Journal of Applied Analysis and Computation
Volume 11, Number 5, October 2021, 2402–2422
Website:http://www.jaac-online.com
DOI:10.11948/20200404

IDENTIFYING THE INITIAL CONDITION FOR SPACE-FRACTIONAL SOBOLEV EQUATION

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Abstract In this work, a final value problem for a fractional pseudo-parabolic equation is considered. Firstly, we present the regularity of solution. Secondly, we show that this problem is ill-posed in Hadamard's sense. After that we use the quasi–boundary value regularization method to solve this problem. To show that the proposed theoretical results are appropriate, we present an illustrative numerical example.

Keywords Final value problem, fractional pseudo-parabolic equation, Ill-posed problem, convergence estimates, regularization.

MSC(2010) 35K05, 35K99, 47J06, 47H10x.

1. Introduction

In this paper, for we study the linear nonclassical diffusion equation which called pseudo-parabolic equation in the following form

$$u_t(x,t) - m\Delta u_t(x,t) + (-\Delta)^\beta u(x,t) = f(x,t), \text{ for } x \in \Omega, \ t \in (0,T], \tag{1.1}$$

satisfies the following Dirichlet boundary condition

$$u(x,t)|_{\partial\Omega\times(0,T]} = 0. \tag{1.2}$$

- *Initial value problem (IVP):* This problem consists in finding $u(x,t)$ for $t \in (0,T]$ from the initial data

$$u(x,0) = u_0(x), \quad x \in \Omega. \tag{1.3}$$

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- *Final value problem (FVP):* This problem is related to recovering $u(x,t)$ for $t \in [0,T)$ from the terminal data (or final state)

$$u(x,T) = g(x), \quad x \in \Omega. \tag{1.4}$$

Pseudo-parabolic equations have many applications in science and technology, particularly in physical phenomena such as aggregation of populations, seepage of homogeneous fluids through a fissured rock, we can see in [13] and the references of this paper. We have the standard Laplace operator when $\beta = 1$, it means that the operator $(-\Delta)^\beta$ becomes $-\Delta$. In this case, many authors have studied the nonclassical diffusion equation has been studied with many various directions and motivations such as [1, 7, 8, 10, 13, 15, 24, 25]. As far as we know that the results on fractional pseudo-parabolic equation are limited, we can mention them in some few papers, for example [3, 4, 13, 26, 27]. For final value problem for pseudo-parabolic equation (1.1), (1.2) and (1.4), until now, there are still not many research results for this problem (for instance, [18] and its references). In general, the problem statisfies the conditions (1.1), (1.2) and (1.4) is ill-posed problem, namely, a solution do not exist and if a solution exists it does not depend continuously on the data. Moreover, if we have a small error in the perturbed data, it can lead to the blow-up of the solution.

In case $m = 0$ and $\beta = 1$, the problem (1.1) is the parabolic problem. There are many methods which are used to solve the final value problem for the classical parabolic equation. In this area, we can find much research works, e.g., Dang Duc Trong and his group, (see [21–23]).

In case $m > 0$ and $\beta = 1$, the problem (1.1), (1.2) and (1.3) is called by the pseudo-parabolic equation (we can refer to [17, 20]). In [13], Fang and his group studied the time-decay and global existence for the solution of the fractional pseudo-parabolic equation.

For an example of the regularization topic, [2], V.V. Au et al., studied the Problem (1.1) in the cases of globally or locally Lipschitzian source term with the help of modified Lavrentiev and Fourier truncated regularization methods.

In case the observation g satisfying the statistical model. In [9], the authors consider Problem (1.1) in a different point of view of data. They assume that the exact value of g by the observation $\widetilde{g}_{\mathbf{i}} = \widetilde{g}_{i_1,i_2,\dots,i_m}$ satisfying the statistical model

$$\widetilde{g}_{\mathbf{i}} = g + \delta_{\mathbf{i}} \mathcal{X}_{\mathbf{i}}, \tag{1.5}$$

where $\delta_{\mathbf{i}} = \delta_{i_1,i_2,\dots,i_m} > 0$ is bounded by a given positive constant and $\mathcal{X}_{\mathbf{i}} = \mathcal{X}_{i_1,i_2,\dots,i_m} \overset{i.i.d}{\sim} \mathcal{N}(0,1)$ are i.i.d. standard Gaussian random variables, for $\mathbf{i} \in \mathbb{Z}_+^k$.

In this paper, we strongly consider the quasi-boundary value regularization method to solve the problem (1.1), (1.2), and (1.4). In 1983, [16], Showalter who used the quasi-boundary value method for the homogeneous problem which gave stability pretty good than the quasi reversibility method of Lattes and Lions, see [14]. This method has a long history of research. The main idea of this method is adding appropriate items into the final data. By using the above idea, the author Denche-Bessila in [11], solved the backward heat problem by replacing the final condition as follows

$$u(T) + \epsilon u(0) = g, \quad u(T) - \epsilon u'(0) = g.$$

In the present paper, our main target is to provide one regularized solution that is called regularized solution for approximating $u(x,t)$, $t \in (0,T]$. It gives the error estimate between the regularized solution and the sought solution, we also give one numerical example to show the efficiency of the proposed method about the convergent estimate.

The paper is structured as follows. In Section 1, we introduce about the problem which is considered in this work. In next section, we give some preliminary materials to be used later in this paper. Next, we show the regularity of $u(x,t)$ and $\frac{\partial u}{\partial t}(x,t)$ and the ill-posedness of the fractional inverse source problem. In Section 3, we propose a generalized quasi-boundary value regularization method and show information about the error estimate between the sought solution and the regularized solution in Section 4. Finally, one numerical example to test our proposed regularized method is shown in Section 5. Lastly, we give some comments in Section 6 - Conclusion.

2. Preliminary results

We start this section by introducing several definitions and lemmas.

Definition 2.1. Let $\mathcal{L}$ be the operator on the domain $D(-\mathcal{L}) := H_0^1(\Omega) \cap H^2(\Omega)$ and assume that $-\mathcal{L}$ has a eigenvalues λ_k with eigenfunction $e_k \in D(\mathcal{L})$ as follows

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \leq ...$$

and $\lambda_k \to \infty$ as $k \to \infty$. An example of $\mathcal{L}$ is the negative Laplacian operator $-\Delta$ on $L^2(\Omega)$, we have

$$\mathcal{L} e_k(x) = \lambda_k e_k(x), \quad x \in \Omega, \text{ and } e_k(x) = 0, x \in \partial\Omega.$$

Definition 2.2 (Hilbert scale space)**.** Being doing so, we introduce some suitable Sobolev space, and fix some notation, $\beta > 0$. Let us recall that the spectral problem

$$\mathcal{L}^\beta e_k(x) = \lambda_k^\beta e_k(x), \quad x \in \Omega, \text{ and } e_k(x) = 0, \ x \in \partial\Omega,$$

admits a family of eigenvalues

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \quad \text{for} \quad \lambda_k \to \infty \text{ as } k \to \infty,$$

and the corresponding eigenfunctions $e_k \in H_0^1(\Omega)$. The Hilbert scale space $\mathcal{H}^s$ $(s > 0)$ is defined as follows

$$\mathcal{H}^s(\Omega) := \Big\{ f \in L^2(\Omega) : \sum_{k=1}^{\infty} \lambda_k^{2s} \big| \langle f, e_k \rangle_{L^2(\Omega)} \big|^2 < \infty \Big\}, \tag{2.1}$$

with the norm

$$\|f\|^2_{\mathcal{D}^s(\Omega)} = \sum_{k=1}^{\infty} \lambda_k^{2s} \big| \langle f, e_k \rangle_{\mathcal{L}^2(\Omega)} \big|^2 < \infty. \tag{2.2}$$

For a Hilbert space X, we denote by $L^p(0,T;X)$ and $C([0,T];X)$ the space of the continuous functions $f : [0,T] \to X$, such that

$$\|f\|_{L^p(0,T;X)} = \left(\int_0^T \|f\|_X^p dt \right)^{\frac{1}{p}} < \infty, \quad 1 \leq p < \infty,$$

$$\|f\|_{C([0,T];X)} = \sup_{0\le t\le T} \|f(t)\|_X < \infty,$$

and

$$\|f\|_{L^\infty(0,T;X)} = \operatorname*{ess\,sup}_{0\le t\le T} \|f(t)\|_X < \infty, \quad p=\infty.$$

The definition of the negative fractional power $\mathcal{L}^{-s}$ with $s>0$ can be founded in [6]. Its domain $\mathcal{D}(\mathcal{L}^{-s})$ is a Hilbert space endowed with the inner product $\langle\cdot,\cdot\rangle$ taken between $\mathcal{D}(\mathcal{L}^{-s})$ and $\mathcal{D}(\mathcal{L}^{s})$. This generates the norm

$$\|\nu\|_{\mathcal{D}(\mathcal{L}^{-s})} = \left(\sum_{k=1}^{\infty} \left|\langle \nu, e_k\rangle\right|^2 \lambda_k^{-2s}\right)^{\frac{1}{2}}. \tag{2.3}$$

Let $L^\infty(0,T;\mathbf{G}_{\beta,m}(\Omega))$ (see [12]) be the following space

$$L^\infty(0,T;\mathbf{G}_{\beta,m}(\Omega)) = \Big\{\nu \in L^\infty(0,T;L^2(\Omega)), \quad \sup_{0\le t\le T} \mathrm{e}^{2tm^{-1}\lambda_k^{\beta-1}} \left|\langle \nu, e_k\rangle\right|^2 < \infty\Big\}, \tag{2.4}$$

for any $\beta>1$ and $m>0$.

Lemma 2.1. *For $0<\alpha<\mathrm{e}T$, by denoting $b(x) = \left(\alpha x + \mathrm{e}^{-xT}\right)^{-1}$, we have*

$$b(x) \le \frac{T}{\alpha\big(1+\ln\big(\frac{T}{\alpha}\big)\big)} \le \frac{T}{\alpha\ln\big(\frac{T}{\alpha}\big)}. \tag{2.5}$$

Proof. The proof of Lemma 2.1 can be found in [21]. ☐

Lemma 2.2. *For any $0<t\le T$, by denoting $\xi_k(m,\beta) = \frac{\lambda_k^\beta}{1+m\lambda_k}$ and $\widetilde{T} = \max\{1,T\}$, we have*

$$\mathcal{D}_k(T,t,\beta) = \frac{\mathrm{e}^{-t\xi_k(m,\beta)}}{\alpha\xi_k(m,\beta) + \mathrm{e}^{-T\xi_k(m,\beta)}}, \tag{2.6}$$

then we obtain

$$\mathcal{D}_k(T,t,\beta) \le \widetilde{T}\Big(\alpha\ln\big(\frac{T}{\alpha}\big)\Big)^{\frac{t}{T}-1}. \tag{2.7}$$

Proof. We can see the same proof in Lemma 3.1 of [19]. ☐

Lemma 2.3. *Let $0<\beta\le 1$, for any $0<t\le\zeta\le T$, by using the inequality $\mathrm{e}^{-z}\le 1$ for all z, $\mu_k = (1+m\lambda_k)^{-1}$ and $\xi_k(m,\beta)$ is defined in the Lemma 2.2, we can find that*

$$\int_t^T \left\|\sum_{k=1}^{\infty} \mu_k \mathrm{e}^{(\zeta-t)\xi_k(m,\beta)} \langle f(\cdot,\zeta), e_k(\cdot)\rangle\right\|^2_{L^2(\Omega)} d\zeta \le \frac{T\mathrm{e}^{2Tm^{-1}\lambda_1^{\beta-1}}}{(1+m\lambda_1)^2} \|f\|_{L^\infty(0,T;L^2(\Omega))}. \tag{2.8}$$

Proof. First of all, we deduce that

$$\left\|\sum_{k=1}^{\infty} \mu_k \mathrm{e}^{(\zeta-t)\xi_k(m,\beta)} \langle f(\cdot,\zeta), e_k(\cdot)\rangle\right\|^2_{L^2(\Omega)} = \sum_{k=1}^{\infty} \frac{\mathrm{e}^{2(\zeta-t)\lambda_k^\beta(1+m\lambda_k)^{-1}}}{\left(1+m\lambda_k\right)^2} \left|\langle f(\cdot,\zeta), e_k(\cdot)\rangle\right|^2$$

$$= \frac{e^{2Tm^{-1}\lambda_1^{\beta-1}}}{(1+m\lambda_1)^2} \sum_{k=1}^{\infty} \left| \left\langle f(\cdot,\zeta), e_k(\cdot) \right\rangle \right|^2. \tag{2.9}$$

Then, we have

$$\int_t^T \left\| \sum_{k=1}^{\infty} \mu_k e^{(\zeta-t)\xi_k(m,\beta)} \left\langle f(\cdot,\zeta), e_k(\cdot) \right\rangle \right\|_{L^2(\Omega)}^2 d\zeta \le \frac{e^{2Tm^{-1}\lambda_1^{\beta-1}}}{(1+m\lambda_1)^2} \int_t^T \|f(\cdot,\zeta)\|_{L^2(\Omega)}^2 d\zeta$$
$$\le \frac{Te^{2Tm^{-1}\lambda_1^{\beta-1}}}{(1+m\lambda_1)^2} \|f\|_{L^\infty(0,T;L^2(\Omega))}^2. \tag{2.10}$$

□

3. Main Result

3.1. Regularity of problem (1.1) for $0 < \beta \le 1$

The goal of this section, we study the final value problem (1)–(4) in first case $0 < \beta \le 1$ and second case $\beta > 1$. In more detail, $0 < \beta \le 1$, we show the regularity of $u(x,t)$ and $\frac{\partial u}{\partial t}(x,t)$. Moreover, in case $\beta > 1$, we show that the problem (1)–(4) is ill-posed. After that we present a modified quasi boundary value method to regularize problem (1)–(3) and convergent rate.

3.2. Homogeneous

The theorem below is the first major result in this section.

Theorem 3.1. *Let u be the solution of the problem (1.1) with $f = 0$, $g \in \mathcal{D}^s(\Omega) \cap \mathcal{D}^{s+\beta}(\Omega)$ for $s > 0$, we obtained the following estimate:*

a) For all $t \in (0,T)$, it gives

$$\|u(\cdot,t)\|_{\mathcal{D}^s(\Omega)} \le e^{(T-t)m^{-1}\lambda_1^{\beta-1}} \|g\|_{\mathcal{D}^s(\Omega)},$$
$$\left\| \frac{du}{dt}(\cdot,t) \right\|_{\mathcal{D}^{s+1}(\Omega)} \le m^{-1} e^{(T-t)m^{-1}\lambda_1^{\beta-1}} \|g\|_{\mathcal{D}^{s+\beta}(\Omega)}. \tag{3.1}$$

b) At $t = 0$, we get

$$\|u(\cdot,0)\|_{\mathcal{D}^s(\Omega)} \le e^{Tm^{-1}\lambda_1^{\beta-1}} \|g\|_{\mathcal{D}^s(\Omega)}. \tag{3.2}$$

Proof. a) For all $t \in (0,T)$, we have

$$u(x,t) = \sum_{k=1}^{\infty} e^{(T-t)\xi_k(m,\beta)} \left\langle g(\cdot), e_k(\cdot) \right\rangle e_k(x), \tag{3.3}$$

which leads to

$$\|u(\cdot,t)\|_{\mathcal{D}^s(\Omega)} = \left\| \sum_{k=1}^{\infty} \lambda_k^s e^{(T-t)\xi_k(m,\beta)} \left\langle g(\cdot), e_k(\cdot) \right\rangle e_k(\cdot) \right\|$$

$$\leq \mathrm{e}^{(T-t)m^{-1}\lambda_1^{\beta-1}} \Big\| \sum_{k=1}^{\infty} \lambda_k^s \big\langle g(\cdot), e_k(\cdot) \big\rangle e_k(\cdot) \Big\|$$
$$\leq \mathrm{e}^{(T-t)m^{-1}\lambda_1^{\beta-1}} \|g\|_{\mathcal{D}^s(\Omega)}. \tag{3.4}$$

To prove the second estimate, we take the derivative on the variable t of $u(x,t)$, we receive

$$\begin{aligned}
\Big\| \frac{du}{dt}(\cdot,t) \Big\|_{\mathcal{D}^{s+1}(\Omega)} &= \Big\| \sum_{k=1}^{\infty} \lambda_k^{s+1} \xi_k(m,\beta) \mathrm{e}^{(T-t)\xi_k(m,\beta)} \big\langle g(\cdot), e_k(\cdot) \big\rangle e_k(\cdot) \Big\| \\
&\leq m^{-1} \Big\| \sum_{k=1}^{\infty} \lambda_k^{s+1} \lambda_k^{\beta-1} \mathrm{e}^{(T-t)\xi_k(m,\beta)} \big\langle g(\cdot), e_k(\cdot) \big\rangle e_k(\cdot) \Big\| \\
&\leq m^{-1} \Big\| \sum_{k=1}^{\infty} \lambda_k^{s+\beta} \mathrm{e}^{(T-t)\xi_k(m,\beta)} \big\langle g(\cdot), e_k(\cdot) \big\rangle e_k(\cdot) \Big\| \\
&\leq m^{-1} \mathrm{e}^{(T-t)m^{-1}\lambda_1^{\beta-1}} \Big\| \sum_{k=1}^{\infty} \lambda_k^{s+\beta} \big\langle g(\cdot), e_k(\cdot) \big\rangle e_k(\cdot) \Big\| \\
&\leq m^{-1} \mathrm{e}^{(T-t)m^{-1}\lambda_1^{\beta-1}} \|g\|_{\mathcal{D}^{s+\beta}(\Omega)}.
\end{aligned} \tag{3.5}$$

Now, for a sequence $\{t_h\} \subset (0,T]$ such that $t_h \to t$ when $h \to \infty$, we get

$$\begin{aligned}
&\lim_{h\to\infty} \|u(\cdot,t_h) - u(\cdot,t)\|_{\mathcal{D}^s(\Omega)} \\
=& \lim_{h\to\infty} \Big\| \sum_{k=1}^{\infty} \lambda_k^s \Big(\mathrm{e}^{(T-t_h)\xi_k(m,\beta)} - \mathrm{e}^{(T-t)\xi_k(m,\beta)} \Big) \big\langle g(\cdot), e_k(\cdot) \big\rangle e_k(\cdot) \Big\| \\
\leq& \lim_{h\to\infty} \Big\| \sum_{k=1}^{\infty} \lambda_k^s \mathrm{e}^{T\xi_k(m,\beta)} \Big(\mathrm{e}^{-t_h\xi_k(m,\beta)} - \mathrm{e}^{-t\xi_k(m,\beta)} \Big) \big\langle g(\cdot), e_k(\cdot) \big\rangle e_k(\cdot) \Big\| \\
\leq& \mathrm{e}^{Tm^{-1}\lambda_1^{\beta-1}} \lim_{h\to\infty} \Big\| \sum_{k=1}^{\infty} \lambda_k^s \Big(\mathrm{e}^{-t_h\xi_k(m,\beta)} - \mathrm{e}^{-t\xi_k(m,\beta)} \Big) \big\langle g(\cdot), e_k(\cdot) \big\rangle e_k(\cdot) \Big\|.
\end{aligned} \tag{3.6}$$

Because of the inequality $\big| \mathrm{e}^{-p} - \mathrm{e}^{-q} \big| \leq |p-q|, \ \forall p,q \geq 0$, one has

$$\begin{aligned}
&\lim_{h\to\infty} \|u(\cdot,t_h) - u(\cdot,t)\|_{\mathcal{D}^s(\Omega)} \\
\leq& \mathrm{e}^{Tm^{-1}\lambda_1^{\beta-1}} m^{-1}\lambda_1^{\beta-1} \Big\| \sum_{k=1}^{\infty} \lambda_k^s \big\langle g(\cdot), e_k(\cdot) \big\rangle e_k(\cdot) \Big\| \lim_{h\to\infty} \big| t_h - t \big| \\
\leq& \mathrm{e}^{Tm^{-1}\lambda_1^{\beta-1}} m^{-1}\lambda_1^{\beta-1} \|g\|_{\mathcal{D}^s(\Omega)} \lim_{h\to\infty} \big| t_h - t \big|.
\end{aligned} \tag{3.7}$$

From (3.7), if $t_h \to t$ as $h \to \infty$, then the right hand side to 0 which leads to $u \in C((0,T]; \mathcal{D}^s(\Omega))$. Next, we can assert that

$$\begin{aligned}
&\lim_{h\to\infty} \Big\| \frac{du}{dt}(\cdot,t_h) - \frac{du}{dt}(\cdot,t) \Big\|_{\mathcal{D}^{s+1}(\Omega)} \\
\leq& \lim_{h\to\infty} m^{-1} \Big\| \sum_{k=1}^{\infty} \lambda_k^{s+\beta} \mathrm{e}^{(T-t)\xi_k(m,\beta)} \big\langle g(\cdot), e_k(\cdot) \big\rangle e_k(\cdot) \Big\|
\end{aligned}$$

$$\leq m^{-1}\mathrm{e}^{Tm^{-1}\lambda_1^{\beta-1}} \lim_{h\to\infty} \Big\| \sum_{k=1}^{\infty} \lambda_k^{s+\beta} \Big(\mathrm{e}^{-t_h\xi_k(m,\beta)} - \mathrm{e}^{-t\xi_k(m,\beta)}\Big) \big\langle g(\cdot), e_k(\cdot)\big\rangle e_k(\cdot) \Big\|$$

$$\leq m^{-1}\mathrm{e}^{Tm^{-1}\lambda_1^{\beta-1}} \lambda_1^{\beta-1} \|g\|_{\mathcal{D}^{s+\beta}(\Omega)} \lim_{h\to\infty} \big|t_h - t\big|. \tag{3.8}$$

From (3.8), this yields $\lim_{h\to\infty} \Big\| \frac{du}{dt}(\cdot, t_h) - \frac{du}{dt}(\cdot, t) \Big\|_{\mathcal{D}^{s+1}(\Omega)} = 0$. Hence, we can conclude that $u \in C^1((0,T]; \mathcal{D}^{s+1}(\Omega))$.

b) At $t = 0$, using (3.18), it can be written as follows:

$$\|u(\cdot,0)\|_{\mathcal{D}^s(\Omega)} \leq \mathrm{e}^{Tm^{-1}\lambda_1^{\beta-1}} \Big\| \sum_{k=1}^{\infty} \lambda_k^s \big\langle g(\cdot), e_k(\cdot)\big\rangle e_k(\cdot) \Big\| \leq \mathrm{e}^{Tm^{-1}\lambda_1^{\beta-1}} \|g\|_{\mathcal{D}^s(\Omega)}. \tag{3.9}$$

The proof is completed. □

3.3. Nonhomogeneous

Since the problem (1)–(2), see [13], we carry on the mild solution of the initial value problem as follows

$$u(x,t) = \sum_{k=1}^{\infty} \Big(\mathrm{e}^{-t\xi_k(m,\beta)} \big\langle u_0(\cdot), e_k(\cdot)\big\rangle\Big) e_k(x) + \sum_{k=1}^{\infty} \Big(\mu_k \int_0^t \mathrm{e}^{-(t-\zeta)\xi_k(m,\beta)} \big\langle f(.,\zeta), e_k(\cdot)\big\rangle d\zeta\Big) e_k(x). \tag{3.10}$$

Next, from (3.10), by letting $t = T$ and using the condition (1.4), we can establish a representation formula for the solution problem (1)–(3), then by a simple computation, one has

$$u(x,t) = \sum_{k=1}^{\infty} \Big(\mathrm{e}^{(T-t)\xi_k(m,\beta)} \big\langle g(\cdot), e_k(\cdot)\big\rangle\Big)\ e_k(x) - \sum_{k=1}^{\infty} \Big(\mu_k \int_t^T \mathrm{e}^{(\zeta-t)\xi_k(m,\beta)} \big\langle f(.,\zeta), e_k(\cdot)\big\rangle\ d\zeta\ \Big) e_k(x). \tag{3.11}$$

Theorem 3.2. *Let u be the solution of the problem (1.1), assume that $g \in L^2(\Omega)$, and $f \in L^\infty(0,T; L^2(\Omega))$ then the following estimates*

- *For all $t \in (0,T)$, we get*

$$\big\|u(\cdot,t)\big\|_{L^2(\Omega)} \leq \mathrm{e}^{(T-t)m^{-1}\lambda_1^{\beta-1}} \|g\|_{L^2(\Omega)} + \frac{\sqrt{T}\mathrm{e}^{Tm^{-1}\lambda_1^{\beta-1}}}{(1+m\lambda_1)} \|f\|_{L^\infty(0,T;L^2(\Omega))},$$

$$\Big\|\frac{\partial u_\alpha}{\partial t}(.,t)\Big\|_{L^2(\Omega)} \leq \sqrt{3}\Big[\frac{\lambda_1^{\beta-1}\mathrm{e}^{(T-t)m^{-1}\lambda_1^{-1}}}{m}\|g\|_{L^2(\Omega)} + \Big(\frac{\sqrt{T}\lambda_1^{\beta-}\mathrm{e}^{Tm^{-1}\lambda_1^{\beta-1}}}{(1+m\lambda_1)} + (m\lambda_1)^{-1}\Big)\|f\|_{L^\infty(0,T;L^2(\Omega))}\Big]. \tag{3.12}$$

- *At $t = 0$, we have*

$$\big\|u(\cdot,0)\big\|_{L^2(\Omega)} \le \mathrm{e}^{Tm^{-1}\lambda_1^{\beta-1}}\|g\|_{L^2(\Omega)} + \frac{\sqrt{T}\mathrm{e}^{Tm^{-1}\lambda_1^{\beta-1}}}{(1+m\lambda_1)}\|f\|_{L^\infty(0,T;L^2(\Omega))}. \tag{3.13}$$

Proof. a) For all $t \in (0,T)$, using the inequality $(p+q)^2 \le 2(p^2+q^2)$, it gives

$$\big\|u(\cdot,t)\big\|^2_{L^2(\Omega)} \le \underbrace{2\Big\|\sum_{k=1}^{\infty}\mathrm{e}^{(T-t)\xi_k(m,\beta)}\big|\big\langle g(\cdot), e_k(\cdot)\big\rangle\big|\Big\|^2_{L^2(\Omega)}}_{\mathcal{A}_1} + \underbrace{2\Big\|\sum_{k=1}^{\infty}\int_t^T \mu_k e^{(\zeta-t)\xi_k(m,\beta)}\big\langle f(\cdot,\zeta), e_k(\cdot)\big\rangle\, d\zeta\, e_k(\cdot)\Big\|^2_{L^2(\Omega)}}_{\mathcal{A}_2}. \tag{3.14}$$

To prove the inequality (3.14), we perform the following two steps
Step 1: Estimate of $\mathcal{A}_1$, we deduce that

$$\begin{aligned}\mathcal{A}_1 &\le 2\Big\|\sum_{k=1}^{\infty}\mathrm{e}^{(T-t)\xi_k(m,\beta)}\big\langle g(\cdot), e_k(\cdot)\big\rangle e_k(\cdot)\Big\|^2_{L^2(\Omega)} \\ &\le 2\mathrm{e}^{2(T-t)m^{-1}\lambda_1^{\beta-1}}\sum_{k=1}^{\infty}\big|\big\langle g(\cdot), e_k(\cdot)\big\rangle\big|^2 \le 2\mathrm{e}^{2(T-t)m^{-1}\lambda_1^{\beta-1}}\|g\|^2_{L^2(\Omega)}.\end{aligned} \tag{3.15}$$

Step 2: Using the Lemma 2.3, we can find that

$$\begin{aligned}\mathcal{A}_2 &\le 2\Big\|\sum_{k=1}^{\infty}\int_t^T \mu_k \mathrm{e}^{(\zeta-t)\xi_k(m,\beta)}\big\langle f(\cdot,\zeta), e_k(\cdot)\big\rangle d\zeta\Big\|^2_{L^2(\Omega)} \\ &\le 2\frac{T\mathrm{e}^{2Tm^{-1}\lambda_1^{\beta-1}}}{(1+m\lambda_1)^2}\|f\|^2_{L^\infty(0,T;L^2(\Omega))}.\end{aligned} \tag{3.16}$$

Combining (3.14), (3.15) and (3.16), we obtain that

$$\big\|u(\cdot,t)\big\|_{L^2(\Omega)} \le \sqrt{2}\mathrm{e}^{(T-t)m^{-1}\lambda_1^{\beta-1}}\|g\|_{L^2(\Omega)} + \sqrt{2}\frac{\sqrt{T}\mathrm{e}^{Tm^{-1}\lambda_1^{\beta-1}}}{(1+m\lambda_1)}\|f\|_{L^\infty(0,T;L^2(\Omega))}. \tag{3.17}$$

b) Next, using the inequality $(p+q+r)^2 \le 3(p^2+q^2+r^2),\ \forall p,q,r \ge 0$, we get

$$\Big\|\frac{\partial u_\alpha}{\partial t}(.,t)\Big\|^2_{L^2(\Omega)} \le \underbrace{3\Big\|\sum_{k=1}^{\infty}\xi_k(m,\beta)\mathrm{e}^{(T-t)\xi_k(m,\beta)}\big\langle g(\cdot), e_k(\cdot)\big\rangle\Big\|^2_{L^2(\Omega)}}_{\mathcal{B}_1^2} + \underbrace{3\Big\|\sum_{k=1}^{\infty}\mu_k\xi_k(m,\beta)\int_t^T \mathrm{e}^{(\zeta-t)\xi_k(m,\beta)}\big\langle f(.,\zeta), e_k(\cdot)\big\rangle\, d\zeta\Big\|^2_{L^2(\Omega)}}_{\mathcal{B}_2^2} + \underbrace{3\Big\|\sum_{k=1}^{\infty}\mu_k\big\langle f(.,t), e_k(\cdot)\big\rangle\Big\|^2_{L^2(\Omega)}}_{\mathcal{B}_3^2}. \tag{3.18}$$

Step 1: Because of $\xi_k(m,\beta) = \lambda_k^{\beta}(1+m\lambda_k)^{-1} \le m^{-1}\lambda_k^{\beta-1} \le m^{-1}\lambda_1^{\beta-1}$. Hence, we have the estimation for $\mathcal{B}_1^2$ as follow:

$$\begin{aligned}\mathcal{B}_1^2 &\le 3m^{-2}\lambda_1^{2\beta-2}\mathrm{e}^{2(T-t)m^{-1}\lambda_1^{\beta-1}} \sum_{k=1}^{\infty} \left|\left\langle g(\cdot), e_k(\cdot)\right\rangle\right|^2 \\ &\le 3m^{-2}\lambda_1^{2\beta-2}\mathrm{e}^{2(T-t)m^{-1}\lambda_1^{\beta-1}} \|g\|^2_{L^2(\Omega)}. \end{aligned} \tag{3.19}$$

Step 2: Applying the Lemma 2.3, estimate of $\mathcal{B}_2^2$, we get

$$\begin{aligned}\mathcal{B}_2^2 &\le 3\bigg(\sum_{k=1}^{\infty}\mu_k\xi_k(m,\beta)\int_t^T \mathrm{e}^{(\zeta-t)\xi_k(m,\beta)}\big\langle f(.,\zeta), e_k(\cdot)\big\rangle\, d\zeta\bigg)^2 \\ &\le 3\frac{T\lambda_1^{2\beta-2}\mathrm{e}^{2Tm^{-1}\lambda_1^{\beta-1}}}{(1+m\lambda_1)^4}\|f\|^2_{L^\infty(0,T;L^2(\Omega))}. \end{aligned} \tag{3.20}$$

Step 3: In here, we note that for $0 \le t \le T$ there holds

$$\left|f_k(t)\right|^2 \le \sum_{k=1}^{\infty}\left|\left\langle f(.,t), e_k(\cdot)\right\rangle\right|^2 \le \|f\|^2_{L^\infty(0,T;L^2(\Omega))}. \tag{3.21}$$

Hence, we conclude that $\mathcal{B}_3^2$ can be bounded as follows

$$\mathcal{B}_3^2 \le 3(m\lambda_1)^{-2}\|f\|^2_{L^\infty(0,T;L^2(\Omega))}. \tag{3.22}$$

Combining (3.19), (3.20) and (3.22), we receive (3.12) holds

$$\begin{aligned}\left\|\frac{\partial u_\alpha}{\partial t}(.,t)\right\|_{L^2(\Omega)} \le & \sqrt{3}\bigg[\frac{\lambda_1^{\beta-1}\mathrm{e}^{(T-t)m^{-1}\lambda_1^{-1}}}{m}\|g\|_{L^2(\Omega)} \\ & + \Big(\frac{\sqrt{T}\lambda_1^{\beta-}\mathrm{e}^{Tm^{-1}\lambda_1^{\beta-1}}}{(1+m\lambda_1)} + (m\lambda_1)^{-1}\Big)\|f\|_{L^\infty(0,T;L^2(\Omega))}\bigg]. \end{aligned} \tag{3.23}$$

b) At $t = 0$, by replacing $t = 0$ in (3.17), it enables us to write

$$\|u(.,0)\|_{L^2(\Omega)}\sqrt{2}\mathrm{e}^{Tm^{-1}\lambda_1^{\beta-1}}\|g\|_{L^2(\Omega)} + \sqrt{2}\frac{\sqrt{T}\mathrm{e}^{Tm^{-1}\lambda_1^{\beta-1}}}{(1+m\lambda_1)}\|f\|_{L^\infty(0,T;L^2(\Omega))}. \tag{3.24}$$

□

4. Regularization and error estimate of (1.1) by a modified quasi boundary value method when $\beta > 1$

4.1. Construction of a regularization problem

In fact, from (3.11), we know that $\xi_k(m,\beta) = \lambda_k^{\beta}(1+m\lambda_k)^{-1} \approx \frac{\lambda_k^{\beta-1}}{\lambda_k^{-1}+m}$. Assume that fixed $\beta > 1$, when $k \to \infty$, it gives $\lambda_k^{\beta}(1+m\lambda_k)^{-1} \to \infty$. From (3.11), we receive $\exp\big((T-t)\lambda_k^{\beta}(1+m\lambda_k)^{-1}\big)$ and $\exp\big((s-t)\lambda_k^{\beta}(1+m\lambda_k)^{-1}\big)$ tends to infinity quickly. Small errors in high-frequency components can blow up and completely destroy the solution for $0 < t \le T$. Therefore, recovering $u(x,t)$ from the data $g(x)$

is severely ill-posed. Hence, we employ the modified quasi boundary value method to established a regularized problem, namely

$$\begin{cases} u_{\alpha,t}(x,t)-m\Delta u_{\alpha,t}(x,t)+(-\Delta)^{\beta}u_{\alpha}(x,t)=\sum_{k=1}^{\infty}\mu_k\frac{\mathrm{e}^{-T\xi_k(m,\beta)}}{\alpha\xi_k(m,\beta)+\mathrm{e}^{-T\xi_k(m,\beta)}}f(x,t), \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (x,t)\in\Omega\times(0,T), \\ u_{\alpha}(x,t)=0, \qquad\qquad\qquad\qquad\qquad x\in\partial\Omega,\ t\in(0,T], \\ u_{\alpha}(x,T)=\sum_{k=1}^{\infty}\frac{\mathrm{e}^{-T\xi_k(m,\beta)}}{\alpha\xi_k(m,\beta)+\mathrm{e}^{-T\xi_k(m,\beta)}}g(x), \qquad x\in\Omega. \end{cases} \tag{4.1}$$

In this section, we propose the following regularized solutions as follows:

$$\begin{aligned} u_{\alpha}(x,t)=&\sum_{k=1}^{\infty}\left(\frac{\mathrm{e}^{-t\xi_k(m,\beta)}}{\alpha\ \xi_k(m,\beta)+\mathrm{e}^{-T\xi_k(m,\beta)}}\big\langle g(\cdot),e_k(\cdot)\big\rangle\right)\ e_k(x) \\ &-\sum_{k=1}^{\infty}\left(\mu_k\int_t^T\frac{\mathrm{e}^{(\zeta-t-T)\xi_k(m,\beta)}}{\alpha\ \xi_k(m,\beta)+\mathrm{e}^{-T\xi_k(m,\beta)}}\big\langle f(.,\zeta),e_k(\cdot)\big\rangle\ d\zeta\ \right)e_k(x), \end{aligned} \tag{4.2}$$

where α is the regularization parameter. Next, taking the derivative of the function $u(x,t)$ according to the variable t, we have

$$\begin{aligned} u_{\alpha,t}(x,t)=&\sum_{k=1}^{\infty}\left(-\xi_k(m,\beta)\frac{\mathrm{e}^{-t\xi_k(m,\beta)}}{\alpha\xi_k(m,\beta)+\mathrm{e}^{-T\xi_k(m,\beta)}}\big\langle g(\cdot),e_k(\cdot)\big\rangle\right)e_k(x) \\ &+\sum_{k=1}^{\infty}\left(\mu_k\xi_k(m,\beta)\int_t^T\frac{\mathrm{e}^{(\zeta-t-T)\xi_k(m,\beta)}}{\alpha\ \xi_k(m,\beta)+\mathrm{e}^{-T\xi_k(m,\beta)}}\big\langle f(.,\zeta),e_k(\cdot)\big\rangle\ d\zeta\right)e_k(x) \\ &+\sum_{k=1}^{\infty}\left(\mu_k\frac{\mathrm{e}^{-T\xi_k(m,\beta)}}{\alpha\xi_k(m,\beta)+\mathrm{e}^{-T\xi_k(m,\beta)}}\big\langle f(.,t),e_k(\cdot)\big\rangle\right)e_k(x), \end{aligned} \tag{4.3}$$

and

$$u_{\alpha}(x,T)=\sum_{k=1}^{\infty}\left(\frac{\mathrm{e}^{-T\xi_k(m,\beta)}}{\alpha\ \xi_k(m,\beta)+\mathrm{e}^{-T\xi_k(m,\beta)}}\big\langle g(\cdot),e_k(\cdot)\big\rangle\right)\ e_k(x). \tag{4.4}$$

Theorem 4.1. *For any $0<t\le\zeta\le T$. Let $u_{\alpha}^1(x,t)$ be a regularized solution as (4.2) corresponding to the final data $g_1(x)$ and $f(x,t)$. Similarly, let $u_{\alpha}^2(x,t)$ be a regularized solution as (4.2) corresponding to the final data $g_2(x)$ and $f(x,t)$, then one has*

$$\big\|u_{\alpha}^1(\cdot,t)-u_{\alpha}^2(\cdot,t)\big\|_{L^2(\Omega)}\le\left(\frac{T}{\alpha\ln(\frac{T}{\alpha})}\right)\big\|g_1-g_2\big\|_{L^2(\Omega)}. \tag{4.5}$$

Proof. From (4.2), we get

$$\begin{aligned} u_{\alpha}^1(x,t)=&\sum_{k=1}^{\infty}\left(\frac{\mathrm{e}^{-t\xi_k(m,\beta)}}{\alpha\ \xi_k(m,\beta)+\mathrm{e}^{-T\xi_k(m,\beta)}}\big\langle g_1(\cdot),e_k(\cdot)\big\rangle\right)e_k(x) \\ &-\sum_{k=1}^{\infty}\left(\mu_k\int_t^T\frac{\mathrm{e}^{(\zeta-t-T)\xi_k(m,\beta)}}{\alpha\ \xi_k(m,\beta)+\mathrm{e}^{-T\xi_k(m,\beta)}}\big\langle f(.,\zeta),e_k(x)\big\rangle\ d\zeta\ \right)e_k(x), \end{aligned}$$

$$u_\alpha^2(x,t) = \sum_{k=1}^{\infty}\left(\frac{e^{-t\xi_k(m,\beta)}}{\alpha\ \xi_k(m,\beta)+e^{-T\xi_k(m,\beta)}}\big\langle g_2(\cdot), e_k(\cdot)\big\rangle\right)e_k(x)$$

$$-\sum_{k=1}^{\infty}\left(\mu_k\int_t^T \frac{e^{(\zeta-t-T)\xi_k(m,\beta)}}{\alpha\ \xi_k(m,\beta)+e^{-T\xi_k(m,\beta)}}\big\langle f(.,\zeta), e_k(x)\big\rangle\ d\zeta\ \right)e_k(x)\cdot \quad (4.6)$$

From (4.6), we get

$$\big\|u_\alpha^1(\cdot,t)-u_\alpha^2(\cdot,t)\big\|_{L^2(\Omega)}^2 \le \sum_{k=1}^{\infty}\left(\frac{e^{-t\xi_k(m,\beta)}}{\alpha\ \xi_k(m,\beta)+e^{-T\xi_k(m,\beta)}}\right)^2\big(g_{1,k}(\cdot)-g_{2,k}(\cdot)\big)^2. \quad (4.7)$$

From (4.7), using the Lemma 2.1, we get

$$\|\mathcal{V}_1\|_{L^2(\Omega)}^2 \le \left(\frac{T}{\alpha\ln(\frac{T}{\alpha})}\right)\sum_{k=1}^{\infty}\big|\big\langle g_{1,k}(\cdot)-g_{2,k}(\cdot), e_k(\cdot)\big\rangle\big|^2 \le \left(\frac{T}{\alpha\ln(\frac{T}{\alpha})}\right)\|g_1-g_2\|_{L^2(\Omega)}^2. \quad (4.8)$$

Hence, we can find that

$$\big\|u_\alpha^1(\cdot,t)-u_\alpha^2(\cdot,t)\big\|_{L^2(\Omega)} \le \left(\frac{T}{\alpha\ln(\frac{T}{\alpha})}\right)\big\|g_1-g_2\big\|_{L^2(\Omega)}. \quad (4.9)$$

The completes the proof of Theorem 4.1. □

Theorem 4.2. *For any $g(x)\in L^2(\Omega)$, we prove $u_\alpha(x,T)\to g(x)\in L^2(\Omega)$ when $\alpha\to 0$. Let $\varepsilon>0$, choose $\mathcal{N}$ such that $\sum_{k=\mathcal{N}+1}^{\infty}\big|\big\langle g(\cdot), e_k(\cdot)\big\rangle\big|^2 \le \frac{\varepsilon^2}{4}$, we receive*

$$\begin{aligned}\big\|u_\alpha(\cdot,T)-g(\cdot)\big\|_{L^2(\Omega)}^2 &\le 2\sum_{k=1}^{\mathcal{N}}\left(\frac{e^{-T\xi_k(m,\beta)}}{\alpha\xi_k(m,\beta)+e^{-T\xi_k(m,\beta)}}-1\right)^2\big|\big\langle g(\cdot), e_k(\cdot)\big\rangle\big|^2+\frac{\varepsilon^2}{2}\\ &\le 2\sum_{k=1}^{\mathcal{N}}\left(\frac{\alpha\xi_k(m,\beta)}{\alpha\xi_k(m,\beta)+e^{-T\xi_k(m,\beta)}}\right)^2\big|\big\langle g(\cdot), e_k(\cdot)\big\rangle\big|^2+\frac{\varepsilon^2}{2}\\ &\le 2\alpha^2\sum_{k=1}^{\mathcal{N}}\xi_k^2(m,\beta)e^{2T\xi_k(m,\beta)}\big|\big\langle g(\cdot), e_k(\cdot)\big\rangle\big|^2+\frac{\varepsilon^2}{2}\cdot \end{aligned} \quad (4.10)$$

By taking α such that $\alpha<\frac{\varepsilon}{2}\Big(m^{-2}\sum_{k=1}^{\mathcal{N}}e^{2Tm^{-1}\lambda_k^{\beta-1}}\lambda_k^{2\beta-2}\big|\big\langle g(\cdot), e_k(\cdot)\big\rangle\big|^2\Big)^{-1}$, we get $\big\|u_\alpha(\cdot,T)-g(\cdot)\big\|_{L^2(\Omega)}\le\varepsilon$ which the proof of the theorem is complete.

Theorem 4.3. *Let $g(\cdot)\in L^2(\Omega)$, and $f\in L^2(0,T;\mathcal{D}^1(\Omega))$. If the sequence $u_\alpha(x,0)$ converges in $L^2(\Omega)$, then u is a unique solution of the problem (1.1). Furthermore, we prove that $u_\alpha(x,t)$ converges to $u(x,t)$ as α tends to zero uniformly in t.*

Proof. Suppose that $\lim_{\alpha\to 0}u_\alpha(x,0)=u_0(x)$ exists. Let

$$u(x,t)=\sum_{k=1}^{\infty}\Big(e^{-t\xi_k(m,\beta)}u_{0,k}+\int_0^t e^{(\zeta-t)\xi_k(m,\beta)}\big\langle f(\cdot,\zeta), e_k(\cdot)\big\rangle d\zeta\Big)e_k(x),$$

where $u_{0,k} = \langle u(\cdot,0), e_k(\cdot)\rangle$. We can see that $u(x,t)$ satisfies (1.1), then $u_\alpha(x,t)$ has the formula as follows

$$u_\alpha(x,t)=\sum_{k=1}^{\infty}\Big(\mathrm{e}^{-t\xi_k(m,\beta)}u_{\alpha,0,k}+\int_0^t \frac{\mathrm{e}^{(\zeta-t-T)\xi_k(m,\beta)}}{\alpha\xi_k(m,\beta)+\mathrm{e}^{-T\xi_k(m,\beta)}}\langle f(.,\zeta), e_k(\cdot)\rangle(\zeta)d\zeta\Big)e_k(x),$$

where $u_{\alpha,0,k} = \langle u_\alpha(\cdot,0), e_k(\cdot)\rangle$. Using the inequality $(p+q)^2 \le 2(p^2+q^2)$, one has

$$\begin{aligned}
&\|u_\alpha(\cdot,t)-u(\cdot,t)\|^2_{L^2(\Omega)}\\
\le&\sum_{k=1}^{\infty}(u_{\alpha,0,k}-u_{0,k})^2+t^2\sum_{k=1}^{\infty}\Big(\int_0^t \frac{\mathrm{e}^{(2\zeta-2t)\xi_k(m,\beta)}\ \alpha^2\lambda_k^2}{(\alpha\xi_k(m,\beta)+\mathrm{e}^{-T\xi_k(m,\beta)})^2}\big|\langle f(\cdot,\zeta), e_k(\cdot)\rangle\big|^2 d\zeta\Big)\\
\le&\|u_{\alpha,0}(\cdot)-u_0(\cdot)\|^2_{L^2(\Omega)}+\frac{T^4}{\big(1+\ln(\frac{T}{\alpha})\big)^2}\int_0^T\sum_{k=1}^{\infty}\lambda_k^2\big|\langle f(\cdot,\zeta), e_k(\cdot)\rangle\big|^2 d\zeta\\
\le&\|u_{\alpha,0}(\cdot)-u_0(\cdot)\|^2_{L^2(\Omega)}+\frac{T^4}{\Big(1+\ln(\frac{T}{\alpha})\Big)^2}\|f\|^2_{L^\infty(0,T;\mathcal{D}^1(\Omega))}.
\end{aligned}$$

Hence, $\lim_{\alpha\to 0} u_\alpha(x,t) = u(x,t)$. We have $\lim_{\alpha\to 0} u_\alpha(x,T) = u(x,T)$. Applying the theorem 4.2, it gives $u(x,T) = g(x)$. Therefore, $u(x,t)$ is the unique solution of the problem (4.1). In addittion, we see that $u_\alpha(x,t)$ tends to $u(x,t)$ uniformly in t. □

Theorem 4.4. *Assume that* $\|g_\delta(\cdot)–g(\cdot)\|_{L^2(\Omega)}\le\delta$ *and* $\|f_\delta(\cdot,t)–f(\cdot,t)\|_{L^\infty(0,T;\mathbf{G}_{\beta,m}(\Omega))}\le\delta$, *then we can assert that*

$$\|u_\alpha^\delta(.,t)-u(.,t)\|_{L^2(\Omega)}\le\alpha^{\frac{t}{T}}\Big(\ln\big(\frac{T}{\alpha}\big)\Big)^{\frac{t}{T}-1}\Big(\sqrt{2\widetilde{T}}+\sqrt{2T}\widetilde{T}^{3/2}(m\lambda_1)^{-1}+\sqrt{\mathcal{Q}_3}\Big), \tag{4.11}$$

where $\mathcal{Q}_3$ *is defined in* (4.25).

Proof. Using the triangle inequality, we get

$$\|u_\alpha^\delta(.,t)-u(.,t)\|_{L^2(\Omega)}\le\underbrace{\|u_\alpha^\delta(.,t)-u_\alpha(.,t)\|_{L^2(\Omega)}}_{\text{Applying Lemma 4.1}}+\underbrace{\|u_\alpha(.,t)-u(.,t)\|_{L^2(\Omega)}}_{\text{Applying Lemma 4.2}}. \tag{4.12}$$

Lemma 4.1. *For any* $0<t\le\zeta\le T$, *and* $g, g_\delta\in L^2(\Omega)$, $f, f_\delta\in L^\infty(0,T;\mathbf{G}_{\beta,m}(\Omega))$ *such that*

$$\|g_\delta(\cdot)-g(\cdot)\|_{L^2(\Omega)}\le\delta,\quad \|f_\delta(\cdot,t)-f(\cdot,t)\|_{L^\infty(0,T;\mathbf{G}_{\beta,m}(\Omega))}\le\delta, \tag{4.13}$$

then we have

$$\|u_\alpha^\delta(.,t)-u_\alpha(.,t)\|_{L^2(\Omega)}\le\Big(\sqrt{Q}_1\|g_\delta-g\|_{L^2(\Omega)}+\sqrt{Q}_2\|f_\delta-f\|_{L^\infty(0,T;\mathbf{G}_{m,\beta}(\Omega))}\Big). \tag{4.14}$$

Proof. From (4.2), we proposed the regularized solution with g_δ, f_δ as follows

$$u_\alpha^\delta(x,t)=\sum_{k=1}^{\infty}\Big(\frac{\mathrm{e}^{-t\xi_k(m,\beta)}}{\alpha\ \xi_k(m,\beta)+\mathrm{e}^{-T\xi_k(m,\beta)}}\langle g_\delta(\cdot), e_k(\cdot)\rangle\Big)\ e_k(x)$$

$$-\sum_{k=1}^{\infty}\left(\mu_k\int_t^T\frac{e^{(\zeta-t-T)\xi_k(m,\beta)}}{\alpha\ \xi_k(m,\beta)+e^{-T\xi_k(m,\beta)}}\big\langle f_\delta(.,\zeta),e_k(\cdot)\big\rangle\ d\zeta\right)e_k(x)\cdot \quad (4.15)$$

From (4.2) and (4.15), it gives

$$\begin{aligned}&\big\|u^\delta_\alpha(.,t)-u_\alpha(.,t)\big\|^2_{L^2(\Omega)}\\ &\le \overbrace{2\sum_{k=1}^{\infty}\left(\frac{e^{-t\xi_k(m,\beta)}}{\alpha\xi_k(m,\beta)+e^{-T\xi_k(m,\beta)}}\big\langle g_\delta(\cdot)-g(\cdot),e_k(\cdot)\big\rangle\right)^2}^{\mathcal{K}_1^2}\\ &\quad+\overbrace{2\sum_{k=1}^{\infty}\left(\mu_k\int_t^T\frac{e^{(\zeta-t-T)\xi_k(m,\beta)}}{\alpha\ \xi_k(m,\beta)+e^{-T\xi_k(m,\beta)}}\big\langle f_\delta(.,\zeta)-f(.,\zeta),e_k(\cdot)\big\rangle d\zeta\right)^2}^{\mathcal{K}_2^2}\cdot \end{aligned}\quad (4.16)$$

Applying the Lemma 2.2, we can find that

$$\mathcal{K}_1^2\le 2\widetilde{T}^2\Big(\alpha\ln\big(\frac{T}{\alpha}\big)\Big)^{2\frac{t}{T}-2}\|g_\delta-g\|^2_{L^2(\Omega)}. \quad (4.17)$$

Next, we have $\mathcal{K}_2$ has the following estimates:

$$\begin{aligned}\mathcal{K}_2^2&\le 2\sum_{k=1}^{\infty}\left(\mu_k\frac{e^{-t\xi_k(m,\beta)}}{\alpha\xi_k(m,\beta)+e^{-T\xi_k(m,\beta)}}\int_t^T e^{(\zeta-T)\xi_k(m,\beta)}\big\langle f_\delta(.,\zeta)-f(.,\zeta),e_k(\cdot)\big\rangle d\zeta\right)^2\\ &\le 2(m\lambda_1)^{-2}\widetilde{T}^2\Big(\alpha\ln\big(\frac{T}{\alpha}\big)\Big)^{2\frac{t}{T}-2}\sum_{k=1}^{\infty}\left(\int_t^T e^{(\zeta-T)\xi_k(m,\beta)}\big\langle f_\delta(.,\zeta)-f(.,\zeta),e_k(\cdot)\big\rangle d\zeta\right)^2\\ &\le 2T(m\lambda_1)^{-2}\widetilde{T}^2\Big(\alpha\ln\big(\frac{T}{\alpha}\big)\Big)^{2\frac{t}{T}-2}\int_0^T\Big|\sum_{k=1}^{\infty}e^{\zeta\xi_k(m,\beta)}\big\langle f_\delta(.,\zeta)-f(.,\zeta),e_k(\cdot)\big\rangle\Big|^2 d\zeta\\ &\le 2T(m\lambda_1)^{-2}\widetilde{T}^2\Big(\alpha\ln\big(\frac{T}{\alpha}\big)\Big)^{2\frac{t}{T}-2}\\ &\quad\times\int_0^T\sum_{k=1}^{\infty}\sup_{0<\zeta<T}e^{2\zeta m^{-1}\lambda_k^{\beta-1}}\Big|\big\langle f_\delta(.,\zeta)-f(.,\zeta),e_k(\cdot)\big\rangle\Big|^2 d\zeta\\ &\le 2T(m\lambda_1)^{-2}\widetilde{T}^2\Big(\alpha\ln\big(\frac{T}{\alpha}\big)\Big)^{2\frac{t}{T}-2}\|f_\delta-f\|^2_{L^\infty(0,T;\mathbf{G}_{\beta,m}(\Omega))}.\end{aligned}\quad (4.18)$$

Combining (4.17) and (4.18), we conclude that

$$\big\|u^\delta_\alpha(.,t)-u_\alpha(.,t)\big\|_{L^2(\Omega)}\le\Big(\sqrt{Q}_1\big\|g_\delta-g\big\|_{L^2(\Omega)}+\sqrt{Q}_2\big\|f_\delta-f\big\|_{L^\infty(0,T;\mathbf{G}_{\beta,m}(\Omega))}\Big), \quad (4.19)$$

where $Q_1=2\widetilde{T}^2\Big(\alpha\ln\big(\frac{T}{\alpha}\big)\Big)^{2\frac{t}{T}-2}$ and $Q_2=2T(m\lambda_1)^{-2}\widetilde{T}^2\Big(\alpha\ln\big(\frac{T}{\alpha}\big)\Big)^{2\frac{t}{T}-2}$. □

Lemma 4.2. *Suppose that $g_\delta,g\in L^2(\Omega)$ such that $\|g_\delta-g\|_{L^2(\Omega)}\le\delta$, $u(x,t)$ be the exact solution same as* (3.11) *satisfying $u(.,0)\in\mathcal{D}^{\beta-1}(\Omega)$ (for any $\beta>1$), and $\mathcal{J}_1=\sum_{k=1}^{\infty}\lambda_k^{2\beta-4}\int_0^T\big|e^{\zeta\xi_k(m,\beta)}\big\langle f(\cdot,\zeta),e_k(\cdot)\big\rangle\big|^2 d\zeta$, then we have*

$$\|u_\alpha(.,t)-u(.,t)\|_{L^2(\Omega)}\le\alpha^{\frac{t}{T}}\Big(\ln\big(\frac{T}{\alpha}\big)\Big)^{\frac{t}{T}-1}\Big(2\widetilde{T}m^{-2}\|u(\cdot,0)\|^2_{\mathcal{D}^{\beta-1}(\Omega)}+2\widetilde{T}m^{-4}T\mathcal{J}_1\Big)^{1/2}. \quad (4.20)$$

Proof. From (3.11) and (4.2) we get

$$
\begin{aligned}
&u_\alpha(x,t)-u(x,t)\\
=&\sum_{k=1}^{\infty}\Big(\mathrm{e}^{T\xi_k(m,\beta)}-\frac{1}{\alpha\ \xi_k(m,\beta)+\mathrm{e}^{-T\xi_k(m,\beta)}}\Big)\mathrm{e}^{-t\xi_k(m,\beta)}\big\langle g(\cdot),e_k(\cdot)\big\rangle e_k(x)\\
&+\sum_{k=1}^{\infty}\Big(\mathrm{e}^{-t\xi_k(m,\beta)}-\frac{\mathrm{e}^{(-t-T)\xi_k(m,\beta)}}{\alpha\xi_k(m,\beta)+e^{-T\xi_k(m,\beta)}}\Big)\mu_k\\
&\times\int_t^T \mathrm{e}^{\zeta\xi_k(m,\beta)}\big\langle f(.,\zeta),e_k(\cdot)\big\rangle d\zeta\, e_k(x)\\
=&\sum_{k=1}^{\infty}\frac{\alpha\mathrm{e}^{-t\xi_k(m,\beta)}}{\alpha\ \xi_k(m,\beta)+\mathrm{e}^{-T\xi_k(m,\beta)}}\Big(\xi_k(m,\beta)\mathrm{e}^{T\xi_k(m,\beta)}\big\langle g(\cdot),e_k(\cdot)\big\rangle\\
&+\mu_k\xi_k(m,\beta)\int_t^T \mathrm{e}^{\zeta\xi_k(m,\beta)}\big\langle f(.,\zeta),e_k(\cdot)\big\rangle d\zeta\Big)e_k(x)\cdot
\end{aligned}
\tag{4.21}
$$

From (3.11) at $t=0$, combining some basic transformation, we get

$$
\begin{aligned}
&\xi_k(m,\beta)\mathrm{e}^{T\xi_k(m,\beta)}\big\langle g(x),e_k(x)\big\rangle+\mu_k\xi_k(m,\beta)\int_t^T \mathrm{e}^{\zeta\xi_k(m,\beta)}\big\langle f(.,\zeta),e_k(\cdot)\big\rangle d\zeta\\
=&\xi_k(m,\beta)u(\cdot,0)-\mu_k\int_0^t \xi_k(m,\beta)\mathrm{e}^{\zeta\xi_k(m,\beta)}\big\langle f(.,\zeta),e_k(\cdot)\big\rangle d\zeta.
\end{aligned}
\tag{4.22}
$$

Therefore, we receive

$$
\begin{aligned}
&\|u_\alpha(\cdot,t)-u(\cdot,t)\|^2_{L^2(\Omega)}\\
\le&2\sum_{k=1}^{\infty}\left|\frac{\alpha\mathrm{e}^{-t\xi_k(m,\beta)}}{\alpha\ \xi_k(m,\beta)+\mathrm{e}^{-T\xi_k(m,\beta)}}\right|^2\\
&\times\left|\xi_k(m,\beta)u(\cdot,0)+\mu_k\int_0^t \xi_k(m,\beta)\mathrm{e}^{\zeta\xi_k(m,\beta)}\big\langle f(.,\zeta),e_k(\cdot)\big\rangle d\zeta\right|^2\\
\le&2\sum_{k=1}^{\infty}\left|\frac{\alpha\mathrm{e}^{-t\xi_k(m,\beta)}}{\alpha\ \xi_k(m,\beta)+\mathrm{e}^{-T\xi_k(m,\beta)}}\right|^2\left|m^{-1}\lambda_k^{\beta-1}u(\cdot,0)\right|^2\\
&+2\sum_{k=1}^{\infty}\left|\frac{\alpha\mathrm{e}^{-t\xi_k(m,\beta)}}{\alpha\ \xi_k(m,\beta)+\mathrm{e}^{-T\xi_k(m,\beta)}}\right|^2\\
&\times\left|m^{-4}\lambda_k^{2\beta-4}T\int_0^T\left|\mathrm{e}^{\zeta\xi_k(m,\beta)}\big\langle f(.,\zeta),e_k(\cdot)\big\rangle\right|^2 d\zeta\right|\\
\le&2m^{-2}\widetilde{T}^2\alpha^{2\frac{t}{T}}\Big(\ln(\frac{T}{\alpha})\Big)^{2\frac{t}{T}-2}\sum_{k=1}^{\infty}\left|\lambda_k^{\beta-1}u(\cdot,0)\right|^2\\
&+2m^{-4}T\widetilde{T}^2\alpha^{2\frac{t}{T}}\Big(\ln(\frac{T}{\alpha})\Big)^{2\frac{t}{T}-2}\sum_{k=1}^{\infty}\lambda_k^{2\beta-4}\int_0^T\left|\mathrm{e}^{\zeta\xi_k(m,\beta)}\big\langle f(.,\zeta),e_k(\cdot)\big\rangle\right|^2 d\zeta\\
\le&2\widetilde{T}^2\alpha^{2\frac{t}{T}}\Big(\ln\big(\frac{T}{\alpha}\big)\Big)^{2\frac{t}{T}-2}\Big(m^{-2}\|u(\cdot,0)\|^2_{\mathcal{D}^{\beta-1}(\Omega)}+m^{-4}T\mathcal{J}_1\Big),
\end{aligned}
\tag{4.23}
$$

whereby $\mathcal{J}_1=\sum\limits_{k=1}^{\infty}\lambda_k^{2\beta-4}\int_0^T\left|\mathrm{e}^{\zeta\xi_k(m,\beta)}\big\langle f(\cdot,\zeta),e_k(\cdot)\big\rangle\right|^2 d\zeta.$ □

Combining (4.19) and (4.23), by choosing $\alpha = \delta$, we can conclude that

$$\begin{aligned}\left\|u_\alpha^\delta(.,t) - u(.,t)\right\|_{L^2(\Omega)} &\le \left\|u_\alpha^\delta(.,t) - u_\alpha(.,t)\right\|_{L^2(\Omega)} + \left\|u_\alpha(.,t) - u(.,t)\right\|_{L^2(\Omega)} \\ &\le \alpha^{\frac{t}{T}}\Big(\ln\big(\frac{T}{\alpha}\big)\Big)^{\frac{t}{T}-1}\Big(\sqrt{2\widetilde{T}} + \sqrt{2T}\widetilde{T}^{3/2}(m\lambda_1)^{-1} + \sqrt{\mathcal{Q}_3}\Big),\end{aligned} \tag{4.24}$$

where

$$\mathcal{Q}_3 = 2\widetilde{T}\Big(m^{-2}\|u(\cdot,0)\|^2_{\mathcal{D}^{\beta-1}(\Omega)} + m^{-4}T\mathcal{J}_1\Big). \tag{4.25}$$

□

5. Numerical simulation

In this section, we show a numerical example to verify the usefulness of the proposed method. The numerical example is constructed in the final value problem with the linear source function with the regularization result base on the modified quasi-boundary value method. After that, we also propose a convergence estimate between the exact and regularized solutions. Before presenting the results, we install some numerical methods to assist in the examples as follows. Firstly, throughout this section we choose $\Omega = (0,\pi),\ T = 1$. The discrete form of our problem is as follows: We divide the domain $(0,\pi)\times(0,1)$ into N_x and N_t sub-intervals of equal length $\triangle_x$ and $\triangle_t$, where $\triangle_x = \dfrac{\pi}{N_x}$ and $\triangle_t = \dfrac{1}{N_t}$, respectively ($M$ and N are the numbers of partitions on the x-axis and t-axis). Then we denote by $f_i^k = f(x_i, t_k)$, where $x_i = (i-1)\triangle_x$ and $t_j = (j-1)\triangle_t$ for $i = 1,2,...,N_x+1;\ j = 1,2,...,N_t+1$. Next, to calculate the integrals, we use the Simpson approximation method as follows

$$\int_a^b f(\zeta)\,d\zeta \approx \frac{\varepsilon_n}{3}f(\zeta_0) + \frac{2\varepsilon_n}{3}\sum_{j=1}^{n/2-1} f(\zeta_{2j}) + \frac{4\varepsilon_n}{3}\sum_{j=1}^{n/2} f(\zeta_{2j-1}) + \frac{\varepsilon_n}{3}f(\zeta_n), \tag{5.1}$$

where $\varepsilon_n = \dfrac{b-a}{n}$. The Matlab code used to calculate this approximation Simpson's Rule Integration was written by Juan Camilo Medina (2020). In addition, we have the orthogonal basis in $L^2(0,\pi)$ is $e_k(x) = \sqrt{\dfrac{2}{\pi}}\sin(kx)$ and the eigenvalues $\lambda_k = k^2,\ k = 1,2,...$, the scalar product of f and g in $L^2(0,\pi)$ is given by

$$\langle f,h\rangle_{L^2(0,\pi)} = \int_0^\pi fh\,dx = \texttt{simpsons(f*h,0,pi,n)}.$$

In this section, we choose functions as follows:

$$g(x) = \big(\sin(x) + \sin(\frac{x}{2})\big),\quad f(x,t) = \big(\sin(x) + \sin(2x)\big)\exp(-t^{2\beta}). \tag{5.2}$$

Next, we consider the problem (1.1) with the noisy model

$$\|g_\delta(\cdot) - g(\cdot)\|_{L^2(0,\pi)} \le \delta,\quad \|f_\delta(\cdot,t) - f(\cdot,t)\|_{L^\infty(0,1;(0,\pi))} \le \delta. \tag{5.3}$$

The couple of (g_δ, f_δ) which are measured data with the following model random noise

$$\delta\text{rand} = \delta(2\text{rand}(\cdot) - 1), \quad g_\delta = g + \delta\text{rand}, \quad f_\delta = f\big(1 + \delta\text{rand}\big). \tag{5.4}$$

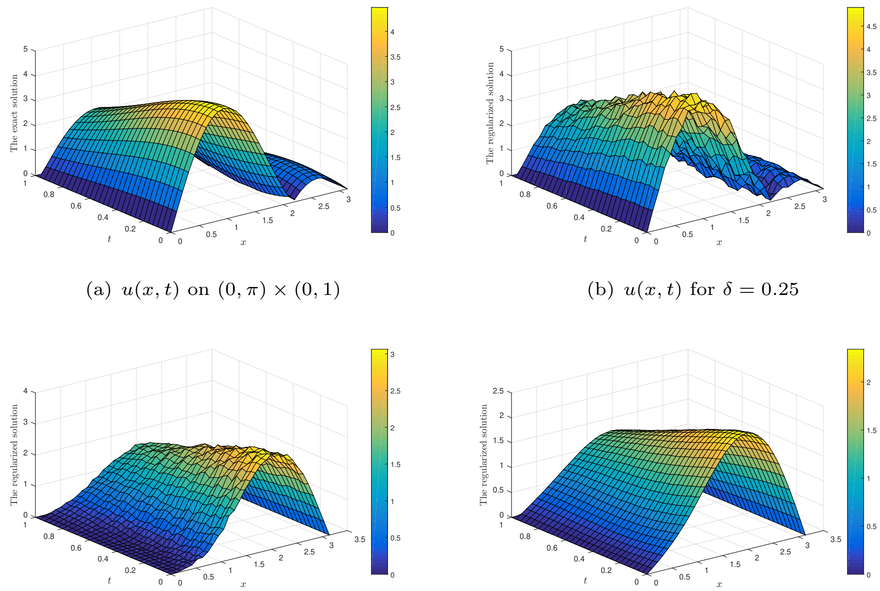
Then we have the regularized solution in the following form

$$\begin{aligned}
&u_\alpha^\delta(x,t)\\
=&\frac{2}{\pi}\sum_{k=1}^{N_k}\left(\frac{\mathrm{e}^{-t\frac{k^{2\beta}}{1+mk^2}}}{\alpha(\delta)\,\frac{k^{2\beta}}{1+mk^2}+\mathrm{e}^{-T\frac{k^{2\beta}}{1+mk^2}}}\int_0^\pi g_\delta(x)\sin(kx)dx\right)\sin(kx)\\
&-\frac{2}{\pi}\sum_{k=1}^{N_k}\left(\frac{1}{1+mk^2}\int_t^1\frac{\mathrm{e}^{(\zeta-t-T)\frac{k^{2\beta}}{1+mk^2}}}{\alpha(\delta)\,\frac{k^{2\beta}}{1+mk^2}+\mathrm{e}^{-T\frac{k^{2\beta}}{1+mk^2}}}\int_0^\pi f_\delta(x,\zeta)\sin(kx)dxd\zeta\right)\\
&\times\sin(kx).
\end{aligned} \tag{5.5}$$

We also calculate the root mean square error $E^\alpha(t)$ to analyze the error between the numerically obtained solution $u_\alpha^\delta(x,t)$ and the exact solution $u(x,t)$ and as follows:

$$E^\alpha(t) = \frac{\sqrt{\sum_{i=1}^{N_x+1}\left|u_\alpha^\delta(x_i,t) - u(x_i,t)\right|^2}}{\sqrt{\sum_{i=1}^{N_x+1}\left|u(x_i,t)\right|^2}}. \tag{5.6}$$

While implementing these numerical examples using the Matlab program, by choosing $m = 0.8$, $\beta = 1.5$, because of in this case, the problem (1.1) os ill-posed. Since the above tables and figures, we show some results as follows. In Figure 1, we show the comparison the convergent estimate between exact solution and the regularized solution in the case 3D. In Figure 2, we give the information for convergent estimate between exact solution and its approximation by quasi boundary value method and the corresponding errors with $\alpha(\delta) = \delta = 0.25$, $\alpha(\delta) = \delta = 0.1$ and $\alpha(\delta) = \delta = 0.01$, respectively. In Figure 3, we give information the convergent estimate between exact solution and its approximation by quasi boundary value method and the corresponding errors with $\alpha(\delta) = \delta = 0.25$, $\alpha(\delta) = \delta = 0.1$ and $\alpha(\delta) = \delta = 0.01$, respectively. In Table 1, it shows the estimation between the exact solution and regularized solution by quasi boundary value in two case $t = 0.5$ and $t = 0.7$. Finally, in Table 2, in the case fixed $m = 1$ and $\beta = 1$, with different $\alpha = \delta$ tends to zero. It is clear that, in Tables 1, 2, the convergence level of the method used in the theory is still equivalent. Therefore, we infer that when $\alpha(\delta)$ goes to zero, the regularized solution tends to the exact solution pretty good.



(a) $u(x,t)$ on $(0,\pi)\times(0,1)$

(b) $u(x,t)$ for $\delta = 0.25$

(c) $u(x,t)$ for $\delta = 0.1$

(d) $u(x,t)$ for $\delta = 0.01$

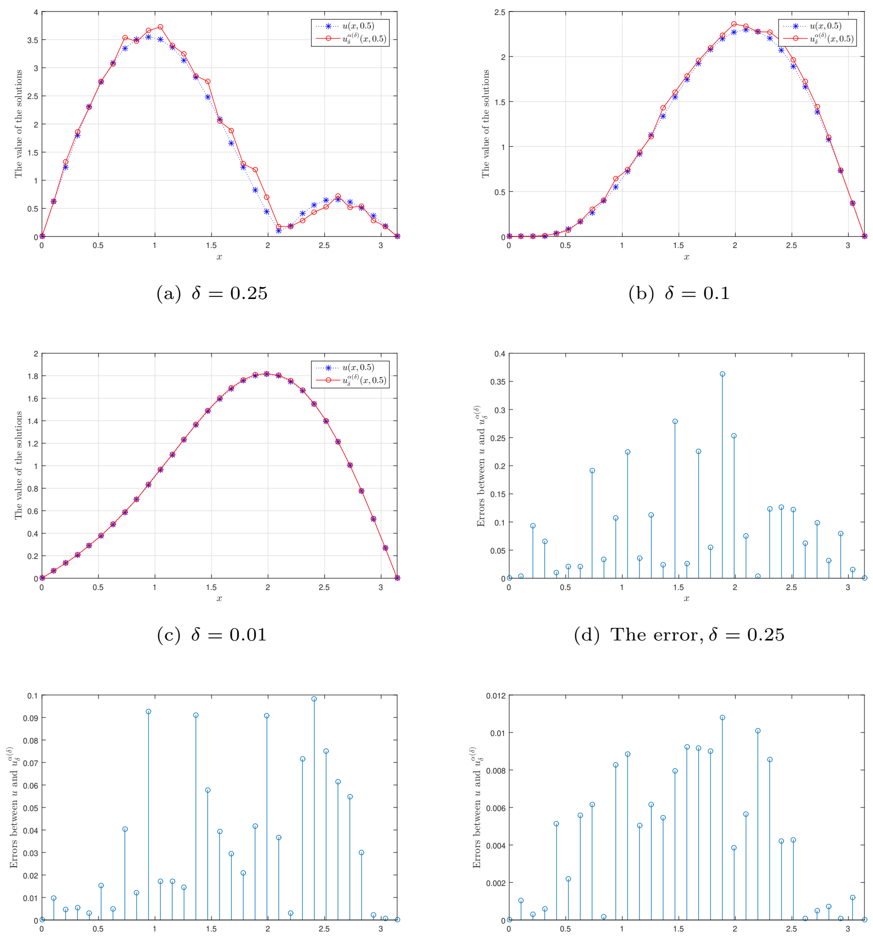
Figure 1. A comparison the regularity by time variable for $m = 0.8, \beta = 1.5$.

Table 1. The error estimation according to the regularity by time variable for $t \in \{0.5, 0.7\}$ and $\alpha = \delta = 0.25, \alpha = \delta = 0.1$ and $\alpha = \delta = 0.01$.

t	$N_x = N_t = 30, N_k = 2, m = 0.8, \beta = 1.5$		
	$E^{0.25}(t)$	$E^{0.1}(t)$	$E^{0.01}(t)$
0.5	0.023475752	0.0082489906	1.04E-03
0.7	0.026470059	0.0107564740	8.07E-04

Table 2. The error estimation according to the regularity by time variable for $t \in \{0.22, 0.44, 0.66, 0.88\}$.

t	$N_x = N_t = 30, N_k = 2, m = 1, \beta = 1$		
	$E^{0.5}(t)$	$E^{0.05}(t)$	$E^{0.005}(t)$
0.22	0.058813211	0.004610596	5.37E-04
0.44	0.055182770	0.004464875	4.26E-04
0.66	0.059215408	0.035225510	3.75E-04
0.88	0.065312409	0.003959612	3.64E-04



(a) $\delta = 0.25$ (b) $\delta = 0.1$

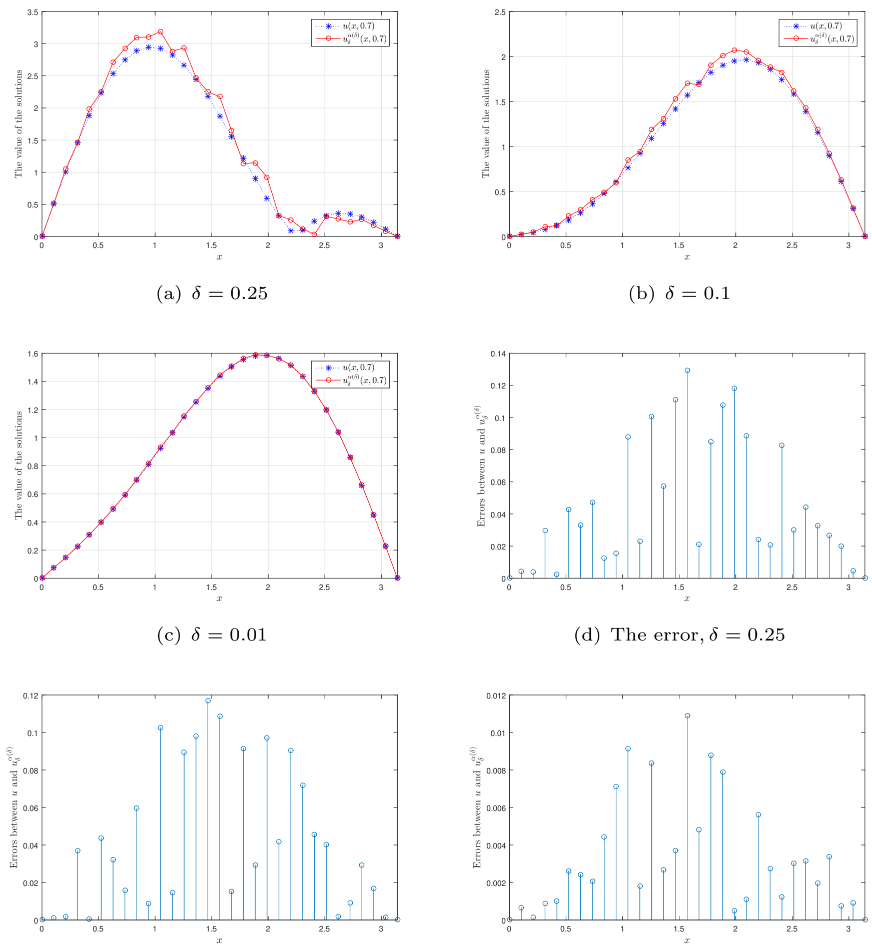
(c) $\delta = 0.01$ (d) The error, $\delta = 0.25$

(e) The error, $\delta = 0.1$ (f) The error, $\delta = 0.01$

Figure 2. A comparison the regularity by time variable for $t = 0.5,\ \alpha = \delta = 0.25, \alpha = \delta = 0.1$ and $\alpha = \delta = 0.01$, at $\beta = 1.5,\ m = 0.8$.

6. Conclusions

In this work, we use the quasi–boundary value method to regularize the fractional pseudo-parabolic equation. We prove that this problem is ill-posed in the sense of Hadamard through on an example and we also carry out the regularity of solution and derivative by time of solution. After that, we proposed the quasi boundary value method to regularize this problem, it gives the information about the convergent estimate between the regularized solution and the exact solution. In addition, by giving the numerical example, we shown that the proposed regularized method is effective.



(a) $\delta = 0.25$

(b) $\delta = 0.1$

(c) $\delta = 0.01$

(d) The error, $\delta = 0.25$

(e) The error, $\delta = 0.1$

(f) The error, $\delta = 0.01$

Figure 3. A comparison the regularity by time variable for $t = 0.7,\ \alpha = \delta = 0.25, \alpha = \delta = 0.1$ and $\alpha = \delta = 0.01$, at $\beta = 1.5,\ m = 0.8$.

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