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SHARP ASYMPTOTIC RESULTS FOR THIRD-ORDER LINEAR DELAY DIFFERENTIAL EQUATIONS

John R. Graef[1,†], Irena Jadlovská[2], and Ercan Tunç[3]

Abstract In the paper, the authors propose an effective Kneser-type oscillation test for Property A for linear third-order delay differential equations that ensures that any nonoscillatory solution converges to zero asymptotically. The result is sharp when applied to Euler-type delay differential equation and improves all existing results reported in the literature.

Keywords Third-order differential equation, delay, almost oscillation, oscillation, Property A.

MSC(2010) 34C10, 34K11.

1. Introduction

Consider the third-order linear delay differential equation

$$y'''(t) + q(t)y(\tau(t)) = 0, \quad t \geq t_0 > 0, \tag{1.1}$$

where $q(t) \in C([t_0, \infty))$ is nonnegative and does not eventually vanish identically, and the delay function $\tau(t) \in C([t_0, \infty))$ satisfies $\tau(t) \leq t$ and $\tau(t) \to \infty$ as $t \to \infty$.

By a *solution* of (1.1) we mean a three times differentiable real-valued function y satisfying (1.1) for all large t. We restrict our attention to those solutions of (1.1) that satisfy the condition $\sup\{|y(t)| : T \leq t < \infty\} > 0$ for any large $T \geq t_0$, and we tacitly assume that equation (1.1) possesses such solutions.

As is customary, a nontrivial solution of (1.1) is termed *oscillatory* or *nonoscillatory* according to whether it does or does not have infinitely many zeros. Equation (1.1) is is called oscillatory if all its solutions are oscillatory.

The investigation of qualitative properties of (1.1) is important for applications since such equations are considered as valuable tools in the modelling of many phenomena in different areas of applied mathematics and physics (see [16]). In particular, oscillation theory of third-order differential equations with variable coefficients has attracted a great deal of attention over the last three decades as is evidenced by the extensive research in the area.

†The corresponding author. Email: John-Graef@utc.edu(J. R. Graef)
1Department of Mathematics, University of Tennessee at Chattanooga, Chattanooga, TN 37403, USA
2Department of Mathematics and Theoretical Informatics, Faculty of Electrical Engineering and Informatics, Technical University of Kosice, Letna 9, 042 00 Kosice, Slovakia and Mathematical Institute, Slovak Academy of Sciences, Gresakova 6, 040 01 Kosice, Slovakia
3Department of Mathematics, Faculty of Arts and Sciences, Tokat Gaziosmanpaşa University, 60240, Tokat, Turkey

It follows from a classical result of Kiguradze [21, Lemma 1.1] that the set $\mathcal{N}$ of all positive nonoscillatory solutions of (1.1) can be divided into the following two classes:

$$\mathcal{N}_0 = \{y(t) : (\exists t_1 \geq t_0)(\forall t \geq t_1)(y(t) > 0,\, y'(t) < 0,\, y''(t) > 0)\}\,,$$
$$\mathcal{N}_2 = \{y(t) : (\exists t_1 \geq t_0)(\forall t \geq t_1)(y(t) > 0,\, y'(t) > 0,\, y''(t) > 0)\}\,.$$

Solutions belonging to the class $\mathcal{N}_0$ are called Kneser solutions. It is known that in the absence of a delay in (1.1), i.e., if $\tau(t) = t$, the class $\mathcal{N}_0$ is always nonempty (see, e.g., [18]). Therefore, results for third-order equations have been often accomplished by introducing the concept of the so-called Property A. We say that equation (1.1) has *Property A* if any solution y of (1.1) is either oscillatory or is a Kneser type solution tending to zero as $t \to \infty$ (see [21]).

To name the situation where $\mathcal{N} = \mathcal{N}_0$, we will say equation (1.1) is *almost oscillatory*. On the other hand, if the class $\mathcal{N}_2$ is nonempty, we will say that (1.1) is nonoscillatory.

Comparison principles have been especially powerful tools in oscillation theory since Sturm's initial contribution to the subject. Their underlying feature is to deduce the oscillatory properties of the given equation from those of a simpler one whose oscillatory behavior is already known. Euler-type differential equations and their various generalizations often serve as suitable comparison equations. Perhaps the most familiar situation is the one for the second-order linear Euler equation

$$y''(t) + \frac{q_0}{t^2} y(t) = 0, \quad q_0 > 0, \tag{1.2}$$

which is oscillatory if and only if

$$q_0 > \max\{-x(x-1) : 0 < x < 1\} = \frac{1}{4}.$$

In 1893, A. Kneser [22] was the first to use Sturmian comparison methods and equation (1.2) to show that the linear equation

$$y''(t) + q(t)y(t) = 0 \tag{1.3}$$

is oscillatory if

$$\liminf_{t\to\infty} t^2 q(t) > \frac{1}{4}$$

and nonoscillatory if

$$\limsup_{t\to\infty} t^2 q(t) < \frac{1}{4}.$$

An extension of Kneser's oscillation criterion to the third-order ordinary differential equation (which is the special case of (1.1) with $\tau(t) = t$)

$$y'''(t) + q(t)y(t) = 0 \tag{1.4}$$

was given by M. Hanan in 1961 [17, Theorem 5.7] and essentially kindled the current interest in investigating the oscillatory and asymptotic behavior of third-order differential equations.

Using the third-order Euler equation

$$y'''(t) + \frac{q_0}{t^3} y(t) = 0, \quad q_0 > 0, \tag{1.5}$$

for comparison purposes, which is almost oscillatory if and only if

$$q_0 > \max\{-x(x-1)(x-2) : 1 < x < 2\} = \frac{2}{3\sqrt{3}},$$

Hanan showed that (1.4) is almost oscillatory if

$$\liminf_{t\to\infty} t^3 q(t) > \frac{2}{3\sqrt{3}} \tag{1.6a}$$

and nonoscillatory (in the sense that all its solutions are nonoscillatory) if

$$\limsup_{t\to\infty} t^3 q(t) < \frac{2}{3\sqrt{3}}. \tag{1.6b}$$

The constant $2/(3\sqrt{3})$ corresponding to $1/4$ for the second-order equation (1.3) is the best possible for (1.4) in the sense that all solutions of (1.5) are nonoscillatory if $q_0 \le 2/(3\sqrt{3})$.

A natural question that arises is how to extend Hanan's Kneser-type criterion (1.6) from the ordinary equation (1.4) to the delay equation (1.1). To test the strength of the results, we shall use the delay Euler-type equation

$$y'''(t) + \frac{q_0}{k^2 t^3} y(kt) = 0, \quad q_0 > 0, \quad k \in (0,1]. \tag{1.7}$$

Applying Mahfoud's comparison theorem [24, Theorem 1], we see that (1.7) is almost oscillatory if

$$q_0 > \frac{2}{3\sqrt{3}}. \tag{1.8}$$

For nonoscillation, we use the trial solution $y(t) = t^x$, $x \in (1,2)$ to see that unbounded solutions $y \in \mathcal{N}_2$ of (1.7) exist if x satisfies the indicial equation

$$f(x) := -x(x-1)(x-2)k^{2-x} = q_0, \tag{1.9}$$

that is, if

$$q_0 \le \max\{f(x) : 1 < x < 2\} =: m_2. \tag{1.10}$$

Clearly, there is the gap $(m_2, 2/(3\sqrt{3})]$ between the almost oscillation and nonoscillation of (1.7) in case $k < 1$. To the best of our knowledge, this gap cannot be completely filled by any existing results for (1.1) obtained by different techniques; see the papers [1–6, 8–11, 13–15, 23, 26, 28] and the references cited therein or the recent monographs of Padhi and Pati [25] and Saker [27] for extensive bibliographies on the subject.

The purpose of the paper is to establish an efficient criterion for detecting Property A for equation (1.1) that is sharp in the sense that it gives a necessary and sufficient condition for the delay Euler equation (1.7) to be almost oscillatory (or, more precisely, to have Property A). Our motivation comes from the recent papers [12, 19, 20], where a similar technique lead to obtaining sharp oscillation results for second-order half-linear differential equations with deviating arguments. A major advantage of this technique is its potential for investigating more general nonlinear differential equations.

2. Main results

The main result in this paper is the following.

Theorem 2.1. *Let*

$$\lambda_* := \liminf_{t\to\infty} \frac{t}{\tau(t)}.$$

If

$$\liminf_{t\to\infty} \tau^2(t)tq(t) > \begin{cases} 0, & \text{for } \lambda_* = \infty, \\ M_2, & \text{for } \lambda_* < \infty, \end{cases} \tag{2.1}$$

where M_2 is defined by

$$M_2 := \max\left\{-x(x-1)(x-2)\lambda_*^{x-2} : 1 < x < 2\right\}, \tag{2.2}$$

then equation (1.1) *has Property A.*

Note that for $\tau(t) = t$, Theorem 2.1 reduces to the oscillation part of Hanan's result [17, Theorem 5.7] (see (1.6)). Moreover, as an immediate consequence of Theorem 2.1 and the nonoscillation condition (1.10), we obtain the following sharp result for the delay Euler equation (1.7).

Corollary 2.1. *The Euler equation* (1.7) *has Property A if and only if*

$$q_0 > M_2 = m_2. \tag{2.3}$$

Remark 2.1. The particular case $\lambda_* = \infty$ applies, for example, if the function τ is of the form $\tau(t) = t^k$, $k \in (0,1)$. As an illustration of Theorem 2.1 in this special case, we can conclude that the equation

$$y'''(t) + \frac{q_0}{t^{1+2/k}} y\left(t^k\right) = 0, \quad q_0 > 0, \quad 0 < k < 1, \tag{2.4}$$

has Property A for any q_0.

Remark 2.2. In many results for third-order delay differential equations including (1.1) as a particular case, the authors require that the delay function $\tau(t)$ is strictly increasing. This is not needed in Theorem 2.1.

Remark 2.3. In general, results in the literature consist of two independent conditions, the first one ensuring that any Kneser solution tends to zero, and the second one for the nonexistence of solutions in the class $\mathcal{N}_2$. To the contrary, our Theorem 2.1 is a single-condition criterion guaranteeing the same property.

Open problem. It is known that the delay argument $\tau(t)$ can cause the oscillation of all solutions of (1.1), that is, $\mathcal{N}_0 = \mathcal{N}_2 = \emptyset$. In a recent work [7], the authors show that the delay Euler equation (1.7) is oscillatory if and only if

$$q_0 > \max\{m_0, m_2\},$$

where m_0 and m_2 are local maxima of $f(x)$ defined by (1.9) on $(-\infty, 0)$ and $(1,2)$, respectively. In view of this, the open problem is to show whether

$$\liminf_{t\to\infty} \tau^2(t)tq(t) > \max\{M_0, M_2\},$$

where $\lambda_* < \infty$ and

$$M_0 := \max\left\{-x(x-1)(x-2)\lambda_*^{x-2} : -\infty < x < 0\right\}.$$

is sufficient or not for the oscillation of (1.1).

3. Auxiliary results and proof of the main theorem

We assume that all functional inequalities hold eventually, that is, they are satisfied for all t large enough. As usual, and without loss of generality, we can assume from now on that nonoscillatory solutions of (1.1) are eventually positive.

In the sequel, we will make use of the constants

$$\beta_* := \liminf_{t\to\infty} \frac{\tau^2(t)tq(t)}{2} \quad \text{and} \quad \lambda_* := \liminf_{t\to\infty} \frac{t}{\tau(t)}.$$

The following lemmas require the positivity of β_* as does Theorem 2.1. Clearly, for any $\beta \in (0, \beta_*)$ and $\lambda \in (1, \lambda_*)$ for $\tau(t) < t$, and $\lambda = \lambda_*$ for $\tau(t) = t$, there is a $t_1 \ge t_0$ large enough such that

$$\frac{\tau^2(t)tq(t)}{2} \ge \beta \quad \text{and} \quad \frac{t}{\tau(t)} \ge \lambda, \quad t \ge t_1. \tag{3.1}$$

We will make use of these facts in our proofs.

Lemma 3.1. *Assume that $\beta_* > 0$. If y is a solution of* (1.1) *belonging to $\mathcal{N}_0$, then* $\lim_{t\to\infty} y(t) = 0$.

Proof. Let $y(t) \in \mathcal{N}_0$ and choose $t_1 \ge t_0$ so that $y(\tau(t)) > 0$ on $[t_1, \infty)$. Clearly, there exists a finite number ℓ such that $\lim_{t\to\infty} y(t) = \ell \ge 0$. Assume that $\ell > 0$. Then there exists $t_2 \ge t_1$ such that $y(\tau(t)) \ge \ell$ for $t \ge t_2$. Using this and (3.1) in (1.1), we see that

$$-y'''(t) = q(t)y(\tau(t)) \ge \frac{2\beta}{\tau^2(t)t} y(\tau(t)) \ge \frac{2\beta\ell}{\tau^2(t)t} \ge \frac{2\beta\ell}{t^3}\frac{t^2}{\tau^2(t)} \ge \frac{2\beta\ell\lambda^2}{t^3}.$$

Integrating the above inequality twice from t to ∞, we have

$$-y'(t) \ge \frac{\beta\ell\lambda^2}{t}. \tag{3.2}$$

Integrating (3.2) from t_2 to t, we arrive at

$$y(t_2) \ge y(t) + \beta\ell\lambda^2 \ln\frac{t}{t_2} \to \infty \quad \text{as } t \to \infty,$$

which contradicts the fact that y is bounded. Hence $\ell = 0$ and this proves the lemma. □

In our next lemma, we prove some basic but important properties of solutions belonging to the class $\mathcal{N}_2$.

Lemma 3.2. *Assume that $\beta_* > 0$ and let y be a solution* (1.1) *belonging to $\mathcal{N}_2$. Then for t sufficiently large:*

(i) $\lim_{t\to\infty} y(t) = \lim_{t\to\infty} y'(t) = \infty$*;*

(ii) $\lim_{t\to\infty} y''(t) = \lim_{t\to\infty} y'(t)/t = \lim_{t\to\infty} y(t)/t^2 = 0$*;*

(iii) *$y'(t)/t$ is decreasing and $y'(t) > ty''(t)$;*

(iv) *$y(t)/t^2$ is decreasing and $y(t) > ty'(t)/2$.*

Proof. Let $y \in \mathcal{N}_2$ and choose $t_1 \geq t_0$ such that $y(\tau(t)) > 0$ for $t \geq t_1$.

(i) First, note that $\beta_* > 0$ implies

$$\int_{t_0}^{\infty} \tau^2(s)q(s)\mathrm{d}s = \infty \tag{3.3}$$

since

$$\int_{t_1}^{t} \tau^2(s)q(s)\mathrm{d}s \geq 2\beta \int_{t_1}^{t} \frac{\mathrm{d}s}{s} = 2\beta \ln \frac{t}{t_1} \to \infty \quad \text{as } t \to \infty.$$

Now y' is increasing and positive, so $y'(t) \geq y'(t_1) =: \ell$ for $t \geq t_1$. Integrating from t_1 to t, we obtain

$$y(t) \geq \ell(t - t_1). \tag{3.4}$$

Letting t to ∞, it is obvious that $y \to \infty$. Employing (3.4) in (1.1) yields

$$-y'''(t) \geq \ell q(t)(\tau(t) - t_1) \tag{3.5}$$

and integrating (3.5) from t to ∞, we obtain

$$y''(t) \geq \ell \int_{t}^{\infty} q(s)(\tau(s) - t_1)\mathrm{d}s.$$

Integrating the last inequatity from t_1 to t, interchanging the order of integration, and using (3.3) in resulting inequality, we arrive at

$$\begin{aligned} y'(t) &\geq \ell \int_{t_1}^{t} \int_{u}^{\infty} q(s)(\tau(s) - t_1)\mathrm{d}s\mathrm{d}u = \int_{t_1}^{t} q(s)(\tau(s) - t_1)(s - t_1)\mathrm{d}s \\ &\geq \int_{t_1}^{t} q(s)(\tau(s) - t_1)^2\mathrm{d}s \to \infty \quad \text{as} \quad t \to \infty \end{aligned}$$

which finishes the proof of (i).

(ii) Since y'' is a positive decreasing function,

$$\lim_{t \to \infty} y''(t) = \ell \geq 0.$$

If $\ell > 0$, then $y''(t) \geq \ell > 0$ and so $y(t) \geq \ell(t - t_1)^2/2$. Using this in (1.1), we have

$$y'''(t) \geq q(t)y(\tau(t)) \geq \frac{\ell}{2}q(t)(\tau(t) - t_1)^2. \tag{3.6}$$

Integrating from t_1 to t gives

$$y''(t) \geq \frac{\ell}{2} \int_{t_1}^{t} q(s)(\tau(s) - t_1)^2\mathrm{d}s \to \infty \quad \text{as} \quad t \to \infty,$$

which is a contradiction. Hence, $\ell = 0$. Applying l'Hôspital's rule and using (i), we see that (ii) holds.

(iii) Again using the fact that y'' is positive and decreasing, it follows that

$$y'(t) = y'(t_1) + \int_{t_1}^{t} y''(s)\mathrm{d}s \geq y'(t_1) + y''(t)(t - t_1).$$

In view of (ii), there is a $t_2 > t_1$ such that $y'(t_1) > y''(t)t_1$ for $t \geq t_2$. Thus,

$$y'(t) > ty''(t), \quad t \geq t_2,$$

and consequently,

$$\left(\frac{y'(t)}{t}\right)' = \frac{y''(t)t - y'(t)}{t^2} < 0,$$

which proves (iii).

(iv) In view of the fact that $y'(t)/t$ is a decreasing function tending to zero, we see that

$$\begin{aligned} y(t) &= y(t_1) + \int_{t_1}^{t} \frac{y'(s)}{s} s\mathrm{d}s \geq y(t_1) + \frac{y'(t)}{t}\left(\frac{t^2}{2} - \frac{t_1^2}{2}\right) \\ &= \frac{y'(t)t}{2} + y(t_1) - \frac{y'(t)t_1^2}{2t} > \frac{y'(t)t}{2} \end{aligned}$$

for $t \geq t_3$ for some $t_3 \geq t_2$. Therefore,

$$\left(\frac{y(t)}{t^2}\right)' = \frac{y'(t)t^2 - 2ty(t)}{t^4} < 0,$$

which proves (iv) and completes the proof of the lemma. □

Our next lemma provides some additional properties of solutions in the class $\mathcal{N}_2$.

Lemma 3.3. *Assume that $\beta_* > 0$ and $y \in \mathcal{N}_2$. Then, for any $\beta \in (0, \beta_*)$ and t sufficiently large:*

(v) $y'(t)/t^{1-\beta}$ is decreasing and $ty''(t) < (1-\beta)y'(t)$;

(vi) $\beta < 1$;

(vii) $\lim_{t\to\infty} y'(t)/t^{1-\beta} = 0$;

(viii) $y(t)/t^{2-\beta}$ is decreasing and $y(t) > y'(t)t/(2-\beta)$.

Proof. Let $y \in \mathcal{N}_2$ with $y(\tau(t)) > 0$ for $t \geq t_1 \geq t_0$.

(v) Define the function

$$z(t) := y'(t) - ty''(t),$$

which is positive by (iii). Differentiating z, and using (1.1) and (3.1), we see that

$$z'(t) = (y' - ty'')' = -ty'''(t) = tq(t)y(\tau(t)) \geq 2\beta \frac{y(\tau(t))}{\tau^2(t)}. \tag{3.7}$$

Using (iv), we have

$$z'(t) \geq 2\beta \frac{y(t)}{t^2} \geq \beta \frac{y'(t)}{t}$$

for $t \geq t_2$ for some $t_2 \geq t_1$. Integrating from t_2 to t and using the fact that $y'(t)/t$ is decreasing (see (iii)), there exists $t_3 \geq t_2$ such that

$$z(t) = z(t_2) + \beta \int_{t_2}^{t} \frac{y'(s)}{s}\mathrm{d}s \geq z(t_2) + \beta \frac{y'(t)}{t}(t - t_2) > \beta y'(t), \quad t \geq t_3. \tag{3.8}$$

That is,

$$ty''(t) < (1-\beta)y'(t), \quad t \geq t_3.$$

Hence, for $t \geq t_3$,

$$\left(\frac{y'(t)}{t^{1-\beta}}\right)' = \frac{y''(t)t^{1-\beta} - (1-\beta)t^{-\beta}y'(t)}{t^{2(1-\beta)}} = \frac{y''(t)t - (1-\beta)y'(t)}{t^{2-\beta}} < 0, \tag{3.9}$$

so part (v) holds.

Part (vi) clearly holds in view of (v) and (iii).

To prove (vii), it suffices to show that there is an $\varepsilon > 1$ such that

$$\left(\frac{y'(t)}{t^{1-\varepsilon\beta}}\right)' < 0 \tag{3.10}$$

for large t. Using (3.9), we see that for any $k \in (0,1)$ and t sufficiently large, say $t \geq t_4 \geq t_3$,

$$\begin{aligned} y(t) &= y(t_1) + \int_{t_1}^{t} \frac{y'(s)}{s^{1-\beta}} s^{1-\beta} \mathrm{d}s \\ &\geq y(t_1) + \frac{y'(t)}{t^{1-\beta}} \int_{t_1}^{t} s^{1-\beta} \mathrm{d}s \\ &= y(t_1) + \frac{y'(t)}{t^{1-\beta}} \frac{t^{2-\beta} - t_1^{2-\beta}}{2-\beta} \geq k\frac{y'(t)t}{2-\beta}, \end{aligned} \tag{3.11}$$

which implies

$$\left(\frac{y}{t^{(2-\beta)/k}}\right)' \leq 0, \quad t \geq t_4.$$

Employing the above monotonicity in (3.7) gives

$$z'(t) \geq 2\beta\frac{y(\tau(t))}{\tau^2(t)} = 2\beta\frac{y(\tau(t))\tau^{(2-\beta)/k}(t)}{\tau^{2+(2-\beta)/k}(t)} \geq 2\beta\frac{y(t)\tau^{(2-\beta)/k}(t)}{t^{(2-\beta)/k}\tau^2(t)} = 2\beta\frac{y(t)}{t^2}\left(\frac{t}{\tau(t)}\right)^{2-(2-\beta)/k}.$$

If we choose $k > (2-\beta)/2$, then

$$\left(\frac{t}{\tau(t)}\right)^{2-(2-\beta)/k} \geq \lambda^{2-(2-\beta)/k} \geq 1$$

and using (3.11), we have

$$z'(t) \geq 2\beta\frac{y(t)}{t^2} \geq \frac{2\beta k}{2-\beta}\frac{y'(t)}{t}. \tag{3.12}$$

Integrating the last inequality from t_4 to t and using (iii), we see that there exists $t_5 > t_4$ such that

$$z(t) \geq z(t_3) + \frac{2\beta k}{2-\beta}\frac{y'(t)}{t}(t-t_3) > \frac{2\beta k}{2-\beta}y'(t), \quad t \geq t_5.$$

In view of the definition of z, we conclude that

$$ty''(t) < \left(1 - \frac{2\beta k}{2-\beta}\right)y'(t),$$

which implies that (3.10) holds with $\varepsilon = 2k/(2-\beta) > 1$. This proves (vii).

(viii) Using (vii) in (3.11), there exists $t_6 > t_5$ such that

$$y(t_1) - \frac{y'(t)}{t^{1-\beta}} \frac{t_1^{2-\beta}}{2-\beta} > 0, \quad t \geq t_6,$$

and so (3.11) becomes

$$(2-\beta)y(t) > y'(t)t, \quad t \geq t_6, \tag{3.13}$$

which implies

$$\left(\frac{y(t)}{t^{2-\beta}}\right)' = \frac{y'(t)t^{2-\beta} - (2-\beta)t^{1-\beta}y(t)}{t^{2(2-\beta)}} = \frac{y'(t)t - (2-\beta)y(t)}{t^{3-\beta}} < 0$$

for $t \geq t_6$. This proves (viii) and finishes the proof of the lemma. □

Lemma 3.4. *Assume that $\beta_* > 0$ and $\lambda_* = \infty$. Then $\mathcal{N}_2 = \emptyset$.*

Proof. Suppose, to the contrary, that $y \in \mathcal{N}_2 \neq \emptyset$ and let $t_1 \geq t_0$ be such that $y(\tau(t)) > 0$ for $t \geq t_1$. Since $\lambda_* = \infty$, for any $\lambda > 1$, there exists $t_\lambda \geq t_1$ such that $t/\tau(t) \geq \lambda$ for $t \geq t_\lambda$. From (3.7), (viii), and (3.1), for $t \geq t_2 \geq t_\lambda$,

$$z'(t) \geq 2\beta \frac{y(\tau(t))}{\tau^2(t)} \geq 2\beta \frac{y(t)}{t^2}\left(\frac{t}{\tau(t)}\right)^\beta \geq 2\beta\lambda^\beta \frac{y(t)}{t^2} > \frac{2}{2-\beta}\beta\lambda^\beta \frac{y'(t)}{t}. \tag{3.14}$$

As λ can be arbitrarily large, we can choose it such that $\lambda^\beta > (2-\beta)/2\beta$. Therefore,

$$z'(t) > \frac{y'(t)}{t}.$$

Integrating and using the fact that $y'(t)/t$ is decreasing, we obtain

$$z(t) \geq z(t_1) + \frac{y'(t)}{t}(t - t_1),$$

so $z(t) > y'(t)$ for t sufficiently large, say $t \geq t_2$ for some $t_2 \geq t_1$. This implies $ty''(t) < 0$, which is a contradiction, and completes te proof of the lemma. □

In view of Lemma 3.4, from this point forward it is reasonable to assume that λ_* is a finite constant so that $\mathcal{N}_2 \neq \emptyset$. It is then possible to initiate an iterative procedure that will improve the monotonicity results in Lemma 3.3 and that in turn will lead to a sharp oscillation result for equation (1.1). To this end, let us define the sequence $\{\beta_n\}_{n\in\mathbb{N}_0}$ by

$$\beta_0 := \beta_*, \qquad \beta_n := \frac{2\beta_0\lambda_*^{\beta_{n-1}}}{(2-\beta_{n-1})(1-\beta_{n-1})}. \tag{3.15}$$

By induction, it is easy to verify that if $\beta_i < 1$ for $i = 1, 2, \ldots, n$, then β_{n+1} exists and

$$\frac{\beta_{n+1}}{\beta_n} = \ell_n > 1, \tag{3.16}$$

where

$$\ell_0 := \frac{\beta_1}{\beta_0} = \frac{2\lambda_*^{\beta_0}}{(2-\beta_0)(1-\beta_0)} > 1,$$

$$\ell_n := \frac{\beta_{n+1}}{\beta_n} = \frac{\lambda_*^{\beta_n}(2-\beta_{n-1})(1-\beta_{n-1})}{\lambda_*^{\beta_{n-1}}(2-\beta_n)(1-\beta_n)} > 1.$$

The following lemma is an iterative version of Lemma 3.3.

Lemma 3.5. *Assume that $\beta_* > 0$ and y is a solution of* (1.1) *belonging to the class $\mathcal{N}_2$. Then for any $\varepsilon_n \in (0,1)$ and sufficiently large t:*

(I)$_n$ $y'(t)/t^{1-\varepsilon_n\beta_n}$ *is decreasing and* $ty''(t) < (1-\varepsilon_n\beta_n)y'(t)$*;*

(II)$_n$ $\varepsilon_n\beta_n < 1$*;*

(III)$_n$ $\lim_{t\to\infty} y'(t)/t^{1-\varepsilon_n\beta_n} = 0$*;*

(IV)$_n$ $y(t)/t^{2-\varepsilon_n\beta_n}$ *is decreasing and* $y(t) > y'(t)t/(2-\varepsilon_n\beta_n)$.

Proof. Let $y \in \mathcal{N}_2$ with $y(\tau(t)) > 0$ for $t \ge t_1$ for some $t_1 \ge t_0$. We will proceed by induction on n. For $n = 0$, the conclusion follows from Lemma 3.3, where $\varepsilon_0 = \beta/\beta_*$ and $\lim_{\beta\to\beta_*}\varepsilon_0 = 1$.

Next, assume that (I)$_n$–(IV)$_n$ hold for $n \ge 1$ for $t \ge t_n \ge t_1$. We need to show that they each hold for $n+1$.

(I)$_n$ Using (viii) and (3.1) in (3.7), we have

$$\begin{aligned} z'(t) = (y' - ty'')' &\ge 2\varepsilon_0\beta_0 \frac{y(\tau(t))}{\tau^2(t)} \\ &= 2\varepsilon_0\beta_0 \frac{y(\tau(t))}{\tau^{2-\varepsilon_n\beta_n}(t)\tau^{\varepsilon_n\beta_n}(t)} \\ &\ge 2\varepsilon_0\beta_0 \frac{y(t)}{t^2}\frac{t^{\varepsilon_n\beta_n}}{\tau^{\varepsilon_n\beta_n}(t)} \\ &\ge 2\varepsilon_0\beta_0\lambda^{\varepsilon_n\beta_n}\frac{y(t)}{t^2} \\ &\ge \frac{2\varepsilon_0\beta_0\lambda^{\varepsilon_n\beta_n}}{2-\varepsilon_n\beta_n}\frac{y'(t)}{t}. \end{aligned}$$

Integrating from t_n to t and using (v) and (vii) in the resulting inequality, we see that there exists $t_n' > t_n$ such that

$$\begin{aligned} z(t) &\ge z(t_n) + \frac{2\varepsilon_0\beta_0\lambda^{\varepsilon_n\beta_n}}{2-\varepsilon_n\beta_n}\int_{t_n}^{t}\frac{y'(s)}{s^{1-\varepsilon_n\beta_n}s^{\varepsilon_n\beta_n}}\mathrm{d}s \\ &\ge z(t_n) + \frac{2\varepsilon_0\beta_0\lambda^{\varepsilon_n\beta_n}}{(2-\varepsilon_n\beta_n)(1-\varepsilon_n\beta_n)}\frac{y'(t)}{t^{1-\varepsilon_n\beta_n}}\left(t^{1-\varepsilon_n\beta_n} - t_n^{1-\varepsilon_n\beta_n}\right) \\ &> \frac{2\varepsilon_0\beta_0\lambda^{\varepsilon_n\beta_n}}{(2-\varepsilon_n\beta_n)(1-\varepsilon_n\beta_n)}y'(t) \\ &= \varepsilon_{n+1}\beta_{n+1}y'(t), \quad t \ge t_n', \end{aligned}$$

where

$$\varepsilon_{n+1} = \varepsilon_0\frac{\lambda^{\varepsilon_n\beta_n}}{\lambda_*^{\beta_n}}\frac{(2-\beta_n)(1-\beta_n)}{(2-\varepsilon_n\beta_n)(1-\varepsilon_n\beta_n)}, \quad n \in \mathbb{N}_0.$$

Clearly,

$$\lim_{(\lambda\to\lambda_*)(\beta\to\beta_0)}\varepsilon_{n+1} = 1,$$

which proves $(\mathrm{I})_n$.

$(\mathrm{II})_n$ This clearly holds in view of $(\mathrm{I})_n$ and (iii).

$(\mathrm{III})_n$ As for the case $n = 0$, it will suffice to show that there is $\varepsilon > 1$ such that

$$\left(\frac{y'(t)}{t^{1-\varepsilon\varepsilon_n\beta_n}}\right)' < 0. \tag{3.17}$$

Using $(\mathrm{I})_n$, we see that for any $k \in (0, 1)$, there exists $t''_n \geq t'_n$ so that

$$\begin{aligned} y(t) &= y(t_n) + \int_{t_n}^{t} \frac{y'(s)}{s^{1-\varepsilon_n\beta_n}} s^{1-\varepsilon_n\beta_n} \mathrm{d}s \\ &\geq y(t_n) + \frac{y'(t)}{t^{1-\varepsilon_n\beta_n}} \int_{t_n}^{t} s^{1-\varepsilon_n\beta_n} \mathrm{d}s \\ &= y(t_n) + \frac{y'(t)}{t^{1-\varepsilon_n\beta_n}} \frac{t^{2-\varepsilon_n\beta_n} - t_n^{2-\varepsilon_n\beta_n}}{2-\varepsilon_n\beta_n} \geq k\frac{y'(t)t}{2-\varepsilon_n\beta_n}, \quad t \geq t''_n, \end{aligned} \tag{3.18}$$

that is,

$$\left(\frac{y}{t^{(2-\varepsilon_n\beta_n)/k}}\right)' < 0. \tag{3.19}$$

Then (3.7) and (3.19) imply

$$\begin{aligned} z'(t) &\geq 2\varepsilon_0\beta_0 \frac{y(\tau(t))}{\tau^2(t)} \\ &= 2\varepsilon_0\beta_0 \frac{y(\tau(t))\tau^{(2-\varepsilon_n\beta_n)/k}(t)}{\tau^{2+(2-\varepsilon_n\beta_n)/k}(t)} \\ &\geq 2\varepsilon_0\beta_0 \frac{y(t)}{t^{(2-\varepsilon_n\beta_n)/k}} \frac{\tau^{(2-\varepsilon_n\beta_n)/k}(t)}{\tau^2(t)} \\ &= 2\varepsilon_0\beta_0 \frac{y(t)}{t^2} \frac{\tau^{(2-\varepsilon_n\beta_n)/k}(t)}{\tau^2(t)} \frac{t^2}{t^{(2-\varepsilon_n\beta_n)/k}} \\ &= 2\varepsilon_0\beta_0 \frac{y(t)}{t^2} \left(\frac{t}{\tau(t)}\right)^{2-(2-\varepsilon_n\beta_n)/k}. \end{aligned} \tag{3.20}$$

Now take $k > (2 - \varepsilon_n\beta_n)/2$. Then again by (iv),

$$z'(t) \geq 2\varepsilon_0\beta_0 \frac{y(t)}{t^2} \geq \frac{2\varepsilon_0\beta_0 k}{2-\varepsilon_n\beta_n} \frac{y'(t)}{t}.$$

Integrating from t''_n to t and using (ii), we have

$$z(t) \geq z(t''_n) + \frac{2\varepsilon_0\beta_0 k}{2-\varepsilon_n\beta_n} \frac{y'(t)}{t}(t - t''_n) > \frac{2\varepsilon_0\beta_0 k}{2-\varepsilon_n\beta_n} y'(t), \quad t \geq t'''_n,$$

for some $t'''_n \geq t''_n$. Therefore,

$$ty''(t) < \left(1 - \frac{2\varepsilon_0\beta_0 k}{2-\varepsilon_n\beta_n}\right) y'(t),$$

and so (3.17) holds, where $\varepsilon = 2k/(2 - \varepsilon_n\beta_n) > 1$. This proves $(\mathrm{III})_n$.

$(\mathrm{IV})_n$ From (3.18),

$$(2 - \varepsilon_n\beta_n)y(t) > y'(t)t$$

and so

$$\left(\frac{y(t)}{t^{2-\varepsilon_n\beta_n}}\right)' = \frac{y'(t)t^{2-\varepsilon_n\beta_n} - (2-\varepsilon_n\beta_n)t^{1-\varepsilon_n\beta_n}y(t)}{t^{2(2-\varepsilon_n\beta_n)}} = \frac{y'(t)t-(2-\varepsilon_n\beta_n)y(t)}{t^{3-\varepsilon_n\beta_n}} < 0,$$

completing the proof of the lemma. □

As a consequence of the above lemmas, we can prove the main result in our paper, namely, Theorem 2.1.

Proof of Theorem 2.1. Assume that y is a nonoscillatory solution of (1.1) such that $y(\tau(t)) > 0$ for $t \geq t_1$ for some $t_1 \geq t_0$. Then either $y \in \mathcal{N}_0$ or $y \in \mathcal{N}_2$. By condition (2.1), $\beta_* > 0$, so if $y \in \mathcal{N}_0$, then by Lemma 3.1, $\lim_{t\to\infty} y(t) = 0$.

Now if $y \in \mathcal{N}_2$, we need to consider two cases, namely, $\lambda_* = \infty$ or $\lambda_* < \infty$. If $\lambda_* = \infty$, then by Lemma 3.4 we see that $\mathcal{N}_2 = \emptyset$, and so (1.1) has Property A.

Finally, assume that $\lambda_* < \infty$. We claim that

$$\beta_{n-1} < 1, \quad n \in \mathbb{N}. \tag{3.21}$$

From $(\mathrm{II})_n$, $\varepsilon_n\beta_n < 1$. Since $\varepsilon_n \in (0,1)$ can be chosen arbitrarily, set $\varepsilon_n > 1/\ell_n$, where ℓ_n is defined by (3.16). Then,

$$1 > \varepsilon_n\beta_n = \varepsilon_n\ell_n\beta_{n-1} > \beta_{n-1},$$

which proves the claim. In view of (3.21), we conclude that the sequence $\{\beta_n\}_{n=0}^{\infty}$ defined by (3.15) is increasing and bounded from above, that is, there exists a finite limit

$$\lim_{n\to\infty} \beta_n = y,$$

where $y \in (0,1)$ is a root of the equation

$$y(2-y)(1-y)\lambda_*^{-y} = 2\beta_0.$$

Set $x = 2 - y$. Then $x \in (1,2)$ satisfies

$$-x(1-x)(2-x)\lambda_*^{x-2} = 2\beta_0. \tag{3.22}$$

However, condition (2.1) implies that (3.22) does not possess positive solutions. Hence, $\mathcal{N}_2 = \emptyset$ and the proof is complete. □

As a final remark, we wish to mention that one of the reviewers asked if it would be possible to obtain corresponding non-improvable bounds for solutions in the class $\mathcal{N}_0$. This would in fact be an interesting problem to investigate. The biggest barrier to applying the approach used here is that a lower non-zero bound for a positive solution belonging to the class $\mathcal{N}_0$ is not known due to the alternating signs of its derivatives. If such a bound could be determined, then what the reviewer suggests might be possible.

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