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ANALYTICAL DYNAMICS OF A FRICTION OSCILLATOR UNDER TWO-FREQUENCY EXCITATIONS WITH FLOW BARRIERS*

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Abstract In this paper, the analytical dynamics is investigated in a periodically forced friction oscillator under two-frequency excitations. The nonlinear friction force is approximated by a piecewise linear, kinetic friction model with the static friction force, G-functions are defined through the dot product of the vector fields and the normal vector. By the sign of G-functions, the necessary and sufficient conditions for the flow passibility and the grazing motions to the separation boundary are developed. For the examined system, the boundary possesses flow barriers caused by the static friction force. Because the flow barriers exist on the separation boundary, the singularities of the flow on such a separation boundary will be changed accordingly. Based on the critical values of flow barriers, the necessary and sufficient conditions for the onset and vanishing of the stick motions on the boundary with flow barriers are also developed. Furthermore, the periodic motions of such an oscillator are determined through the corresponding mapping structures. Illustrations of the periodic motions in such a piecewise friction model are given to verify the analytical conditions.

Keywords Discontinuous dynamical systems, flow barriers, passable motions, stick motions, mapping structure.

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1. Introduction

Discontinuous dynamical systems exist widely in industrial applications. The dynamics and responses of such discontinuous systems directly cause the efficiency and destruction of the machines. The early investigation of discontinuous systems in mechanical engineering can be found in 1930's [11–13]. The discontinuity in such dynamical systems is caused by the friction forces. Since then, much research work has been done to investigate the dynamics of discontinuous dynamical systems for its important applications [22–24,28,53]. For piecewise linear systems, Levinson [29]

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used a piecewise linear model to investigate the periodically excited Van der Pol equation, and found infinitely many periodic solutions which cannot be perturbed away. Further results for this piecewise model of the Van der Pol equation were given in [30,31]. Shaw and Holmes [56] investigated a piecewise linear system with a single discontinuity through the mapping techniques and numerically predicted the chaotic motion. Natsiavas [48] investigated the periodic motions and stability for a system with a symmetric, tri-linear spring. Nordmark [49] introduced the grazing mapping to study the non-periodic motions. Kleczka et al. [25] investigated the periodic motions and bifurcations of a piecewise linear oscillator and numerically observed the grazing motion. Leine and van Campen [26] discussed the discontinuous bifurcations of periodic solutions through the Floquet multipliers of periodic solutions. The analytical prediction of periodic responses of piecewise linear systems was given in [27]. Bernardo et al. [6, 7] discussed the normal formal mapping for piecewise smooth dynamical systems with/without sliding. Li et al. [51] proposed new conditions of stability and stabilization for periodic piecewise linear systems, state-feedback controllers with time-varying polynomial controller gain are designed to stabilize an unstable periodic piecewise system.

For discontinuous dynamical system, Filippov [18] presented differential equations with discontinuous right-hand sides, which started from the Coulomb friction oscillator. To investigated the sliding motion along the discontinuous boundary, the differential inclusion was introduced via the set-valued analysis, and the existence and uniqueness of the solution for such a discontinuous differential equation were discussed. The comprehensive discussion of such discontinuous differential equations can be referred to Ref. [19]. According to the theoretical basis for dynamic characteristics composition proposed in [4], Bazhenov et al. [5] further studied bifurcations in discontinuous vibroimpact system and observed phenomena unique for non-smooth systems with discontinuous right-hand side. Since the discontinuity exists widely in engineering and control systems, Aizerman and Pyatniskii [1, 2] extended Filippov's concept and developed a generalized theory for discontinuous dynamical systems. From such a generalized theory, Utkin [58] developed sliding mode control for controlling dynamic systems through the discontinuity. Utkin [59] presented sliding modes and the corresponding variable structure systems, and the theory of automatic control systems described with variable structures and sliding motions was also developed in [60]. Decarlo et al. [14] gave a review on the development of the sliding mode control. In 1988, Filippov systematically presented a geometrical theory of differential equations with discontinuous right-hand sides. From geometrical points of view, Broucke et al. [8] discussed structural stability of piecewise smooth systems. Leine et al. [37] used the Filippov theory to investigate bifurcations for nonlinear discontinuous system. However, the Filippov's theory mainly focused on the existence and uniqueness of the solutions for non-smooth dynamical systems. Such a differential equation theory with discontinuity is still difficult to be used for determining the complexity of discontinuous dynamical systems because the local singularity of a flow to the separation boundary was not discussed. To further investigate the local singularity of a flow in the vicinity of the separation boundary, Luo [38] established a general theory for discontinuous dynamical systems on connectable domains. The local singularity of discontinuous dynamical systems near the separation boundary was discussed. To determine the sink and source flows in discontinuous dynamical systems, Luo introduced the imaginary, sink and source flows in [39]. The detailed discussion of the local singularity and dynamics of discontinuous dynamical systems was presented in Luo [40]. In [41], Luo and Gegg used the local singularity theory to develop the force criteria for the harmonically driven linear oscillator with dry friction, and in [42], Luo and Gegg analytically investigated periodic motions in such an oscillator. From differential geometry points of view, Luo [43] introduced G-function to measure the local singularity, and presented the flow switchability theory in discontinuous dynamical systems. In recent years, the dynamics of discontinuous dynamical systems have been well studied by above G-function and the flow switchability theory. Luo and Thapa [44] investigated the singularity and switchability of periodic motions in a simplified brake system with periodic excitation. Luo and Huang [45] investigated the discontinuous dynamics of a nonlinear friction-induced, periodically forced oscillator, the analytical conditions for motion switchability at the velocity boundary were developed. Guo and Luo [20, 21] used a semi-analytical method to obtain bifurcation trees of periodic motions to chaos in a periodically driven pendulum and Duffing oscillator, mapping structures are developed and the corresponding eigenvalue analysis was carried out for the stability and bifurcation of the periodic motions. Sun and Fu [54] developed the analytical conditions for synchronization of the Van der Pol equation with a sinusoidally forced pendulum using the theory of discontinuous dynamical systems. Fu and Zheng [17] studied the chatter dynamics for a class of second-order impulsive switched systems-a certain of Van der Pol equations, and presented the sufficient conditions to keep the pulse phenomena absent. Also in [62], Zheng and Fu investigated the chatter conditions of a second-order impulsive dynamical system via the method of flow theory in discontinuous systems, and analyzed the dynamical behaviors of flow on separation surface and got general results on chatter criterion. In [63], Zhang and Fu investigated flow switching on corresponding boundaries in a periodic-excited horizontal impact pair with dry friction, the analytical switching conditions on each boundary are developed. Tang and Fu [57] used the theory of discontinuous dynamical system to investigate the period-k solutions of population differential system with state-dependent impulsive effect and obtained the necessary and sufficient conditions for trajectory direction of a population differential system. Sun and Fu [55] studied the discontinuous dynamics of a class of oscillators with asymmetric damping using the flow switchability theory of the discontinuous dynamical systems.

In the above mentioned theories, the discontinuity in discontinuous dynamical systems is based on different vector fields and all the applications were based on discontinuous dynamical systems without flow barriers. In practice, artificial dynamical systems often possess flow barriers on the separation boundary, caused by the static friction force or the impact damping, and the lower and upper limits of the boundary flow barriers control the existence of the boundary flow. For instance, the static friction force is a kind of flow barrier to a discontinuous dynamical system with dynamic friction force. The wall of the impact damper is a permanent flow barrier to the impact ball. Once the flow barriers exist on the separation boundary, the singularities of the flow to such a separation boundary will be changed accordingly. If the G-function of flow barrier is not defined on $S \subset \partial\Omega_{ij}$ with the critical values of the flow barrier, the window of the flow barrier can be formed where no flow barriers on such a portion exist. For the permanent windows of the flow barrier, the flow can always be switched from one domain into another domain via the boundary. However, for instantaneous windows of the flow barrier, the flow may not be switched at the boundary for the next moment. Therefore, the above mentioned theories for discontinuous dynamical systems cannot be used. In 2007, Luo [33] introduced the flow barriers in discontinuous dynamical systems. In 2008, Luo and Zwiegart [34] used the non-smooth dynamical theory to investigate dynamic behaviors of a periodically forced, nonlinear friction oscillator, and the studied dynamical system has the flow barrier due to the static friction force. The force criteria for the onset and vanishing of stick motions were developed through the input and output flow forces, and the analytical conditions for the grazing bifurcation to the boundary were also presented. In [35,36], Luo systematically presented the theory of flow barriers in discontinuous dynamical systems. The dynamical behaviors of the flow to the boundary with flow barriers were discussed, and a periodically forced friction oscillator with flow barriers were studied for a better understanding of flow barriers in practical problems, and this sample problem can be applied to the cutting dynamics in manufacturing and brake system in automobile industry, which shows the flow barrier theory provide a useful tool to design desired dynamical systems to satisfy engineering-oriented complex systems, furthermore, the flow barrier theory provide a theoretic base to develop control theory and stability.

The dynamics of dry friction damped systems has been studied for many years, the early research on dry friction oscillators can be found in [11, 12]. Since then, much work has been done in this field. Using an incremental harmonic balance method, Pierre et al. [52] presented a multi-harmonic frequency domain analysis of Coulomb damped systems using an incremental harmonic balance method. Feeny and Moon [15] investigated the geometry of chaotic attractors for dry friction oscillators experimentally and numerically. In [46], an analytical method of calculating Lyapunov exponents for non-linear dynamic systems with discontinuity was presented and was applied to the analysis of a Coulomb damped oscillator. Oestreich et al. [50] employed a one-dimensional map to discuss bifurcation and stability of a non-smooth friction oscillator on a moving base, and the response of a dry friction oscillator on a moving base was also analyzed by Andreaus and Casini [3], with emphasis laid on the influence of the base speed and the friction modelling on the system response. Van De Vrande et al. [61] computed both stable and unstable periodic solutions for the stick-slip vibration of an autonomous system with dry friction. Fan et al. [16] studied the dynamical behaviors of a friction induced oscillator with switching control law through the flow switching theory of discontinuous dynamical systems, and the analytical conditions of the passable motion, stick motion, sliding motion and grazing motion are presented. M. Pascal [47] considered a system of two masses connected by linear springs and in contact with a belt moving at a constant velocity, several periodic orbits including contact against the fixed obstacle followed by slip and stick phases are obtained in analytical form. However, in all of the research mentioned above, either no excitation or only a single harmonic excitation was assumed. In practice, multi-excitations can exist in various vibration systems with dry friction, and they may have a dramatic impact on the system's dynamic characteristics. In [9], a mass-spring friction oscillator subjected to two harmonic disturbing forces with different frequencies was investigated for the first time. The focus of this paper was to study the effect of the two-frequency excitation along with Coulomb damping on dynamical behaviors of the oscillator. Due to the two frequency excitations, the amplitude-frequency curve of system appears different near the resonance. For the one-stop motion, the amplitude does not peak near the natural frequency, instead it peaks at such a frequency value where both excitation frequencies are much less than the natural frequency. Also, the one-stop motion exists only in the frequency range where the larger excitation frequency is less than the natural frequency. Moreover, for the non-stop motion, there exists a jump phenomenon at the resonance frequency for the phase angle response of the oscillator due to the two-frequency excitation. Thus, the dynamic response of the oscillator subjected to two frequency excitations demonstrates characteristics significantly different from those due to a single frequency excitation. Next, in [10], the dynamic behavior of a dry friction oscillator subjected to two harmonic excitations on a moving belt with constant velocity was investigated, with focus laid on bifurcation analysis to get the influence of the two-frequency excitation upon the qualitative features of system dynamics. It was found that the ratio between the two excitation frequencies has a significant impact upon the system dynamics, as the ratio is increased, more likely periodic motions will occur than chaotic motions. However, in [9, 10], the complex dynamical behaviors of the oscillator, such as the local singularity of the flow, the analytical prediction of periodic motions, etc were not discussed.

In this paper, the dynamics of the dry friction oscillator subjected to two harmonic excitations with different frequencies on a moving belt is further investigated. In this examined system, the kinetic friction force is a nonlinear function strongly dependent on the relative velocity between the mass and the belt, and the maximum static friction force is different from the kinetic friction force at the zero relative velocity. Such a difference causes the vector fields of the dynamical system to have the flow barrier. The flow barriers existing in the vector field of the dynamical systems will lead to more difficulty to investigate such dynamical systems. To avoid computational errors , the nonlinear kinetic friction force is modeled by a piecewise linear friction instead of the full nonlinear model in this paper. Different domains and boundaries are defined in phase space for such piecewise linear friction model. Using the flow switchability theory of the discontinuous dynamical systems, the analytical conditions for the flow passibility and the grazing motions are developed. Because of the flow barriers existing on the separation boundary, the force criteria for the onset and vanishing of stick motions on the discontinuous boundary are also developed using the flow barrier theory of the discontinuous dynamical systems. Further, The mapping techniques are used to determine the periodic motions of the friction oscillator. The numerical simulations of the periodic motions in such a piecewise friction model are given to verify the analytical conditions.

2. A friction oscillator with flow barriers

Consider a periodically forced friction oscillator in Figure 1(a). The dynamical system consists of a mass m, a damper of viscous damping coefficient c, and two springs with the stiffness k_1 and k_2, respectively. The moving mass rests on the horizontal belt surface travelling with a constant speed V. The mass is subjected to two harmonic excitations with the same amplitude P but different frequencies ω_1 and ω_2. The coordinate system (x, t) is absolute with displacement x and time t. The nonlinear friction force is approximated by a piecewise linear model, as shown

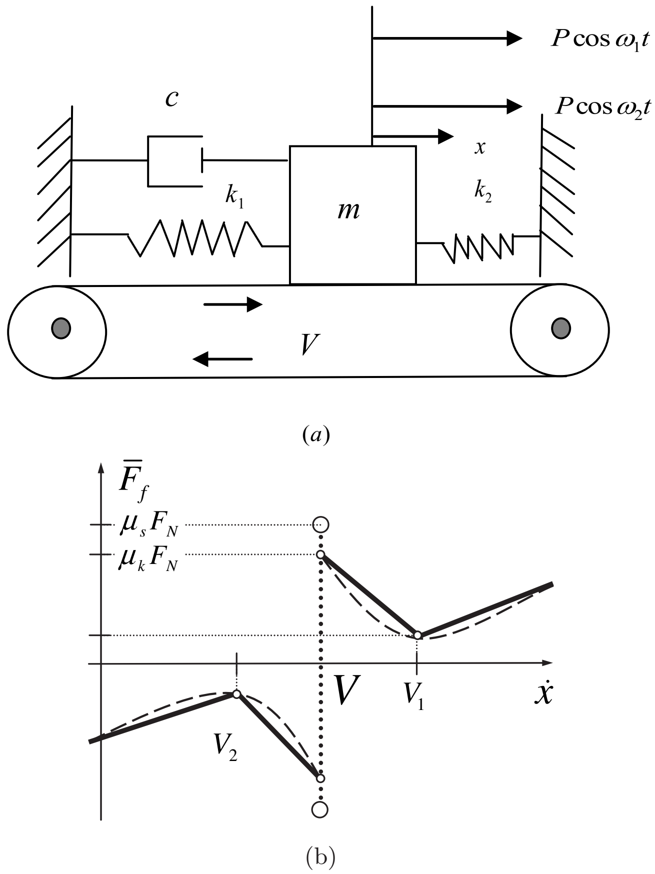


Figure 1. (a) The friction-induced oscillator and (b) piecewise linear friction force model.

in Figure 1(b). The piecewise linear friction force is given by

$$\bar{F}_f(\dot{x}) \begin{cases} = \mu_1(\dot{x} - V_1) - \mu_2(V_1 - V) + F_N\mu_k, & \dot{x} \in [V_1, +\infty), \\ = -\mu_2(\dot{x} - V) + F_N\mu_k, & \dot{x} \in (V, V_1), \\ \in [-\mu_s F_N, \mu_s F_N], & \dot{x} = V, \\ = -\mu_3(\dot{x} - V) - F_N\mu_k, & \dot{x} \in (V_2, V), \\ = \mu_4(\dot{x} - V_2) - \mu_3(V_2 - V) - F_N\mu_k, & \dot{x} \in (-\infty, V_2], \end{cases} \tag{2.1}$$

where $\dot{x} \triangleq \frac{dx}{dt}$.$\mu_s, \mu_k$ and F_N are the static and kinetic friction coefficient and a normal force to the contact surface, respectively. The coefficients $\mu_j (j = 1, 2, 3, 4)$ are the slope for friction force with velocity. For this problem, the normal force $F_N = mg$ and g is the gravitational acceleration. The static friction force is in the interval of $[-\mu_s F_N, \mu_s F_N]$. The amplitude of the static friction force is $\mu_s F_N$. The dynamic friction forces just for the beginning of the relative motion are $\pm\mu_k F_N$. Two boundaries for the piecewise continuity of the friction force are at $\dot{x} = V_1$ and $\dot{x} = V_2$. The third boundary is at $\dot{x} = V$. For this boundary, the dynamical friction force for the passable motion is discontinuous. The relative motion does not exist when the mass and the belt stick together. Only when the nonfriction force is greater than the static friction force, the relative motion between the mass and the belt can start. If the mass sticks on the belt surface, the nonfriction force per unit mass in the x-direction is determined by

$$F_{nf} = Q_0 \cos\omega_1 t + Q_0 \cos\omega_2 t - 2aV - bx \qquad \text{for } \dot{x} = V, \tag{2.2}$$

where $Q_0 = P/m, a = c/2m$ and $b = (k_1 + k_2)/m$. For the stick motion, the nonfriction force is less than the maximum static friction force, i.e., $|F_{nf}| \le F_{f_s}$ and $F_{f_s} = \mu_s F_N/m$. The mass does not have any relative motion to the belt. Therefore, no acceleration exists because the belt speed is constant, i.e.,

$$\ddot{x} = 0, \qquad \text{for } \dot{x} = V. \tag{2.3}$$

If the nonfriction force is greater than the maximum static friction force, i.e., $|F_{nf}| > F_{f_s}$, the non-stick motion occurs. For the non-stick motion, the total force per unit mass is

$$F = Q_0 \cos\omega_1 t + Q_0 \cos\omega_2 t - F_{f_k} - 2a\dot{x} - bx \qquad \text{for } \dot{x} \ne V, \tag{2.4}$$

where $F_{f_k} = \bar{F}_f/m$, for $\dot{x} \ne V$. Therefore, the equation of the non-stick motion for such a dynamical system with a piecewise linear friction is

$$\ddot{x} + 2a\dot{x} + bx = Q_0 \cos\omega_1 t + Q_0 \cos\omega_2 t - F_{f_k} \qquad \text{for } \dot{x} \ne V. \tag{2.5}$$

3. Domains and boundaries

For simplicity, the state vector and the corresponding vector field for such a system are introduced as

$$\mathbf{x} \triangleq (x, \dot{x})^T \equiv (x, y)^T \text{ and } \mathbf{F} \triangleq (y, F)^T. \tag{3.1}$$

The discontinuities in this dynamical system are caused by the jumping from the static to dynamical friction forces and the piecewise linear dynamical friction model. As discussed before, there are four velocity regions caused by the three velocity boundaries. Therefore, the phase space can be partitioned into four sub-domains by the three velocity boundaries. Such a phase space partition is sketched in Figure 2. Among the three velocity boundaries, the friction force jumping is as a main discontinuity at $\dot{x} = V$. Thus the naming of the sub-domains in phase space starts from the domain near the main discontinuous boundary $\dot{x} = V$. In fact , the sub-domains can be named arbitrarily. Based on the direction of trajectories of mass motion in phase space, the corresponding boundaries are also named, as shown in Figure 2. The boundary with the friction force jumping is depicted by a dotted line. The remaining boundaries are represented by two dashed lines, respectively. The named domains and the oriented boundaries are expressed by

$$\begin{aligned} &\Omega_1 = \{(x,y)|y \in (V, V_1)\}, \quad \Omega_2 = \{(x,y)|y \in (V_1, +\infty)\}, \\ &\Omega_3 = \{(x,y)|y \in (V_2, V)\}, \quad \Omega_4 = \{(x,y)|y \in (-\infty, V_2)\}. \end{aligned} \tag{3.2}$$

$$\partial\Omega_{\alpha\beta} = \{(x,y)|\varphi_{\alpha\beta}(x,y) \equiv y - V_\rho = 0\}, \tag{3.3}$$

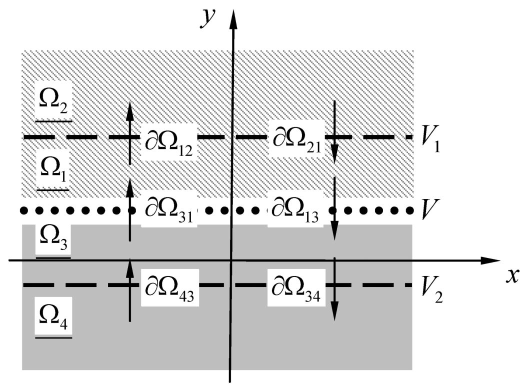


Figure 2. Phase plane partition and oriented boundaries.

where $\rho = 1$ if $\alpha, \beta \in \{1, 2\}, \rho = 0$ if $\alpha, \beta \in \{1, 3\}$ and $\rho = 2$ if $\alpha, \beta \in \{3, 4\}$. $V_0 \equiv V$. The subscript $(.)_{\alpha\beta}$ means the boundary from Ω_α to Ω_β. The domains are accessible for a specific vector field. On the boundary $\partial\Omega_{13}$ or $\partial\Omega_{31}$, the vector fields are C^0-discontinuous, but on the boundaries $\partial\Omega_{12}$ and $\partial\Omega_{34}$, the vector fields are C^0-continuous. Based on the definitions of the boundaries and domains in phase space, the equations of motion in Eqs. (2.3) and (2.5) are written as

$$\dot{\mathbf{x}} = \mathbf{F}^{(j)}(\mathbf{x}, t), \quad j \in \{0, 1, 2, 3, 4\}, \tag{3.4}$$

where

$$\left.\begin{aligned} &\mathbf{F}^{(0)}(\mathbf{x}, t) = (V, 0)^T \quad \text{on } \partial\Omega_{13} \text{ or } \partial\Omega_{31}, \\ &\mathbf{F}^{(j)}(\mathbf{x}, t) = (y, F^{(j)}(\mathbf{x}, t))^T \quad \text{in } \Omega_j, \ j \in \{1, 2, 3, 4\}. \end{aligned}\right\} \tag{3.5}$$

$$F^{(j)}(\mathbf{x}, t) = Q_0 \cos \omega_1 t + Q_0 \cos \omega_2 t - F_{f_k}^{(j)}(\mathbf{x}, t) - 2a_j \dot{x} - b_j x. \tag{3.6}$$

From Eq.(2.1), the dynamical friction forces per unit mass can be expressed by

$$\left.\begin{aligned} &F_{f_k}^{(2)}(\mathbf{x}, t) = \nu_1(y - V_1) - \nu_2(V_1 - V) + F_N\nu_k, \ y \in [V_1, +\infty), \\ &F_{f_k}^{(1)}(\mathbf{x}, t) = -\nu_2(y - V) + F_N\nu_k, \ y \in (V, V_1), \\ &F_{f_k}^{(3)}(\mathbf{x}, t) = -\nu_3(y - V) - F_N\nu_k, \ y \in (V_2, V), \\ &F_{f_k}^{(4)}(\mathbf{x}, t) = \nu_4(y - V_2) - \nu_3(V_2 - V) - F_N\nu_k, \ y \in (-\infty, V_2], \end{aligned}\right\} \tag{3.7}$$

where $\nu_i = \mu_i/m$ $(i = 1, 2, 3, 4)$ and $\nu_k = \mu_k/m$ are the slope coefficients of friction forces and dynamical friction coefficient per unit mass.

4. Switchability conditions

Using the flow switchability theory and the flow barrier theory of the discontinuous dynamical systems in [35, 43], the analytical conditions for the flow passibility, the onset and vanishing of the stick motions on the separation boundary with flow barriers will be developed in this section.

4.1. Basic Theory

Before discussing the analytical switching conditions for the complex motions in the friction oscillator, concepts of G-functions and the boundary flow barrier, the fundamental theory on the sink flow to the boundary with flow barriers will be first presented.

Definition 4.1. Consider a dynamical system $\dot{\mathbf{x}}^{(\alpha)} \equiv \mathbf{F}^{(\alpha)}(\mathbf{x}^{(\alpha)}, t, \mathbf{p}_\alpha) \in R^n$ in domain $\Omega_\alpha(\alpha \in \{i,j\})$ which has a flow $\mathbf{x}_t^{(\alpha)} = \Phi(t_0, \mathbf{x}_0^{(\alpha)}, \mathbf{p}_\alpha, t)$ with an initial condition $(t_0, \mathbf{x}_0^{(\alpha)})$ and on the boundary $\partial\Omega_{ij} = \{\mathbf{x}|\varphi_{ij}(\mathbf{x}, t, \boldsymbol{\lambda}) = 0,\ \varphi_{ij}$ is C^r-continuous$(r \geq 1)\} \subset R^{n-1}$, there is a flow $\mathbf{x}_t^{(0)} = \Phi(t_0, \mathbf{x}_0^{(0)}, \boldsymbol{\lambda}, t)$ with an initial condition $(t_0, \mathbf{x}_0^{(0)})$. The $0-$order G-functions of the flow $\mathbf{x}_t^{(\alpha)}$ to the flow $\mathbf{x}_t^{(0)}$ on the boundary in the normal direction of the boundary $\partial\Omega_{ij}$ are defined as

$$\begin{aligned} G_{\partial\Omega_{ij}}^{(\alpha)}(\mathbf{x}_t^{(0)}, t_\pm, \mathbf{x}_{t\pm}^{(\alpha)}, \mathbf{p}_\alpha, \boldsymbol{\lambda}) &= G_{\partial\Omega_{ij}}^{(0,\alpha)}(\mathbf{x}_t^{(0)}, t_\pm, \mathbf{x}_{t\pm}^{(\alpha)}, \mathbf{p}_\alpha, \boldsymbol{\lambda}) \\ &= D_{\mathbf{x}_t^{(0)}}^t \mathbf{n}_{\partial\Omega_{ij}}^T . (\mathbf{x}_{t\pm}^{(\alpha)} - \mathbf{x}_t^{(0)}) +^t \mathbf{n}_{\partial\Omega_{ij}}^T . (\dot{\mathbf{x}}_{t\pm}^{(\alpha)} - \dot{\mathbf{x}}_t^{(0)}). \end{aligned} \tag{4.1}$$

Definition 4.2. Consider a dynamical system $\dot{\mathbf{x}}^{(\alpha)} \equiv \mathbf{F}^{(\alpha)}(\mathbf{x}^{(\alpha)}, t, \mathbf{p}_\alpha) \in R^n$ in domain $\Omega_\alpha(\alpha \in \{i,j\})$ which has a flow $\mathbf{x}_t^{(\alpha)} = \Phi(t_0, \mathbf{x}_0^{(\alpha)}, \mathbf{p}_\alpha, t)$ with an initial condition $(t_0, \mathbf{x}_0^{(\alpha)})$ and on the boundary $\partial\Omega_{ij} = \{\mathbf{x}|\varphi_{ij}(\mathbf{x}, t, \boldsymbol{\lambda}) = 0,\ \varphi_{ij}$ is C^r-continuous$(r \geq 1)\} \subset R^{n-1}$, there is a flow $\mathbf{x}_t^{(0)} = \Phi(t_0, \mathbf{x}_0^{(0)}, \boldsymbol{\lambda}, t)$ with an initial condition $(t_0, \mathbf{x}_0^{(0)})$. The $1-$order G-functions of the flow $\mathbf{x}_t^{(\alpha)}$ to the boundary flow $\mathbf{x}_t^{(0)}$ in the normal direction of the boundary $\partial\Omega_{ij}$ are defined as

$$\begin{aligned} G_{\partial\Omega_{ij}}^{(1,\alpha)}(\mathbf{x}_t^{(0)}, t_\pm, \mathbf{x}_{t\pm}^{(\alpha)}, \mathbf{p}_\alpha, \boldsymbol{\lambda}) =& D_{\mathbf{x}_t^{(0)}}^2 {}^t\mathbf{n}_{\partial\Omega_{ij}}^T . (\mathbf{x}_{t\pm}^{(\alpha)} - \mathbf{x}_t^{(0)}) + 2D_{\mathbf{x}_t^{(0)}}^t \mathbf{n}_{\partial\Omega_{ij}}^T . (\dot{\mathbf{x}}_{t\pm}^{(\alpha)} - \dot{\mathbf{x}}_t^{(0)}) \\ &+^t \mathbf{n}_{\partial\Omega_{ij}}^T . (\ddot{\mathbf{x}}_{t\pm}^{(\alpha)} - \ddot{\mathbf{x}}_t^{(0)}). \end{aligned} \tag{4.2}$$

In the above definitions, the total derivative $D_{\mathbf{x}_t^{(0)}}(\cdot) = \frac{\partial(\cdot)}{\partial \mathbf{x}_t^{(0)}} \dot{\mathbf{x}}_t^{(0)} + \frac{\partial(\cdot)}{\partial t}$ and the normal vector of the boundary surface $\partial\Omega_{ij}$ at point $\mathbf{x}^{(0)}(t)$ is given by

$$^t\mathbf{n}_{\partial\Omega_{ij}}(\mathbf{x}^{(0)}, t, \boldsymbol{\lambda}) = \bigtriangledown\varphi_{ij}(\mathbf{x}^{(0)}, t, \boldsymbol{\lambda}) = \left(\frac{\partial\varphi_{ij}}{\partial x_1^{(0)}}, \frac{\partial\varphi_{ij}}{\partial x_2^{(0)}}, \cdots, \frac{\partial\varphi_{ij}}{\partial x_n^{(0)}}\right)_{(t,\mathbf{x}^{(0)})}^T. \tag{4.3}$$

Consider the flow contacts with the boundary at time t_m, i.e., $\mathbf{x}_{t_m}^{(\alpha)} = \mathbf{x}_{t_m}^{(0)} = \mathbf{x}_m$, and the boundary $\partial\Omega_{ij}$ is linear independent of time t, then

$$\begin{aligned} G_{\partial\Omega_{ij}}^{(0,\alpha)}(\mathbf{x}_{t_m}^{(0)}, t_{m\pm}, \mathbf{x}_{tm\pm}^{(\alpha)}, \mathbf{p}_\alpha, \boldsymbol{\lambda}) &= G_{\partial\Omega_{ij}}^{(0,\alpha)}(\mathbf{x}_m, t_{m\pm}, \mathbf{p}_\alpha, \boldsymbol{\lambda}) \\ &=^t \mathbf{n}_{\partial\Omega_{ij}}^T . \dot{\mathbf{x}}_t^{(\alpha)}|_{(\mathbf{x}_m^{(0)}, t_{m\pm}, \mathbf{x}_m^{(\alpha)})}, \\ G_{\partial\Omega_{ij}}^{(1,\alpha)}(\mathbf{x}_{t_m}^{(0)}, t_{m\pm}, \mathbf{x}_{tm\pm}^{(\alpha)}, \mathbf{p}_\alpha, \boldsymbol{\lambda}) &= G_{\partial\Omega_{ij}}^{(1,\alpha)}(\mathbf{x}_m, t_{m\pm}, \mathbf{p}_\alpha, \boldsymbol{\lambda}) \\ &=^t \mathbf{n}_{\partial\Omega_{ij}}^T . \ddot{\mathbf{x}}_t^{(\alpha)}|_{(\mathbf{x}_m^{(0)}, t_{m\pm}, \mathbf{x}_m^{(\alpha)})}. \end{aligned} \tag{4.4}$$

The time $t_{m\pm} = t_m \pm 0$ indicates responses in domains rather than on the boundaries, and t_{m-}, t_{m+} are the time before approaching and after departing the separation boundary, respectively.

To investigate the flow property to the boundary with flow barriers, the G-functions for the flow barrier are given as follows.

Definition 4.3. For a discontinuous dynamical system $\dot{\mathbf{x}}^{(\alpha)} \equiv \mathbf{F}^{(\alpha)}(\mathbf{x}^{(\alpha)}, t, \mathbf{p}_\alpha)$ in domain $\Omega_\alpha(\alpha \in \{i,j\})$, there is a point $\mathbf{x}^{(0)}(t_m) \equiv \mathbf{x}_m \in \partial\Omega_{ij} = \{\mathbf{x}|\varphi_{ij}(\mathbf{x}, t, \boldsymbol{\lambda}) = 0\}$ at time t_m between two adjacent domains $\Omega_\alpha(\alpha = i, j)$. There is a vector field $\mathbf{F}^{(\rho\succ\gamma)}(\mathbf{x}^{(\lambda)}, t, \boldsymbol{\pi}_\lambda, q^{(\lambda)})$ for $q^{(\lambda)} \in [q_1^{(\lambda)}, q_2^{(\lambda)}]$ $(\rho, \gamma \in \{0, i, j\}, \lambda \in \{i, j\}$ and $\rho \neq \gamma$ if $\rho \neq 0)$ on the boundary $\partial\Omega_{ij}$. For the point $\mathbf{x}^{(\rho)}(t_m) = \mathbf{x}_m$, the G-function of the vector field is defined as

$$G_{\partial\Omega_{ij}}^{(\rho\succ\gamma)}(\mathbf{x}_m, t_{m\pm}, \boldsymbol{\pi}_\lambda, \boldsymbol{\lambda}, q^{(\lambda)}) \equiv \mathbf{n}_{\partial\Omega_{ij}}^T(\mathbf{x}^{(0)}, t, \boldsymbol{\lambda}).[\mathbf{F}^{(\rho\succ\gamma)}(\mathbf{x}^{(\lambda)}, t, \boldsymbol{\pi}_\lambda, q^{(\lambda)}) - \mathbf{F}^{(0)}(\mathbf{x}^{(0)}, t, \boldsymbol{\lambda})]\,|_{(\mathbf{x}_m^{(\lambda)}, \mathbf{x}_m^{(0)}, t_{m\pm})}\,. \tag{4.5}$$

The higher order G-function of the vector field $\mathbf{F}^{(\rho\succ\gamma)}(\mathbf{x}^{(\lambda)}, t, \boldsymbol{\pi}_\lambda, q^{(\lambda)})$ is defined for $k_\lambda = 0, 1, 2, \cdots$ as

$$\begin{aligned} &G_{\partial\Omega_{ij}}^{(k_\lambda,\rho\succ\gamma)}(\mathbf{x}_m, t_{m\pm}, \boldsymbol{\pi}_\lambda, \boldsymbol{\lambda}, q^{(\lambda)}) \\ &= \sum_{r=1}^{k_\lambda+1} C_{k_\lambda+1}^r D_0^{k_\lambda+1-r} \mathbf{n}_{\partial\Omega_{ij}}^T(\mathbf{x}^{(0)}, t, \boldsymbol{\lambda}).[D_\lambda^{r-1}\mathbf{F}^{(\rho\succ\gamma)}(\mathbf{x}^{(\lambda)}, t, \boldsymbol{\pi}_\lambda, q^{(\lambda)}) \\ &\quad - D_0^{r-1}\mathbf{F}^{(0)}(\mathbf{x}^{(0)}, t, \boldsymbol{\lambda})]|_{(\mathbf{x}_m^{(\lambda)}, \mathbf{x}_m^{(0)}, t_{m\pm})}. \end{aligned} \tag{4.6}$$

For simplicity in discussion, the following notations and sign function are adopted.

$$\begin{aligned} &G_{\partial\Omega_{ij}}^{(k_\alpha,\alpha)}(\mathbf{x}_m, t_{m\pm}) \equiv G_{\partial\Omega_{ij}}^{(k_\alpha,\alpha)}(\mathbf{x}_m, t_{m\pm}, \mathbf{P}_\alpha, \boldsymbol{\lambda}), \\ &G_{\partial\Omega_{ij}}^{(k_\lambda,\rho\succ\gamma)}(\mathbf{x}_m, q^{(\lambda)}) \equiv G_{\partial\Omega_{ij}}^{(k_\lambda,\rho\succ\gamma)}(\mathbf{x}_m, t_{m\pm}, \boldsymbol{\pi}_\lambda, \boldsymbol{\lambda}, q^{(\lambda)}). \end{aligned} \tag{4.7}$$

$$\hbar_\alpha = \begin{cases} +1 & \text{for } \mathbf{n}_{\partial\Omega_{ij}}^T \to \Omega_\beta, \\ -1 & \text{for } \mathbf{n}_{\partial\Omega_{ij}}^T \to \Omega_\alpha. \end{cases} \tag{4.8}$$

Definition 4.4. For a discontinuous dynamical system $\dot{\mathbf{x}}^{(\alpha)} \equiv \mathbf{F}^{(\alpha)}(\mathbf{x}^{(\alpha)}, t, \mathbf{p}_\alpha)$ in domain $\Omega_\alpha(\alpha \in \{i,j\})$, there is a point $\mathbf{x}^{(0)}(t_m) \equiv \mathbf{x}_m \in \partial\Omega_{ij} = \{\mathbf{x}|\varphi_{ij}(\mathbf{x}, t, \boldsymbol{\lambda}) = 0\}$ at time t_m between two adjacent domains $\Omega_\alpha(\alpha = i, j)$. Suppose there is a vector field of $\mathbf{F}^{(0\succ 0_\alpha)}(\mathbf{x}^{(\alpha)}, t, \boldsymbol{\pi}_\alpha, q^{(\alpha)})$ for $q^{(\alpha)} \in [q_1^{(\alpha)}, q_2^{(\alpha)}]$ on the boundary $\partial\Omega_{ij}$ with

$$0 \in [\hbar_\alpha G_{\partial\Omega_{ij}}^{(0\succ 0_\alpha)}(\mathbf{x}_m, q_2^{(\alpha)}), \hbar_\alpha G_{\partial\Omega_{ij}}^{(0\succ 0_\alpha)}(\mathbf{x}_m, q_1^{(\alpha)})] \subset R. \tag{4.9}$$

The two possible leaving flows in the source flow satisfy

$$\begin{aligned} &\hbar_\alpha G_{\partial\Omega_{ij}}^{(\alpha)}(\mathbf{x}_m, t_{m+}) < 0 \quad \text{and} \\ &\hbar_\alpha G_{\partial\Omega_{ij}}^{(\beta)}(\mathbf{x}_m, t_{m+}) > 0, \end{aligned} \tag{4.10}$$

$(\alpha, \beta \in \{i, j\}$ and $\alpha \neq \beta)$. The vector field of $\mathbf{F}^{(0\succ 0_\alpha)}(\mathbf{x}^{(\alpha)}, t, \boldsymbol{\pi}_\alpha, q^{(\alpha)})$ is called the flow barrier of the boundary flow in the source flow on the α-side if the following conditions are satisfied. The two critical values of $\mathbf{F}^{(0\succ 0_\alpha)}(\mathbf{x}^{(\alpha)}, t, \boldsymbol{\pi}_\alpha, q_\sigma^{(\alpha)})$ for $\sigma = 1, 2$ are called the lower and upper limits of the boundary flow barriers on the α-side.

(i) The boundary flow of $\mathbf{x}^{(0)}$ cannot be switched to the leaving flow of $\mathbf{x}^{(\alpha)}$ if

$$\begin{aligned}&\mathbf{x}^{(0)}(t_m) = \mathbf{x}^{(0\succ 0_\alpha)}(t_{m\pm}, q_\sigma^{(\alpha)}) = \mathbf{x}_m \text{ for } \sigma = 1,2;\\ &\hbar_\alpha G_{\partial\Omega_{ij}}^{(0\succ 0_\alpha)}(\mathbf{x}_m, q_1^{(\alpha)}) > 0 \text{ and } \hbar_\alpha G_{\partial\Omega_{ij}}^{(0\succ 0_\alpha)}(\mathbf{x}_m, q_2^{(\alpha)}) < 0.\end{aligned} \tag{4.11}$$

(ii) The boundary flow of $\mathbf{x}^{(0)}$ cannot be switched to the leaving flow of $\mathbf{x}^{(\alpha)}$ at the critical points of the flow barrier (i.e., $q^{(\alpha)} = q_\sigma^{(\alpha)},\ \sigma \in \{1,2\}$) if

$$\begin{aligned}&\mathbf{x}^{(0)}(t_m) = \mathbf{x}^{(0\succ 0_\alpha)}(t_{m\pm}, q_\sigma^{(\alpha)}) = \mathbf{x}_m \text{ for } \sigma = 1,2;\\ &\hbar_\alpha G_{\partial\Omega_{ij}}^{(0\succ 0_\alpha)}(\mathbf{x}_m, q_2^{(\alpha)})<0 \text{ and } \hbar_\alpha G_{\partial\Omega_{ij}}^{(s_\alpha,0\succ 0_\alpha)}(\mathbf{x}_m, q_1^{(\alpha)})=0 \text{ for } s_\alpha=0,1,2,\cdots,l_\alpha-1;\\ &\hbar_\alpha \mathbf{n}_{\partial\Omega_{ij}}^T(\mathbf{x}^{(0)}(t_{m+\epsilon})).[\mathbf{x}^{(0\succ 0_\alpha)}(t_{m+\epsilon}, q_1^{(\alpha)}) - \mathbf{x}^{(0)}(t_{m+\epsilon})] > 0.\end{aligned} \tag{4.12}$$

(iii) The boundary flow of $\mathbf{x}^{(0)}$ can be switched to the leaving flow of $\mathbf{x}^{(\alpha)}$ at the critical points of the flow barrier (i.e., $q^{(\alpha)} = q_\sigma^{(\alpha)},\ \sigma \in \{1,2\}$) if

$$\begin{aligned}&\mathbf{x}^{(0)}(t_m) = \mathbf{x}^{(0\succ 0_\alpha)}(t_{m\pm}, q_\sigma^{(\alpha)}) = \mathbf{x}_m \text{ for } \sigma = 1,2;\\ &\hbar_\alpha G_{\partial\Omega_{ij}}^{(0\succ 0_\alpha)}(\mathbf{x}_m, q_2^{(\alpha)})<0 \text{ and } \hbar_\alpha G_{\partial\Omega_{ij}}^{(s_\alpha,0\succ 0_\alpha)}(\mathbf{x}_m, q_1^{(\alpha)})=0 \text{ for } s_\alpha=0,1,2,\cdots,l_\alpha-1;\\ &\hbar_\alpha \mathbf{n}_{\partial\Omega_{ij}}^T(\mathbf{x}^{(0)}(t_{m+\epsilon})).[\mathbf{x}^{(0\succ 0_\alpha)}(t_{m+\epsilon}, q_1^{(\alpha)}) - \mathbf{x}^{(0)}(t_{m+\epsilon})] < 0.\end{aligned} \tag{4.13}$$

Lemma 4.1 (Theorem 10, [35]). *For a discontinuous dynamical system* $\dot{\boldsymbol{x}}^{(\alpha)} \equiv \boldsymbol{F}^{(\alpha)}(\boldsymbol{x}^{(\alpha)}, t, \boldsymbol{p}_\alpha)$ *in domain* $\Omega_\alpha(\alpha \in \{i,j\})$, *there is a point* $\mathbf{x}^{(0)}(t_m) \equiv \mathbf{x}_m \in \partial\Omega_{ij} = \{\boldsymbol{x}|\varphi_{ij}(\boldsymbol{x}, t, \boldsymbol{\lambda}) = 0\}$ *at time* t_m *between two adjacent domains* $\Omega_\alpha(\alpha = i,j)$. *For* $\mathbf{x}_m \in S \subseteq \partial\Omega_{ij}$, *there exist a source flow barrier* $\mathbf{F}^{(0\succ 0_\alpha)}(\mathbf{x}^{(\alpha)}, t, \boldsymbol{\pi}_\alpha, q^{(\alpha)})$ *for* $q^{(\alpha)} \in [q_1^{(\alpha)}, q_2^{(\alpha)}]$ *on the* α*-side of the boundary* $\partial\Omega_{ij}$ *and Eqs.* (4.9) *and* (4.10) *are satisfied.*

(i) The boundary flow of $\mathbf{x}^{(0)}$ *cannot be switched to the leaving flow of* $\mathbf{x}^{(\alpha)}$ *in a source flow on the* α*-side if and only if*

$$\begin{aligned}&\hbar_\alpha G_{\partial\Omega_{ij}}^{(0\succ 0_\alpha)}(\mathbf{x}_m, q_2^{(\alpha)}) < 0 \quad \text{and}\\ &\hbar_\alpha G_{\partial\Omega_{ij}}^{(0\succ 0_\alpha)}(\mathbf{x}_m, q_1^{(\alpha)}) > 0.\end{aligned} \tag{4.14}$$

(ii) The boundary flow of $\mathbf{x}^{(0)}$ *cannot be switched to the leaving flow of* $\mathbf{x}^{(\alpha)}$ *in a source flow at the critical points of the flow barrier on the* α*-side if and only if*

$$\begin{aligned}&\hbar_\alpha G_{\partial\Omega_{ij}}^{(s_\alpha,0\succ 0_\alpha)}(\mathbf{x}_m, q_1^{(\alpha)}) = 0, \ \text{for } s_\alpha = 0,1,2,\cdots,l_\alpha - 1\\ &\hbar_\alpha G_{\partial\Omega_{ij}}^{(l_\alpha,0\succ 0_\alpha)}(\mathbf{x}_m, q_1^{(\alpha)}) > 0 \quad \text{and}\\ &\hbar_\alpha G_{\partial\Omega_{ij}}^{(0\succ 0_\alpha)}(\mathbf{x}_m, q_2^{(\alpha)}) < 0.\end{aligned} \tag{4.15}$$

(iii) The boundary flow of $\mathbf{x}^{(0)}$ *is switched to the leaving flow of* $\mathbf{x}^{(\alpha)}$ *in a source flow at the critical points of the flow barrier on the* α*-side if and only if*

$$\begin{aligned}&\hbar_\alpha G_{\partial\Omega_{ij}}^{(s_\alpha,0\succ 0_\alpha)}(\mathbf{x}_m, q_1^{(\alpha)}) = 0, \ \text{for } s_\alpha = 0,1,2,\cdots,l_\alpha - 1\\ &\hbar_\alpha G_{\partial\Omega_{ij}}^{(l_\alpha,0\succ 0_\alpha)}(\mathbf{x}_m, q_1^{(\alpha)}) < 0 \quad \text{and}\\ &\hbar_\alpha G_{\partial\Omega_{ij}}^{(0\succ 0_\alpha)}(\mathbf{x}_m, q_2^{(\alpha)}) < 0.\end{aligned} \tag{4.16}$$

Lemma 4.2 (Theorem 16, [35]). *For a discontinuous dynamical system* $\dot{\boldsymbol{x}}^{(\alpha)} \equiv \boldsymbol{F}^{(\alpha)}(\boldsymbol{x}^{(\alpha)}, t, \boldsymbol{p}_\alpha)$ *in domain* $\Omega_\alpha(\alpha \in \{i,j\})$*, there is a point* $\mathbf{x}^{(0)}(t_m) \equiv \mathbf{x}_m \in \partial\Omega_{ij} = \{\boldsymbol{x}|\varphi_{ij}(\boldsymbol{x}, t, \boldsymbol{\lambda}) = 0\}$ *at time* t_m *between two adjacent domains* $\Omega_\alpha(\alpha = i, j)$*. Suppose the boundary flow in the sink flow on the boundary is formed under certain conditions. There is a boundary flow barrier of* $\mathbf{F}^{(0\succ 0_\alpha)}(\mathbf{x}^{(\alpha)}, t, \boldsymbol{\pi}_\alpha, q^{(\alpha)})$ *at* $q^{(\alpha)} \in [q_1^{(\alpha)}, q_2^{(\alpha)}]$ *on the* α*-side of the boundary* $\partial\Omega_{ij}$ *for* $\mathbf{x}_m \in S \subseteq \partial\Omega_{ij}$*, with the G-function*

$$0 \in [\hbar_\alpha G^{(0\succ 0_\alpha)}_{\partial\Omega_{ij}}(\mathbf{x}_m, q_2^{(\alpha)}), \hbar_\alpha G^{(0\succ 0_\alpha)}_{\partial\Omega_{ij}}(\mathbf{x}_m, q_1^{(\alpha)})] \subset R \tag{4.17}$$

and also there is a boundary flow barrier of $\mathbf{F}^{(0\succ 0_\beta)}(\mathbf{x}^{(\beta)}, t, \boldsymbol{\pi}_\beta, q^{(\beta)})$ *at* $q^{(\beta)} \in [q_1^{(\beta)}, q_2^{(\beta)}]$ *with the G-function*

$$0 \in [\hbar_\alpha G^{(0\succ 0_\beta)}_{\partial\Omega_{ij}}(\mathbf{x}_m, q_1^{(\beta)}), \hbar_\alpha G^{(0\succ 0_\alpha)}_{\partial\Omega_{ij}}(\mathbf{x}_m, q_2^{(\beta)})] \subset R \tag{4.18}$$

on the β*-side of the boundary* $\partial\Omega_{ij}$ $(\alpha, \beta \in \{i,j\}$ *and* $\alpha \neq \beta)$*. The boundary flow of* $\mathbf{x}^{(0)}$ *disappears on the* α*-side if and only if*

$$\left.\begin{array}{l} \text{both } \hbar_\alpha G^{(\alpha)}_{\partial\Omega_{ij}}(\mathbf{x}_m, t_{m+}) < 0 \\ \text{and } \begin{array}{l} \hbar_\alpha G^{(s_\alpha, 0\succ 0_\alpha)}_{\partial\Omega_{ij}}(\mathbf{x}_m, q_1^{(\alpha)}) = 0 \quad \text{for } s_\alpha = 0,1,2,\cdots,l_\alpha - 1; \\ \hbar_\alpha G^{(l_\alpha, 0\succ 0_\alpha)}_{\partial\Omega_{ij}}(\mathbf{x}_m, q_1^{(\alpha)}) < 0 \end{array} \\ \text{on the } \ \alpha - \text{side}, \end{array}\right\} \tag{4.19}$$

$$\left.\begin{array}{ll} \text{either} & \hbar_\alpha G^{(0\succ 0_\beta)}_{\partial\Omega_{ij}}(\mathbf{x}_m, q_1^{(\beta)}) > 0 \quad \text{but } \hbar_\alpha G^{(\beta)}_{\partial\Omega_{ij}}(\mathbf{x}_m, t_{m+}) < 0 \\ \text{or} & \hbar_\alpha G^{(0\succ 0_\beta)}_{\partial\Omega_{ij}}(\mathbf{x}_m, q_1^{(\beta)}) < 0 \quad \text{or } \hbar_\alpha G^{(s_\beta, 0\succ 0_\beta)}_{\partial\Omega_{ij}}(\mathbf{x}_m, q_1^{(\beta)}) = 0 \\ & \text{for } s_\beta = 0,1,2,\cdots,l_\beta - 1 \quad \text{and} \quad \hbar_\alpha G^{(l_\beta, 0\succ 0_\beta)}_{\partial\Omega_{ij}}(\mathbf{x}_m, q_1^{(\beta)}) < 0 \end{array}\right\} \tag{4.20}$$

on the $\beta - $ *side.*

4.2. Main Results

In the examined systems, according to Eq.(3.7), the two force boundaries relative to $V_{1,2}$ (i.e. $y = V_1$ or V_2) are C_0-continuous. However, the force boundary relative to the velocity V is a discontinuous boundary. If the coming and leaving flow barriers does not exist, the coming and leaving flow vector fields for $\mathbf{x}_m \in \partial\Omega_{\alpha\beta}(\alpha, \beta \in \{1,3\})$ are

$$\mathbf{F}^{(\alpha)}(\mathbf{x}_m, t_{m\pm}) = (y_m, F^{(\alpha)}(\mathbf{x}_m, t_{m\pm}))^T. \tag{4.21}$$

The time t_m reflects the moment for the motion just on the boundary and the time $t_{m\pm} = t_m \pm 0$ represents the flows in the regions instead of the boundary. However, due to the static friction force, the boundary flow barriers on the boundary $\partial\Omega_{13}$ for $\mathbf{x}_m \in \partial\Omega_{\alpha\beta}(\alpha, \beta \in \{1,3\})$ are

$$\begin{aligned} &\mathbf{F}^{(0\succ 0_\alpha)}(\mathbf{x}_m, q^{(\alpha)}) = (y_m, F^{(0\succ 0_\alpha)}(\mathbf{x}_m, t_m, q^{(\alpha)}))^T \\ &F^{(0\succ 0_\alpha)}(\mathbf{x}_m, t_m, q^{(\alpha)}) = Q_0 \cos\omega_1 t_m + Q_0 \cos\omega_2 t_m - F^{(\alpha)}_{f_s}(q^{(\alpha)}) - 2a_\alpha y_m - b_\alpha x_m. \end{aligned} \tag{4.22}$$

The static friction force per unit mass on the boundary $\partial\Omega_{13}$ are

$$F_{f_s}^{(1)}(q^{(1)}) \in (-\infty, F_N\nu_s] \text{ and}$$
$$F_{f_s}^{(3)}(q^{(3)}) \in [-F_N\nu_s, +\infty). \tag{4.23}$$

The lower and upper limits of the boundary flow barriers on the α-side of the boundary $\partial\Omega_{13}$ for $\mathbf{x}_m \in \partial\Omega_{\alpha\beta}(\alpha, \beta \in \{1, 3\})$ are

$$\left.\begin{aligned}
&\mathbf{F}^{(0\succ 0_1)}(\mathbf{x}_m, q_1^{(1)}) = (y_m, F^{(0\succ 0_1)}(\mathbf{x}_m, t_{m+}, q_1^{(1)}))^T,\\
&F^{(0\succ 0_1)}(\mathbf{x}_m, t_{m+}, q_1^{(1)}) = Q_0\cos\omega_1 t_m + Q_0\cos\omega_2 t_m - F_N\nu_s - 2a_1 y_m - b_1 x_m,\\
&\mathbf{F}^{(0\succ 0_1)}(\mathbf{x}_m, q_2^{(1)}) = (y_m, +\infty)^T;\\
&\mathbf{F}^{(0\succ 0_3)}(\mathbf{x}_m, q_1^{(3)}) = (y_m, F^{(0\succ 0_3)}(\mathbf{x}_m, t_{m+}, q_1^{(3)}))^T,\\
&F^{(0\succ 0_3)}(\mathbf{x}_m, t_{m+}, q_1^{(3)}) = Q_0\cos\omega_1 t_m + Q_0\cos\omega_2 t_m + F_N\nu_s - 2a_3 y_m - b_3 x_m,\\
&\mathbf{F}^{(0\succ 0_3)}(\mathbf{x}_m, q_2^{(3)}) = (y_m, -\infty)^T.
\end{aligned}\right\} \tag{4.24}$$

Before discussing of the analytical conditions, the G-functions can be reduced for the special boundary. From Eq. (3.3), the boundaries are straight lines in phase space, which implies that the normal vectors are constant vectors. Using Eq. (4.3), the normal vector for the boundary$\partial\Omega_{\alpha\beta}$ with $\alpha, \beta \in \{1, 2, 3, 4\}$ is

$$\mathbf{n}_{\partial\Omega_{\alpha\beta}} = \mathbf{n}_{\partial\Omega_{\beta\alpha}} = \left(\frac{\partial\varphi_{\alpha\beta}}{\partial x}, \frac{\partial\varphi_{\alpha\beta}}{\partial y}\right)^T = (0, 1)^T. \tag{4.25}$$

The normal vectors of the boundaries ($\partial\Omega_{12}$ and $\partial\Omega_{21}$), ($\partial\Omega_{13}$ and $\partial\Omega_{31}$) and ($\partial\Omega_{34}$ and $\partial\Omega_{43}$) point to the domains Ω_2, Ω_1 and Ω_3, respectively. Thus

$$\left.\begin{aligned}
&G_{\partial\Omega_{\alpha\beta}}^{(\alpha)}(\mathbf{x}^{(\alpha)}, t) = \mathbf{n}_{\partial\Omega_{\alpha\beta}}^T.\mathbf{F}^{(\alpha)}(\mathbf{x}^{(\alpha)}, t) = F^{(\alpha)}(\mathbf{x}^{(\alpha)}, t),\\
&G_{\partial\Omega_{\alpha\beta}}^{(1,\alpha)}(\mathbf{x}^{(\alpha)}, t) = \mathbf{n}_{\partial\Omega_{\alpha\beta}}^T.D\mathbf{F}^{(\alpha)}(\mathbf{x}^{(\alpha)}, t) = DF^{(\alpha)}(\mathbf{x}^{(\alpha)}, t);\\
&G^{(0\succ 0_\alpha)}(\mathbf{x}^{(\alpha)}, t) = \mathbf{n}_{\partial\Omega_{\alpha\beta}}^T.\mathbf{F}^{(0\succ 0_\alpha)}(\mathbf{x}^{(\alpha)}, t) = F^{(0\succ 0_\alpha)}(\mathbf{x}^{(\alpha)}, t),\\
&G^{(1,0\succ 0_\alpha)}(\mathbf{x}^{(\alpha)}, t) = \mathbf{n}_{\partial\Omega_{\alpha\beta}}^T.D\mathbf{F}^{(0\succ 0_\alpha)}(\mathbf{x}^{(\alpha)}, t) = DF^{(0\succ 0_\alpha)}(\mathbf{x}^{(\alpha)}, t),
\end{aligned}\right\} \tag{4.26}$$

where

$$\left.\begin{aligned}
&D\mathbf{F}^{(\alpha)}(\mathbf{x}^{(\alpha)}, t) = (F^{(\alpha)}(\mathbf{x}^{(\alpha)}, t), DF^{(\alpha)}(\mathbf{x}^{(\alpha)}, t))^T,\\
&DF^{(\alpha)}(\mathbf{x}^{(\alpha)}, t)) = \nabla F^{(\alpha)}(\mathbf{x}^{(\alpha)}, t)).\mathbf{F}^{(\alpha)}(\mathbf{x}^{(\alpha)}, t) + \partial_t F^{(\alpha)}(\mathbf{x}^{(\alpha)}, t);\\
&D\mathbf{F}^{(0\succ 0_\alpha)}(\mathbf{x}^{(\alpha)}, t) = (F^{(0\succ 0_\alpha)}(\mathbf{x}^{(\alpha)}, t), DF^{(0\succ 0_\alpha)}(\mathbf{x}^{(\alpha)}, t))^T,\\
&DF^{(0\succ 0_\alpha)}(\mathbf{x}^{(\alpha)}, t) = \nabla F^{(0\succ 0_\alpha)}(\mathbf{x}^{(\alpha)}, t).\mathbf{F}^{(0\succ 0_\alpha)}(\mathbf{x}^{(\alpha)}, t) + \partial_t F^{(0\succ 0_\alpha)}(\mathbf{x}^{(\alpha)}, t).
\end{aligned}\right\} \tag{4.27}$$

From the aforementioned definitions and lemmas, the analytical conditions for the flow switching in the friction oscillator with flow barriers will be developed in the following theorems.

Theorem 4.1. *For the friction oscillator described in Section 2, the force conditions for passable motions on the boundary* $\partial\Omega_{\alpha\beta}$ *are*

$$\left.\begin{aligned} &F^{(\alpha)}(\mathbf{x}_m, t_{m-}) > 0 \textit{ and } F^{(\beta)}(\mathbf{x}_m, t_{m+}) > 0 \textit{ for } (\alpha,\beta) \in \{(1,2),(3,1),(4,3)\};\\ &F^{(\alpha)}(\mathbf{x}_m, t_{m-}) < 0 \textit{ and } F^{(\beta)}(\mathbf{x}_m, t_{m+}) < 0 \textit{ for } (\alpha,\beta) \in \{(2,1),(1,3),(3,4)\}. \end{aligned}\right\} \tag{4.28}$$

Proof. From Luo [45], the necessary and sufficient conditions for the passable motions on the boundary $\partial\Omega_{\alpha\beta}$ are

$$\left.\begin{aligned} \hbar_\alpha G^{(\alpha)}_{\partial\Omega_{\alpha\beta}}(\mathbf{x}_m, t_{m-}) > 0,\\ \hbar_\alpha G^{(\beta)}_{\partial\Omega_{\alpha\beta}}(\mathbf{x}_m, t_{m+}) > 0. \end{aligned}\right\} \tag{4.29}$$

From Eq. (4.26), the 0-order G-function for the boundaries are

$$\left.\begin{aligned} G^{(\alpha)}_{\partial\Omega_{\alpha\beta}}(\mathbf{x}_m, t_{m-}) = F^{(\alpha)}(\mathbf{x}_m, t_{m-}),\\ G^{(\beta)}_{\partial\Omega_{\alpha\beta}}(\mathbf{x}_m, t_{m+}) = F^{(\beta)}(\mathbf{x}_m, t_{m+}). \end{aligned}\right\} \tag{4.30}$$

For $(\alpha,\beta) \in \{(1,2),(3,1),(4,3)\}$, according to Eq. (4.8), the sign function $\hbar_\alpha = 1$. The force conditions for passable motions on the boundary $\partial\Omega_{\alpha\beta}$ are

$$F^{(\alpha)}(\mathbf{x}_m, t_{m-}) > 0 \text{ and } F^{(\beta)}(\mathbf{x}_m, t_{m+}) > 0. \tag{4.31}$$

However, for $(\alpha,\beta) \in \{(2,1),(1,3),(3,4)\}$, $\hbar_\alpha = -1$. Thus, the force conditions for passable motions on the boundary $\partial\Omega_{\alpha\beta}$ are

$$F^{(\alpha)}(\mathbf{x}_m, t_{m-}) < 0 \text{ and } F^{(\beta)}(\mathbf{x}_m, t_{m+}) < 0. \tag{4.32}$$

□

Theorem 4.2. *For the friction oscillator described in Section 2, the force conditions for the stick motion between the mass and the translation belt on the corresponding boundary* $\partial\Omega_{13}$ *are*

$$F^{(1)}(\mathbf{x}_m, t_{m-}) < 0 \textit{ and } F^{(3)}(\mathbf{x}_m, t_{m-}) > 0. \tag{4.33}$$

Proof. From Luo [38], the existence condition of the stick motion on the boundary $\partial\Omega_{\alpha\beta}$ is

$$\left.\begin{aligned} \hbar_\alpha G^{(\alpha)}_{\partial\Omega_{\alpha\beta}}(\mathbf{x}_m, t_{m-}) > 0,\\ \hbar_\alpha G^{(\beta)}_{\partial\Omega_{\alpha\beta}}(\mathbf{x}_m, t_{m-}) < 0. \end{aligned}\right\} \tag{4.34}$$

Note that the normal vector of the boundary $\partial\Omega_{13}$ points to the domain Ω_1, then $\hbar_\alpha = -1$. Combined with Eq. (4.30), the force conditions for the stick motion on the boundary $\partial\Omega_{13}$ can be obtained as Eq. (4.33). □

Theorem 4.3. *For the friction oscillator described in Section 2, the force conditions for the onset of the stick motion on $\partial\Omega_{13}$ are*

$$\left.\begin{aligned}&F^{(1)}(\mathbf{x}_m, t_{m-}) < 0,\\&F^{(3)}(\mathbf{x}_m, t_{m\pm}) = 0 \text{ with } DF^{(3)}(\mathbf{x}_m, t_{m\pm}) < 0\end{aligned}\right\} \text{for } \Omega_1 \to \partial\Omega_{13},$$
$$\left.\begin{aligned}&F^{(3)}(\mathbf{x}_m, t_{m-}) > 0,\\&F^{(1)}(\mathbf{x}_m, t_{m\pm}) = 0 \text{ with } DF^{(1)}(\mathbf{x}_m, t_{m\pm}) > 0\end{aligned}\right\} \text{for } \Omega_3 \to \partial\Omega_{13}. \tag{4.35}$$

Proof. From Luo [38], the switching bifurcation from the nonstick motion to the stick motion is

$$\left.\begin{aligned}&\hbar_\alpha G^{(\alpha)}_{\partial\Omega_{\alpha\beta}}(\mathbf{x}_m, t_{m-}) > 0;\\&\hbar_\alpha G^{(\beta)}_{\partial\Omega_{\alpha\beta}}(\mathbf{x}_m, t_{m\pm}) = 0,\\&\hbar_\alpha G^{(1,\beta)}_{\partial\Omega_{\alpha\beta}}(\mathbf{x}_m, t_{m\pm}) > 0\end{aligned}\right\} \text{ for } \Omega_\alpha \to \partial\Omega_{\alpha\beta}. \tag{4.36}$$

For $\Omega_1 \to \partial\Omega_{13}, \hbar_\alpha = -1$. However, for $\Omega_3 \to \partial\Omega_{13}, \hbar_\alpha = +1$. Combined with Eq. (4.26), the force conditions for the onset of the stick motion on $\partial\Omega_{13}$ can be obtained as Eq. (4.35). □

Once the stick motion is formed under Eq. (4.34), the boundary flow will control the motion on the boundary, which are independent of the vector fields except for the conditions in Eq. (4.34). For the examined system, the boundary flow on the boundary possesses a boundary flow barrier caused by the static friction force. To obtain a new nonstick motion on the moving belt, the non-friction force must be greater than the static friction force. Next, the necessary and sufficient conditions for vanishing of the stick motion are given as follows.

Theorem 4.4. *For the friction oscillator described in Section 2, the force conditions for vanishing of the stick motion are*

$$\left.\begin{aligned}&\text{either } F^{(0\succ 0_1)}(\mathbf{x}_m, q_1^{(1)}) > 0 \text{ but } F^{(1)}(\mathbf{x}_m, t_{m+}) < 0\\&\text{or } F^{(0\succ 0_1)}(\mathbf{x}_m, q_1^{(1)}) < 0\\&\quad \text{or } F^{(0\succ 0_1)}(\mathbf{x}_m, q_1^{(1)}) = 0 \text{ with } DF^{(0\succ 0_1)}(\mathbf{x}_m, q_1^{(1)}) < 0;\end{aligned}\right\} \tag{4.37}$$

$$\left.\begin{aligned}&F^{(3)}(\mathbf{x}_m, t_{m+}) < 0\\&F^{(0\succ 0_3)}(\mathbf{x}_m, q_1^{(3)}) = 0 \text{ with } DF^{(0\succ 0_3)}(\mathbf{x}_m, q_1^{(3)}) < 0\end{aligned}\right\} \tag{4.38}$$

from $\partial\Omega_{13} \to \Omega_3$, and

$$\left.\begin{aligned}&\text{either } F^{(0\succ 0_3)}(\mathbf{x}_m, q_1^{(3)}) < 0 \text{ but } F^{(3)}(\mathbf{x}_m, t_{m+}) > 0\\&\text{or } F^{(0\succ 0_3)}(\mathbf{x}_m, q_1^{(3)}) > 0\\&\quad \text{or } F^{(0\succ 0_3)}(\mathbf{x}_m, q_1^{(3)}) = 0 \text{ with } DF^{(0\succ 0_3)}(\mathbf{x}_m, q_1^{(3)}) > 0;\end{aligned}\right\} \tag{4.39}$$

$$\left.\begin{array}{l} F^{(1)}(\mathbf{x}_m, t_{m+}) > 0 \\ F^{(0\succ 0_1)}(\mathbf{x}_m, q_1^{(1)}) = 0 \; with \; DF^{(0\succ 0_1)}(\mathbf{x}_m, q_1^{(1)}) > 0 \end{array}\right\} \tag{4.40}$$

from $\partial\Omega_{13} \to \Omega_1$.

Proof. From Lemma 4.1 and 4.2, the necessary and sufficient conditions for vanishing of the stick motions on the α-side are

$$\left.\begin{array}{ll} \text{both} & \hbar_\alpha G^{(\alpha)}_{\partial\Omega_{\alpha\beta}}(\mathbf{x}_m, t_{m+}) < 0, \\ \text{and} & \hbar_\alpha G^{(0\succ 0_\alpha)}_{\partial\Omega_{\alpha\beta}}(\mathbf{x}_m, q_1^{(\alpha)}) = 0 \text{ with } \hbar_\alpha G^{(1,0\succ 0_\alpha)}_{\partial\Omega_{\alpha\beta}}(\mathbf{x}_m, q_1^{(\alpha)}) < 0; \end{array}\right\} \tag{4.41}$$

$$\left.\begin{array}{ll} \text{either} & \hbar_\alpha G^{(0\succ 0_\beta)}_{\partial\Omega_{\alpha\beta}}(\mathbf{x}_m, q_1^{(\beta)}) > 0 \text{ but } \hbar_\alpha G^{(\beta)}_{\partial\Omega_{\alpha\beta}}(\mathbf{x}_m, t_{m+}) < 0 \\ \text{or} & \hbar_\alpha G^{(0\succ 0_\beta)}_{\partial\Omega_{\alpha\beta}}(\mathbf{x}_m, q_1^{(\beta)}) < 0 \\ & \text{or } \hbar_\alpha G^{(0\succ 0_\beta)}_{\partial\Omega_{\alpha\beta}}(\mathbf{x}_m, q_1^{(\beta)}) = 0 \text{ with } \hbar_\alpha G^{(1,0\succ 0_\beta)}_{\partial\Omega_{\alpha\beta}}(\mathbf{x}_m, q_1^{(\beta)}) < 0. \end{array}\right\} \tag{4.42}$$

For $\partial\Omega_{13} \to \Omega_3, \hbar_\alpha = +1$. Combined with Eq. (4.26), the force conditions for vanishing of the stick motion are obtained as Eqs. (4.37) and (4.38).
For $\partial\Omega_{13} \to \Omega_1, \hbar_\alpha = -1$. Similarly, one can obtain the force conditions as Eqs. (4.39) and (4.40). □

Theorem 4.5. *For the friction oscillator described in Section 2, the force conditions for grazing motions to the boundary are*

$$\left.\begin{array}{l} F^{(\alpha)}(\mathbf{x}_m, t_{m\pm}) = 0; \\ DF^{(\alpha)}(\mathbf{x}_m, t_{m\pm}) > 0 \; for \; \alpha = 2,1,3 \; on \; \partial\Omega_{\alpha\beta} \in \{\partial\Omega_{21}, \partial\Omega_{13} \; and \; \partial\Omega_{34}\}, \\ DF^{(\alpha)}(\mathbf{x}_m, t_{m\pm}) < 0 \; for \; \alpha = 1,3,4 \; on \; \partial\Omega_{\alpha\beta} \in \{\partial\Omega_{12}, \partial\Omega_{31} \; and \; \partial\Omega_{43}\}. \end{array}\right\} \tag{4.43}$$

Proof. From Luo [45], the necessary and sufficient conditions for grazing motion to the boundary in Eq. (3.3) are

$$\left.\begin{array}{l} G^{(\alpha)}_{\partial\Omega_{\alpha\beta}}(\mathbf{x}_m, t_{m\pm}) = 0 \text{ for } \alpha \neq \beta; \\ G^{(1,\alpha)}_{\partial\Omega_{\alpha\beta}}(\mathbf{x}_m, t_{m\pm}) > 0 \text{ for } \alpha = 2,1,3 \text{ at } \partial\Omega_{\alpha\beta} \in \{\partial\Omega_{21}, \partial\Omega_{13} \text{ and } \partial\Omega_{34}\}, \\ G^{(1,\alpha)}_{\partial\Omega_{\alpha\beta}}(\mathbf{x}_m, t_{m\pm}) < 0 \text{ for } \alpha = 1,3,4 \text{ at } \partial\Omega_{\alpha\beta} \in \{\partial\Omega_{12}, \partial\Omega_{31} \text{ and } \partial\Omega_{43}\}. \end{array}\right\} \tag{4.44}$$

Using the Eq. (4.26), the force conditions for the grazing motions are easily obtained as Eq. (4.43). □

4.3. Generic Mappings and Numerical Simulations

To illustrate the motion with flow barriers in discontinuous dynamical systems, the basic mappings are introduced as in [34]. The mappings are determined by the close-form solution of the differential equation in the corresponding domain. With the initial condition (t_k, x_k, V), the direct integration of Eq. (2.3) yields

$$x = V(t - t_k) - x_k. \tag{4.45}$$

Substitution of Eq. (4.45) into Eq. (3.6) gives the forces for the very small neighborhood of the stick motion in the domains Ω_j ($j \in \{1,3\}$). Because of the static friction jumping, the forces at $(\mathbf{x}_m, t_m)$ for the coming and leaving flows and the boundary flow barriers on the boundary $\partial\Omega_{13}$ are:

$$F^{(j)}(\mathbf{x}_m, t_{m\pm}) = Q_0 \cos\omega_1 t_{m\pm} + Q_0 \cos\omega_2 t_{m\pm} - 2a_j V - b_j x_m - d_j, \tag{4.46}$$

$$F^{(0\succ 0_j)}(\mathbf{x}_m, q_1^{(j)}) = Q_0 \cos\omega_1 t_{m\pm} + Q_0 \cos\omega_2 t_{m\pm} - 2a_j V - b_j x_m - d_j^{(0\succ 0_j)}, \tag{4.47}$$

where $d_1 = -d_3 = \nu_k F_N$ and $d_1^{(0\succ 0_1)} = -d_3^{(0\succ 0_3)} = \nu_s F_N$.

To develop generic mappings, the switching sets on the boundary should be numbered first. The switching set for the discontinuous force boundary is represented by Σ_1, and the other separation boundaries are Σ_2 and Σ_3. The switching sets for the three boundaries are

$$\Sigma_\alpha = \Sigma_\alpha^0 \cup \Sigma_\alpha^+ \cup \Sigma_\alpha^- \quad \text{for } \alpha = 1,2,3. \tag{4.48}$$

The corresponding switching subsets are defined as

$$\left.\begin{aligned} &\Sigma_\alpha^0 = \{(x_i, \Omega t_i) | \dot{x}_i = V_\rho\} \quad \text{and} \\ &\Sigma_\alpha^\pm = \{(x_i, \Omega t_i) | \dot{x}_i = V_\rho^\pm\}, \end{aligned}\right\} \tag{4.49}$$

where $V_\rho^\pm = \lim_{\delta\to 0}(V_\rho \pm \delta)$ for an arbitrary small $\delta > 0$ and $\rho = 0,1,2$ for $\alpha = 1,2,3$.

In phase space, the trajectories in Ω_j starting and ending at the separation boundaries are sketched in Figure 3. The starting and ending points for mapping $P_{j_{\beta\alpha}}$ in Ω_j are $(x_k, \dot{x_k}, t_k)$ on Σ_α and $(x_{k+1}, \dot{x_{k+1}}, t_{k+1})$ on Σ_β, respectively. Notice that the indices $j = 1,2,3,4$ and $\alpha, \beta = 1,2,3$ are for domains and boundaries, respectively. The stick mapping is $P_{0_{11}}$. Thus, from the switching sets, the mappings are defined as

$$\left.\begin{aligned} &P_{1_{11}}: \Sigma_1^+ \xrightarrow{\Omega_1} \Sigma_1^+, \quad P_{1_{22}}: \Sigma_2^- \xrightarrow{\Omega_1} \Sigma_2^-, \\ &P_{2_{22}}: \Sigma_2^+ \xrightarrow{\Omega_2} \Sigma_2^+, \quad P_{3_{11}}: \Sigma_1^- \xrightarrow{\Omega_3} \Sigma_1^-, \\ &P_{3_{33}}: \Sigma_3^+ \xrightarrow{\Omega_3} \Sigma_3^+, \quad P_{4_{33}}: \Sigma_3^- \xrightarrow{\Omega_4} \Sigma_3^- \end{aligned}\right\} \tag{4.50}$$

for the local mappings,

$$\left.\begin{aligned} &P_{1_{21}}: \Sigma_1^+ \xrightarrow{\Omega_1} \Sigma_2^-, \quad P_{1_{12}}: \Sigma_2^- \xrightarrow{\Omega_1} \Sigma_1^+, \\ &P_{3_{31}}: \Sigma_1^- \xrightarrow{\Omega_3} \Sigma_3^+, \quad P_{3_{13}}: \Sigma_3^+ \xrightarrow{\Omega_3} \Sigma_1^- \end{aligned}\right\} \tag{4.51}$$

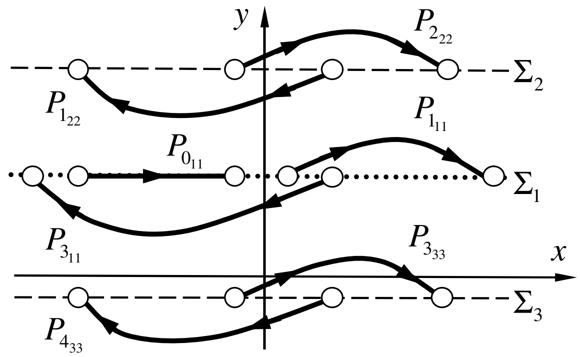
for the global mappings and

$$P_{0_{11}}: \Sigma_1^0 \xrightarrow{\partial\Omega_{13}} \Sigma_1^0 \tag{4.52}$$

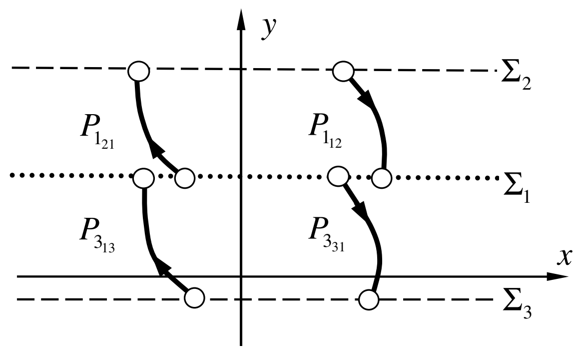
for the stick mapping.

The governing equations of $P_{0_{11}}$ for a sink flow to leave for Ω_j with $j \in \{1,3\}$ are

$$\left.\begin{aligned} &-x_{k+1} + V(t_{k+1} - t_k) + x_k = 0, \\ &Q_0 \cos\omega_1 t_{k+1} + Q_0 \cos\omega_2 t_{k+1} - 2a_j V - b_j[V(t_{k+1} - t_k) + x_k] - d_j^{(0\succ 0_j)} = 0. \end{aligned}\right\} \tag{4.53}$$



(a)



(b)

Figure 3. Regular and stick mappings: (a) local and stick mappings, (b) global mappings.

Since the differential equation in each domain is linear, the closed form solution for such a linear differential equation can be obtained [34]. For the non-stick motion, the governing equations for mapping $P_{j_{\beta\alpha}}$ $(j = 1, 2, 3, 4$ and $\alpha, \beta = 1, 2, 3)$ are

$$\left.\begin{aligned} f_1^{(j_{\beta\alpha})}(x_k, \Omega t_k, x_{k+1}, \Omega t_{k+1}) = 0, \\ f_2^{(j_{\beta\alpha})}(x_k, \Omega t_k, x_{k+1}, \Omega t_{k+1}) = 0. \end{aligned}\right\} \tag{4.54}$$

From the foregoing relations, the periodic motions for such a periodically forced, frictional oscillator can be obtained. The details can be referred to [34].

The parameters $(m = 5, a_{1,2,3,4} = 0.1, b_{1,2,3,4} = 30, \mu_s = 0.5, \mu_k = 0.4, \mu_{1,3} = 0.1, \mu_{2,4} = 0.5$ and $g = 9.8)$ are considered for numerical illustrations. First, consider the non-stick periodic motion pertaining to $P_{4_{33}3_{31}1_{12}2_{22}1_{21}3_{13}} = P_{4_{33}} \circ P_{3_{31}} \circ P_{1_{12}} \circ P_{2_{22}} \circ P_{1_{21}} \circ P_{3_{13}}$. As in [34], the periodic motions in such an oscillator can be obtained using above conditions. The responses of displacement, velocity, the phase plane and force distributions are presented respectively in Figs. 4(a)-(f) for the periodic motion of mapping $P_{4_{33}3_{31}1_{12}2_{22}1_{21}3_{13}}$ with $\omega_1 = 5, \omega_2 = 15, V = 3, V_1 = 4.5, V_2 = 1.5$, and $Q_0 = 70$ and the initial condition $(t_k, x_k, \dot{x}_k) \approx (2.0389, -2.2792, 1.50)$. The responses in different domains are depicted through different colour curves, accordingly. The red filled cycle is the starting points of the periodic motion and the other circles are switching points. The arrows give the direction of the periodic motion. In addition, the corresponding mappings are labeled in plots. The displacement and velocity responses are illustrated in Figure 4(a) and Figure 4(b). The periodic trajectory in phase plane is clearly shown in Figure 4(c). This periodic motion intersects with the boundaries $\partial\Omega_{13}, \partial\Omega_{12}$ and $\partial\Omega_{34}$. Consider the force in Ω_α $(\alpha = 1, 3)$ as

$$\left.\begin{aligned} F^{(1)} &\equiv F^{(1)}(\mathbf{x}, t) = Q_0 \cos\omega_1 t + Q_0 \cos\omega_2 t - 2a_1\dot{x} - b_1 x + \nu_2(\dot{x} - V) - \mu_k g, \\ F^{(3)} &\equiv F^{(3)}(\mathbf{x}, t) = Q_0 \cos\omega_1 t + Q_0 \cos\omega_2 t - 2a_3\dot{x} - b_3 x + \nu_3(\dot{x} - V) + \mu_k g. \end{aligned}\right\} \tag{4.55}$$

Therefore, with $\dot{x}_m = V$, the force conditions on the boundary $\partial\Omega_{31}$ from domain Ω_3 to Ω_1 are from Eq. (4.28) at time t_{m^-} and t_{m^+}

$$\left.\begin{aligned} F_-^{(3)} &\equiv F^{(3)}(\mathbf{x}_m, t_{m^-}) = Q_0 \cos\omega_1 t_{m^-} + Q_0 \cos\omega_2 t_{m^-} - 2a_3 V - b_3 x_m + \mu_k g > 0, \\ F_+^{(1)} &\equiv F^{(1)}(\mathbf{x}_m, t_{m^+}) = Q_0 \cos\omega_1 t_{m^+} + Q_0 \cos\omega_2 t_{m^+} - 2a_1 V - b_1 x_m - \mu_k g > 0; \end{aligned}\right\} \tag{4.56}$$

and the force conditions on the boundary $\partial\Omega_{13}$ from domain Ω_1 to Ω_3 are at time t_{m^-} and t_{m^+}

$$\left.\begin{aligned} F_-^{(1)} &\equiv F^{(1)}(\mathbf{x}_m, t_{m^-}) = Q_0 \cos\omega_1 t_{m^-} + Q_0 \cos\omega_2 t_{m^-} - 2a_1 V - b_1 x_m - \mu_k g < 0, \\ F_+^{(3)} &\equiv F^{(3)}(\mathbf{x}_m, t_{m^+}) = Q_0 \cos\omega_1 t_{m^+} + Q_0 \cos\omega_2 t_{m^+} - 2a_3 V - b_3 x_m + \mu_k g < 0. \end{aligned}\right\} \tag{4.57}$$

Because $a_1 = a_3$ and $b_1 = b_3$, the total force on the boundary $\partial\Omega_{13}$ is discontinuous. But the total force on the boundary $\partial\Omega_{34}$ and $\partial\Omega_{12}$ is continuous. Such force characteristics of the periodic motion can be observed in Figure 4(d). The force distributions along both displacement and velocity are plotted in Figs. 4(e)-(f). Next, consider a stick periodic motion relative to mapping $P_{4_{33}3_{31}0_{11}3_{13}}$ with $\omega_1 = 1, \omega_2 = 3, Q_0 = 10, V = 1.96, V_1 = 2.42, V_2 = 1.5$. The other parameters are the same as in the first example. The equation of motion for the discussed oscillator is

$$m\ddot{x} + c\dot{x} + (k_1 + k_2)x = P\text{cos}\omega_1 t + P\text{cos}\omega_2 t - \bar{F}_f. \tag{4.58}$$

By introducing the notations

$$\omega_0 = \sqrt{\frac{k_1 + k_2}{m}}, \tau = \omega_0 t, \lambda = \frac{c}{m\omega_0}, u_0 = \frac{P}{m\omega_0^2}, x_f = \frac{\bar{F}_f}{(k_1 + k_2)}, \eta = \frac{\omega_1}{\omega_0}, \tag{4.59}$$

Eq. (4.58) can be normalized as

$$x'' + \lambda x' + x = u_0\text{cos}(\eta\tau) + u_0\text{cos}(\frac{\omega_2}{\omega_1}\eta\tau) - x_f, \tag{4.60}$$

where the prime x' indicates differentiation with respect to the non-dimensional time τ. The responses of displacement, velocity, the phase plane and force distributions are presented respectively in Figs. 5(a)-(f) for the periodic motion of mapping

$P_{4_{33}3_{31}0_{11}3_{13}}$ with the initial condition $(\tau_k, x_k, x_k^{'}) \approx (30.9346, -0.0684, 0.2739)$. The response in different domains are depicted through different colour curves, accordingly. The red filled cycle is the starting points of the periodic motion and the other circles are switching points. The arrows give the direction of the periodic motion. In addition, the corresponding mappings are labeled in plots.

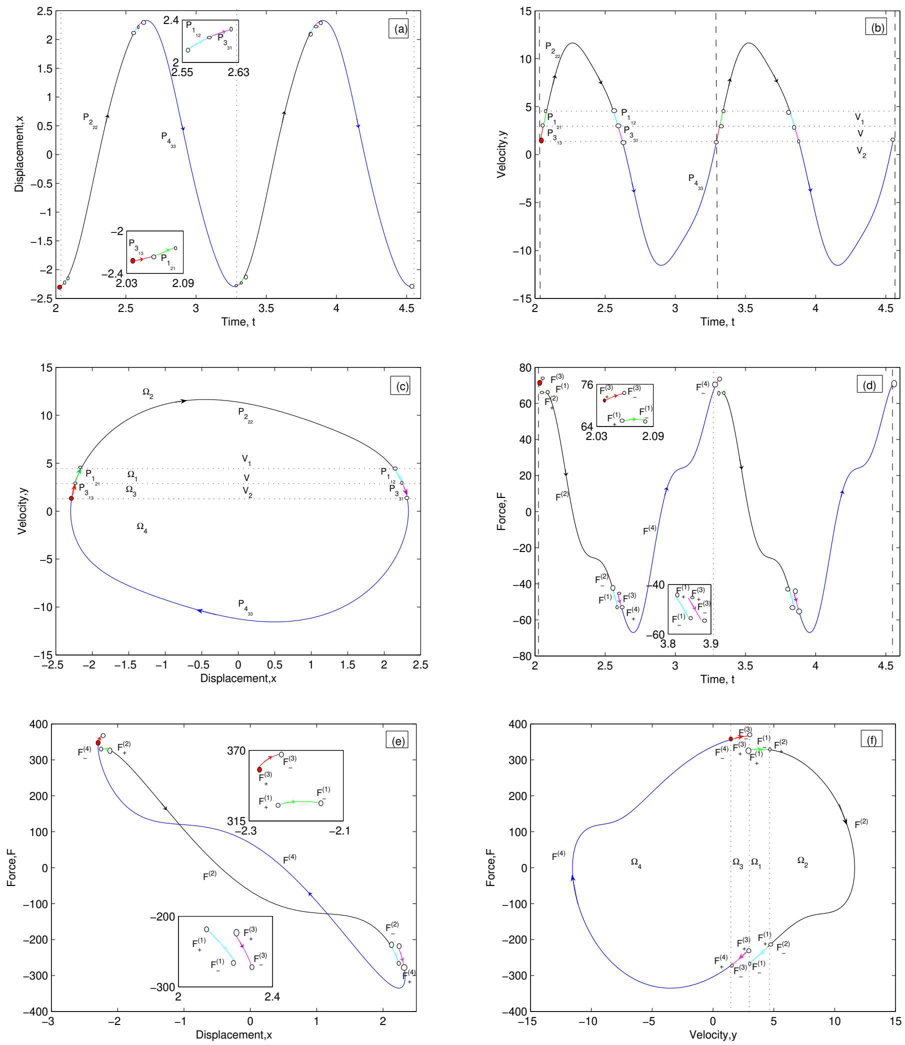


Figure 4. Periodical responses of mapping $P_{4_{33}} \circ P_{3_{31}} \circ P_{1_{12}} \circ P_{2_{22}} \circ P_{1_{21}} \circ P_{3_{13}}$: (a)displacement time history, (b) velocity time history, (c) phase plane, (d) forces time history, (e) force distribution along displacement, (f) force distribution along velocity for $\omega_1 = 5, \omega_2 = 15$ and $Q_0 = 70$ with the initial conditions $(t_k, x_k, \dot{x_k}) = (2.0389, -2.2792, 1.50)$.

In Figure 5(a), the displacement is continuous because the velocity is C^0-continuous. The nonsmoothness of the velocity response is observed in Figure 5(b) because the force is C^0-discontinuous. The stick motion in the velocity response is clearly observed. The stick motion in phase plane is a straight line along the discontinuous boundary, as shown in Figure 5(c). The trajectory of this periodic motion exists in domains Ω_3 and Ω_4. The force description in domains Ω_3 and Ω_4 is given as

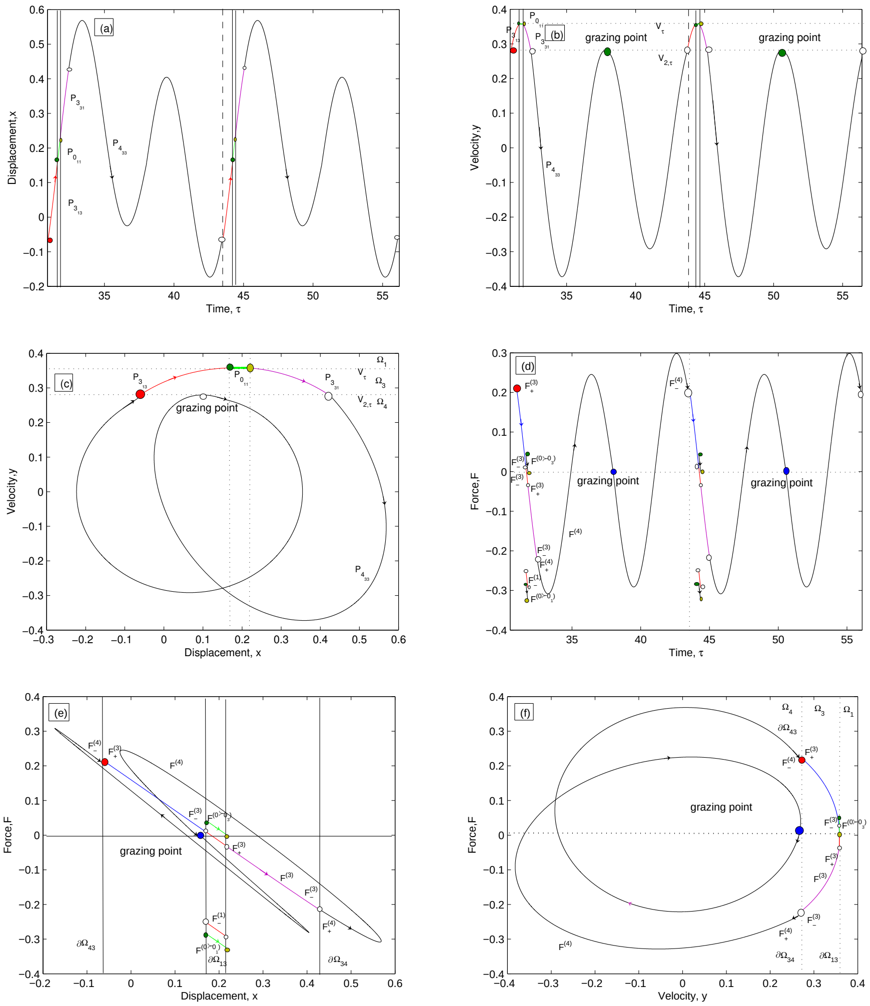


Figure 5. Periodical responses of mapping $P_{4_{33}} \circ P_{3_{31}} \circ P_{0_{11}} \circ P_{3_{13}}$: (a)displacement time history, (b) velocity time history, (c) phase plane, (d) forces time history, (e) force distribution along displacement, (f) force distribution along velocity for $\omega_1 = 1, \omega_2 = 3$ and $Q_0 = 10$ with the initial conditions $(\tau_k, x_k, x_k{}') = (30.9346, -0.0684, 0.2739)$.

$$\left.\begin{aligned} F^{(3)} &\equiv F^{(3)}(\mathbf{x},\tau) = \frac{1}{\omega_0^2} F^{(3)}(\mathbf{x},t) \\ &= \frac{1}{\omega_0^2}(Q_0 \cos\omega_1 t + Q_0 \cos\omega_2 t - 2a_3\dot{x} - b_3 x + \nu_3(\dot{x} - V) + \mu_k g), \\ F^{(4)} &\equiv F^{(4)}(\mathbf{x},\tau) = \frac{1}{\omega_0^2} F^{(4)}(\mathbf{x},t) \\ &= \frac{1}{\omega_0^2}(Q_0 \cos\omega_1 t + Q_0 \cos\omega_2 t - 2a_4\dot{x} - b_4 x - \nu_4(\dot{x} - V_2) + \nu_3(V_2 - V) + \mu_k g). \end{aligned}\right\} \tag{4.61}$$

The force conditions on the boundary $\partial\Omega_{34}$ from domain Ω_3 to Ω_4 are at time

t_{m^-} and t_{m^+}

$$\left.\begin{aligned} F_-^{(3)} &\equiv F^{(3)}(\mathbf{x}_m, \tau_{m^-}) = \frac{1}{\omega_0^2} F^{(3)}(\mathbf{x}_m, t_{m^-}) \\ &= \frac{1}{\omega_0^2}(Q_0 \cos\omega_1 t_{m^-} + Q_0 \cos\omega_2 t_{m^-} - 2a_3 V_2 - b_3 x_m + \nu_3(V_2 - V) + \mu_k g) < 0, \\ F_+^{(4)} &\equiv F^{(4)}(\mathbf{x}_m, \tau_{m^+}) = \frac{1}{\omega_0^2} F^{(4)}(\mathbf{x}_m, t_{m^+}) \\ &= \frac{1}{\omega_0^2}(Q_0 \cos\omega_1 t_{m^+} + Q_0 \cos\omega_2 t_{m^+} - 2a_4 V_2 - b_4 x_m + \nu_3(V_2 - V) + \mu_k g) < 0; \end{aligned}\right\} \tag{4.62}$$

and the force conditions on the boundary $\partial\Omega_{43}$ from domain Ω_4 to Ω_3 are at time t_{m^-} and t_{m^+}

$$\left.\begin{aligned} F_-^{(4)} &\equiv F^{(4)}(\mathbf{x}_m, \tau_{m^-}) = \frac{1}{\omega_0^2} F^{(4)}(\mathbf{x}_m, t_{m^-}) \\ &= \frac{1}{\omega_0^2}(Q_0 \cos\omega_1 t_{m^-} + Q_0 \cos\omega_2 t_{m^-} - 2a_4 V_2 - b_4 x_m + \nu_3(V_2 - V) + \mu_k g) > 0, \\ F_+^{(3)} &\equiv F^{(3)}(\mathbf{x}_m, \tau_{m^+}) = \frac{1}{\omega_0^2} F^{(3)}(\mathbf{x}_m, t_{m^+}) \\ &= \frac{1}{\omega_0^2}(Q_0 \cos\omega_1 t_{m^+} + Q_0 \cos\omega_2 t_{m^+} - 2a_3 V_2 - b_3 x_m + \nu_3(V_2 - V) + \mu_k g) > 0. \end{aligned}\right\} \tag{4.63}$$

Since the friction force on $\partial\Omega_{13}$ is C^0-discontinuous, such a force discontinuity causes the existence of the stick motion along the boundary $\partial\Omega_{13}$. From Eq. (4.33), the condition for the stick motion on $\partial\Omega_{13}$ is

$$\left.\begin{aligned} F_-^{(1)} &\equiv F^{(1)}(\mathbf{x}_m, \tau_{m^-}) = \frac{1}{\omega_0^2} F^{(1)}(\mathbf{x}_m, t_{m^-}) \\ &= \frac{1}{\omega_0^2}(Q_0 \cos\omega_1 t_{m^-} + Q_0 \cos\omega_2 t_{m^-} - 2a_1 V - b_1 x_m - \mu_k g) < 0, \\ F_-^{(3)} &\equiv F^{(3)}(\mathbf{x}_m, \tau_{m^-}) = \frac{1}{\omega_0^2} F^{(3)}(\mathbf{x}_m, t_{m^-}) \\ &= \frac{1}{\omega_0^2}(Q_0 \cos\omega_1 t_{m^-} + Q_0 \cos\omega_2 t_{m^-} - 2a_3 V - b_3 x_m + \mu_k g) > 0. \end{aligned}\right\} \tag{4.64}$$

Because the static and kinetic friction forces are different, the flow barriers exist in this dynamical system. Therefore, once the stick motion appears between the mass and the translation belt, the stick motion disappearance requires

$$\left.\begin{aligned} F^{(0\succ 0_1)} &\equiv F^{(0\succ 0_1)}(\mathbf{x}_m, q_1^{(1)}) \\ &= \frac{1}{\omega_0^2}(Q_0 \cos\omega_1 t_{m^-} + Q_0 \cos\omega_2 t_{m^-} - 2a_1 V - b_1 x_m - \mu_s g) < 0; \\ F^{(0\succ 0_3)} &\equiv F^{(0\succ 0_3)}(\mathbf{x}_m, q_1^{(3)}) \\ &= \frac{1}{\omega_0^2}(Q_0 \cos\omega_1 t_{m\pm} + Q_0 \cos\omega_2 t_{m\pm} - 2a_3 V - b_3 x_m + \mu_s g) = 0, \\ DF^{(0\succ 0_3)} &\equiv DF^{(0\succ 0_3)}(\mathbf{x}_m, q_1^{(3)}) \\ &= \frac{1}{\omega_0^2}(-c_3 V - Q_0\omega_1 \sin(\omega_1 t_{m\pm}) - Q_0\omega_2 \sin(\omega_2 t_{m\pm})) < 0, \end{aligned}\right\} \text{for } \partial\Omega_{13} \to \Omega_3 \tag{4.65}$$

$$\left.\begin{aligned} F^{(0\succ 0_3)} &\equiv F^{(0\succ 0_3)}(\mathbf{x}_m, q_1^{(3)}) \\ &= \frac{1}{\omega_0^2}(Q_0\cos\omega_1 t_{m^-} + Q_0\cos\omega_2 t_{m^-} - 2a_3V - b_3x_m + \mu_s g) > 0; \\ F^{(0\succ 0_1)} &\equiv F^{(0\succ 0_1)}(\mathbf{x}_m, q_1^{(1)}) \\ &= \frac{1}{\omega_0^2}(Q_0\cos\omega_1 t_{m\pm} + Q_0\cos\omega_2 t_{m\pm} - 2a_1V - b_1x_m - \mu_s g) = 0, \\ DF^{(0\succ 0_1)} &\equiv DF^{(0\succ 0_1)}(\mathbf{x}_m, q_1^{(1)}) \\ &= \frac{1}{\omega_0^2}(-c_1V - Q_0\omega_1\sin(\omega_1 t_{m\pm}) - Q_0\omega_2\sin(\omega_2 t_{m\pm})) > 0. \end{aligned}\right\}$$

$$\text{for } \partial\Omega_{13} \to \Omega_1 \tag{4.66}$$

For simplicity, $F^{(0\succ 0_\alpha)} \triangleq F^{(0\succ 0_\alpha)}(\mathbf{x}_m, q_1^{(\alpha)})$ is depicted to observe the force criteria for the disappearance of the stick motion. Since $F_-^{(3)} > 0$ and $F_-^{(1)} < 0$, the stick motion appears on the boundary $\partial\Omega_{13}$ because the conditions in Eq. (4.33) are satisfied. The stick motion disappears at $F^{(0\succ 0_3)} = 0$ and $DF^{(0\succ 0_3)} < 0$, which satisfies the conditions in Eq. (4.65). Such a condition indicates that the nonfriction force must be greater than the static friction force (i.e. flow barrier). Once the relative motion starts between the oscillator and the belt, the kinetic friction force will control the motion in domain Ω_3. So the corresponding force jumps from zero to the negative one (i.e. $F_+^{(3)} < 0$). Such force characteristics of stick motion and switching are presented in Figs. 5(d)-(f). The forces at the switching points on the boundary $\partial\Omega_{34}$ and $\partial\Omega_{43}$ are labeled by $F_\pm^{(3)}$ and $F_\mp^{(4)}$. In addition, such force distributions in domains Ω_3 and Ω_4 are labeled by $F^{(3)}$ and $F^{(4)}$.

5. Conclusion

A periodically forced, friction oscillator under two-frequency excitations is investigated in this paper. The nonlinear friction force is approximated by a piecewise linear, kinetic friction model with the static force. The necessary and sufficient conditions for the passibility of the flow and the grazing motions to the separation boundary are developed. Because of the flow barriers existing on the separation boundary, the singularities of the flow on such a separation boundary will be changed accordingly. Thus, the necessary and sufficient conditions for the onset and vanishing of the stick motions are also developed using the flow barrier theory. The periodic motions of such an oscillator are analytically predicted through the corresponding mapping structures. Illustrations of the periodic motions in such a piecewise friction model are presented to verify the analytical conditions. It is noted that the present analysis provided an effective way to design desired dynamical systems to satisfy engineering-oriented complex systems and the ideas can be extended for control system design.

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