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ON A NEW HILBERT-TYPE INEQUALITY IN THE WHOLE PLANE WITH THE GENERAL HOMOGENEOUS KERNEL*

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Abstract By means of the weight coefficients and the idea of introducing parameters, a new discrete Hilbert -type inequality in the whole plane with the general homogeneous kernel is given, which is an extension of Hardy-Hilbert's inequality. The equivalent form is obtained. The equivalent statements of the best possible constant factor related to several parameters, the operator expressions and a few particular cases are considered.

Keywords Weight coefficient, Hilbert-type inequality, equivalent form, equivalent statement; parameter, operator expression.

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1. Introduction

Suppose that $p > 1, \frac{1}{p} + \frac{1}{q} = 1, a_m, b_n \geq 0, 0 < \sum_{m=1}^{\infty} a_m^p < \infty$ and $0 < \sum_{n=1}^{\infty} b_n^q < \infty$. We have the following Hardy-Hilbert's inequality with the best possible constant factor $\frac{\pi}{\sin(\pi/p)}$ (cf. [4], Theorem 315):

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m b_n}{m+n} < \frac{\pi}{\sin(\pi/p)} \left(\sum_{m=1}^{\infty} a_m^p \right)^{\frac{1}{p}} \left(\sum_{n=1}^{\infty} b_n^q \right)^{\frac{1}{q}}. \tag{1.1}$$

For $p = q = 2$, inequality (1.1) reduces to the well known Hilbert's inequality.

If $f(x), g(y) \geq 0, 0 < \int_0^{\infty} f^p(x)dx < \infty$ and $0 < \int_0^{\infty} g^q(y)dy < \infty$, then we have the following integral analogous of (1.1) named in Hardy-Hilbert's integral inequality:

$$\int_0^{\infty} \int_0^{\infty} \frac{f(x)g(y)}{x+y} dxdy < \frac{\pi}{\sin(\pi/p)} \left(\int_0^{\infty} f^p(x)dx \right)^{\frac{1}{p}} \left(\int_0^{\infty} g^q(y)dy \right)^{\frac{1}{q}}, \tag{1.2}$$

with the best possible constant factor $\frac{\pi}{\sin(\pi/p)}$ (cf. [4], Theorem 316).

In 1998, by introducing an independent parameter $\lambda > 0$, Yang [32, 33] gave an extension of (1.2) (for $p = q = 2$) with the kernel as $\frac{1}{(x+y)^{\lambda}}$ and the best possible

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constant factor $B(\frac{\lambda}{2},\frac{\lambda}{2})$ ($B(u,v)$ $(u,v>0)$ is the beta function). Inequalities (1.1) and (1.2) with their extensions play an important role in analysis and its applications (cf. [1, 2, 5, 6, 15, 20, 25–27, 34, 35, 41]).

The following half-discrete Hilbert-type inequality was provided in 1934 (cf. [4], Theorem 351): If $K(x)$ $(x>0)$ is decreasing, $p>1, \frac{1}{p}+\frac{1}{q}=1, 0<\phi(s)=\int_0^\infty K(x)x^{s-1}dx<\infty, a_n\geq 0, 0<\sum_{n=1}^\infty a_n^p<\infty$, then

$$\int_0^\infty x^{p-2}\left(\sum_{n=1}^\infty K(nx)a_n\right)^p dx<\phi^p(\frac{1}{q})\sum_{n=1}^\infty a_n^p. \tag{1.3}$$

Some new extensions and applications of (1.3) were considered by [7, 21–23, 30, 40] in recent years.

In 2016, by the use of the technique of real analysis, Hong et al. [8] provided some equivalent statements of the extensions of (1.1) with the best possible constant factor related to several parameters. The other similar results about the extensions of (1.1)-(1.3) were given by [9–14, 18, 19, 24, 28, 31, 36–39].

In this paper, following the way of [8], by means of the weight coefficients and the idea of introducing parameters, a new discrete Hilbert-type inequality in the whole plane is given as follows: for $r>1, \frac{1}{r}+\frac{1}{s}=1$,

$$\sum_{|m|=1}^\infty\sum_{|n|=1}^\infty\frac{a_m b_n}{|m|+|n|}<\frac{2\pi}{\sin(\pi/r)}\left(\sum_{|m|=1}^\infty |m|^{\frac{p}{r}-1}a_m^p\right)^{\frac{1}{p}}\left(\sum_{|n|=1}^\infty |n|^{\frac{q}{s}-1}b_n^q\right)^{\frac{1}{q}}, \tag{1.4}$$

which is an extension of (1.1). The more general forms as well as the equivalent forms are obtained. The equivalent statements of the best possible constant factor related to several parameters, the operator expressions and a few particular cases are considered.

2. Some lemmas

In what follows, we suppose that $p>1, \frac{1}{p}+\frac{1}{q}=1, \lambda,\lambda_i\in\mathbf{R}:=(-\infty,\infty), \lambda-\lambda_i, \lambda_i\leq 1$ $(i=1,2)$, $k_\lambda(x,y)$ (≥ 0) is a homogeneous function of degree $-\lambda$, satisfying

$$k_\lambda(ux,uy)=u^{-\lambda}k_\lambda(x,y)\ (u,x,y>0),$$

and $k_\lambda(x,y)$ is strictly decreasing with respect to $x>0$ (resp. $y>0$) , such that

$$k_\lambda(\gamma):=\int_0^\infty k_\lambda(1,u)u^{\gamma-1}du\in\mathbf{R}_+:=(0,\infty)\ (\gamma=\lambda_2,\lambda-\lambda_1). \tag{2.1}$$

We assume that $\widehat{\lambda}_1:=\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q}, \widehat{\lambda}_2:=\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p}, \alpha,\beta\in(-1,1), a_m,b_n\geq 0$ $(|m|,|n|\in\mathbf{N}:=\{1,2,\cdots\})$, satisfying

$$0<\sum_{|m|=1}^\infty(|m|+\alpha m)^{p(1-\widehat{\lambda}_1)-1}a_m^p<\infty \text{ and } 0<\sum_{|n|=1}^\infty(|n|+\beta n)^{q(1-\widehat{\lambda}_2)-1}b_n^q<\infty,$$

where, $\sum_{|j|=1}^\infty\cdots=\sum_{j=-1}^{-\infty}\cdots+\sum_{j=1}^\infty\cdots$ $(j=m,n)$.

Lemma 2.1. *For $\gamma > 0$, we have the following inequalities:*

$$\frac{1}{\gamma}[(1-\alpha)^{-\gamma-1}+(1+\alpha)^{-\gamma-1}] < \sum_{|m|=1}^{\infty}(|m|+\alpha m)^{-\gamma-1} < \frac{1}{\gamma}[(1-\alpha)^{-\gamma-1}+(1+\alpha)^{-\gamma-1}](\gamma+1). \tag{2.2}$$

Proof. By the decreasing property of series, we find

$$\begin{aligned}\sum_{|m|=1}^{\infty}(|m|+\alpha m)^{-\gamma-1} &= \sum_{m=-1}^{\infty}[(1-\alpha)(-m)]^{-\gamma-1}\sum_{m=1}^{\infty}[(1+\alpha)m]^{-\gamma-1}\\ &= \sum_{m=1}^{\infty}[(1-\alpha)m]^{-\gamma-1}\sum_{m=1}^{\infty}[(1+\alpha)m]^{-\gamma-1}\\ &= [(1-\alpha)^{-\gamma-1}+(1+\alpha)^{-\gamma-1}](1+\sum_{m=2}^{\infty}m^{-\gamma-1})\\ &< [(1-\alpha)^{-\gamma-1}+(1+\alpha)^{-\gamma-1}](1+\int_{1}^{\infty}x^{-\gamma-1}dx)\\ &= \frac{1}{\gamma}[(1-\alpha)^{-\gamma-1}+(1+\alpha)^{-\gamma-1}](\gamma+1),\\ \sum_{|m|=1}^{\infty}(|m|+\alpha m)^{-\gamma-1} &= [(1-\alpha)^{-\gamma-1}+(1+\alpha)^{-\gamma-1}]\sum_{m=1}^{\infty}m^{-\gamma-1}\\ &> [(1-\alpha)^{-\gamma-1}+(1+\alpha)^{-\gamma-1}]\int_{1}^{\infty}x^{-\gamma-1}dx\\ &= \frac{1}{\gamma}[(1-\alpha)^{-\gamma-1}+(1+\alpha)^{-\gamma-1}].\end{aligned}$$

Hence, we have (2.2).

The lemma is proved. □

Definition 2.1. We set

$$k(m,n) := k_\lambda(|m|+\alpha m, |n|+\beta n)\ (|m|,|n| \in \mathbf{N}),$$

and define the following weight coefficients:

$$\omega(\lambda_2,m) := (|m|+\alpha m)^{\lambda-\lambda_2}\sum_{|n|=1}^{\infty}k(m,n)(|n|+\beta n)^{\lambda_2-1}\ (|m|\in\mathbf{N}), \tag{2.3}$$

$$\varpi(\lambda_1,n) := (|n|+\beta n)^{\lambda-\lambda_1}\sum_{|m|=1}^{\infty}k(m,n)(|m|+\alpha m)^{\lambda_1-1}\ (|n|\in\mathbf{N}). \tag{2.4}$$

Lemma 2.2. *The following inequalities are valid:*

$$\omega(\lambda_2,m) < \frac{2}{1-\beta^2}k_\lambda(\lambda_2)\ (|m|\in\mathbf{N}), \tag{2.5}$$

$$\varpi(\lambda_1,n) < \frac{2}{1-\alpha^2}k_\lambda(\lambda-\lambda_1)\ (|n|\in\mathbf{N}). \tag{2.6}$$

Proof. For $|m| \in \mathbf{N}$, we set

$$k^{(1)}(m,y) := k_\lambda(|m| + \alpha m, (1-\beta)(-y)), y < 0,$$
$$k^{(2)}(m,y) := k_\lambda(|m| + \alpha m, (1+\beta)y), y > 0,$$

wherefrom, $k^{(1)}(m,-y) := k_\lambda(|m| + \alpha m, (1-\beta)y), y > 0$. We find

$$\begin{aligned}
\omega(\lambda_2, m) &= (|m| + \alpha m)^{\lambda-\lambda_2}\{\sum_{n=-1}^{\infty} k^{(1)}(m,n)[(1-\beta)(-n)]^{\lambda_2-1} \\
&\quad + \sum_{n=1}^{\infty} k^{(2)}(m,n)[(1+\beta)n]^{\lambda_2-1}\} \\
&= (|m| + \alpha m)^{\lambda-\lambda_2}\{\sum_{n=1}^{\infty} k^{(1)}(m,-n)[(1-\beta)n]^{\lambda_2-1} \\
&\quad + \sum_{n=1}^{\infty} k^{(2)}(m,n)[(1+\beta)n]^{\lambda_2-1}\} \\
&= (|m| + \alpha m)^{\lambda-\lambda_2}[(1-\beta)^{\lambda_2-1}\sum_{n=1}^{\infty} k^{(1)}(m,-n)n^{\lambda_2-1} \\
&\quad + (1+\beta)^{\lambda_2-1}\sum_{n=1}^{\infty} k^{(2)}(m,n)n^{\lambda_2-1}].
\end{aligned}$$

In view of the assumptions, for fixed $|m| \in \mathbf{N}$, both $k^{(1)}(m,-y)y^{\lambda_2-1}$ and $k^{(2)}(m,y)y^{\lambda_2-1}$ are strictly decreasing with respect to $y \in (0,\infty)$. By the decreasing property of series, we have

$$\begin{aligned}
\omega(\lambda_2, m) &< (|m| + \alpha m)^{\lambda-\lambda_2}[(1-\beta)^{\lambda_2-1}\int_0^\infty k^{(1)}(m,-y)y^{\lambda_2-1}dy \\
&\quad + (1+\beta)^{\lambda_2-1}\int_0^\infty k^{(2)}(m,y)y^{\lambda_2-1}dy] \\
&= (|m| + \alpha m)^{\lambda-\lambda_2}[(1-\beta)^{\lambda_2-1}\int_0^\infty k_\lambda(|m| + \alpha m, (1-\beta)y)y^{\lambda_2-1}dy \\
&\quad + (1+\beta)^{\lambda_2-1}\int_0^\infty k_\lambda(|m| + \alpha m, (1+\beta)y)y^{\lambda_2-1}dy], \qquad (2.7)
\end{aligned}$$

$$\begin{aligned}
\omega(\lambda_2, m) &> (|m| + \alpha m)^{\lambda-\lambda_2}[(1-\beta)^{\lambda_2-1}\int_1^\infty k_\lambda(|m| + \alpha m, (1-\beta)y)y^{\lambda_2-1}dy \\
&\quad + (1+\beta)^{\lambda_2-1}\int_1^\infty k_\lambda(|m| + \alpha m, (1+\beta)y)y^{\lambda 1_2-1}dy]. \qquad (2.8)
\end{aligned}$$

Setting $u = \frac{(1-\beta)y}{|m|+\alpha m}$ (resp. $u = \frac{(1+\beta)y}{|m|+\alpha m}$) in the first (resp. second) integral of (2.7), we obtain

$$\omega(\lambda_{2)}, m < [(1-\beta)^{-1} + (1+\beta)^{-1}]\int_0^\infty k_\lambda(1,u)u^{\lambda_2-1}du = \frac{2k_\lambda(\lambda_2)}{1-\beta^2}.$$

Hence, we have (2.5).

In the same way, setting $v = \frac{1}{u}$, we obtain

$$\begin{aligned}\varpi(\lambda_1, n) &< \frac{2}{1-\alpha^2}\int_0^\infty k_\lambda(u,1)u^{\lambda_1-1}du \\ &= \frac{2}{1-\alpha^2}\int_0^\infty k_\lambda(1,v)v^{(\lambda-\lambda_1)-1}dv = \frac{2k_\lambda(\lambda-\lambda_1)}{1-\alpha^2},\end{aligned}$$

and then (2.6) follows.

The lemma is proved. □

Lemma 2.3. *The following inequality follows:*

$$\begin{aligned}H &:= \sum_{|m|=1}^{\infty}\sum_{|n|=1}^{\infty} k(m,n)a_m b_n < \frac{2k_\lambda^{\frac{1}{p}}(\lambda_2)k_\lambda^{\frac{1}{q}}(\lambda-\lambda_1)}{(1-\beta^2)^{1/p}(1-\alpha^2)^{1/q}} \\ &\times \left[\sum_{|m|=1}^{\infty}(|m|+\alpha m)^{p(1-\widehat{\lambda}_1)-1}a_m^p\right]^{\frac{1}{p}}\left[\sum_{|n|=1}^{\infty}(|n|+\beta n)^{q(1-\widehat{\lambda}_2)-1}b_n^q\right]^{\frac{1}{q}}. \quad (2.9)\end{aligned}$$

Proof. By Hölder's inequality with weight (cf. [16]), we obtain

$$\begin{aligned}H &= \sum_{|m|=1}^{\infty}\sum_{|n|=1}^{\infty} k(m,n)\left[\frac{(|n|+\beta n)^{(\lambda_2-1)/p}}{(|m|+\alpha n)^{(\lambda_1-1)/q}}a_m\right]\left[\frac{(|m|+\alpha n)^{(\lambda_1-1)/q}}{(|n|+\beta n)^{(\lambda_2-1)/p}}b_n\right] \\ &\leq \left[\sum_{|m|=1}^{\infty}\sum_{|n|=1}^{\infty} k(m,n)\frac{(|n|+\beta n)^{\lambda_2-1}}{(|m|+\alpha n)^{(\lambda_1-1)(p-1)}}a_m^p\right]^{\frac{1}{p}} \\ &\quad\times\left[\sum_{|n|=1}^{\infty}\sum_{|m|=1}^{\infty} k(m,n)\frac{(|m|+\alpha n)^{\lambda_1-1}}{(|n|+\beta n)^{(\lambda_2-1)(q-1)}}b_n^q\right]^{\frac{1}{q}} \\ &= \left[\sum_{|m|=1}^{\infty}\omega(\lambda_2,m)(|m|+\alpha m)^{p(1-\widehat{\lambda}_1)-1}a_m^p\right]^{\frac{1}{p}} \\ &\quad\times\left[\sum_{|n|=1}^{\infty}\varpi(\lambda_1,n)(|n|+\beta n)^{q(1-\widehat{\lambda}_2)-1}b_n^q\right]^{\frac{1}{q}}.\end{aligned}$$

Then by (2.5) and (2.6), we have (2.9).

The lemma is proved. □

Remark 2.1. (i) By (2.9), for $\lambda_1 + \lambda_2 = \lambda$, we find

$$0 < \sum_{|m|=1}^{\infty}(|m|+\alpha m)^{p(1-\lambda_1)-1}a_m^p < \infty,\ 0 < \sum_{|n|=1}^{\infty}(|n|+\beta n)^{q(1-\lambda_2)-1}b_n^q < \infty,$$

and the following inequality:

$$H = \sum_{|m|=1}^{\infty}\sum_{|n|=1}^{\infty} k(m,n)a_m b_n < \frac{2k_\lambda(\lambda_2)}{(1-\beta^2)^{1/p}(1-\alpha^2)^{1/q}}$$

$$\times \left[\sum_{|m|=1}^{\infty} (|m| + \alpha m)^{p(1-\lambda_1)-1} a_m^p\right]^{\frac{1}{p}} \left[\sum_{|n|=1}^{\infty} (|n| + \beta n)^{q(1-\lambda_2)-1} b_n^q\right]^{\frac{1}{q}}. \quad (2.10)$$

In particular, for $\alpha = \beta = 0, a_{-m} = a_m, b_{-n} = b_n$ $(m, n \in \mathbf{N})$ in (2.10), we have

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} k_\lambda(m,n)a_m b_n < k_\lambda(\lambda_2)\left[\sum_{m=1}^{\infty} m^{p(1-\lambda_1)-1}a_m^p\right]^{\frac{1}{p}}\left[\sum_{n=1}^{\infty} n^{q(1-\lambda_2)-1}b_n^q\right]^{\frac{1}{q}}. \quad (2.11)$$

(ii) For $\lambda = 1, \lambda_1 = \frac{1}{q}, \lambda_2 = \frac{1}{p}$ in (2.11), we have

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} k_1(m,n)a_m b_n < k_1(\frac{1}{p})\left(\sum_{m=1}^{\infty} a_m^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty} b_n^q\right)^{\frac{1}{q}}; \quad (2.12)$$

for $\lambda = 1, \lambda_1 = \frac{1}{p}, \lambda_2 = \frac{1}{q}$ in (2.11), we have the dual form of (2.12) as follows:

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} k_1(m,n)a_m b_n < k_1(\frac{1}{q})\left(\sum_{m=1}^{\infty} m^{p-2}a_m^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty} n^{q-2}b_n^q\right)^{\frac{1}{q}}; \quad (2.13)$$

for $p = q = 2$, both (2.12) and (2.13) reduce to the following Hilbert-type inequality:

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} k_1(m,n)a_m b_n < k_1(\frac{1}{2})\left(\sum_{m=1}^{\infty} a_m^2 \sum_{n=1}^{\infty} b_n^2\right)^{\frac{1}{2}}. \quad (2.14)$$

(iii) For $\alpha = \beta = 0, \lambda = 1, k_1(m,n) = \frac{1}{m+n}, \lambda_1 = \frac{1}{r}, \lambda_2 = \frac{1}{s}$ $(r > 1, \frac{1}{r} + \frac{1}{s} = 1)$, (2.9) reduces to (1.4); for $r = q, s = p, a_{-m} = a_m, b_{-n} = b_n$ $(m, n \in \mathbf{N})$, (1.4) reduces to (1.1). Hence, (2.9) is an extension of (1.1).

Lemma 2.4. *The constant factor* $\frac{2k_\lambda(\lambda_2)}{(1-\beta^2)^{1/p}(1-\alpha^2)^{1/q}}$ *in (2.10) is the best possible.*

Proof. For any $\varepsilon > 0$, we set

$$\widetilde{a}_m := (|m| + \alpha m)^{(\lambda_1 - \frac{\varepsilon}{p})-1}, \widetilde{b}_n := (|n| + \beta n)^{(\lambda_2 - \frac{\varepsilon}{q})-1} \ (|m|, |n| \in \mathbf{N}).$$

If there exists a constant $M(\leq \frac{2k_\lambda(\lambda_2)}{(1-\beta^2)^{1/p}(1-\alpha^2)^{1/q}})$, such that (2.10) is valid when we replace $\frac{2k_\lambda(\lambda_2)}{(1-\beta^2)^{1/p}(1-\alpha^2)^{1/q}}$ by M, then in particular, we have

$$\begin{aligned}\widetilde{H} &:= \sum_{|m|=1}^{\infty}\sum_{|n|=1}^{\infty} k_\lambda(|m| + \alpha m, |n| + \beta n)\widetilde{a}_m\widetilde{b}_n \\ &< M\left[\sum_{|m|=1}^{\infty} (|m| + \alpha m)^{p(1-\lambda_1)-1}\widetilde{a}_m^p\right]^{\frac{1}{p}}\left[\sum_{|n|=1}^{\infty} (|n| + \beta n)^{q(1-\lambda_2)-1}\widetilde{b}_n^q\right]^{\frac{1}{q}}.\end{aligned}$$

By Lemma 2.1, we obtain

$$\widetilde{H} < M\left[\sum_{|m|=1}^{\infty} (|m| + \alpha m)^{-\varepsilon-1}\right]^{\frac{1}{p}}\left[\sum_{|n|=1}^{\infty} (|n| + \beta n)^{-\varepsilon-1}\right]^{\frac{1}{q}}$$

$$< \frac{M}{\varepsilon}(\varepsilon+1)[(1-\alpha)^{-\varepsilon-1}+(1+\alpha)^{-\varepsilon-1}]^{\frac{1}{p}}[(1-\beta)^{-\varepsilon-1}+(1+\beta)^{-\varepsilon-1}]^{\frac{1}{q}}.$$

By (2.3) (for $\lambda_1+\lambda_2=\lambda$) and (2.8), replacing λ_2 (resp. λ_1) by $\lambda_2-\frac{\varepsilon}{q}$ (resp. $\lambda_1+\frac{\varepsilon}{q}$), we have

$$\begin{aligned}
&\omega(\lambda_2-\frac{\varepsilon}{q},m)\\
&> (|m|+\alpha m)^{\lambda_1+\frac{\varepsilon}{q}}[(1-\beta)^{(\lambda_2-\frac{\varepsilon}{q})-1}\int_1^\infty k_\lambda(|m|+\alpha m,(1-\beta)y)y^{(\lambda_2-\frac{\varepsilon}{q})-1}dy\\
&\quad +(1+\beta)^{(\lambda_2-\frac{\varepsilon}{q})-1}\int_1^\infty k_\lambda(|m|+\alpha m,(1+\beta)y)y^{(\lambda_2-\frac{\varepsilon}{q})-1}dy].
\end{aligned}$$

Then we find

$$\begin{aligned}
\widetilde{H} &= \sum_{|m|=1}^\infty[\sum_{|n|=1}^\infty k(m,n)(|n|+\beta n)^{(\lambda_2-\frac{\varepsilon}{q})-1}](|m|+\alpha m)^{(\lambda_1-\frac{\varepsilon}{p})-1}\\
&= \sum_{|m|=1}^\infty \omega(\lambda_2-\frac{\varepsilon}{q},m)(|m|+\alpha m)^{-\varepsilon-1}\\
&> (1-\beta)^{(\lambda_2-\frac{\varepsilon}{q})-1}\sum_{|m|=1}^\infty(|m|+\alpha m)^{(\lambda_1-\frac{\varepsilon}{p})-1}\\
&\quad \times\int_1^\infty k_\lambda(|m|+\alpha m,(1-\beta)y)y^{(\lambda_2-\frac{\varepsilon}{q})-1}dy\\
&\quad +(1+\beta)^{(\lambda_2-\frac{\varepsilon}{q})-1}\sum_{|m|=1}^\infty(|m|+\alpha m)^{(\lambda_1-\frac{\varepsilon}{p})-1}\\
&\quad \times\int_1^\infty k_\lambda(|m|+\alpha m,(1+\beta)y)y^{(\lambda_2-\frac{\varepsilon}{q})-1}dy\\
&= (1-\beta)^{(\lambda_2-\frac{\varepsilon}{q})-1}\\
&\quad \times\int_1^\infty\sum_{|m|=1}^\infty k_\lambda(|m|+\alpha m,(1-\beta)y)(|m|+\alpha m)^{(\lambda_1-\frac{\varepsilon}{p})-1}y^{(\lambda_2-\frac{\varepsilon}{q})-1}dy\\
&\quad +(1+\beta)^{(\lambda_2-\frac{\varepsilon}{q})-1}\\
&\quad \times\int_1^\infty\sum_{|m|=1}^\infty k_\lambda(|m|+\alpha m,(1+\beta)y)(|m|+\alpha m)^{(\lambda_1-\frac{\varepsilon}{p})-1}y^{(\lambda_2-\frac{\varepsilon}{q})-1}dy\\
&= \sum_{i=1}^4 H_i,
\end{aligned}$$

where, we indicate

$$\begin{aligned}
H_1 :=&(1-\beta)^{(\lambda_2-\frac{\varepsilon}{q})-1}(1-\alpha)^{(\lambda_1-\frac{\varepsilon}{p})-1}\\
&\times\int_1^\infty\sum_{m=1}^\infty k_\lambda((1-\alpha)m,(1-\beta)y)m^{(\lambda_1-\frac{\varepsilon}{p})-1}y^{(\lambda_2-\frac{\varepsilon}{q})-1}dy,\\
H_2 :=&(1-\beta)^{(\lambda_2-\frac{\varepsilon}{q})-1}(1+\alpha)^{(\lambda_1-\frac{\varepsilon}{p})-1}
\end{aligned}$$

$$\times \int_1^\infty \sum_{m=1}^\infty k_\lambda((1+\alpha)m,(1-\beta)y)m^{(\lambda_1-\frac{\varepsilon}{p})-1}y^{(\lambda_2-\frac{\varepsilon}{q})-1}dy,$$

$$H_3 := (1+\beta)^{(\lambda_2-\frac{\varepsilon}{q})-1}(1-\alpha)^{(\lambda_1-\frac{\varepsilon}{p})-1}$$
$$\times \int_1^\infty \sum_{m=1}^\infty k_\lambda((1-\alpha)m,(1+\beta)y)m^{(\lambda_1-\frac{\varepsilon}{p})-1}y^{(\lambda_2-\frac{\varepsilon}{q})-1}dy,$$

$$H_4 := (1+\beta)^{(\lambda_2-\frac{\varepsilon}{q})-1}(1+\alpha)^{(\lambda_1-\frac{\varepsilon}{p})-1}$$
$$\times \int_1^\infty \sum_{m=1}^\infty k_\lambda((1+\alpha)m,(1+\beta)y)m^{(\lambda_1-\frac{\varepsilon}{p})-1}y^{(\lambda_2-\frac{\varepsilon}{q})-1}dy.$$

For fixed x, setting $u=\frac{(1-\beta)y}{(1-\alpha)x}$ in the following, since $\frac{2}{1-\eta}\geq 1$ $(\eta=\alpha,\beta)$, by Fubini theorem (cf. [17]), we obtain

$$\begin{aligned}
H_1 &> (1-\beta)^{\lambda_2-\frac{\varepsilon}{q}-1}(1-\alpha)^{\lambda_1-\frac{\varepsilon}{p}-1}\\
&\quad\times \int_{\frac{2}{1-\alpha}}^\infty \Big[\int_{\frac{2}{1-\beta}}^\infty k_\lambda((1-\alpha)x,(1-\beta)y)x^{(\lambda_1-\frac{\varepsilon}{p})-1}y^{(\lambda_2-\frac{\varepsilon}{q})-1}dy\Big]dx\\
&= \frac{(1-\alpha)^{-\varepsilon-1}}{1-\beta}\int_{\frac{2}{1-\alpha}}^\infty x^{-\varepsilon-1}\int_{\frac{2}{(1-\alpha)x}}^\infty k_\lambda(1,u)u^{\lambda_2-\frac{\varepsilon}{q}-1}du dx\\
&= \frac{1}{(1-\beta)(1-\alpha)}\Big[\int_2^\infty v^{-\varepsilon-1}\int_{\frac{2}{v}}^2 k_\lambda(1,u)u^{\lambda_2-\frac{\varepsilon}{q}-1}du dv\\
&\quad+\int_2^\infty v^{-\varepsilon-1}\int_2^\infty k_\lambda(1,u)u^{\lambda_2-\frac{\varepsilon}{q}-1}du dv\Big]\\
&= \frac{1}{(1-\beta)(1-\alpha)}\Big[\int_0^2\Big(\int_{\frac{2}{v}}^\infty v^{-\varepsilon-1}dv\Big)k_\lambda(1,u)u^{\lambda_2-\frac{\varepsilon}{q}-1}du\\
&\quad+\frac{1}{\varepsilon 2^\varepsilon}\int_2^\infty k_\lambda(1,u)u^{\lambda_2-\frac{\varepsilon}{q}-1}du\Big]\\
&= \widetilde{H}_1 := \frac{1}{\varepsilon 2^\varepsilon(1-\beta)(1-\alpha)}G,\\
G &:= \int_0^2 k_\lambda(1,u)u^{\lambda_2+\frac{\varepsilon}{p}-1}du+\int_2^\infty k_\lambda(1,u)u^{\lambda_2-\frac{\varepsilon}{q}-1}du.
\end{aligned}$$

In the same way, we can find that

$$H_2 > \widetilde{H}_2 := \frac{1}{\varepsilon 2^\varepsilon(1-\beta)(1+\alpha)}G,$$
$$H_3 > \widetilde{H}_3 := \frac{1}{\varepsilon 2^\varepsilon(1+\beta)(1-\alpha)}G,$$
$$H_4 > \widetilde{H}_4 := \frac{1}{\varepsilon 2^\varepsilon(1+\beta)(1+\alpha)}G.$$

In view of the above results, we have

$$\frac{4G}{2^\varepsilon(1-\beta^2)(1-\alpha^2)} = \varepsilon\sum_{i=1}^4 \widetilde{H}_i < \varepsilon\sum_{i=1}^4 H_i < \varepsilon\widetilde{H}$$

$$< M(\varepsilon+1)[(1-\alpha)^{-\varepsilon-1}+(1+\alpha)^{-\varepsilon-1}]^{\frac{1}{p}}[(1-\beta)^{-\varepsilon-1}+(1+\beta)^{-\varepsilon-1}]^{\frac{1}{q}}.$$

For $\varepsilon \to 0$, by Fatou lemma (cf. [17]), we find

$$\frac{4}{(1-\beta^2)(1-\alpha^2)}k_\lambda(\lambda_2) \leq \frac{2M}{(1-\alpha^2)^{1/p}(1-\beta^2)^{1/q}},$$

namely, $\frac{2k_\lambda(\lambda_2)}{(1-\beta^2)^{1/p}(1-\alpha^2)^{1/q}} \leq M$, which means that $M = \frac{2k_\lambda(\lambda_2)}{(1-\beta^2)^{1/p}(1-\alpha^2)^{1/q}}$ is the best possible constant factor of (2.10).

The lemma is proved. □

Remark 2.2. (i) In view of Lemma 2.4, the particular constant factors in (2.11)-(2.14) are also the best possible.

(ii) Since $\widehat{\lambda}_1 = \frac{\lambda-\lambda_2}{p} + \frac{\lambda_1}{q}, \widehat{\lambda}_2 = \frac{\lambda-\lambda_1}{q} + \frac{\lambda_2}{p}$, we find $\widehat{\lambda}_1 \leq \frac{1}{p} + \frac{1}{q} = 1, \widehat{\lambda}_2 \leq 1$,

$$\widehat{\lambda}_1 + \widehat{\lambda}_2 = \frac{\lambda-\lambda_2}{p} + \frac{\lambda_1}{q} + \frac{\lambda-\lambda_1}{q} + \frac{\lambda_2}{p} = \lambda.$$

By Hölder's inequality (cf. [16]), it follows that

$$\begin{aligned}
0 &< k_\lambda(\widehat{\lambda}_2) = k_\lambda(\frac{\lambda_2}{p} + \frac{\lambda-\lambda_1}{q}) \\
&= \int_0^\infty k_\lambda(1,u)u^{\frac{\lambda_2}{p}+\frac{\lambda-\lambda_1}{q}-1}du = \int_0^\infty k_\lambda(1,u)(u^{\frac{\lambda_2-1}{p}})(u^{\frac{\lambda_2}{p}+\frac{\lambda-\lambda_1-1}{q}})du \\
&\leq \left(\int_0^\infty k_\lambda(1,u)u^{\lambda_2-1}du\right)^{\frac{1}{p}}\left(\int_0^\infty k_\lambda(1,u)u^{\lambda-\lambda_1-1}du\right)^{\frac{1}{q}} \\
&= k_\lambda^{\frac{1}{p}}(\lambda_2)k_\lambda^{\frac{1}{q}}(\lambda-\lambda_1) < \infty. \qquad (2.15)
\end{aligned}$$

In view of (2.10), for $\lambda_i = \widehat{\lambda}_i$ $(i=1,2)$, we have

$$\begin{aligned}
H <& \frac{2k_\lambda(\widehat{\lambda}_2)}{(1-\beta^2)^{1/p}(1-\alpha^2)^{1/q}} \\
&\times \left[\sum_{|m|=1}^\infty (|m|+\alpha m)^{p(1-\widehat{\lambda}_1)-1}a_m^p\right]^{\frac{1}{p}}\left[\sum_{|n|=1}^\infty (|n|+\beta n)^{q(1-\widehat{\lambda}_2)-1}b_n^q\right]^{\frac{1}{q}}. \qquad (2.16)
\end{aligned}$$

Lemma 2.5. *If the constant factor* $\frac{2k_\lambda^{\frac{1}{p}}(\lambda_2)k_\lambda^{\frac{1}{q}}(\lambda-\lambda_1)}{(1-\beta^2)^{1/p}(1-\alpha^2)^{1/q}}$ *in (2.9) is the best possible, then we have* $\lambda_1+\lambda_2=\lambda$.

Proof. If the constant factor $\frac{2k_\lambda^{\frac{1}{p}}(\lambda_2)k_\lambda^{\frac{1}{q}}(\lambda-\lambda_1)}{(1-\beta^2)^{1/p}(1-\alpha^2)^{1/q}}$ in (2.9) is the best possible, then by (2.16), we have the following inequality:

$$\frac{2k_\lambda^{\frac{1}{p}}(\lambda_2)k_\lambda^{\frac{1}{q}}(\lambda-\lambda_1)}{(1-\beta^2)^{1/p}(1-\alpha^2)^{1/q}} \leq \frac{2k_\lambda(\widehat{\lambda}_2)}{(1-\beta^2)^{1/p}(1-\alpha^2)^{1/q}} \ (\in \mathbf{R}_+),$$

namely, $k_\lambda^{\frac{1}{p}}(\lambda_2)k_\lambda^{\frac{1}{q}}(\lambda-\lambda_1) \leq k_\lambda(\widehat{\lambda}_2)$, which means that (2.15) keeps the form of equality.

We observe that (2.15) keeps the form of equality if and only if there exist constants A and B, such that they are not both zero and (cf. [16]) $Au^{\lambda_2-1} = Bu^{\lambda-\lambda_1-1}$ *a.e.* in $\mathbf{R}_+$. Assuming that $A \neq 0$, it follows that $u^{\lambda_1+\lambda_2-\lambda-1} = B/A$ *a.e.* in $\mathbf{R}_+$, and then $\lambda_1+\lambda_2-\lambda = 0$, namely, $\lambda_1+\lambda_2 = \lambda$.

The lemma is proved. □

3. Main results

Theorem 3.1. *Inequality (2.9) is equivalent to the following Hilbert-type inequality in the whole plane:*

$$\begin{aligned} L &:= \left[\sum_{|n|=1}^{\infty} (|n|+\beta n)^{p\widehat{\lambda}_2-1}\left(\sum_{|m|=1}^{\infty} k(m,n)a_m\right)^p\right]^{\frac{1}{p}} \\ &< \frac{2k_\lambda^{\frac{1}{p}}(\lambda_2)k_\lambda^{\frac{1}{q}}(\lambda-\lambda_1)}{(1-\beta^2)^{1/p}(1-\alpha^2)^{1/q}}\left[\sum_{|m|=1}^{\infty}(|m|+\alpha m)^{p(1-\widehat{\lambda}_1)-1}a_m^p\right]^{\frac{1}{p}}. \end{aligned} \tag{3.1}$$

Proof. Suppose that (3.1) is valid. By Hölder's inequality (cf. [16]), we find

$$\begin{aligned} H &= \sum_{|n|=1}^{\infty}\left[(|n|+\beta n)^{\widehat{\lambda}_2-\frac{1}{p}}\sum_{|m|=1}^{\infty} k(m,n)a_m\right]\left[(|n|+\beta n)^{\frac{1}{p}-\widehat{\lambda}_2}b_n\right] \\ &\le L\left[\sum_{|n|=1}^{\infty}(|n|+\beta n)^{q(1-\widehat{\lambda}_2)-1}b_n^q\right]^{\frac{1}{q}}. \end{aligned} \tag{3.2}$$

Then by (3.1), we obtain (2.9).

On the other hand, assuming that (2.9) is valid, we set

$$b_n := (|n|+\beta n)^{p\widehat{\lambda}_2-1}\left(\sum_{|m|=1}^{\infty} k(m,n)a_m\right)^{p-1}, |n| \in \mathbf{N}.$$

Then we have

$$L^p = \sum_{|n|=1}^{\infty}(|n|+\beta n)^{q(1-\widehat{\lambda}_2)-1}b_n^q = H. \tag{3.3}$$

If $L = 0$, then (3.1) is naturally valid; if $L = \infty$, then it is impossible that makes (3.1) valid, namely, $L < \infty$. Suppose that $0 < L < \infty$. By (2.9), it follows that

$$\begin{aligned} L^p &= \sum_{|n|=1}^{\infty}(|n|+\beta n)^{q(1-\widehat{\lambda}_2)-1}b_n^q \\ &= H < \frac{2k_\lambda^{\frac{1}{p}}(\lambda_2)k_\lambda^{\frac{1}{q}}(\lambda-\lambda_1)}{(1-\beta^2)^{1/p}(1-\alpha^2)^{1/q}}\left[\sum_{|m|=1}^{\infty}(|m|+\alpha m)^{p(1-\widehat{\lambda}_1)-1}a_m^p\right]^{\frac{1}{p}} L^{p-1}, \end{aligned}$$

$$L = \left[\sum_{|n|=1}^{\infty} (|n| + \beta n)^{q(1-\widehat{\lambda}_2)-1} b_n^q \right]^{\frac{1}{p}}$$
$$< \frac{2k_\lambda^{\frac{1}{p}}(\lambda_2) k_\lambda^{\frac{1}{q}}(\lambda - \lambda_1)}{(1-\beta^2)^{1/p}(1-\alpha^2)^{1/q}} \left[\sum_{|m|=1}^{\infty} (|m| + \alpha m)^{p(1-\widehat{\lambda}_1)-1} a_m^p \right]^{\frac{1}{p}},$$

namely, (3.1) follows, which is equivalent to (2.9).

The theorem is proved. □

Theorem 3.2. *The following statements (i), (ii), (iii), (iv) and (v) are equivalent:*

(i) Both $k_\lambda^{\frac{1}{p}}(\lambda_2) k_\lambda^{\frac{1}{q}}(\lambda - \lambda_1)$ and $k_\lambda(\frac{\lambda_2}{p} + \frac{\lambda - \lambda_1}{q})$ are independent of p, q;

(ii) $k_\lambda^{\frac{1}{p}}(\lambda_2) k_\lambda^{\frac{1}{q}}(\lambda - \lambda_1) \le k_\lambda(\frac{\lambda_2}{p} + \frac{\lambda - \lambda_1}{q})$;

(iii) $\lambda_1 + \lambda_2 = \lambda$;

(iv) $\frac{2k_\lambda^{\frac{1}{p}}(\lambda_2) k_\lambda^{\frac{1}{q}}(\lambda - \lambda_1)}{(1-\beta^2)^{1/p}(1-\alpha^2)^{1/q}}$ is the best possible constant factor of (2.9);

(v) $\frac{2k_\lambda^{\frac{1}{p}}(\lambda_2) k_\lambda^{\frac{1}{q}}(\lambda - \lambda_1)}{(1-\beta^2)^{1/p}(1-\alpha^2)^{1/q}}$ is the best possible constant factor of (3.1).

If the statement (iii) follows, namely, $\lambda_1 + \lambda_2 = \lambda$, then we have (2.10) and the following equivalent inequality with the best possible constant factor $\frac{2k_\lambda(\lambda_2)}{(1-\beta^2)^{1/p}(1-\alpha^2)^{1/q}}$:

$$\left[\sum_{|n|=1}^{\infty} (|n| + \beta n)^{p\lambda_2 - 1} \left(\sum_{|m|=1}^{\infty} k(m,n) a_m \right)^p \right]^{\frac{1}{p}}$$
$$< \frac{2k_\lambda(\lambda_2)}{(1-\beta^2)^{1/p}(1-\alpha^2)^{1/q}} \left[\sum_{|m|=1}^{\infty} (|m| + \alpha m)^{p(1-\lambda_1)-1} a_m^p \right]^{\frac{1}{p}}. \tag{3.4}$$

In particular, for $\alpha = \beta = 0, a_{-m} = a_m, b_{-n} = b_n$ $(m, n \in \mathbf{N})$ in (3.4), we have the following inequality with the best possible constant factor equivalent to (2.11):

$$\left[\sum_{n=1}^{\infty} n^{p\lambda_2 - 1} \left(\sum_{m=1}^{\infty} k(m,n) a_m \right)^p \right]^{\frac{1}{p}} < k_\lambda(\lambda_2) \left[\sum_{m=1}^{\infty} m^{p(1-\lambda_1)-1} a_m^p \right]^{\frac{1}{p}}. \tag{3.5}$$

Proof. $(i) => (ii)$. Since both $k_\lambda^{\frac{1}{p}}(\lambda_2) k_\lambda^{\frac{1}{q}}(\lambda - \lambda_1)$ and $k_\lambda(\frac{\lambda_2}{p} + \frac{\lambda - \lambda_1}{q})$ are independent of p, q, we find

$$k_\lambda^{\frac{1}{p}}(\lambda_2) k_\lambda^{\frac{1}{q}}(\lambda - \lambda_1) = \lim_{q \to \infty} \lim_{p \to 1^+} k_\lambda^{\frac{1}{p}}(\lambda_2) k_\lambda^{\frac{1}{q}}(\lambda - \lambda_1) = k_\lambda(\lambda_2).$$

Then by Fatou lemma (cf. [17]), we have the following inequality:

$$k_\lambda(\frac{\lambda_2}{p} + \frac{\lambda - \lambda_1}{q})$$
$$= \lim_{q \to \infty} k_\lambda(\lambda_2 + \frac{\lambda - \lambda_1 - \lambda_2}{q}) \ge k_\lambda(\lambda_2 + \lim_{q \to \infty} \frac{\lambda - \lambda_1 - \lambda_2}{q})$$
$$= k_\lambda(\lambda_2) = k_\lambda^{\frac{1}{p}}(\lambda_2) k_\lambda^{\frac{1}{q}}(\lambda - \lambda_1).$$

$(ii) => (iii)$. If $k_\lambda^{\frac{1}{p}}(\lambda_2)k_\lambda^{\frac{1}{q}}(\lambda-\lambda_1) \le k_\lambda(\frac{\lambda_2}{p}+\frac{\lambda-\lambda_1}{q})$, then (2.15) keeps the form of equality. By the proof of Lemma 2.5, it follows that $\lambda_1+\lambda_2=\lambda$.

$(iii) => (i)$. If $\lambda_1+\lambda_2=\lambda$, then we have

$$k_\lambda^{\frac{1}{p}}(\lambda_2)k_\lambda^{\frac{1}{q}}(\lambda-\lambda_1) = k_\lambda(\frac{\lambda_2}{p}+\frac{\lambda-\lambda_1}{q}) = k_\lambda(\lambda_2).$$

Both $k_\lambda^{\frac{1}{p}}(\lambda_2)k_\lambda^{\frac{1}{q}}(\lambda-\lambda_1)$ and $k_\lambda(\frac{\lambda_2}{p}+\frac{\lambda-\lambda_1}{q})$ are independent of p, q.

Hence, we have $(i) <=> (ii) <=> (iii)$.

$(iii) <=> (iv)$. By Lemma 2. 4 and Lemma 2.5, we obtain the conclusions.

$(iv) <=> (v)$. If the constant factor in (2.9) is the best possible, then so is constant factor in (3.1) . Otherwise, by (3.2), we would reach a contradiction that the constant factor in (2.9) is not the best possible. On the other-hand, if the constant factor in (3.1) is the best possible, then so is constant factor in (2.9). Otherwise, by (3.3), we would reach a contradiction that the constant factor in (3.1) is not the best possible.

Therefore, the statements (i), (ii), (iii), (iv) and (v) are equivalent.

The theorem is proved. □

4. Operator expressions

We set functions:

$$\varphi(m) := (|m|+\alpha m)^{p(1-\widehat{\lambda}_1)-1}, \psi(n) := (|n|+\beta n)^{q(1-\widehat{\lambda}_2)-1},$$

where from, $\psi^{1-p}(n) = (|n|+\beta n)^{p\widehat{\lambda}_2-1}$ $(|m|, |n| \in \mathbf{N})$.

Define the following real normed spaces:

$$l_{p,\varphi} := \left\{ a = \{a_m\}_{|m|=1}^{\infty} : ||a||_{p,\varphi} := \left(\sum_{|m|=1}^{\infty} \varphi(m)|a_m|^p \right)^{\frac{1}{p}} < \infty \right\},$$

$$l_{q,\psi} := \left\{ b = \{b_n\}_{|n|=1}^{\infty} : ||b||_{q,\psi} := \left(\sum_{|n|=1}^{\infty} \psi(n)|b_n|^q \right)^{\frac{1}{q}} < \infty \right\},$$

$$l_{p,\psi^{1-p}} := \left\{ c = \{c_n\}_{|n|=1}^{\infty} : ||c||_{p,\psi^{1-p}} := \left(\sum_{|n|=1}^{\infty} \psi^{1-p}(n)|c_n|^q \right)^{\frac{1}{p}} < \infty \right\}.$$

Assuming that $a \in l_{p,\varphi}$, setting

$$c = \{c_n\}_{|n|=1}^{\infty}, c_n := \sum_{|m|=1}^{\infty} k(m,n)a_m, |n| \in \mathbf{N},$$

we can rewrite (3.1) as follows:

$$||c||_{p,\psi^{1-p}} < \frac{2k_\lambda^{\frac{1}{p}}(\lambda_2)k_\lambda^{\frac{1}{q}}(\lambda-\lambda_1)}{(1-\beta^2)^{1/p}(1-\alpha^2)^{1/q}}||a||_{p,\varphi} < \infty,$$

namely, $c \in l_{p,\psi^{1-p}}$.

Definition 4.1. Define a Hilbert-type operator $T : l_{p,\varphi} \to l_{p,\psi^{1-p}}$ as follows: For any $a \in l_{p,\varphi}$, there exists a unique representation $Ta = c \in l_{p,\psi^{1-p}}$, satisfying for any $|n| \in \mathbf{N}, Ta(n) = c_n$. Define the formal inner product of Ta and $b \in l_{q,\psi}$ as follows:

$$(Ta, b) := \sum_{|n|=1}^{\infty} \left(\sum_{|m|=1}^{\infty} k(m,n) a_m \right) b_n = H,$$

$$||T|| = \sup_{a(\neq 0) \in l_{p,\varphi}} \frac{||Ta||_{p,\psi^{1-p}}}{||a||_{p,\varphi}}.$$

By Theorem 3.1 and Theorem 3.2, we have

Theorem 4.1. *If $a \in l_{p,\varphi}, b \in l_{q,\psi}, ||a||_{p,\varphi}, ||b||_{q,\psi} > 0$, then we have the following equivalent inequalities:*

$$(Ta, b) < \frac{2k_{\lambda}^{\frac{1}{p}}(\lambda_2) k_{\lambda}^{\frac{1}{q}}(\lambda - \lambda_1)}{(1-\beta^2)^{1/p}(1-\alpha^2)^{1/q}} ||a||_{p,\varphi} ||b||_{q,\psi}, \tag{4.1}$$

$$||Ta||_{p,\psi^{1-p}} < \frac{2k_{\lambda}^{\frac{1}{p}}(\lambda_2) k_{\lambda}^{\frac{1}{q}}(\lambda - \lambda_1)}{(1-\beta^2)^{1/p}(1-\alpha^2)^{1/q}} ||a||_{p,\varphi} \tag{4.2}$$

Moreover, $\lambda_1 + \lambda_2 = \lambda$ if and only if the constant factor

$$\frac{2k_{\lambda}^{\frac{1}{p}}(\lambda_2) k_{\lambda}^{\frac{1}{q}}(\lambda - \lambda_1)}{(1-\beta^2)^{1/p}(1-\alpha^2)^{1/q}}$$

in (4.1) and (4.2) is the best possible, namely,

$$||T|| = \frac{2k_{\lambda}(\lambda_2)}{(1-\beta^2)^{1/p}(1-\alpha^2)^{1/q}}. \tag{4.3}$$

Example 4.1. (i) For $\lambda > 0, \lambda_i \in (0, \lambda) \cap (0, 1]$ $(i = 1, 2)$, setting $k_{\lambda}(x, y) = \frac{1}{(x+y)^{\lambda}}$ $(x, y > 0)$, then

$$k(m,n) = \frac{1}{(|m| + \alpha m + |n| + \beta n)^{\lambda}} \quad (|m|, |n| \in \mathbf{N}).$$

$k_{\lambda}(x, y) x^{\lambda_1 - 1}$ (resp. $k_{\lambda}(x, y) y^{\lambda_2 - 1}$) is strictly decreasing with respect to $x > 0$ (resp. $y > 0$), such that

$$k_{\lambda}(\gamma) = \int_0^{\infty} \frac{u^{\gamma-1}}{(1+u)^{\lambda}} du = B(\gamma, \lambda - \gamma) \in \mathbf{R}_+ \ (\gamma = \lambda_2, \lambda - \lambda_1).$$

By Theorem 4.1, it follows that $\lambda_1 + \lambda_2 = \lambda$ if and only if

$$||T|| = \frac{2B(\lambda_1, \lambda_2)}{(1-\beta^2)^{1/p}(1-\alpha^2)^{1/q}}.$$

(ii) For $\lambda > 0, \lambda_i \in (0, \lambda) \cap (0, 1]$ $(i = 1, 2)$, setting $k_{\lambda}(x, y) = \frac{\ln(x/y)}{x^{\lambda} - y^{\lambda}}$ $(x, y > 0)$, then

$$k(m,n) = \frac{\ln[(|m| + \alpha m)/(|n| + \beta n)]}{(|m| + \alpha m)^{\lambda} - (|n| + \beta n)^{\lambda}} \quad (|m|, |n| \in \mathbf{N}).$$

$k_\lambda(x,y)x^{\lambda_1-1}$ (resp. $k_\lambda(x,y)y^{\lambda_2-1}$) is strictly decreasing with respect to $x>0$ (resp. $y>0$), such that

$$k_\lambda(\gamma)=\int_0^\infty \frac{u^{\gamma-1}\ln u}{u^\lambda-1}du=[\frac{\pi}{\lambda\sin(\pi\gamma/\lambda)}]^2\in\mathbf{R}_+\ (\gamma=\lambda_2,\lambda-\lambda_1).$$

By Theorem 4.1, it follows that $\lambda_1+\lambda_2=\lambda$ if and only if

$$||T||=\frac{2}{(1-\beta^2)^{1/p}(1-\alpha^2)^{1/q}}[\frac{\pi}{\lambda\sin(\pi\lambda_2/\lambda)}]^2.$$

(iii) For $0<\eta+\lambda_i<1$ $(i=1,2)$, $\lambda+2\eta>\min_{i=1,2}\{0,\eta+\lambda_i\}$, setting $k_\lambda(x,y)=\frac{(\min\{x,y\})^\eta}{(\max\{x,y\})^{\lambda+\eta}}$ $(x,y>0)$, then

$$k(m,n)=\frac{(\min\{|m|+\alpha m,|n|+\beta n\})^\eta}{(\max\{|m|+\alpha m,|n|+\beta n\})^{\lambda+\eta}}\ (|m|,|n|\in\mathbf{N}),$$

$$k_\lambda(x,y)x^{\lambda_1-1}=\frac{(\min\{x,y\})^\eta x^{\lambda_1-1}}{(\max\{x,y\})^{\lambda+\eta}}=\begin{cases}x^{\eta+\lambda_1-1},0<x<y,\\ \frac{y^\eta}{x^{\lambda+\eta-\lambda_1+1}},x\ge y\end{cases}$$

(resp. $k_\lambda(x,y)y^{\lambda_2-1}$) is strictly decreasing with respect to $x>0$ (resp. $y>0$), such that

$$\begin{aligned}k_\lambda(\gamma)&=\int_0^\infty\frac{(\min\{1,u\})^\eta u^{\gamma-1}}{(\max\{1,u\})^{\lambda+\eta}}du=\int_0^1 u^{\eta+\gamma-1}du+\int_1^\infty\frac{u^{\gamma-1}}{u^{\lambda+\eta}}du\\&=\frac{\lambda+2\eta}{(\eta+\gamma)(\lambda+\eta-\gamma)}\in\mathbf{R}_+\ (\gamma=\lambda_2,\lambda-\lambda_1).\end{aligned}$$

By Theorem 4.1, it follows that $\lambda_1+\lambda_2=\lambda$ if and only if

$$||T||=\frac{2}{(1-\beta^2)^{1/p}(1-\alpha^2)^{1/q}}\frac{\lambda+2\eta}{(\eta+\lambda_1)(\eta+\lambda_2)}.$$

Example 4.2. (i) In view of the following expression for the cotangent function (cf. [3]):

$$\cot x=\frac{1}{x}+\sum_{k=1}^\infty\left(\frac{1}{x-\pi k}+\frac{1}{x+\pi k}\right)\ (x\in(0,\pi)),$$

for $b\in(0,1)$, by Lebesgue term by term theorem (cf. [17]), we obtain

$$\begin{aligned}A_b:&=\int_0^\infty\frac{u^{b-1}}{1-u}du=\int_0^1\frac{u^{b-1}}{1-u}du+\int_1^\infty\frac{u^{b-1}}{1-u}du\\&=\int_0^1\frac{u^{b-1}}{1-u}du-\int_0^1\frac{v^{-b}}{1-v}dv=\int_0^1\frac{u^{b-1}-u^{-b}}{1-u}du\\&=\int_0^1\sum_{k=1}^\infty\left(\frac{1}{k+b}-\frac{1}{k+1-b}\right)\\&=\pi\left[\frac{1}{\pi b}+\sum_{k=1}^\infty\left(\frac{1}{\pi b-\pi k}+\frac{1}{\pi b+\pi k}\right)\right]=\pi\cot\pi b\in\mathbf{R}.\end{aligned}$$

(ii) For $\lambda, \eta > 0$, we set the homogeneous function of degree $-\lambda$ as follows:

$$k_\lambda^{(\eta)}(x,y) := \frac{x^\eta - y^\eta}{x^{\lambda+\eta} - y^{\lambda+\eta}} \ (x, y > 0),$$

satisfying $k_\lambda^{(\eta)}(v,v) := \frac{\eta}{(\lambda+\eta)v^\lambda}$ $(v > 0)$. Then we have

$$k_\lambda^{(\eta)}(m,n) := \frac{(|m| + \alpha m)^\eta - (|n| + \beta n)^\eta}{(|m| + \alpha m)^{\lambda+\eta} - (|n| + \beta n)^{\lambda+\eta}} \ (|m|, |n| \in \mathbf{N}).$$

It follows that $k_\lambda^{(\eta)}(x,y)$ is a positive and continuous function with respect to $x, y > 0$. For $x \neq y$, we find

$$\frac{\partial}{\partial x} k_\lambda^{(\eta)}(x,y) = -x^{\eta-1}(x^{\lambda+\eta} - y^{\lambda+\eta})^{-2}\varphi(x,y),$$

where, we set the following differentiable function:

$$\varphi(x,y) := \lambda x^{\lambda+\eta} - (\lambda+\eta)y^\eta x^\lambda + \eta y^{\lambda+\eta} \ (x, y > 0).$$

We find that for $0 < x < y$, $\frac{\partial}{\partial x}\varphi(x,y) = \lambda(\lambda+\eta)x^{\lambda-1}(x^\eta - y^\eta) < 0$; for $x > y$. $\frac{\partial}{\partial x}\varphi(x,y) > 0$. It follows that $\varphi(x,y)$ is strictly decreasing (resp. increasing) with respect to $x < y$ (resp. $x > y$). Since $\varphi(y,y) = \min_{x>0}\varphi(x,y) = 0$ $(y > 0)$, then $\varphi(x,y) > 0$ $(x \neq y)$, namely, $\frac{\partial}{\partial x} k_\lambda^{(\eta)}$ $(x,y) < 0$ $(x \neq y)$. Therefore, in view of $k_\lambda^{(\eta)}(x,y)$ is continuous at $x = y$, we conform that $k_\lambda^{(\eta)}(x,y)$ $(y > 0)$ is strictly decreasing with respect to $x > 0$. In the same way, we can show that $k_\lambda^{(\eta)}(x,y)$ $(x > 0)$ is also strictly decreasing with respect to $y > 0$. Hence, for $\lambda_i \in (0,\lambda) \cap (0,1]$ $(i = 1,2)$, $k_\lambda^{(\eta)}(x,y)x^{\lambda_1-1}$ (resp. $k_\lambda^{(\eta)}(x,y)y^{\lambda_2-1}$) is strictly decreasing with respect to $x > 0$ (resp. $y > 0$) .

(iii) Since $k_\lambda^{(\eta)}(x,y) > 0$, by (i), for $\gamma = \lambda_2, \lambda - \lambda_1$, we obtain (cf. [29])

$$\begin{aligned}
k_\lambda(\gamma) &= \int_0^\infty k_\lambda^{(\eta)}(1,u)u^{\gamma-1}du = \int_0^\infty \frac{1-u^\eta}{1-u^{\lambda+\eta}}u^{\gamma-1}du \\
&\overset{v=u^{\lambda+\eta}}{=} \frac{1}{\lambda+\eta}\left(\int_0^\infty \frac{v^{\frac{\gamma}{\lambda+\eta}-1}}{1-v}dv - \int_0^\infty \frac{v^{\frac{\gamma+\eta}{\lambda+\eta}-1}}{1-v}dv\right) \\
&= \frac{\pi}{\lambda+\eta}\left[\cot(\frac{\pi\gamma}{\lambda+\eta}) - \cot(\frac{\pi(\lambda+\gamma)}{\lambda+\eta})\right] \\
&= \frac{\pi}{\lambda+\eta}\left[\cot(\frac{\pi\gamma}{\lambda+\eta}) + \cot(\frac{\pi(\lambda-\gamma)}{\lambda+\eta})\right] \in \mathbf{R}_+.
\end{aligned}$$

By Theorem 4.1, it follows that $\lambda_1 + \lambda_2 = \lambda$ if and only if

$$||T|| = \frac{2}{(1-\beta^2)^{1/p}(1-\alpha^2)^{1/q}} \cdot \frac{\pi}{\lambda+\eta}\left[\cot(\frac{\pi\lambda_1}{\lambda+\eta}) + \cot(\frac{\pi\lambda_2}{\lambda+\eta})\right].$$

5. Conclusions

In this paper, by means of the weight coefficients and the idea of introduced parameters, a new discrete Hilbert-type inequality in the whole plane is obtained in

Lemma 2.2, which is an extension of (1.1). The equivalent form is given in Theorem 3.1. The equivalent statements of the best possible constant factor related to several parameters are considered in Theorem 3.2. The operator expressions, some particular cases are provided in Theorem 4.1 and Example 4.1-4.2. The lemmas and theorems provide an extensive account of this type of inequalities.

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