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A NEW GENERALIZATION OF $\mathcal{F}$-METRIC SPACES AND SOME FIXED POINT THEOREMS AND AN APPLICATION*

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Abstract In this paper, we extend $\mathcal{F}$-metric spaces to more general spaces, named generalized $\mathcal{F}$-metric spaces and establish some fixed point theorems via comparison function, F-contraction, Geraghty contraction and JS-contraction in the setting of generalized $\mathcal{F}$-metric spaces. Our results generalize many present theorems.

Keywords $\mathcal{F}$-metric space, fixed point, Geraghty contraction, F-contraction.

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1. Introduction

In recent years, the notions of metric spaces have been extended in many directions [10,12,23–26] for example controlled metric spaces [14] and doulbe controlled metric spaces [1]. Recently Jleli and Samet [10] introduced a new generalization of metric space named $\mathcal{F}$-metric space, and soon many scholars considered the $\mathcal{F}$-metric space [8, 13, 15, 17, 20]. Inspired by [10], we extend it to a more general space.

Let $\mathcal{F}$ be the set of functions $f : (0, +\infty) \to \mathbf{R}$ satisfying the following conditions:

($\mathcal{F}_1$) f is non-decreasing, i. e. $0 < s < t \Rightarrow f(s) \le f(t)$;

($\mathcal{F}_2$) for every sequence $\{t_n\} \subset (0, +\infty)$, we have

$$\lim_{n\to\infty} t_n = 0 \Leftrightarrow \lim_{n\to\infty} f(t_n) = -\infty.$$

For example, $f_1(t) = \ln t$, $f_2(t) = \frac{-1}{t}$, $f_1, f_2 \in \mathcal{F}$.

Definition 1.1 ([10]). Let X be a nonempty set and $D : X \times X \to [0, +\infty)$ be a given mapping. If there exist a constant $\alpha \ge 0$ and a function $f \in \mathcal{F}$ such that, for all $x, y \in X$, the following conditions hold:

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(D_1) $D(x,y)=0$ iff $x=y$;

(D_2) $D(x,y)=D(y,x)$;

(D_3) for every $N\in\{2,3,4,\cdots\}$ and for every $(x_i)_{i=1}^N\subset X$ with $(x_1,x_N)=(x,y)$, we have

$$D(x,y)>0\Rightarrow f(D(x,y))\leq f\left(\sum_{i=1}^{N-1}D(x_i,x_{i+1})\right)+\alpha,$$

then D is said to be an $\mathcal{F}$-metric on X, and the pair (X,D) is said to be an $\mathcal{F}$-metric space.

We introduce the following definition which generalizes the $\mathcal{F}$-metric space.

Definition 1.2. Let X be a nonempty set and $D:X\times X\to[0,+\infty)$ be a given mapping. If there exist a constant $\alpha\geq 0$ and a function $f\in\mathcal{F}$ such that, for all $x,y,z\in X$, the following conditions hold:

(D_1) $D(x,y)=0$ iff $x=y$;

(D_2) $D(x,y)=D(y,x)$;

(D_3^*) $D(x,y)>0\Rightarrow f(D(x,y))\leq f(D(x,z)+D(z,y))+\alpha$,

then D is said to be a generalized $\mathcal{F}$-metric on X, and the pair (X,D) is said to be a generalized $\mathcal{F}$-metric space.

Every $\mathcal{F}$-metric on X is a generalized $\mathcal{F}$-metric on X, because from (D_3) we get

$$D(x,y)>0\Rightarrow f(D(x,y))\leq f(D(x,z)+D(z,y))+\alpha.$$

Then D satisfies (D_3^*).

Every metric is a generalized $\mathcal{F}$-metric, because that $d(x,y)\leq d(x,z)+d(z,y)$ yields to $\ln(d(x,y))\leq\ln(d(x,z)+d(z,y))+0$ for $d(x,y)>0$. Then d satisfies (D_3^*) with $f(t)=\ln t$ and $\alpha=0$.

To show the range of generalized $\mathcal{F}$-metric spaces are really larger than $\mathcal{F}$-metric spaces, we recall the definitions of s-relaxed$_p$ metric space and b-metric space as follows.

Definition 1.3 ([6])**.** Let X be a nonempty set and $D:X\times X\to[0,+\infty)$ be a given mapping satisfying (D_1),(D_2), and

(S) there exists $s\geq 1$ such that for every $(x,y)\in X\times X$, $N\in\{2,3,4,\cdots\}$, and for every $(x_i)_{i=1}^N\subset X$ with $(x_1,x_N)=(x,y)$, we have $D(x,y)\leq s\left(\sum_{i=1}^{N-1}D(x_i,x_{i+1})\right)$.

Then D is said to be an s-relaxed$_p$ metric on X, and the pair (X,D) is said to be an s-relaxed$_p$ metric space.

Every s-relaxed$_p$ metric space is an $\mathcal{F}$-metric space with $f=\ln x$ and $\alpha=\ln s$.

Definition 1.4 ([4])**.** Let X be a nonempty set and $D:X\times X\to[0,+\infty)$ be a given mapping satisfying (D_1), (D_2), and

(G) there exists $s\geq 1$ such that for every $(x,y,z)\in X\times X\times X$, we have $D(x,y)\leq s(D(x,z)+D(z,y))$.

Then D is said to be a b-metric on X, and the pair (X, D) is said to be a b-metric space.

Every b-metric is a generalized $\mathcal{F}$-metric with $f(t) = \ln t$ and $\alpha = \ln s$.

Every s-relaxed$_p$-metric on X is a b-metric on X, because from (S) we get

$$D(x,y)) \leq D(x,z) + D(z,y),$$

which shows that D satisfies (G).

The following examples show that there are b-metric spaces that are not s-relaxed$_p$ metric spaces. So there are generalized $\mathcal{F}$-metric spaces (for example, some $b-$metric spaces) that are not $\mathcal{F}$-metric spaces (for example, some s-relaxed$_p$ metric spaces).

Example 1.1 (Proposition 2.1 in [10]). Let $X = [0,1]$, and let $d : X \times X \to [0, +\infty)$ be a mapping defined by $d(x,y) = (x-y)^2, (x,y) \in X \times X$. It is well known that d is a b-metric on X with coefficient $K = 2$. But d is not an s-relaxed$_p$ metric, because

$$d(0,1) > K(d(0, \frac{1}{n}) + d(\frac{1}{n}, \frac{2}{n}) + \cdots + d(\frac{n-1}{n}, \frac{n}{n})) = \frac{K}{n} \to 0, \text{ as } n \to \infty.$$

From $(\mathcal{F}_2)$ we get

$$f(d(0, \frac{1}{n}) + d(\frac{1}{n}, \frac{2}{n}) + \cdots + d(\frac{n-1}{n}, \frac{n}{n})) + \alpha = f(\frac{1}{n}) + \alpha \to -\infty, \text{ as } n \to \infty.$$

Thus, d on X is not an $\mathcal{F}$-metric, but a generalized $\mathcal{F}$-metric.

Example 1.2 (a case of Example 11 in [18]). Let $X = \{\log_2 2, \log_2 3, \log_2 4, \cdots\}$, $n \in \mathbb{N}$, $K \in (1, \infty)$, $a_n = \frac{1}{(2K)^n}$, $f(n) = -[-\log_2 n]$, $g(n) = (2n - 2^{f(n)})K^{f(n)} + (2^{f(n)} - n)K^{f(n)-1}$,

$$d(\log_2 n, \log_2 m) = \begin{cases} 0, & n = m; \\ g(n-m)a_k, & 2^k \leq m < n \leq 2^{k+1}; \\ d(\log_2 m, \log_2 2^{j+1}) + & \sum\limits_{i=j+1}^{k-1} d(i, i+1) + d(\log_2 2^k, \log_2 n), \\ & 2^j \leq m < 2^{j+1} \leq 2^k < n \leq 2^{k+1}; \\ d(\log_2 m, \log_2 n), & n < m. \end{cases}$$

It was proved in [18] that d is a b-metric on X, $\sum\limits_{i=2}^{\infty} d(\log_2 i, \log_2(i+1)) < \infty$ and $d(n, n+1) = 1$. It implies $\sum\limits_{i=2^n}^{2^{n+1}-1} d(\log_2 i, \log_2(i+1)) \to 0$, as $n \to \infty$. We get

$$d(n, n+1) = 1 > K \left(\sum_{i=2^n}^{2^{n+1}-1} d(\log_2 i, \log_2(i+1)) \right) \to 0, \text{ as } n \to \infty.$$

Then, d is not an s-relaxed$_p$ metric. It is easy to get d on X is a generalized $\mathcal{F}$-metric, not an $\mathcal{F}$-metric.

The following example shows that the generalized $\mathcal{F}$-metric spaces are really more extensive than b-metric spaces.

Example 1.3. Let $X = \mathbb{R}$, $a > 0$, $b > 0$, $D : X \times X \to [0, +\infty)$ given by

$$D(x,y) = \begin{cases} ae^{b(|x-y|)}, & \text{if } x \neq y, \\ 0, & \text{if } x = y. \end{cases}$$

Then, $D(3n, 0) = ae^{3nb}$, $D(3n, n) = ae^{2nb}$, and $D(0, n) = ae^{nb}$. If $D(3n, 0) \leq s(D(3n, n) + D(n, 0))$, i. e. $ae^{3nb} \leq s(ae^{2nb} + ae^{nb})$, then $e^{nb} \leq s(1 + \frac{1}{e^{nb}})$. It is a contradiction if we take $n \to \infty$. Therefore, D is not a b-metric.

Next, we show that D is a generalized $\mathcal{F}$-metric. Let $f(t) = \frac{-1}{t}$. For given $(x, y) \in X \times X$ with $D(x, y) > 0$, for every $z \in X$ we have

$$\begin{aligned} \frac{1}{a} + f(D(x,z) + D(z,y)) - f(D(x,y)) &= \frac{1}{a} - \frac{1}{D(x,z) + D(z,y)} + \frac{1}{ae^{b(|x-y|)}} \\ &\geq \frac{1}{a} - \frac{1}{a} + \frac{1}{ae^{b(|x-y|)}} \geq 0. \end{aligned}$$

Therefore, we have

$$f(D(x,y)) \leq f(D(x,z) + D(z,y)) + \frac{1}{a}.$$

Then D is a generalized $\mathcal{F}$-metric on X with $f(t) = \frac{-1}{t}$ and $\alpha = \frac{1}{a}$.

In [3,10], a natural topology defined on $\mathcal{F}$-metric spaces was discussed. However we think that they actually discussed a natural topology on generalized $\mathcal{F}$-metric spaces. In [17], Som, Petrusel et al. proved the metrizability of $\mathcal{F}$-metric spaces, and actually proved the metrizability of generalized $\mathcal{F}$-metric spaces.

Definition 1.5. Let (X, D) be a generalized $\mathcal{F}$-metric space. For every $x_0 \in X$ and $r > 0$, the ball with centre x_0 and radius r is defined by

$$B(x_0, r) = \{y \in X : D(x_0, y) < r\}.$$

Definition 1.6. Let (X, D) be a generalized $\mathcal{F}$-metric space. A subset $\mathcal{O}$ of X is said to be $\mathcal{F}$-open if for every $x \in \mathcal{O}$, there is some $r > 0$ such that $B(x, r) \subset \mathcal{O}$. We say that a subset $\mathcal{C}$ of X is $\mathcal{F}$-closed if $X \setminus \mathcal{C}$ is $\mathcal{F}$-open. We denote the family of all $\mathcal{F}$-open subsets of X by $\tau_{\mathcal{F}}$.

Proposition 1.1. *Let (X, D) be a generalized $\mathcal{F}$-metric space. Then $\tau_{\mathcal{F}}$ is a topology on X.*

Definition 1.7. Let (X, D) be a generalized $\mathcal{F}$-metric space.

1. A sequence $\{x_n\}$ is said to be $\mathcal{F}$-Cauchy if, for any $\epsilon > 0$, there exists a positive integer n_0 such that, for all $m, n \geq n_0$, $D(x_n, x_m) < \epsilon$;
2. A sequence $\{x_n\}$ is said to be $\mathcal{F}$-convergent to a point $x \in X$ if, for any $\epsilon > 0$, there exists a positive integer n_0 such that, for all $n \geq n_0$, $D(x, x_n) < \epsilon$;
3. A $\mathcal{F}$-metric space is called $\mathcal{F}$-complete if every $\mathcal{F}$-Cauchy sequence is $\mathcal{F}$-convergent in X.

2. Fixed point results in generalized $\mathcal{F}$-metric spaces

Lemma 2.1. *Let (X, D) be a generalized $\mathcal{F}$-metric space. If a sequence $\{x_n\} \subset X$ has a limit in X, then the limit is unique.*

Proof. We assume $x, y \in X$ are both limits of $\{x_n\}$ as $n \to \infty$. If $D(x,y) \neq 0$, from the definition of generalized $\mathcal{F}$-metric space, we get

$$f(D(x,y)) \leq f(D(x,x_n) + D(x_n,y)) + \alpha.$$

By virtue of $(\mathcal{F}_2)$, we derive that $\lim\limits_{n\to\infty} f(D(x,y)) = -\infty$. This contradicts $f(D(x,y)) < +\infty$. Hence we get $D(x,y) = 0$, i.e. $x = y$. □

2.1. Fixed point results via comparison functions

Let $\phi^n(x)$ denote the n-th iteration of ϕ in the follows.

Let Φ_1 be the family of functions $\phi : [0,+\infty) \to [0,+\infty)$ satisfying:

1. $s < t \Rightarrow \phi(s) \leq \phi(t)$;
2. $\sum\limits_{n=1}^{\infty} \phi^n(x) < \infty$, for all $x > 0$.

Let Φ_2 be the family of functions $\phi : [0,+\infty) \to [0,+\infty)$ satisfying:

1. $s < t \Rightarrow \phi(s) \leq \phi(t)$;

2*. $\lim\limits_{n\to\infty} \phi^n(x) = 0$, for all $x > 0$.

Remark 2.1. If $\sum\limits_{n=1}^{\infty} \phi^n(x) < \infty$, for all $x > 0$, then $\lim\limits_{n\to\infty} \phi^n(x) = 0$, for every $x > 0$. Thus, $\Phi_1 \subset \Phi_2$, i.e., the class of Φ_2 is larger than the class of Φ_1. In what follows, a function $\phi \in \Phi_2$ is called a comparison function.

For example, $\phi_1(t) = kt, k \in (0,1)$, $\phi_2(t) = \frac{t}{1+t}$. $\phi_1^n(t) = k^n t \to 0$, $\phi_2^n(t) = \frac{t}{1+nt} \to 0$, as $n \to \infty$.

It is easy to check that the following lemma holds.

Lemma 2.2. *If $\phi \in \Phi_2$, then the following are satisfied:*

1. *$\phi(t) < t$, for all $t > 0$;*
2. *$\phi(0) = 0$.*

Lemma 2.3. *Let (X, D) be a generalized $\mathcal{F}$-metric space with $(f, \alpha) \in \mathcal{F} \times [0,+\infty)$. If there exists a functioin $\phi \in \Phi_2$, such that a sequence $\{x_n\}$ satisfies*

$$D(x_n, x_{n+1}) \leq \phi(D(x_{n-1}, x_n)), \tag{2.1}$$

then $\{x_n\}$ is an $\mathcal{F}$-Cauchy sequence.

Proof. From

$$D(x_n, x_{n+1}) \leq \phi(D(x_{n-1}, x_n)) \leq \phi^{n-1}(D(x_0, x_1)),$$

we get $\lim\limits_{n\to\infty} D(x_n, x_{n+1}) = 0$. We want to show by induction in m that, for all $m \in \{1, 2, 3, \cdots\}$

$$\lim_{n\to\infty} D(x_n, x_{n+m}) = 0. \tag{2.2}$$

It is obvious that (2.2) holds for $m = 1$. Assume that (2.2) is satisfied for some $m \in \{1, 2, 3, \cdots\}$. Since

$$D(x_n, x_{n+m+1}) > 0 \Rightarrow f(D(x_n, x_{n+m+1})) \leq f(D(x_n, x_{n+m}) + D(x_{n+m}, x_{n+m+1})) + \alpha,$$

and

$$D(x_n, x_{n+m}) + D(x_{n+m}, x_{n+m+1}) \to 0, \text{ as } n \to \infty,$$

we have

$$f(D(x_n, x_{n+m}) + D(x_{n+m}, x_{n+m+1})) \to -\infty, \text{ as } n \to \infty.$$

From $(\mathcal{F}_2)$ we get

$$\lim_{n\to\infty} f(D(x_n, x_{n+m+1})) = -\infty,$$

i.e.,

$$\lim_{n\to\infty} D(x_n, x_{n+m+1}) = 0.$$

Hence, (2.2) holds for all $m \geq 1$. Thus, the sequence $\{x_n\}$ is an $\mathcal{F}$-Cauchy sequence. □

Theorem 2.1. *Let (X, D) be an $\mathcal{F}$-complete generalized $\mathcal{F}$-metric space with $(f, \alpha) \in \mathcal{F} \times [0, +\infty)$. Let $T : X \to X$ be a mapping. If there exists a function $\phi \in \Phi_2$ such that*

$$D(Tx, Ty) \leq \phi(D(x, y)), \tag{2.3}$$

for all $x, y \in X$, then T has a unique fixed point in X.

Proof. Let $x_0 \in X$ be an arbitrary element. Let $\{x_n\}$ be the sequence defined by $x_n = T^n x_0, n = 1, 2, \cdots$. If there exists some $N \in \{0, 1, 2, \cdots\}$ such that $T^N x_0 = T^{N+1} x_0$, then T has a fixed point $T^N x_0$. Next we assume that for every $n \in \{0, 1, 2, \cdots\}$, $T^n x_0 \neq T^{n+1} x_0$. From (2.3), we obtain

$$D(x_n, x_{n+1}) = D(Tx_{n-1}, Tx_n) \leq \phi(D(x_{n-1}, x_n)),$$

which implies that $\{x_n\}$ is an $\mathcal{F}$-Cauchy sequence. Since the generalized $\mathcal{F}$-metric space is $\mathcal{F}$-complete then there exists an $x \in X$ such that $\lim_{n\to\infty} D(x_n, x) = 0$. From

$$D(Tx_n, Tx) \leq \phi(D(x_n, x)) \leq D(x_n, x) \to 0, \text{ as } n \to \infty,$$

we see that Tx is also a limit of sequence $\{x_n\}$. From the uniqueness of limit of sequence in generalized $\mathcal{F}$-metric space, we have $Tx = x$. If T has another fixed point y, then

$$D(Tx, Ty) \leq \phi(D(x, y)) < D(x, y),$$

which is a contradiction. □

Corollary 2.1. *Let (X, D) be an $\mathcal{F}$-complete generalized $\mathcal{F}$-metric space with $(f, \alpha) \in \mathcal{F} \times [0, +\infty)$. Let $T : X \to X$ be a mapping. If there exists $k \in [0, 1)$ such that for all $x, y \in X$,*

$$D(Tx, Ty) \leq kD(x, y),$$

then T has a unique fixed point in X.

Proof. Let $\phi(t) = kt$ in Theorem 2.1. □

Corollary 2.2 ([10]). *Let (X, D) be an $\mathcal{F}$-complete $\mathcal{F}$-metric space with $(f, \alpha) \in \mathcal{F} \times [0, +\infty)$. Let $T : X \to X$ be a mapping. If there exists $k \in [0, 1)$ such that for all $x, y \in X$,*

$$D(Tx, Ty) \leq kD(x, y),$$

then T has a unique fixed point in X.

Proof. $\mathcal{F}$-metric space is a generalized $\mathcal{F}$-metric space, thus the conditions of Theorem 2.1 are satisfied. □

Corollary 2.3 ([2]). *Let (X, D) be a complete b-metric space. Let $T : X \to X$ be a mapping. If there exists $k \in [0, 1)$ such that for all $x, y \in X$,*

$$D(Tx, Ty) \leq kD(x, y),$$

then T has a unique fixed point in X.

Proof. Because b-metric space is a generalized $\mathcal{F}$-metric space, the conditions of Theorem 2.1 are satisfied. □

Corollary 2.4 (Banach type contraction). *Let (X, D) be a complete metric space. Let $T : X \to X$ be a mapping. If there exists $k \in [0, 1)$ such that for all $x, y \in X$,*

$$D(Tx, Ty) \leq kD(x, y),$$

then T has a unique fixed point in X.

Proof. Metric space is a generalized $\mathcal{F}$-metric space, then the conditions of Theorem 2.1 are satisfied. □

Corollary 2.5. *Let (X, D) be a $\mathcal{F}$-complete $\mathcal{F}$-metric space with $(f, \alpha) \in \mathcal{F} \times [0, +\infty)$. Let $T : X \to X$ be a mapping. If there exists $\phi \in \Phi_1$ such that for all $x, y \in X$,*

$$D(Tx, Ty) \leq \phi(D(x, y)),$$

then T has a unique fixed point in X.

Proof. Because $\Phi_1 \subset \Phi_2$ and $\mathcal{F}$-metric space is a generalized $\mathcal{F}$-metric space, the conditions of Theorem 2.1 are satisfied. □

2.2. Fixed point results using F-contractions

In this section we use the theorems of semimetric to get some fixed point theorems on $\mathcal{F}$-metric spaces. Now, we need to recall the concept of semimetric space.

Definition 2.1. Let X be a nonempty set and $d : X \times X \to [0, +\infty)$ be a given mapping. Suppose that for all $x, y \in X$, (D_1) and (D_2) are satisfied. Then d is said to be a semimetric on X, and the pair (X, d) is said to be a semimetric space.

Definition 2.2. Let (X, d) be a semimetric space.

1. A sequence $\{x_n\}$ is said to be Cauchy if $\lim\limits_{n\to\infty} \sup\{d(x_m, x_n) : m > n\} = 0$;
2. A sequence $\{x_n\}$ is said to converge to a point $x \in X$ if $\lim\limits_{n\to\infty} d(x, x_n) = 0$;
3. X is said to be complete if every Cauchy sequence converges in X.

Lemma 2.4. *Let (X, D) be a generalized $\mathcal{F}$-metric space. Then the following holds:*
(D_4) For any $\varepsilon > 0$, there exists $\delta > 0$ such that $D(x, z) < \delta$ and $D(z, y) < \delta$ imply $D(x, y) < \varepsilon$.

Proof. Let $\varepsilon > 0$. By $(\mathcal{F}_2)$, for $f(\varepsilon) - \alpha$, there exists $\delta > 0$ such that $0 < t < \delta$ implies $f(t) < f(\varepsilon) - \alpha$. By (D_3^*), $D(x, z) < \frac{\delta}{2}$ and $D(z, y) < \frac{\delta}{2}$ imply

$$f(D(x, y)) \leq f(D(x, z) + D(z, y)) + \alpha < f(\varepsilon),$$

From $(\mathcal{F}_1)$, we get $D(x, y) < \varepsilon$. □

Lemma 2.5 ([19]). *Let (X,d) be a complete semimetric space. Assume (D_4) is satisfied. Let $T : X \to X$ be a mapping. Assume that there exists a function $F : (0,\infty) \to \mathbb{R}$ and a real number $\tau \in (0,\infty)$ satisfying $(\mathcal{F}_2)$ and*

$$\tau + F(d(Tx,Ty)) \le F(d(x,y)).$$

Then T has a unique fixed point z. Moreover, $\{T^n x\}$ converges to z for all $x \in X$.

Theorem 2.2. *Let (X,d) be an $\mathcal{F}$-complete generalized $\mathcal{F}$-metric space. Let $T : X \to X$ be a mapping. Assume that there exists a function $F : (0,\infty) \to \mathbb{R}$ and a real number $\tau \in (0,\infty)$ satisfying $(\mathcal{F}_2)$ and*

$$\tau + F(d(Tx,Ty)) \le F(d(x,y)).$$

Then T has a unique fixed point z. Moreover, $\{T^n x\}$ converges to z for all $x \in X$.

Proof. Firstly generalized $\mathcal{F}$-metric spaces are semimetric spaces. Secondly from Lemma 2.4 (D_4) holds. By Lemma 2.5 we obtain the desired result. □

2.3. Fixed Point Results Using Geraghty Contractions

The Geraghty contraction was originated from Geraghty [7], and was advanced in many aspects [11, 21, 22]. Now we apply it to generalized $\mathcal{F}$-metric spaces.

Let Γ be the family of functions $\gamma : [0,+\infty) \to (-\infty, 0]$ such that:

$$\limsup_{n\to\infty} \gamma(t_n) = 0 \Rightarrow \lim_{n\to\infty} t_n = 0.$$

For example $\gamma_1(x) = -x$, $\gamma_2(x) = -x^3$, $\gamma_1, \gamma_2 \in \Gamma$.

Definition 2.3. Let (X,D) be a generalized $\mathcal{F}$-metric space with $(f,\alpha) \in \mathcal{F} \times [0,+\infty)$. Let $T : X \to X$ be a mapping. If for all $x,y,z \in X$ there exists a function $\gamma \in \Gamma$ satisfying

$$D(Tx,Ty) > 0 \Rightarrow f(D(Tx,Ty)) \le \gamma(D(x,y)) + f(D(x,y)) - \alpha, \tag{2.4}$$

then the mapping T is called an $\mathcal{F}$-Geraghty contraction.

Theorem 2.3. *Let (X,D) be an $\mathcal{F}$-complete generalized $\mathcal{F}$-metric space with $(f,\alpha) \in \mathcal{F} \times [0,+\infty)$. The mapping $T : X \to X$ is an $\mathcal{F}$-Geraghty contraction and f is increasing and continuous. Then T has a unique fixed point p, and for all $x \in X$, the sequence $\{T^n x\}$ converges to p.*

Proof. Let $x_0 \in X$ be an arbitrary element. Let $\{x_n\}$ be the sequence defined by $x_n = T^n x_0$, $n \in \{0,1,2,\cdots\}$. If there exists some $N \in \{0,1,2,\cdots\}$ such that $T^N x_0 = T^{N+1} x_0$ then T has a fixed point. Next we assume for every $n \in \{0,1,2,\ldots\}$, $T^n x_0 \ne T^{n+1} x_0$. From (2.4) we get

$$\begin{aligned} f(D(x_{n+1},x_{n+2})) \le& \gamma(D(x_n,x_{n+1})) + f(D(x_n,x_{n+1})) - \alpha \\ \le& f(D(x_n,x_{n+1})). \end{aligned} \tag{2.5}$$

From the increasing property of f, we have $D(x_n,x_{n+1}) \le D(x_{n-1},x_n)$. There exists a $r \ge 0$ such that $\lim_{n\to\infty} D(x_n,x_{n+1}) = r$. If $r > 0$ from (2.5) and the continuity of f we get

$$f(r) \le \limsup_{n\to\infty} \gamma(D(x_n,x_{n+1})) + f(r) - \alpha.$$

From $0 \leq \alpha \leq \limsup_{n\to\infty} \gamma(D(x_n, x_{n+1})) \leq 0$, we get $\lim_{n\to\infty} D(x_n, x_{n+1}) = 0$, which is a contradiction. So $r = 0$.

Next, we prove that the sequence $\{x_n\}$ is an $\mathcal{F}$-Cauthy sequence. Suppose the contrary, i.e., there exists $\epsilon > 0$ for which we can find two subsequences $\{x_{n_i}\}$ and $\{x_{m_i}\}$ such that m_i is the smallest index for which

$$i \leq n_i \leq m_i \text{ and } D(x_{n_i}, x_{m_i}) \geq \epsilon.$$

This means that

$$D(x_{n_i}, x_{m_i-1}) < \epsilon.$$

On the one hand, from the increasing property of f we get

$$f(\epsilon) \leq f(D(x_{n_i}, x_{m_i})) \leq f(D(x_{n_i}, x_{n_i+1}) + D(x_{n_i+1}, x_{m_i})) + \alpha,$$

then

$$f^{-1}(f(\epsilon) - \alpha) \leq \limsup_{i\to\infty} D(x_{n_i+1}, x_{m_i}).$$

From the increasing property of f, we get

$$\begin{aligned} f(\epsilon) - \alpha \leq& f(\limsup_{i\to\infty} D(x_{n_i+1}, x_{m_i})) \\ =& \limsup_{i\to\infty} f(D(x_{n_i+1}, x_{m_i})). \end{aligned} \tag{2.6}$$

On the other hand,

$$\begin{aligned} f(D(x_{n_i+1}, x_{m_i})) &\leq \gamma(D(x_{n_i}, x_{m_i-1})) + f(D(x_{n_i}, x_{m_i-1})) - \alpha \\ &\leq \gamma(D(x_{n_i}, x_{m_i-1})) + f(\epsilon) - \alpha. \end{aligned} \tag{2.7}$$

Combining (2.6) and (2.7) we get

$$0 \leq \limsup_{i\to\infty} \gamma(D(x_{n_i}, x_{m_i-1})),$$

which implies $\limsup_{i\to\infty} \gamma(D(x_{n_i}, x_{m_i-1})) = 0$, i.e. $\lim_{i\to\infty} D(x_{n_i}, x_{m_i-1}) = 0$. From

$$f(D(x_{n_i}, x_{m_i})) \leq f(D(x_{n_i}, x_{m_i-1}) + D(x_{m_i-1}, x_{m_i})) + \alpha,$$

we get $\lim_{i\to\infty} D(x_{n_i}, x_{m_i}) = 0$, a contradiction. So $\{T^n x\}$ is an $\mathcal{F}$-Cauthy sequence. From the $\mathcal{F}$-complete of the generalized $\mathcal{F}$-metric space, there exists an $x \in X$ such that $\lim_{n\to\infty} D(x_n, x) = 0$.

From

$$f(D(Tx_n, Tx)) \leq \gamma(D(x_n, x)) + f(D(x_n, x)) - \alpha,$$

we get $f(D(Tx_n, Tx)) \to -\infty$, i.e. $D(Tx_n, Tx) \to 0$ as $n \to \infty$. Tx is also a limit of sequence $\{x_n\}$. From the uniqueness of limit of sequence in generalized $\mathcal{F}$-metric space, we have $Tx = x$.

If T has another fixed point $y \in X$, and $D(x, y) > 0$, then

$$f(D(Tx, Ty)) \leq \gamma(D(x, y)) + f(D(x, y)) - \alpha,$$

which implies $\alpha \leq \gamma(D(x,y))$. If $\alpha > 0$, a contradiction. If $\alpha = 0$, from the proposition of γ we have $D(x,y) = 0$, $x = y$, a contradiction. □

Let $\mathcal{B}$ be the family of functions $\beta : [0, +\infty) \to [0, 1)$ such that:

$$\limsup_{n\to\infty} \beta(t_n) = 1 \Rightarrow \lim_{n\to\infty} t_n = 0.$$

For example, $\beta_1(x) = \begin{cases} e^{-x}, & x > 0, \\ 0, & x = 0. \end{cases}$, $\beta_2(x) = \begin{cases} \frac{1}{1+\frac{1}{100}x}, & x > 0, \\ 0, & x = 0. \end{cases}$, $\beta_1(x), \beta_2(x) \in \mathcal{B}$.

Corollary 2.6 ([7]). *If (X, D) is a complete metric space and a mapping $T : X \to X$ satisfies*

$$D(Tx, Ty) \leq \beta(D(x,y))D(x,y), \text{ for all } x, y \in X, \tag{2.8}$$

where $\beta \in \mathcal{B}$, then T has a unique fixed point p and for any $x \in X$, the sequence $\{T^n x\}$ converges to p.

Proof. Because the metric space is a generalized $\mathcal{F}$-metric space with $f(x) = \ln x$ and $\alpha = 0$. From (2.8) we get

$$D(Tx, Ty) > 0 \Rightarrow \ln(D(Tx, Ty)) \leq \ln(\beta(D(x,y))) + \ln(D(x,y)).$$

Form $\lim_{n\to\infty} \ln(\beta(x_n)) = 0 \Rightarrow \lim_{n\to\infty} \beta(x_n) = 1 \Rightarrow \lim_{n\to\infty} x_n = 0$, we obtain $\gamma(x) = \ln \beta(x) \in \Gamma$. So it can be concluded that

$$D(Tx, Ty) > 0 \Rightarrow f(D(Tx, Ty)) \leq \gamma(D(x,y)) + f(D(x,y)) - 0.$$

All the conditions of Theorem 2.3 are satisfied. □

Corollary 2.7 ([5]). *If (X, D) is a complete b-metric space with coefficient $s \geq 1$ and a mapping $T : X \to X$ satisfies*

$$D(Tx, Ty) \leq \frac{\beta(D(x,y))}{s} D(x,y), \text{ for all } x, y \in X, \tag{2.9}$$

where $\beta \in \mathcal{B}$, then T has a unique fixed point p and for any $x \in X$, the sequence $\{T^n x\}$ converges to p.

Proof. Because b-metric space is a generalized $\mathcal{F}$-metric space with $f(x) = \ln x$ and $\alpha = \ln s$. From (2.9) we get

$$D(Tx, Ty) > 0 \Rightarrow \ln(D(Tx, Ty)) \leq \ln(\beta(D(x,y))) + \ln(D(x,y)) - \ln(s).$$

$\lim_{n\to\infty} \ln(\beta(x_n)) = 0 \Rightarrow \lim_{n\to\infty} \beta(x_n) = 1 \Rightarrow \lim_{n\to\infty} x_n = 0$. So $\gamma(x) = \ln \beta(x) \in \Gamma$. All the conditions of Theorem 2.3 are satisfied. □

Example 2.1. Let $X = [0, 1]$ and $D : X \times X \to [0, \infty]$ be defined by $D(x,y) = (x - y)^2$, for all $x, y \in [0, 1]$. It is easy to check that (X, D) is a b-metric space with parameter $s = 2$. So (X, D) is also a generalized $\mathcal{F}$-metric space with $f(x) = \ln x$ and $\alpha = \ln 2$. Set $Tx = \frac{x^2}{8}$ for all $x \in X$, $\beta(t) = \frac{1}{16}$ and $\gamma(t) = \ln(\beta(t)) = -\ln 16$, for all $t > 0$. We get

$$D(Tx, Ty) = \frac{1}{64}(x + y)^2(x - y)^2 \leq \frac{1}{32}(x - y)^2 = \frac{\frac{1}{16}}{2} D(x,y),$$

then

$$\ln(D(Tx,Ty)) \le \ln\frac{1}{16} + \ln(D(x,y)) - \ln 2.$$

It is easy to know the conditions of Theorem 2.3 are satisfied. Hence T has a fixed point 0.

2.4. Fixed point results related to JS-contractions

The JS-contraction was originated from Jleli et al. [9], and was advanced [16]. Now we use the same principle in generalized $\mathcal{F}$-metric spaces.

Let Θ be the family of functions $\theta : \mathbf{R} \to (\mathbf{1}, \infty)$ satisfying:

- increasing;
- $\lim_{n\to\infty} \theta(t_n) = 1 \Leftrightarrow \lim_{n\to\infty} t_n = -\infty$.

For example, $\theta(t) = 1 + e^t \in \Theta$.

Theorem 2.4. *Let (X, D) be an $\mathcal{F}$-complete generalized $\mathcal{F}$-metric space with $f \in \mathcal{F}$, $\alpha \ge 0$ and let $T : X \to X$ be a given mapping. Suppose that f is increasing and there exist $\theta \in \Theta$ and $k \in (0, 1)$ such that*

$$\theta(f(D(Tx,Ty))) \le [\theta(f(D(x,y)) - \alpha)]^k, \text{ for all } x,y \in X. \tag{2.10}$$

Then T has a unique fixed point.

Proof. Let $x_0 \in X$ be an arbitrary element. Let $\{x_n\}$ be a sequence defined by $x_n = T^n x_0$, $n =\in \{0, 1, 2, \dots\}$. If there exists some $N \in \{0, 1, 2, \dots\}$ such that $T^N x_0 = T^{N+1} x_0$, then T has a fixed point. Next we assume for every $n \in \{0, 1, 2, \dots\}$, $T^n x_0 \ne T^{n+1} x_0$.

Step 1. We will show that $\lim_{n\to\infty} D(x_n, x_{n+1}) = 0$. By (2.10) we get

$$\begin{aligned}\theta(f(D(x_n,x_{n+1}))) &\le \theta(f(D(x_{n-1},x_n)) - \alpha)^k\\ &\le \theta(f(D(x_{n-1},x_n)))^k\\ &\le \theta(f(D(x_0,x_1))^{k^n}.\end{aligned}$$

With $\lim_{n\to\infty} k^n = 1$, we have $\lim_{n\to\infty} \theta(f(D(x_n, x_{n+1}))) = 1$, $\lim_{n\to\infty} f(D(x_n, x_{n+1})) = -\infty$, i.e. $\lim_{n\to\infty} D(x_n, x_{n+1}) = 0$.

Step 2. Next, we prove that the sequence $\{x_n\}$ is an $\mathcal{F}$-Cauthy sequence. Suppose the contrary, i.e., there exists $\epsilon > 0$ for which we can find two subsequences $\{x_{n_i}\}$ and $\{x_{m_i}\}$ such that m_i is the smallest index for which

$$i \le n_i \le m_i \text{ and } D(x_{n_i}, x_{m_i}) \ge \epsilon.$$

These mean that

$$D(x_{n_i}, x_{m_i-1}) < \epsilon.$$

On the one hand, from the increasing property of f we get

$$f(\epsilon) \le f(D(x_{n_i}, x_{m_i})) \le f(D(x_{n_i}, x_{n_i+1}) + D(x_{n_i+1}, x_{m_i})) + \alpha.$$

So we get

$$f^{-1}(f(\epsilon)-\alpha) \le \limsup_{i\to\infty} D(x_{n_i+1}, x_{m_i}),$$

then

$$\begin{aligned} f(\epsilon)-\alpha \le& f(\limsup_{i\to\infty} D(x_{n_i+1}, x_{m_i})) \\ =& \limsup_{i\to\infty} f(D(x_{n_i+1}, x_{m_i})). \end{aligned} \tag{2.11}$$

On the other hand,

$$\begin{aligned} \theta(f(D(x_{n_i+1}, x_{m_i}))) \le& \theta(f(D(x_{n_i}, x_{m_i-1}))-\alpha)^k \\ \le& \theta(f(\epsilon)-\alpha)^k. \end{aligned} \tag{2.12}$$

Combining (2.11) and (2.12) we get

$$\theta(f(\epsilon)-\alpha) \le \theta(f(\epsilon)-\alpha)^k.$$

It is in contradiction with $k \in (0,1)$. So the sequence $\{T^n x\}$ is an $\mathcal{F}$-Cauthy sequence. From the completeness of generalized $\mathcal{F}$-metric space, there exists a point, assuming $p \in X$ is the limit of $\{x_n\}$. From

$$\theta(f(D(Tx_n, Tp))) \le \theta(f(D(x_n, p))-\alpha)^k, \tag{2.13}$$

we get Tp is also a limit of $\{x_n\}$. From the uniqueness of the limit in generalized $\mathcal{F}$-metric space, we get $Tp = p$. □

Example 2.2. Let $X = [0,4]$, $D(x,y) = (x-y)^2$. The (X,D) is a b-metric with coefficient $s = 2$. Let $Tx = \frac{x}{2\sqrt{2}}$. There exist $f(t) = \ln t$, $\theta(t) = 1+e^t$, $\alpha = \ln 2$, $k = \frac{1}{2}$ such that

$$\theta(f(D(Tx,Ty))) \le \theta(f(D(x,y))-\alpha)^k,$$

i.e.,

$$\begin{aligned} D(x,y) \le 16 \Rightarrow& 1+\frac{1}{4}D(x,y)+\frac{1}{64}D(x,y)^2 \le 1+\frac{D(x,y)}{2} \\ \Rightarrow& (1+\frac{1}{8}D(x,y))^2 \le 1+\frac{D(x,y)}{2} \\ \Rightarrow& 1+e^{\ln(D(Tx,Ty))} \le (1+e^{\ln(D(x,y))-\ln 2})^{\frac{1}{2}}. \end{aligned}$$

Thus, the conditions of Theorem 2.4 are satisfied, T has a fixed point 0.

3. Application

In this section, we apply our results to solve the first order periodic boundary value problem:

$$\begin{cases} x'(t) = f(t, x(t)), \quad t \in [0,T], \\ x(0) = x(T). \end{cases} \tag{3.1}$$

where $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function on $[0,T]$. Problem (3.1) can be rewritten as

$$\begin{cases} x'(t) + \lambda x(t) = f(t, x(t)) + \lambda x(t), \\ x(0) = x(T). \end{cases}$$

It is equivalent to the integral equation

$$x(t) = \int_0^T G(t,s)(f(s,x(s)) + \lambda x(s))ds,$$

where G is the Green's function given as

$$G(t,s) = \begin{cases} \frac{e^{\lambda(T+s-t)}}{e^{\lambda T}-1}, & 0 \le s \le t \le T, \\ \frac{e^{\lambda(s-t)}}{e^{\lambda T}-1}, & 0 \le t \le s \le T. \end{cases}$$

It is easy to see that

$$\int_0^T G(t,s)ds = \frac{1}{\lambda}.$$

Let $C(I)$ denote the set of continuous functions on $I := [0,T]$. We define an operator $T : C(I) \to C(I)$ as

$$Tx(t) = \int_0^T G(t,s)(f(s,x(s)) + \lambda x(s))ds. \tag{3.2}$$

Theorem 3.1. *If there exists $\lambda > 0$ such that, for every $x, y \in C(I)$ and $s \in I$,*

$$|f(s,x(s)) + \lambda x(s) - f(s,y(s)) - \lambda y(s)| \le \lambda \sqrt{\frac{|x(s)-y(s)|^2}{1+|x(s)-y(s)|^2}}, \tag{3.3}$$

then Problem (3.1) has a unique solution in $C(I)$.

Proof. Let $D(x,y) = \max\limits_{t\in[0,T]}\{|x(t)-y(t)|^2\}$. Then $(C(I), D)$ is an $\mathcal{F}$-complete generalized $\mathcal{F}$-metric space.

$$\begin{aligned} D(Tx,Ty) &= \max_{t\in[0,T]}\{|Tx(t)-Ty(t)|^2\} \\ &\le \max_{t\in[0,T]}\left\{\left(\int_0^T G(t,s)|f(s,x(s)) + \lambda x(s) - f(s,y(s)) - \lambda y(s)|ds\right)^2\right\} \\ &\le \max_{t\in[0,T]}\left\{\left(\int_0^T G(t,s)\lambda\sqrt{\frac{|x(s)-y(s)|^2}{1+|x(s)-y(s)|^2}}ds\right)^2\right\} \\ &\le \max_{t\in[0,T]}\left\{\left(\int_0^T G(t,s)ds\right)^2 \cdot \max_{s\in[0,T]}\left\{\lambda\sqrt{\frac{|x(s)-y(s)|^2}{1+|x(s)-y(s)|^2}}\right\}^2\right\} \\ &\le \max_{t\in[0,T]}\left\{\frac{1}{\lambda^2}\cdot\lambda^2\cdot\max_{s\in[0,T]}\left\{\frac{|x(s)-y(s)|^2}{1+|x(s)-y(s)|^2}\right\}\right\} \\ &\le \frac{\max\limits_{s\in[0,T]}|x(s)-y(s)|^2}{1+\max\limits_{s\in[0,T]}|x(s)-y(s)|^2} \le \frac{D(x,y)}{1+D(x,y)}. \end{aligned}$$

The conditions of Theorem 2.1 are satisfied with $\phi(t) = \frac{t}{1+t}$. T has a fixed point in $C(I)$, i.e. Question (3.1) has a unique solution in $C(I)$. $\square$

Conclusion

We introduce a generalized $\mathcal{F}$-metric space and prove the existence of fixed point theorems via comparison function, F-contraction, Geraghty contraction and JS-contraction in generalized $\mathcal{F}$-metric space. Our results improve and generalize some results in metric space and b-metric space. Generalized $\mathcal{F}$-metric spaces may be further considered.

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