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A STRONG CONVERGENCE HALPERN-TYPE INERTIAL ALGORITHM FOR SOLVING SYSTEM OF SPLIT VARIATIONAL INEQUALITIES AND FIXED POINT PROBLEMS*

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Abstract In this paper, we propose a new Halpern-type inertial extrapolation method for approximating common solutions of the system of split variational inequalities for two inverse-strongly monotone operators, the variational inequality problem for monotone operator, and the fixed point of composition of two nonlinear mappings in real Hilbert spaces. We establish that the proposed method converges strongly to an element in the solution set of the aforementioned problems under certain mild conditions. In addition, we present some numerical experiments to show the efficiency and applicability of our method in comparison with some related methods in the literature. This result improves and generalizes many recent results in this direction in the literature.

Keywords Generalized system of variational inequality problem, split feasibility problem, inertial iterative scheme, firmly nonexpansive, fixed point problem.

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1. Introduction

Let C be a nonempty closed convex subset of a real Hilbert H with inner product $\langle \cdot, \cdot \rangle$, induced norm $\| \cdot \|$ and $A : H \to H$ be an operator. The classical Variational Inequality Problem (shortly, VIP) is to find $x \in C$ such that

$$\langle Ax, y - x \rangle \geq 0 \ \forall \ y \in C. \tag{1.1}$$

The notion of VIP was introduced by Stampacchia [36] and Fichera [13, 14] for modeling problems arising from mechanics and for solving Signorini problem. It is well-known that many problems in economics, mathematical sciences, mathematical physics can be formulated as VIP (see for instance, [17, 21, 22] and references therein). The set of solution of VIP (1.1) is denoted by Ω.

Due to the fruitful applications of VIPs, many researchers have developed different iterative algorithm to approximate the solution of (1.1). For example, the author in [19] introduced the following iterative process:

$$x_{n+1} = P_C(I - \lambda_n A)x_n. \tag{1.2}$$

It has been established that if A is strongly monotone and Lipschitz continuous, then the iterative scheme (1.2) converges strongly under some suitable conditions. In addition, if A is inverse strongly monotone, the iterative scheme (1.2) converges weakly under some suitable conditions. When the condition on A is relaxed (say, e.g., to monotone), the method (1.2) fails to converge (even weakly) to a solution of the VIP. An attempt to overcome this setback was made by Korpelevich [23]. He introduced the following extragradient type algorithm and established a weak convergence of their iterative method when A is monotone and Lipschitz continuous in the finite-dimensional Euclidean space as follows:

$$\begin{cases} x_1 \in C, \\ y_n = P_C(x_n - \lambda A x_n), \\ x_{n+1} = P_C(x_n - \lambda A y_n) \ \ \forall \ \ n \in \mathbb{N}. \end{cases} \tag{1.3}$$

Under some suitable conditions, he established that the sequence $\{x_n\}$ generated iteratively by (1.3) converges to an element in the solution set Ω. Since then, many authors have studied the VIP in Hilbert spaces using different approach, (see [1, 16, 20, 34] and the references therein). However, in all of these approaches, the convergence of their methods were obtained under the monotonicity and Lipschitz continuity assumptions of the underlying operator A. It is sometimes challenging or even impossible to calculate the Lipschitz constant of the given monotone operator, thus, making their methods very difficult in applications. Finding an iterative algorithm which uses a single projection operator with limited number of evaluation of the cost function for solving the VIP has become a fruitful effort in the last decade. It is worth noting that several researchers have attempt to generalized the concept of VIP in the literature. In the light of this fact, Verma [44] introduced a System of Nonlinear Variational Inequalities Problems (SNVIP) and in 2008, Ceng et. al. [7] generalized the concept of SNVIP introduced by Verma by introducing a new system of variational inequalities problem (SNVIP) as follows: Find $(x, y) \in C \times C$ such that

$$\begin{cases} \langle \eta A_2(y) + x - y, z - x\rangle \geq 0 \ \ \forall\, z \in C, \\ \langle \gamma A_1(x) + y - x, z - y\rangle \geq 0 \ \ \forall\, z \in C, \end{cases} \tag{1.4}$$

where $A_1, A_2 : C \to H$ are two mappings and η, γ are two constants. It is easy to see that if $A_1 = A_2$, then (1.4) becomes, find $(x, y) \in C \times C$ such that

$$\begin{cases} \langle \eta A_1(y) + x - y, z - x\rangle \geq 0 \ \ \forall\, z \in C, \\ \langle \gamma A_1(x) + y - x, z - y\rangle \geq 0 \ \ \forall\, z \in C, \end{cases} \tag{1.5}$$

which is the SNVIP introduced by Verma in [44]. More so, if $x = y$, then (1.4) reduces to (1.1). In the light of generalizing the notion of SNVIP introduced by Verma [44] and Ceng et al. [7], in 2017, Sahu et. al. [31] introduced a new system of variational inequalities problem involving two nonempty closed and convex subsets C and D of a real Hilbert space H as follows: Find $(x, y) \in C \times D$ such that

$$\begin{cases} \langle \eta A_2(y) + x - y, z - x\rangle \geq 0 \ \ \forall\, z \in D, \\ \langle \gamma A_1(x) + y - x, z - y\rangle \geq 0 \ \ \forall\, z \in C, \end{cases} \tag{1.6}$$

where $A_1 : C \to H, A_2 : D \to H$ are σ_1, σ_2 inversely strongly monotone operators, respectively and $\eta, \gamma > 0$ are constants. In [32], Sahu introduced the notion of alternating point technique and established that the SNVIP (1.6) is equivalent to the following alternating points formulation. Find $(x, y) \in C \times D$ such that

$$\begin{cases} y = P_D(I - \lambda_1 A_1)x, \\ x = P_C(I - \lambda_2 A_2)y. \end{cases} \tag{1.7}$$

The SNVIP is denoted by $SNVI(\{A_1, C\}, \{A_2, D\})$ and the set of solution of (1.6) is denoted by Γ.

The concept of Split Feasibility Problem (SFP) was introduced by Censor and Elfving [10] in the framework of finite-dimensional Hilbert spaces. The SFP is finding

$$x \in C \text{ such that } Bx \in D, \tag{1.8}$$

where C and D are nonempty, closed and convex subsets of real Hilbert spaces H_1 and H_2 respectively, and $B : H_1 \to H_2$ is a bounded linear operator. The SFP finds real life applications in image recovery, signal processing, control theory, data compression, computer tomography and so on (see [6, 8], and references therein). Therefore, it constitutes for a lot of work being done by researchers in various abstract spaces (see [38, 45]). In 2009, Censor and Segal [9] further extend the notion of SFP by introducing the concept of Split Common Fixed Point Problem (SCFP), which is finding

$$x \in F(T) \quad \text{ such that } Bx \in F(S), \tag{1.9}$$

where $F(T),\ F(S)$ denote the set of fixed points of T and S respectively, $T : C \to C,\ S : D \to D$ are two nonlinear operators and $B : H_1 \to H_2$ is a bounded linear operator. Motivated by the results of Takahashi *et al.* [39], Censor *et al.* [11] and Tian and Jiang [41] introduced and studied the following Generalized Split Feasibility Problem (GSFP) over the solution set of VIP, which is to find

$$x \in C \ \text{ such that } \ \langle A(x), \ \ y - x\rangle \geq 0 \ \ \forall y \in C \ \text{ and } \ Bx \in F(S), \tag{1.10}$$

where C is a nonempty, closed and convex subset of H_1, $B : H_1 \to H_2$ is a bounded linear operator, $A : C \to H_1$ is a single-valued operator and $S : H_2 \to H_2$ is a nonexpansive mapping. Tian and Jiang [40] proposed the following iterative algorithm for finding solutions of the GSFP (1.10): Let $x_1 \in C$, define the sequence $\{x_n\}$, $\{y_n\}$ and $\{t_n\}$ by

$$\begin{cases} y_n = P_C(x_n - \tau_n B^*(I - S)Bx_n), \\ t_n = P_C(y_n - \lambda_n A(y_n)), \\ x_{n+1} = P_C(y_n - \lambda_n A(t_n)), \end{cases} \tag{1.11}$$

for each $n \in \mathbb{N}$, where $\{\tau_n\} \subset [a, b]$ for some $a, b \in \left(0, \frac{1}{\|B\|^2}\right)$ and $\{\lambda_n\} \subset [c, d]$ for some $c, d \in \left(0, \frac{1}{L}\right)$, $S : H_2 \to H_2$ is a nonexpansive mapping, $A : C \to H_1$ is a monotone and L-Lipschitz continuous. They proved that the sequence $\{x_n\}$ generated by Algorithm (1.11) converges weakly to the solution of the following problem: Find $x \in C$ such that

$$\langle Ax, y - x\rangle \geq 0 \ \ \forall\ y \in C \text{ and } \text{ such that } Bx \in F(S).$$

Since strong convergence is more desirable than weak convergence. Tian and Jiang in [41] extends Algorithm (1.11) and obtained a strong convergent result. They defined the iterative algorithm as follows: Let $x_1 \in C$, define the sequence $\{x_n\}$, $\{y_n\}, \{w_n\}$ and $\{t_n\}$ by

$$\begin{cases} y_n = P_C(x_n - \tau_n B^*(I - S)Bx_n), \\ t_n = P_C(y_n - \lambda_n A(y_n)), \\ w_n = P_C(y_n - \lambda_n A(t_n)), \\ x_{n+1} = \alpha_n h(x_n) + (1 - \alpha_n)w_n, \end{cases} \tag{1.12}$$

where $h : H \to H$ is a contraction, $\{\alpha_n\} \subset (0, 1)$ and S, B, A are the same as defined in Algorithm (1.11).

Remark 1.1. In Algorithm (1.11) and Algorithm (1.12) the underlying operator A is not inversely strongly monotone but monotone and L-Lipschitz continuous. However, researchers tends to reduce the number of metric projection in an iterative algorithm due to its negative effect on the convergence rate of the iterative scheme. Thus, we are back to the question of whether there is an iterative method designed to solve the VIP (1.1) or GSFP (1.10), where the underlying operator is monotone and with a minimum number of metric projections.

On the other hand, the inertial extrapolation method has proven to be an effective way for accelerating the rate of convergence of iterative algorithms. The technique was introduced in 1964 and is based on a discrete version of a second order dissipative dynamical system [26,29]. The inertial type algorithms use its two previous iterates to obtain its next iterate [2, 24]. For details on the inertia extrapolation, see [1, 3–5] and the references therein. In 2018, Dong et al. [12] proposed an inertial type iterative algorithm for approximating the solution of (1.1). The method is of the form:

$$\begin{cases} x_0, x_1 \in H, \\ w_n = x_n + \theta_n(x_n - x_{n-1}), \\ y_n = P_C(w_n - \lambda A w_n), \\ d(w_n, y_n) = w_n - y_n - \lambda(Aw_n - Au_n), \\ x_{n+1} = w_n - \zeta \eta_n d(w_n, y_n), \end{cases} \tag{1.13}$$

where $\zeta \in (0,2), \lambda \in (0, \frac{1}{L})$ and

$$\eta_n = \begin{cases} \frac{\phi(w_n, y_n)}{d(w_n, y_n)}, & \text{if } d(w_n, y_n) \neq 0, \\ 0, & \text{if } d(w_n, y_n) = 0, \end{cases} \tag{1.14}$$

where $\phi(w_n, y_n) = \langle w_n - y_n, d(w_n, y_n) \rangle$. They established that the sequence $\{x_n\}$ converges weakly to an element of Ω.

In 2020, Sahu et al. [33] introduced an inertial type iterative algorithm for approximating the solution of a class of variational inequality problems for monotone operators and system of nonlinear variational inequalities problems for two inverse strongly monotone operators. The propose algorithm is of the form:

$$\begin{cases} x_0, x_1 \in C, \\ u_n = x_n + \theta_n(x_n - x_{n-1}), \\ v_n = P_C(I - \mu_2 A_2) P_D (I - \mu_1 A_1) u_n, \\ z_n = P_C(v_n - A v_n), \\ y_n = (1 - \lambda_n) v_n + \lambda_n z_n, \\ \text{where } \ \lambda_n = l^{m_n} \text{ and } m_n \text{ is the smallest nonnegative integer m such that} \\ \langle A v_n - A(1 - l^m) v_n + l^m z_n), v_n - z_n \rangle \leq \mu \|v_n - z_n\|^2, \\ x_{n+1} = (1 - \beta_n) u_n + \beta_n P_C(v_n - \gamma_n d_n) \ \ \forall\, n \in \mathbb{N}, \end{cases} \tag{1.15}$$

where $\mu_1 \in (0, 2\eta_2)$ and $\mu_2 \in (0, 2\eta_2)$, $d_n = v_n - z_n + \frac{1}{\lambda_n} A(y_n)$ and $\gamma_n = \rho(1 - \mu)\frac{\|v_n - z_n\|}{\|d_n\|^2}$ if $d_n \neq 0$ and $\gamma_n = 0$ if $d_n = 0$. They established that the sequence $\{x_n\}$ converges weakly to an element of the solution set of (1.1) and (1.6).

Remark 1.2. In Algorithm (1.13), the underlying operator A is monotone and L-Lipschitz continuous, since strong convergence is more desirable than weak convergence. It is therefore natural to ask if Algorithm (1.13) can be further modified to get a strong convergence and with weaker operator.

Remark 1.3. In Algorithm (1.15), the underlying operator A for VIP is not inversely strongly monotone nor monotone and L-Lipschitz continuous. Thus, this algorithm provides an affirmative answer to the question of developing an iterative algorithm that can approximate a VIP in which the underlying operator is monotone. However, the number of metric projection in the iterative algorithm will slow down the rate of convergence. Thus, we are back to the question of whether there is an iterative method designed to solve the VIP (1.1) in which the underlying operator is monotone and SNVIP (1.6), with minimum metric projection.

Remark 1.4. Having highlight the setbacks in the above algorithms, one of the purpose of this paper is to address the setbacks in the above algorithms by introducing an iterative algorithm together with an inertial extrapolation method based on altering point technique. The following are our contributions in the paper.

1. Observe that Algorithm 3 can be viewed as a single projection method for solving the classical VIP (1.1) and a fixed point problem of composition of two mappings in one space H_1 in which the underlying operator A is just monotone. In addition, a double projection alternating points formulation method under a bounded linear operator B for solving SNVIP (1.6) in another space H_2 with no extra projection. A notable advantage of this method (Algorithm 3) for solving VIP is that the strongly inversely monotonicity and Lipschitz continuity of the operator A usually used in other papers to guarantee convergence, is removed and no extra projection required under this setting (see for example, [27,33] and the references therein).

2. The choice of stepsize $\sigma_n := \left(\epsilon, \frac{\|(G_D-I)Bv_n\|^2}{\|B^*(G_D-I)Bv_n\|^2} - \epsilon\right)$ used in Algorithm 3 does not require the knowledge of the operator norms $\|B\|$. It is known that Algorithms with parameters depending on operator norm are not easy in practice to execute (see, [40,41] and the references therein).

3. In addition, in establishing our results, the condition $\sum_{n=1}^{\infty} \theta_n \|x_n - x_{n-1}\|^2 < \infty$ as used by authors who use inertial type algorithm (see [25] and the references therein) to solve some optimization problems is not used in this work.

The rest of this paper is organized as follows: In Section 2, we recall some useful definitions and results needed to establish the main result in this paper. In Section 3, we present our proposed method and highlight some of its advantages over existing methods in this area of study. In Section 4, we establish strong convergence result of our method. In Section 5, we present some examples and numerical experiments of the proposed method in comparison with Algorithm 1.13 and Algorithm 1.15 to show the efficiency and applicability of our method in the framework of infinite and finite dimensional Hilbert spaces. In Section 6, we make some concluding remarks. The result in this paper generalizes, unifies and extends other corresponding results in the literature.

2. Preliminaries

In this section, we begin by recalling some known and useful results which are needed in the sequel.

Let H be a real Hilbert space. The set of fixed point of T will be denoted by $F(T)$, that is $F(T) = \{x \in H : Tx = x\}$. We denote strong and weak convergence by "$\to$" and "$\rightharpoonup$", respectively. For any $x, y \in H$ and $\alpha \in [0, 1]$, it is well-known that

$$\langle x, y\rangle = \frac{1}{2}(\|x\|^2 + \|y\|^2 - \|x - y\|^2). \tag{2.1}$$

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle. \tag{2.2}$$

$$\|\alpha x + (1-\alpha)y\|^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x-y\|^2. \tag{2.3}$$

Let H be a real Hilbert space and C a nonempty, closed and convex subset of H. For any $u \in H$, there exists a unique point $P_C u \in C$ such that

$$\|u - P_C\| \leq \|u - y\| \quad \forall y \in C.$$

P_C is called the metric projection of H onto C. It is well-known that P_C is a nonexpansive mapping and that P_C satisfies

$$\langle x - y, P_C x - P_C y\rangle \geq \|P_C x - P_C y\|^2,$$

for all $x, y \in H$. Furthermore, $P_C x$ is characterized by the properties $P_C x \in C$,

$$\langle x - P_C x, P_C x - y\rangle \geq 0$$

for all $y \in C$ and

$$\|x - y\|^2 \geq \|x - P_C x\|^2 + \|y - P_C x\|^2$$

for all $x \in H$ and $y \in C$.

Lemma 2.1 ([33]). *Let C be a nonempty closed convex subset in Hilbert space H. The vector u is the projection of the vector x onto C if and only if*

$$\langle u - x, y - y\rangle \ \forall \ y \in C. \tag{2.4}$$

Definition 2.1. Let $T : H \to H$ be an operator. Then the operator T is called

(a) L-Lipschitz continuous if

$$\|Tx - Ty\| \leq L\|x - y\|,$$

where $L > 0$ and $x, y \in H$. If $L = 1$, Then T is called nonexpansive. Also, if $y \in F(T)$ and $L = 1$, Then T is called quasi-nonexpansive.

(b) monotone if

$$\langle Tx - Ty, x - y\rangle \geq 0, \quad \forall x, y \in H.$$

(c) firmly nonexpansive if

$$\|Tx - Ty\|^2 \leq \langle Tx - Ty, x - y\rangle, \quad \forall \ x, y \in H,$$

or equivalently

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 - \|(I - T)x - (I - T)y\|^2, \quad \forall \ x, y \in H,$$

(d) k-inverse strongly monotone (k-ism) if there exists $k > 0$, such that

$$\langle Tx - Ty, x - y\rangle \geq k\|Tx - Ty\|^2, \quad \forall \ x, y \in H.$$

(e) α-strongly quasi-nonexpansive mapping with $\alpha > 0$ if

$$\|Tx - z\|^2 \leq \|x - z\|^2 - \alpha\|x - Tx\|^2, \ \forall \ z \in F(T), x \in H. \tag{2.5}$$

It is well-known that for any nonexpansive mapping T, the set of fixed point is closed and convex. Also, T satisfies the following inequality

$$\langle (x-Tx)-(y-Ty), Ty-Tx\rangle \leq \frac{1}{2}\|(Tx-x)-(Ty-y)\|^2,\ \forall\ x,y\in H. \quad (2.6)$$

Thus, for all $x\in H$ and $x^*\in F(T)$, we have that

$$\langle x-Tx, x^*-Tx\rangle \leq \frac{1}{2}\|Tx-x\|^2,\ \forall\ x,y\in H. \quad (2.7)$$

Lemma 2.2 ([33]). *Let D_1 and D_2 be nonempty closed and convex subset of H. $A_1 : D_1 \to H$ and $A_2 : D_2 \to H$ be nonlinear operator. Let ν_1 and ν_2 be positive constants and $G_D : D_1 \to D_1$ be an operator defined by*

$$G_D x = P_{D_1}(I-\nu_2 A_2)P_{D_2}(I-\nu_1 A_1)x,\ \forall\ x\in D_1. \quad (2.8)$$

Let $(x,y)\in D_1\times D_2$. Then

$$\begin{aligned}(x,y) \text{ is a solution of SNVIP (1.6)} \Leftrightarrow\ & (x,y)\in\ Alt(P_{D_1}(I-\nu_2 A_2)P_{D_2}(I-\nu_1 A_1))\\ \Leftrightarrow\ & G_D x = x. \quad (2.9)\end{aligned}$$

Lemma 2.3 ([33]). *Let D_1 and D_2 be nonempty closed and convex subset of H. $A_1 : D_1 \to H_2$ and $A_2 : D_2 \to H_2$ are ϱ_1 and ϱ_2 inversely strongly monotone operator with $\nu_1\in(0,2\varrho_1)$ and $\nu_2\in(0,2\varrho_2)$, then, the operator G_D defined by (2.8) is a nonexpansive operator.*

Lemma 2.4 ([18]). *Let C be a closed and convex subset of a Hilbert space H and $T : C\to C$ be nonexpansive mapping with $F(T)\neq\emptyset$. If $\{x_n\}$ is a sequence in C weakly converging to p and if $\{(I-T)x_n\}$ converges strongly to q, then $(I-T)p = p$. In particular, if $q = 0$, then $p\in F(T)$.*

Lemma 2.5 ([37]). *Let C be a nonempty, closed and convex subset of H. Let $A : C\to H$ be a continuous, monotone mapping and $w\in C$. Then*

$$w\in\Omega \quad \textit{if and only if} \quad \langle Ax, x-w\rangle \geq 0, \quad \forall x\in C. \quad (2.10)$$

Lemma 2.6 ([28]). *Let H be a real Hilbert space. Then for all $x,y,z\in H$ and $\alpha,\beta,\gamma\in[0,1]$ with $\alpha+\beta+\gamma = 1$, we have*

$$||\alpha x+\beta y+\gamma z||^2 = \alpha||x||^2+\beta||y||^2+\gamma||z||^2-\alpha\beta||x-y||^2-\alpha\gamma||x-z||^2-\beta\gamma||y-z||^2.$$

Lemma 2.7 ([30]). *Let $\{a_n\}$ be a sequence of positive real numbers, $\{\alpha_n\}$ be a sequence of real number in $(0,1)$ such that $\sum_{n=1}^{\infty}\alpha_n = \infty$ and $\{d_n\}$ be a sequence of real numbers. Suppose that*

$$a_{n+1}\leq (1-\alpha_n)a_n+\alpha_n d_n, n\geq 1.$$

If $\limsup_{k\to\infty} d_{n_k}\leq 0$ for all subsequences $\{a_{n_k}\}$ of $\{a_n\}$ satisfying the condition

$$\liminf_{k\to\infty}\{a_{n_k+1}-a_{n_k}\}\geq 0,$$

then, $\lim\limits_{n\to\infty} a_n = 0$.

3. Proposed Algorithm

In this section, we present our proposed method. We begin with the following assumptions under which our strong convergence is obtained.

Assumption A. Suppose the following hold:

1. The set C is nonempty closed and convex subset of H_1 and the sets D_1, D_2 are nonempty closed and convex subsets of H_2.
2. $A : H_1 \to H_1$ is a monotone operator.
3. $A_1 : D_1 \to H_2$ and $A_2 : D_2 \to H_2$ are ϱ_1 and ϱ_2 inversely strongly monotone operators with $\nu_1 \in (0, 2\varrho_1)$ and $\nu_2 \in (0, 2\varrho_2)$ and $G_D : D_1 \to D_1$ is an operator defined by
$$G_D x = P_{D_1}(I - \nu_2 A_2) P_{D_2}(I - \nu_1 A_1)x \ \forall \ x \in D_1.$$
4. $T_1 : H_1 \to H_1$ is an α-strongly quasi-nonexpansive mapping and $T_2 : H_1 \to H_1$ is a firmly nonexpansive mapping.
5. $B : H_1 \to H_2$ is a bounded linear operator and the solution set $Sol = \{x^* \in \Omega \cap F(T_1 \circ T_2) : Bx^* \in \mathit{\Gamma}\}$ is nonempty, where Ω is the solution set for VIP (1.1) and $\mathit{\Gamma}$ is the solution set for SNVIP (1.6).

We present the following algorithm.

Algorithm 3

Initialization: Given $\lambda_0 > 0$ and $\eta_n, \beta_n, \alpha_n, \mu \in (0, 1)$, for all $n \in \mathbb{N}$, such that $\eta_n \leq \beta_n \leq \alpha_n$ with $\eta_n + \beta_n + \alpha_n = 1$. Let $x_0, x_1, u \in H_1$ be arbitrary.

Iterative step:

Step 1: Given the iterates x_{n-1} and x_n for all $n \in \mathbb{N}$, choose θ_n such that $0 \leq \theta_n \leq \bar{\theta}_n$, where

$$\bar{\theta}_n = \begin{cases} \min\left\{\theta, \frac{\epsilon_n}{||x_n - x_{n-1}||}\right\}, & \text{if } x_n \neq x_{n-1}, \\ \theta, & \text{otherwise}, \end{cases} \tag{3.1}$$

where $\theta > 0$ and $\{\epsilon_n\}$ is a positive sequence such that $\epsilon_n = \circ(\alpha_n)$.

Step 2. Set
$$w_n = x_n + \theta_n(x_n - x_{n-1}).$$

Then, compute

$$u_n = P_C(w_n - \lambda_n A w_n) \ \text{ and } \ v_n = w_n - \tau_n b_n, \tag{3.2}$$

where $b_n := w_n - u_n - \lambda_n(Aw_n - Au_n)$,

$$\tau_n = \begin{cases} \frac{\langle w_n - u_n, b_n \rangle}{\|b_n\|^2}, & \text{if } b_n \neq 0 \\ 0, & \text{otherwise}. \end{cases} \tag{3.3}$$

and

$$\lambda_{n+1} = \begin{cases} \min\left\{\lambda_n, \frac{\mu\|w_n - u_n\|}{\|Aw_n - Au_n\|}\right\}, & \text{if } Aw_n \neq Au_n, \\ \lambda_n, & \text{otherwise}. \end{cases} \tag{3.4}$$

Step 3. Compute
$$y_n = v_n + \sigma_n B^*(G_D - I)Bv_n,$$
where $\sigma_n := \left(\epsilon, \frac{\|(G_D-I)Bv_n\|^2}{\|B^*(G_D-I)Bv_n\|^2} - \epsilon\right)$, if $G_D Bv_n \neq Bv_n$, otherwise, $\sigma_n = \varepsilon$ (ε being a small non-negative real number).
Step 4. Compute
$$x_{n+1} = \alpha_n u + \beta_n x_n + \eta_n (T_1 \circ T_2) y_n. \tag{3.5}$$

Stopping criterion: If $w_n = u_n = v_n = y_n = x_n$, then stop, otherwise, set $n := n+1$ and go back to **Step 1.**

Remark 3.1. Note that in Algorithm 3, it is easy to compute step 1 since the value of $\|x_n - x_{n-1}\|$ is a prior knowledge before choosing θ_n. More so, it is easy to see that the sequence $\{\lambda_n\}$ generated by (3.4) is non-increasing and bounded below by $\min\left\{\lambda_0, \frac{\mu}{L}\right\}$. Hence, the limit $\lim_{n\to\infty} \lambda_n$ exists which we denote by λ, i.e., $\lim_{n\to\infty} \lambda_n = \lambda > 0$. In addition, if $u_n = w_n$ or $b_n = 0$ in Algorithm 3, then, $u_n \in \Omega$. Indeed, using the expression of b_n and (3.4), we have that
$$\begin{aligned}
\|b_n\| &= \|w_n - u_n - \lambda_n(Aw_n - Au_n)\| \\
&= \|w_n - u_n + \lambda_n(Au_n - Aw_n)\| \\
&\leq \|w_n - u_n\| + \lambda_n \|Aw_n - Au_n\| \\
&\leq (1 + \frac{\lambda_n \mu}{\lambda_{n+1}})\|w_n - u_n\|.
\end{aligned} \tag{3.6}$$

Also, we have that
$$\begin{aligned}
\|b_n\| &= \|w_n - u_n - \lambda_n(Aw_n - Au_n)\| \\
&\geq \|w_n - u_n\| - \lambda_n \|Aw_n - Au_n\| \\
&= (1 - \frac{\lambda_n \mu}{\lambda_{n+1}})\|w_n - u_n\|.
\end{aligned} \tag{3.7}$$

Using (3.6) and (3.7), we obtain
$$(1 - \frac{\lambda_n \mu}{\lambda_{n+1}})\|w_n - u_n\| \leq \|b_n\| \leq (1 + \frac{\lambda_n \mu}{\lambda_{n+1}})\|w_n - u_n\| \;\; \forall\, n \in \mathbb{N}. \tag{3.8}$$

Note that $\lim_{n\to\infty}(1 - \frac{\lambda_n \mu}{\lambda_{n+1}}) = 1 - \mu > 0$. Thus, there exists an integer $N \in \mathbb{N}$ such that for $n \geq N$,
$$(1-\mu)\|w_n - u_n\| \leq \|b_n\| \leq (1+\mu)\|w_n - u_n\| \quad \forall n \geq N.$$

Hence $w_n = u_n$ if and only if $b_n = 0$. Thus, if $w_n = u_n$ or $b_n = 0$ for all $n \geq N$, then $w_n = u_n$ and so
$$u_n = P_C(u_n - \lambda_n A u_n)$$
that is $u_n \in \Omega$.

Also, if $y_n = v_n = w_n$, then, $G_D B w_n = B w_n$ and consequently $w_n \in \Gamma$. If $y_n = v_n = w_n$, it is easy to see that
$$w_n = w_n - \sigma_n B^*(G_D - I)Bw_n \Rightarrow \sigma_n B^*(G_D - I)Bw_n = 0 \Rightarrow B^*(G_D - I)Bw_n = 0.$$

That is

$$G_D B w_n = B w_n + \overline{x}, \tag{3.9}$$

where $0 = B^* \overline{x}$. Suppose that $p \in \Gamma$, using (3.9) and the nonexpansivity of G_D, we have that

$$\begin{aligned} \|Bw_n - Bp\|^2 &= \|Bw_n - Bp\|^2 + 2\langle w_n - p, B^*\overline{x}\rangle \\ &= \|Bw_n - Bp\|^2 + 2\langle Bw_n - Bp, \overline{x}\rangle \\ &= \|Bw_n + \overline{x} - Bp\|^2 - \|\overline{x}\|^2 \\ &= \|G_D Bw_n - G_D Bp\|^2 - \|\overline{x}\|^2 \\ &\leq \|Bw_n - Bp\|^2 - \|\overline{x}\|^2, \end{aligned} \tag{3.10}$$

which implies that $\|\overline{x}\|^2 = 0$ and so, we have $\overline{x} = 0$. It follows from (3.9) that $G_D B w_n = B w_n$ and consequently $w_n \in \Gamma$.

Remark 3.2. It is easy to see from (3.1) that $\lim\limits_{n\to\infty} \frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| = 0$.

Proof. Since, $\{\epsilon_n\}$ is a positive sequence such that $\epsilon_n = \circ(\alpha_n)$, which means that $\lim\limits_{n\to\infty} \frac{\epsilon_n}{\alpha_n} = 0$. Clearly, we have that that $\theta_n\|x_n - x_{n-1}\| \leq \epsilon_n$ for all $n \in \mathbb{N}$, which together with $\lim\limits_{n\to\infty} \frac{\epsilon_n}{\alpha_n} = 0$, it follows that

$$\lim_{n\to\infty} \frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| \leq \lim_{n\to\infty} \frac{\epsilon_n}{\alpha_n} = 0.$$

□

4. Convergence Analysis

In this section, we provide the convergence analysis of our Algorithm 3. We begin by proving the following important lemmas.

Lemma 4.1. *Let H be a real Hilbert space and C be a nonempty closed and convex subset of H. Let $T_1 : C \to C$ be α-strongly quasi-nonexpansive mapping and $T_2 : C \to C$ be firmly nonexpansive mapping. Then $F(T_1 \circ T_2) = F(T_1) \cap F(T_2)$.*

Proof. We need to establish that $F(T_1 \circ T_2) \subseteq F(T_1) \cap F(T_2)$ and $F(T_1) \cap F(T_2) \subseteq F(T_1 \circ T_2)$. It is easy to see that $F(T_1) \cap F(T_2) \subseteq F(T_1 \circ T_2)$. We now establish that $F(T_1 \circ T_2) \subseteq F(T_1) \cap F(T_2)$. Let $y \in F(T_1 \circ T_2)$ and $x \in F(T_1) \cap F(T_2)$, we have

$$\begin{aligned} \|y - x\|^2 &= \|T_1(T_2 y) - T_1 x\|^2 \\ &\leq \|T_2 y - x\|^2 - \alpha\|T_2 y - T_1(T_2 y)\|^2 \\ &\leq \|T_2 y - x\|^2. \end{aligned} \tag{4.1}$$

Also, using (4.1), we have

$$\begin{aligned} \|T_2 y - x\|^2 &= \langle T_2 y - x, y - x\rangle \\ &= \frac{1}{2}\|T_2 y - x\|^2 + \frac{1}{2}\|y - x\|^2 - \frac{1}{2}\|T_2 y - y\|^2 \\ &\leq \frac{1}{2}\|T_2 y - x\|^2 + \frac{1}{2}\|T_2 y - x\|^2 - \frac{1}{2}\|T_2 y - y\|^2, \end{aligned} \tag{4.2}$$

which implies that $\|T_2y - y\|^2 = 0 \Rightarrow \|T_2y - y\| = 0 \Rightarrow T_2y = y$. Using this fact, we have that

$$y = (T_1 \circ T_2)y = T_1(T_2y) = T_1y \Rightarrow y \in F(T_1) \cap F(T_2). \tag{4.3}$$

Hence, $F(T_1 \circ T_2) \subseteq F(T_1) \cap F(T_2)$, and so $F(T_1 \circ T_2) = F(T_1) \cap F(T_2)$. □

Lemma 4.2. *Let H be a real Hilbert space and C be a nonempty closed and convex subset of H. Let $T_1 : C \to C$ be α-strongly quasi-nonexpansive mapping and $T_2 : C \to C$ be firmly nonexpansive mapping. Then, $T_1 \circ T_2$ is a quasi-nonexpansive mapping.*

Proof. Let $x \in C$ and $y \in F(T_1 \circ T_2)$, using Lemma 4.1, we have that $y \in F(T_1) \cap F(T_2)$, which implies that $y = T_1y$ and $y = T_2y$. Now, observe that

$$\begin{aligned}
\|(T_1 \circ T_2)x - y\|^2 &= \|T_1(T_2x) - T_1y\|^2 \\
&\le \|T_2x - y\|^2 - \alpha\|T_2x - T_1(T_2x)\|^2 \\
&\le \|T_2x - T_2y\|^2 \\
&\le \|x - y\|^2 - \|(x-y) - (T_2x - T_2y)\|^2 \\
&= \|x - y\|^2 - \|x - T_2x\|^2 \\
&\le \|x - y\|^2.
\end{aligned}$$

□

Lemma 4.3. *Let $\{x_n\}$ be a sequence generated by Algorithm 3. Then under the Assumptions (3), we have that $\{x_n\}$ is bounded.*

Proof. Let $p \in Sol$ and since $\lim\limits_{n\to\infty} \frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| = 0$, there exists $N_1 > 0$ such that $\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| \le N_1$. Then from Algorithm 3, we have

$$\begin{aligned}
\|w_n - p\|^2 &= \|x_n + \theta_n(x_n - x_{n-1}) - p\|^2 \\
&= \|x_n - p\|^2 + 2\theta_n\langle x_n - p, x_n - x_{n-1}\rangle + \theta_n^2\|x_n - x_{n-1}\|^2 \\
&\le \|x_n - p\|^2 + 2\theta_n\|x_n - p\|\|x_n - x_{n-1}\| + \theta_n^2\|x_n - x_{n-1}\|^2 \\
&= \|x_n - p\|^2 + \theta_n\|x_n - x_{n-1}\|[2\|x_n - p\| + \theta_n\|x_n - x_{n-1}\|] \\
&= \|x_n - p\|^2 + \theta_n\|x_n - x_{n-1}\|[2\|x_n - p\| + \alpha_n\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\|] \\
&\le \|x_n - p\|^2 + \theta_n\|x_n - x_{n-1}\|[2\|x_n - p\| + \alpha_nN_1] \\
&\le \|x_n - p\|^2 + \theta_n\|x_n - x_{n-1}\|N_2,
\end{aligned} \tag{4.4}$$

where $N_2 := 2\|x_n - p\| + \alpha_nN_1$.

Since $u_n = P_C(w_n - \lambda_nAw_n)$ and $p \in \Omega$, then by Lemma 2.1, we have

$$\langle u_n - p, u_n - w_n + \lambda_nAw_n\rangle \le 0.$$

Using the monotonicity of A and the fact that $p \in Sol$, we obtain

$$\begin{aligned}
\langle u_n - p, b_n\rangle &= \langle u_n - p, w_n - u_n - \lambda_nAw_n\rangle + \lambda_n\langle u_n - p, Au_n\rangle \\
&\ge \lambda_n\langle u_n - p, Au_n\rangle
\end{aligned}$$

$$= \lambda_n \langle u_n - p, Au_n - Ap\rangle + \lambda_n \langle u_n - p, Ap\rangle \geq 0.$$

Thus, we have

$$\begin{aligned}\langle w_n - p, b_n\rangle &= \langle w_n - u_n, b_n\rangle + \langle u_n - p, b_n\rangle \\ &\geq \langle w_n - u_n, b_n\rangle. \end{aligned} \tag{4.5}$$

Hence from (4.5) and the condition on τ_n (3.3), we have

$$\begin{aligned}\|v_n - p\|^2 &= \|w_n - \tau_n b_n - p\|^2 \\ &= \|w_n - p\|^2 + \tau_n^2\|b_n\|^2 - 2\tau_n\langle w_n - p, b_n\rangle \\ &\leq \|w_n - p\|^2 + \tau_n^2\|b_n\|^2 - 2\tau_n\langle w_n - u_n, b_n\rangle \\ &\leq \|w_n - p\|^2 + \tau_n^2\|b_n\|^2 - 2\tau_n^2\|b_n\|^2 \\ &= \|w_n - p\|^2 - \|\tau_n b_n\|^2 \\ &\leq \|w_n - p\|^2. \end{aligned} \tag{4.6}$$

Furthermore, using Algorithm 3, (2.7) and condition on σ_n, we have that

$$\begin{aligned}\|y_n - p\|^2 &= \|v_n + \sigma_n B^*(G_D - I)Bv_n - p\|^2 \\ &= \|v_n - p\|^2 + \sigma_n^2\|B^*(G_D - I)Bv_n\|^2 + 2\sigma_n\langle v_n - p, B^*(G_D - I)Bv_n\rangle \\ &= \|v_n - p\|^2 + \sigma_n^2\|B^*(G_D - I)Bv_n\|^2 + 2\sigma_n\langle Bv_n - Bp, (G_D - I)Bv_n\rangle \\ &= \|v_n - p\|^2 + \sigma_n^2\|B^*(G_D - I)Bv_n\|^2 \\ &\quad + 2\sigma_n\langle Bv_n - G_DBv_n + G_DBv_n - Bp, (G_D - I)Bv_n\rangle \\ &= \|v_n - p\|^2 + \sigma_n^2\|B^*(G_D - I)Bv_n\|^2 + 2\sigma_n\langle G_DBv_n - Bp, G_DBv_n - Bv_n\rangle \\ &\quad - 2\sigma_n\|(G_D - I)Bv_n\|^2 \\ &\leq \|v_n - p\|^2 + \sigma_n^2\|B^*(G_D - I)Bv_n\|^2 + \sigma_n\|G_DBv_n - Bp\|^2 \\ &\quad - 2\sigma_n\|(G_D - I)Bv_n\|^2 \\ &= \|v_n - p\|^2 - \sigma_n[\|G_DBv_n - Bp\|^2 - \sigma_n\|B^*(G_D - I)Bv_n\|^2] \\ &\leq \|v_n - p\|^2. \end{aligned} \tag{4.7}$$

Lastly, using Algorithm 3, Lemma 4.2, we have

$$\begin{aligned}\|x_{n+1} - p\|^2 &= \|\alpha_n u + \beta_n x_n + \eta_n(T_1 \circ T_2)y_n - p\|^2 \\ &\leq \alpha_n\|u - p\|^2 + \beta_n\|x_n - p\|^2 + \eta_n\|(T_1 \circ T_2)y_n - p\|^2 \\ &\leq \alpha_n\|u - p\|^2 + \beta_n\|x_n - p\|^2 + \eta_n\|y_n - p\|^2 \\ &\leq \alpha_n\|u - p\|^2 + \beta_n\|x_n - p\|^2 + \eta_n\|v_n - p\|^2 \\ &\leq \alpha_n\|u - p\|^2 + \beta_n\|x_n - p\|^2 + \eta_n\|w_n - p\|^2 \\ &\leq \alpha_n\|u - p\|^2 + (1 - \alpha_n - \eta_n)\|x_n - p\|^2 \\ &\quad + \eta_n(\|x_n - p\|^2 + \theta_n\|x_n - x_{n-1}\|N_2) \\ &\leq \alpha_n\|u - p\|^2 + (1 - \alpha_n)\|x_n - p\|^2 + \theta_n\|x_n - x_{n-1}\|N_2 \\ &\leq \max\{\|x_n - p\|^2, \|u - p\|^2\} + \theta_n\|x_n - x_{n-1}\|N_2 \\ &\leq \max\{\max\{\|x_n - p\|^2, \|u - p\|^2\} + \theta_{n-1}\|x_{n-1} - x_{n-2}\|N_2, \|u - p\|^2\} \end{aligned}$$

$$+ \theta_n \|x_n - x_{n-1}\| N_2$$
$$= \max\{\|x_n - p\|^2, \|u - p\|^2\} + \alpha_{n-1}\frac{\theta_{n-1}}{\alpha_{n-1}}\|x_{n-1} - x_{n-2}\|N_2$$
$$+ \alpha_n \frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\|N_2, \tag{4.8}$$

using the fact that $\lim_{n\to\infty} \frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| = 0$, there exists $N_1 > 0$ such that $\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| \leq N_1$, (4.8) becomes

$$\begin{aligned}\|x_{n+1} - p\|^2 &\leq \max\{\|x_n - p\|^2, \|u - p\|^2\} + N_4\\ &\leq \max\{\|x_0 - p\|^2, \|u - p\|^2\} + N_4,\end{aligned}$$

where $N_4 = N_3N_2 + N_2N_1$, thus $\{x_n\}$ generated by Algorithm 3 is bounded. □

We now present our strong convergence result as follows.

Theorem 4.1. *Let $\{x_n\}$ be the sequence generated by Algorithm 3. Suppose the control parameters satisfy $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty, 0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$, and $0 < \liminf_{n\to\infty} \eta_n \leq \limsup_{n\to\infty} \eta_n < 1, \liminf_{n\to\infty} \lambda_n > 0$. Then, $\{x_n\}$ converges strongly to $p \in P_{Sol}u$.*

Proof. Let $p \in \Gamma$. From (2.2), we have

$$\begin{aligned}\|x_{n+1} - p\|^2 &= \|\alpha_n u + \beta_n x_n + \eta_n(T_1 \circ T_2)y_n - p\|^2\\
&= \|\alpha_n(u - p) + \beta_n(x_n - p) + \eta_n((T_1 \circ T_2)y_n - p)\|^2\\
&\leq \|\beta_n(x_n - p) + \eta_n((T_1 \circ T_2)y_n - p)\|^2 + 2\alpha_n\langle u - p, x_{n+1} - p\rangle\\
&\leq \beta_n^2\|x_n - p\|^2 + \eta_n^2\|(T_1 \circ T_2)y_n - p\|^2 + 2\beta_n\eta_n\|x_n - p\|\|(T_1 \circ T_2)y_n - p\|\\
&\quad + 2\alpha_n\langle u - p, x_{n+1} - p\rangle\\
&\leq \beta_n^2\|x_n - p\|^2 + \eta_n^2\|y_n - p\|^2 + 2\beta_n\eta_n\|x_n - p\|\|y_n - p\|\\
&\quad + 2\alpha_n\langle u - p, x_{n+1} - p\rangle\\
&\leq \beta_n^2\|x_n - p\|^2 + \eta_n^2\|y_n - p\|^2 + \beta_n\eta_n[\|x_n - p\|^2 + \|y_n - p\|^2]\\
&\quad + 2\alpha_n\langle u - p, x_{n+1} - p\rangle\\
&= \beta_n(\beta_n + \eta_n)\|x_n - p\|^2 + \eta_n(\eta_n + \beta_n)\|y_n - p\|^2 + 2\alpha_n\langle u - p, x_{n+1} - p\rangle\\
&= \beta_n(1 - \alpha_n)\|x_n - p\|^2 + \eta_n(1 - \alpha_n)\|y_n - p\|^2 + 2\alpha_n\langle u - p, x_{n+1} - p\rangle\\
&\leq \beta_n(1 - \alpha_n)\|x_n - p\|^2 + \eta_n(1 - \alpha_n)\|v_n - p\|^2 + 2\alpha_n\langle u - p, x_{n+1} - p\rangle\\
&\leq \beta_n(1 - \alpha_n)\|x_n - p\|^2 + \eta_n(1 - \alpha_n)\|w_n - p\|^2 + 2\alpha_n\langle u - p, x_{n+1} - p\rangle\\
&\leq \beta_n(1 - \alpha_n)\|x_n - p\|^2 + \eta_n(1 - \alpha_n)[\|x_n - p\|^2 + \theta_n\|x_n - x_{n-1}\|N_2]\\
&\quad + 2\alpha_n\langle u - p, x_{n+1} - p\rangle\\
&\leq (1 - \alpha_n)^2\|x_n - p\|^2 + \alpha_n(1 - \alpha_n)[\theta_n\|x_n - x_{n-1}\|N_2]\\
&\quad + 2\alpha_n\langle u - p, x_{n+1} - p\rangle\\
&\leq (1 - \alpha_n)\|x_n - p\|^2 + \alpha_n[\theta_n\|x_n - x_{n-1}\|N_2 + 2\langle u - p, x_{n+1} - p\rangle]\\
&= (1 - \alpha_n)\|x_n - p\|^2 + \alpha_n[\theta_n\|x_n - x_{n-1}\|N_2 + 2\langle u - p, x_{n+1} - p\rangle]\\
&= (1 - \alpha_n)\|x_n - p\|^2 + \alpha_n\delta_n,\end{aligned} \tag{4.9}$$

where $\delta_n := \theta_n\|x_n - x_{n-1}\|N_2 + 2\langle u - p, x_{n+1} - p\rangle$. According to Lemma 2.7, to conclude our proof, it is sufficient to establish that $\limsup_{k\to\infty} \delta_{n_k} \leq 0$ for every subsequence $\{\|x_{n_k} - p\|\}$ of $\{\|x_n - p\|\}$ satisfying the condition:

$$\liminf_{k\to\infty}\{\|x_{n_k+1} - p\| - \|x_{n_k} - p\|\} \geq 0. \tag{4.10}$$

To establish that $\limsup_{k\to\infty}\delta_{n_k} \leq 0$, we suppose that for every subsequence $\{\|x_{n_k} - p\|\}$ of $\{\|x_n - p\|\}$ such that (4.10) holds. Then,

$$\begin{aligned}
&\liminf_{k\to\infty}\{\|x_{n_k+1} - p\|^2 - \|x_{n_k} - p\|^2\}\\
&=\liminf_{k\to\infty}\{(\|x_{n_k+1} - p\| - \|x_{n_k} - p\|)(\|x_{n_k+1} - p\| + \|x_{n_k} - p\|)\} \geq 0.
\end{aligned} \tag{4.11}$$

Now, using Algorithm 3 and Lemma 4.2, we have

$$\begin{aligned}
&\|x_{n+1} - p\|^2\\
&= \|\alpha_n u + \beta_n x_n + \eta_n(T_1 \circ T_2)y_n - p\|^2\\
&= \|\alpha_n(u - p) + \beta_n(x_n - p) + \eta_n((T_1 \circ T_2)y_n - p)\|^2\\
&\leq \alpha_n\|u - p\|^2 + \beta_n\|x_n - p\|^2 + \eta_n\|(T_1 \circ T_2)y_n - p\|^2 - \beta_n\eta_n\|(T_1 \circ T_2)y_n - x_n\|^2\\
&\leq \beta_n\|x_n - p\|^2 + \eta_n\|y_n - p\|^2 - \beta_n\eta_n\|(T_1 \circ T_2)y_n - x_n\|^2 + \alpha_n\|u - p\|^2\\
&\leq \beta_n\|x_n - p\|^2 + \eta_n\|v_n - p\|^2 - \beta_n\eta_n\|(T_1 \circ T_2)y_n - x_n\|^2 + \alpha_n\|u - p\|^2\\
&\leq \beta_n\|x_n - p\|^2 + \eta_n\|w_n - p\|^2 - \beta_n\eta_n\|(T_1 \circ T_2)y_n - x_n\|^2 + \alpha_n\|u - p\|^2\\
&\leq \eta_n[\|x_n - p\|^2 + \theta_n\|x_n - x_{n-1}\|N_2] + \beta_n\|x_n - p\|^2 - \beta_n\eta_n\|(T_1 \circ T_2)y_n - x_n\|^2\\
&\quad + \alpha_n\|u - p\|^2\\
&= (\eta_n + \beta_n)\|x_n - p\|^2 + \eta_n\theta_n\|x_n - x_{n-1}\|N_2 + \alpha_n\|u - p\|^2\\
&\quad - \beta_n\eta_n\|(T_1 \circ T_2)y_n - x_n\|^2\\
&\leq \|x_n - p\|^2 + \alpha_n\theta_n\|x_n - x_{n-1}\|N_2 + \alpha_n\|u - p\|^2 - \beta_n\eta_n\|(T_1 \circ T_2)y_n - x_n\|^2.
\end{aligned} \tag{4.12}$$

Thus from (4.11), we obtain

$$\begin{aligned}
&\limsup_{k\to\infty}[\beta_{n_k}\eta_{n_k}\|(T_1 \circ T_2)y_{n_k} - x_{n_k}\|^2]\\
&\leq \limsup_{k\to\infty}[\|x_{n_k} - p\|^2 - \|x_{n_k+1} - p\|^2\\
&\quad + \alpha_{n_k}\theta_{n_k}\|x_{n_k} - x_{n_k-1}\|N_2 + \alpha_{n_k}\|u - p\|^2]\\
&\leq -\liminf_{k\to\infty}[\|x_{n_k} - p\|^2 - \|x_{n_k+1} - p\|^2] \leq 0,
\end{aligned} \tag{4.13}$$

which gives

$$\lim_{k\to\infty}\|(T_1 \circ T_2)y_{n_k} - x_{n_k}\| = 0. \tag{4.14}$$

Also, using Algorithm 3 and Lemma 4.2, we have

$$\begin{aligned}
&\|x_{n+1} - p\|^2\\
&= \|\alpha_n u + \beta_n x_n + \eta_n(T_1 \circ T_2)y_n - p\|^2\\
&= \|\alpha_n(u - p) + \beta_n(x_n - p) + \eta_n((T_1 \circ T_2)y_n - p)\|^2\\
&\leq \alpha_n\|u - p\|^2 + \beta_n\|x_n - p\|^2 + \eta_n\|(T_1 \circ T_2)y_n - p\|^2
\end{aligned}$$

$$
\begin{aligned}
&\le \alpha_n\|u-p\|^2+\beta_n\|x_n-p\|^2+\eta_n\|y_n-p\|^2\\
&\le \alpha_n\|u-p\|^2+\beta_n\|x_n-p\|^2+\eta_n\|v_n-p\|^2+\sigma_n^2\|B^*(G_D-I)Bv_n\|^2\\
&\quad-\sigma_n\|(G_D-I)Bv_n\|^2\\
&\le \alpha_n\|u-p\|^2+\beta_n\|x_n-p\|^2+\eta_n\|w_n-p\|^2-\epsilon^2\|B^*(G_D-I)Bv_n\|^2\\
&\le \eta_n[\|x_n-p\|^2+\theta_n\|x_n-x_{n-1}\|N_2]+\beta_n\|x_n-p\|^2+\alpha_n\|u-p\|^2\\
&\quad-\epsilon^2\|B^*(G_D-I)Bv_n\|^2\\
&\le (\eta_n+\beta_n)\|x_n-p\|^2+\eta_n\theta_n\|x_n-x_{n-1}\|N_2+\alpha_n^2\|u-p\|^2-\epsilon^2\|B^*(G_D-I)Bv_n\|^2\\
&\le \|x_n-p\|^2+\alpha_n\theta_n\|x_n-x_{n-1}\|N_2+\alpha_n^2\|u-p\|^2-\epsilon^2\|B^*(G_D-I)Bv_n\|^2
\end{aligned}
$$

this implies from (4.11)

$$
\begin{aligned}
\limsup_{k\to\infty}[\epsilon^2\|B^*(G_D-I)Bv_{n_k}\|^2] &\le \limsup_{k\to\infty}[\|x_{n_k}-p\|^2-\|x_{n_k+1}-p\|^2\\
&\quad+\alpha_{n_k}\theta_{n_k}\|x_{n_k}-x_{n_k-1}\|N_2+\alpha_{n_k}\|u-p\|^2]\\
&\le -\liminf_{k\to\infty}[\|x_{n_k}-p\|^2-\|x_{n_k+1}-p\|^2]\le 0,
\end{aligned}
\tag{4.15}
$$

which gives

$$
\lim_{k\to\infty}\|B^*(G_D-I)Bv_{n_k}\|=0. \tag{4.16}
$$

Also, using Algorithm 3 and Lemma 4.2, we have

$$
\begin{aligned}
&\|x_{n+1}-p\|^2\\
&=\|\alpha_n u+\beta_n x_n+\eta_n(T_1\circ T_2)y_n-p\|^2\\
&=\|\alpha_n(u-p)+\beta_n(x_n-p)+\eta_n((T_1\circ T_2)y_n-p)\|^2\\
&\le \alpha_n\|u-p\|^2+\beta_n\|x_n-p\|^2+\eta_n\|(T_1\circ T_2)y_n-p\|^2\\
&\le \alpha_n\|u-p\|^2+\beta_n\|x_n-p\|^2+\eta_n\|y_n-p\|^2\\
&\le \alpha_n\|u-p\|^2+\beta_n\|x_n-p\|^2+\eta_n\|v_n-p\|^2+\sigma_n^2\|B^*(G_D-I)Bv_n\|^2\\
&\quad-\sigma_n\|(G_D-I)Bv_n\|^2\\
&\le \alpha_n\|u-p\|^2+\beta_n\|x_n-p\|^2+\eta_n\|w_n-p\|^2+\sigma_n^2\|B^*(G_D-I)Bv_n\|^2\\
&\quad-\sigma_n\|(G_D-I)Bv_n\|^2\\
&\le \|x_n-p\|^2+\alpha_n\theta_n\|x_n-x_{n-1}\|N_2+\alpha_n\|u-p\|^2+\sigma_n^2\|B^*(G_D-I)Bv_n\|^2\\
&\quad-\sigma_n\|(G_D-I)Bv_n\|^2
\end{aligned}
\tag{4.17}
$$

this implies from (4.11)

$$
\begin{aligned}
&\limsup_{k\to\infty}[\sigma_n\|(G_D-I)Bv_{n_k}\|^2]\\
&\le \limsup_{k\to\infty}[\|x_{n_k}-p\|^2-\|x_{n_k+1}-p\|^2\\
&\quad+\alpha_{n_k}\theta_{n_k}\|x_{n_k}-x_{n_k-1}\|N_2+\alpha_{n_k}\|u-p\|^2+\sigma_n^2\|B^*(G_D-I)Bv_{n_k}\|^2]\\
&\le -\liminf_{k\to\infty}[\|x_{n_k}-p\|^2-\|x_{n_k+1}-p\|^2]\le 0,
\end{aligned}
\tag{4.18}
$$

which gives

$$\lim_{k\to\infty}\|(G_D - I)Bv_{n_k}\| = 0. \tag{4.19}$$

Furthermore, using (4.16), we have that

$$\lim_{k\to\infty}\|y_{n_k} - v_{n_k}\| = \lim_{k\to\infty}\|B^*(G_D - I)Bv_{n_k}\| = 0. \tag{4.20}$$

Also, using Algorithm 3, (3.2) and Lemma 4.2, we have

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \|\alpha_n u + \beta_n x_n + \eta_n(T_1 \circ T_2)y_n - p\|^2 \\
&= \|\alpha_n(u-p) + \beta_n(x_n - p) + \eta_n((T_1 \circ T_2)y_n - p)\|^2 \\
&\leq \alpha_n\|u-p\|^2 + \beta_n\|x_n - p\|^2 + \eta_n\|(T_1 \circ T_2)y_n - p\|^2 \\
&\leq \alpha_n\|u-p\|^2 + \beta_n\|x_n - p\|^2 + \eta_n\|y_n - p\|^2 \\
&\leq \alpha_n\|u-p\|^2 + \beta_n\|x_n - p\|^2 + \eta_n\|v_n - p\|^2 \\
&\leq \alpha_n\|u-p\|^2 + \beta_n\|x_n - p\|^2 + \eta_n\|w_n - p\|^2 - \eta_n\|\tau_n b_n\|^2 \\
&\leq \|x_n - p\|^2 + \alpha_n\theta_n\|x_n - x_{n-1}\|N_2 + \alpha_n\|u-p\|^2 - \eta_n\|w_n - v_n\|^2.
\end{aligned} \tag{4.21}$$

This implies from (4.11)

$$\begin{aligned}
\limsup_{k\to\infty}[\eta_{n_k}\|w_{n_k} - v_{n_k}\|^2] &\leq \limsup_{k\to\infty}[\|x_{n_k} - p\|^2 - \|x_{n_k+1} - p\|^2 \\
&\quad + \alpha_{n_k}\theta_{n_k}\|x_{n_k} - x_{n_k-1}\|N_2 + \alpha_{n_k}\|u-p\|^2] \\
&\leq -\liminf_{k\to\infty}[\|x_{n_k} - p\|^2 - \|x_{n_k+1} - p\|^2] \leq 0,
\end{aligned} \tag{4.22}$$

which gives

$$\lim_{k\to\infty}\|w_{n_k} - v_{n_k}\| = 0. \tag{4.23}$$

Now, observe that

$$\begin{aligned}
\langle w_{n_k} - u_{n_k}, b_{n_k}\rangle &= \langle w_{n_k} - u_{n_k}, w_{n_k} - u_{n_k} - \lambda_{n_k}(Aw_{n_k} - Au_{n_k})\rangle \\
&= \|w_{n_k} - u_{n_k}\|^2 - \langle w_{n_k} - u_{n_k}, \lambda_{n_k}(Aw_{n_k} - Au_{n_k})\rangle \\
&\geq \|w_{n_k} - u_{n_k}\|^2 - \lambda_{n_k}\|w_{n_k} - u_{n_k}\|\|Aw_{n_k} - Au_{n_k}\| \\
&\geq \|w_{n_k} - u_{n_k}\|^2 - \frac{\mu\lambda_{n_k}}{\lambda_{n_k+1}}\|w_{n_k} - u_{n_k}\|^2 \\
&= (1 - \frac{\mu\lambda_{n_k}}{\lambda_{n_k+1}})\|w_{n_k} - u_{n_k}\|^2
\end{aligned} \tag{4.24}$$

which implies that

$$\begin{aligned}
\|w_{n_k} - u_{n_k}\|^2 &\leq \frac{1}{(1 - \frac{\mu\lambda_{n_k}}{\lambda_{n_k+1}})}\langle w_{n_k} - u_{n_k}, b_{n_k}\rangle \\
&= \frac{1}{(1 - \frac{\mu\lambda_{n_k}}{\lambda_{n_k+1}})}\eta_{n_k}\|b_{n_k}\|^2
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{(1-\frac{\mu\lambda_{n_k}}{\lambda_{n_k+1}})}\eta_{n_k}\|b_{n_k}\|\|w_{n_k}-u_{n_k}-\lambda_{n_k}(Aw_{n_k}-Au_{n_k})\| \\
&\le \frac{1}{(1-\frac{\mu\lambda_{n_k}}{\lambda_{n_k+1}})}\eta_{n_k}\|b_{n_k}\|[\|w_{n_k}-u_{n_k}\|+\lambda_{n_k}\|Au_{n_k}-Aw_{n_k}\|] \\
&\le \frac{(1+\frac{\mu\lambda_{n_k}}{\lambda_{n_k+1}})}{(1-\frac{\mu\lambda_{n_k}}{\lambda_{n_k+1}})}\eta_{n_k}\|b_{n_k}\|\|w_{n_k}-u_{n_k}\| \\
&\le \frac{(1+\frac{\mu\lambda_{n_k}}{\lambda_{n_k+1}})}{(1-\frac{\mu\lambda_{n_k}}{\lambda_{n_k+1}})}\|w_{n_k}-v_{n_k}\|\|w_{n_k}-u_{n_k}\|. \qquad (4.25)
\end{aligned}
$$

Using (4.23), we have that

$$\lim_{k\to\infty}\|w_{n_k}-u_{n_k}\|=0. \qquad (4.26)$$

It is easy to see that, as $k\to\infty$, we have

$$\|w_{n_k}-x_{n_k}\|=\theta_{n_k}||x_{n_k}-x_{n_k-1}||=\beta_{n_k}\cdot\frac{\theta_{n_k}}{\beta_{n_k}}||x_{n_k}-x_{n_k-1}||\to 0. \qquad (4.27)$$

In addition, we have that

$$\|u_{n_k}-x_{n_k}\|\le\|u_{n_k}-w_{n_k}\|+\|w_{n_k}-x_{n_k}\|\to 0 \text{ as } k\to\infty. \qquad (4.28)$$
$$\|v_{n_k}-x_{n_k}\|\le\|v_{n_k}-w_{n_k}\|+\|w_{n_k}-x_{n_k}\|\to 0 \text{ as } k\to\infty. \qquad (4.29)$$
$$\|y_{n_k}-x_{n_k}\|\le\|v_{n_k}-x_{n_k}\|+\sigma_n\|B^*(G_D-I)v_{n_k}\|\to 0 \text{ as } k\to\infty. \qquad (4.30)$$
$$\|(T_1\circ T_2)y_{n_k}-y_{n_k}\|\le\|(T_1\circ T_2)y_{n_k}-x_{n_k}\|+\|x_{n_k}-y_{n_k}\|\to 0 \text{ as } k\to\infty. \qquad (4.31)$$

From the Algorithm 3 and (4.14), observe that

$$
\begin{aligned}
\|x_{n_k+1}-(T_1\circ T_2)y_{n_k}\| &= \|\alpha_n u+\beta_n x_n+\eta_n(T_1\circ T_2)y_{n_k}-(T_1\circ T_2)y_{n_k}\| \\
&\le \alpha_{n_k}\|u-(T_1\circ T_2)y_{n_k}\|+\beta_{n_k}\|x_{n_k}-(T_1\circ T_2)y_{n_k}\| \\
&\quad +\eta_{n_k}\|(T_1\circ T_2)y_{n_k}-(T_1\circ T_2)y_{n_k}\|\to 0 \text{ as } k\to\infty. \qquad (4.32)
\end{aligned}
$$

Using (4.32) and (4.14), it is easy to see that

$$\|x_{n_k+1}-x_{n_k}\|\le\|x_{n_k+1}-(T_1\circ T_2)y_{n_k}\|+\|(T_1\circ T_2)y_{n_k}-x_{n_k}\|\to 0 \text{ as } k\to\infty. \qquad (4.33)$$

Since $\{x_{n_k}\}$ is bounded, there exists a subsequence $\{x_{n_{k_j}}\}$ of $\{x_{n_k}\}$ such that $\{x_{n_{k_j}}\}$ converges weakly to $x^*\in H_1$. By (4.14), (4.28), (4.29), (4.30) and (4.31), we have that the subsequences $\{w_{n_{k_j}}\}$ of $\{w_{n_k}\}$, $\{u_{n_{k_j}}\}$ of $\{u_{n_k}\}$, $\{v_{n_{k_j}}\}$ of $\{v_{n_k}\}$, and $\{y_{n_{k_j}}\}$ of $\{y_{n_k}\}$, all converge weakly to x^* respectively. Hence, with (4.14) and by the demiclosedness of $(T_1\circ T_2)$ (Lemma 2.4) and Lemma 4.1, we have that $x^*\in F(T_1\circ T_2)=F(T_1)\cap F(T_2)$. Furthermore, we show that $x^*\in\Omega$. Since $u_{n_k}=P_C(w_{n_k}-\lambda_{n_k}Aw_{n_k})$, then

$$\langle w_{n_k}-\lambda_{n_k}Aw_{n_k}-u_{n_k},u-u_{n_k}\rangle\le 0 \quad \forall u\in C.$$

Thus

$$\begin{aligned}\langle w_{n_k} - u_{n_k}, u - u_{n_k}\rangle &\leq \lambda_{n_k}\langle Aw_{n_k}, u - u_{n_k}\rangle \\ &= \lambda_{n_k}\langle Aw_{n_k}, w_{n_k} - u_{n_k}\rangle + \lambda_{n_k}\langle Aw_{n_k}, u - w_{n_k}\rangle \quad \forall u \in C.\end{aligned} \tag{4.34}$$

Fix $u \in C$ and let $k \to \infty$ in the last inequality, since $\|w_{n_k} - u_{n_k}\| \to 0$ and $\liminf_{k\to\infty} \lambda_{n_k} > 0$, we have

$$0 \leq \liminf_{k\to\infty}\langle Aw_{n_k}, u - w_{n_k}\rangle, \quad \forall u \in C. \tag{4.35}$$

Since A is monotone, then

$$\langle Au, u - w_{n_k}\rangle \geq \langle Aw_{n_k}, u - w_{n_k}\rangle \quad \forall u \in C.$$

Taking liminf of both sides, we get

$$\liminf_{k\to\infty}\langle Au, u - w_{n_k}\rangle \geq \liminf_{k\to\infty}\langle Aw_{n_k}, u - w_{n_k}\rangle \geq 0, \quad \forall u \in C.$$

More so, since $w_{n_k} \rightharpoonup x^*$, then it follows from (4.35) that

$$\langle Au, u - x^*\rangle = \lim_{k\to\infty}\langle Au, u - w_{n_k}\rangle = \liminf_{k\to\infty}\langle Au, u - w_{n_k}\rangle \geq 0.$$

Therefore using Lemma 2.5, we get $x^* \in \Omega$.
Next, we show that $Bx^* \in \mathit{\Gamma}$. Observe that

$$\begin{aligned}&\|(G - I)Bx^*\|^2 \\ &= \langle GBx^* - Bx^*, GBx^* - Bx^*\rangle \\ &= \langle GBx^* - Bx^*, GBx^* - GBv_{n_k} + GBv_{n_k} - Bw_{n_k} + Bv_{n_k} - Bx^*\rangle \\ &= \langle GBx^* - Bx^*, Bv_{n_k} - Bx^*\rangle + \langle GBx^* - Bx^*, GBv_{n_k} - Bv_{n_k}\rangle \\ &\quad + \langle GBx^* - Bx^*, GBp - GBv_{n_k}\rangle \\ &\leq \langle GBx^* - Bx^*, Bv_{n_k} - Bx^*\rangle + \langle GBx^* - Bx^*, GBv_{n_k} - Bv_{n_k}\rangle.\end{aligned} \tag{4.36}$$

Since B is a bounded linear operator, we have that $\lim\limits_{n\to\infty}\|Bv_{n_k} - Bx^*\| = 0$. Hence using (4.19), we have that $\|(G - I)Bx^*\| = 0$, as such, we obtain that $Bx^* \in \mathit{\Gamma}$.
Since $\{x_{n_k}\}$ is bounded, it follows that there exists a subsequence $\{x_{n_{k_j}}\}$ of $\{x_{n_k}\}$ that converges weakly to x^* such that

$$\limsup_{k\to\infty}\langle u - p, x_{n_k} - p\rangle = \lim_{j\to\infty}\langle u - p, x_{n_{k_j}} - p\rangle = \langle u - p, x^* - p\rangle. \tag{4.37}$$

Hence, since $p = P_\Gamma u$, we have obtain from (4.37) that

$$\limsup_{k\to\infty}\langle u - p, x_{n_k} - p\rangle = \langle u - p, x^* - p\rangle \leq 0, \tag{4.38}$$

which implies that

$$\limsup_{k\to\infty}\langle u - p, x_{n_k+1} - p\rangle \leq 0. \tag{4.39}$$

Using using our assumption and (4.38), we have that $\limsup_{k\to\infty}\delta_{n_k} := \beta_n\frac{\theta_n}{\beta_n}\|x_n - x_{n-1}\|N_2 + 2\langle u - p, x_{n+1} - p\rangle \leq 0$. Thus, the last part of Lemma 2.7 is achieved. Hence, we have that $\lim\limits_{n\to\infty}\|x_n - p\| = 0$. Thus, $\{x_n\}$ converges strongly to $p \in P_{Sol}u$.

□

Remark 4.1. We emphasize here, some of the advantages of our Algorithm 3.

1. Our method provide a solution to the setbacks noted in Algorithm 1.13 and Algorithm 1.15. In addition, our method is more applicable and converges faster than the methods of [31, 33, 40].
2. The implementation of previous algorithms in this direction require at least a prior estimate of the norm of the bounded linear operator B which is very difficult in practice. Moreover, the stepsize defined by this process is often too small and detriorates the convergence of the method. In Algorithm 3, the stepsize is determined self-adaptively and does not require the prior estimate of the norm of the bounded linear operator.
3. Another notable advantage of Algorithm 3 for solving the VIP is that the the conditions of strongly inversely monotonicity or Lipschitz continuity of the operator A usually used in other papers to guarantee convergence is removed and no extra projection required under this setting (see for example, [27, 33] and the references therein). Note that for Algorithm 3, we only assumed that A is a monotone operator on H_1.
4. The sequence generated by the proposed method converges strongly to a solution set of the aforementioned problems in real Hilbert spaces. In addition, the strong convergence analysis of our proposed method does not rely on the usual "Two Cases Approach" widely used in many papers to guarantee strong convergence (see for example, [15, 42, 43]).

5. Numerical Examples

In this section, we present some numerical experiments to illustrate the performance of the algorithm.

Example 5.1. Let $H_1 = H_2 = \mathbb{R}^3$ endowed with norm $\|\cdot\| : \mathbb{R}^3 \to \mathbb{R}$ defined by $\|x\| = \left(\sum_{i=1}^{3} |x_i|^2\right)^{\frac{1}{2}}$ and inner product $\langle \cdot, \cdot \rangle : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ defined by $\langle x, y \rangle = \sum_{i=1}^{3} x_i y_i$ for all $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ and $y = (y_1, y_2, y_3) \in \mathbb{R}^3$. Let $C = [-10, 10] \times [-10, 10] \times [-10, 10]$, $D_1 = [-5, 5] \times [-5, 5] \times [-5, 5]$ and $D_2 = [-10, 10] \times [-10, 10] \times [-10, 10]$. Let $A : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by

$$Ax = (2x_1, 2x_2, 2x_3) \quad \text{for} \quad x \in \mathbb{R}^3$$

and let $A_1 : D_1 \to \mathbb{R}^3$ and $A_2 : D_2 \to \mathbb{R}^3$ be defined by

$$A_1 x = \left(\frac{x_1 - 2}{3}, \frac{2x_2 - 4}{5}, \frac{x_3 - 2}{4} \right) \quad \forall x \in D_1,$$

and

$$A_2 x = \left(x_1 - 2, \frac{x_2 - 2}{3}, \frac{x_3 - 3}{5} \right) \quad x \in D_2.$$

It is easy to see that A is monotone and A_1, A_2 are 1-inverse strongly monotone with $\nu_1, \nu_2 \in (0, 2)$ respectively. More so, let $B : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $Bx = 3x$ for all $x \in \mathbb{R}^3$. Then B is a bounded linear operator. Let $T_1, T_2 : \mathbb{R}^3 \to \mathbb{R}^3$

be defined by $T_1x = \frac{x-5}{2}$ and $T_2x = \frac{x}{2}$. It is easy to see that T_1 is 1-strongly quasi-nonexpansive mapping and T_2 is firmly nonexpansive. Choose $\mu = 0.38, \theta = 0.1, \alpha_n = \frac{1}{n+1}, \epsilon_n = \frac{1}{(n+1)^2}, \beta_n = \frac{3n}{8(n+1)}, \eta_n = 1 - \alpha_n - \beta_n, u = (10, -10, 10)'$. It is easy to verify that all hypothesis of Theorem 4.1 are satisfied. Moreover, $Sol = \{0\}$. We use different choices of x_0, x_1 and test the convergence of our algorithm with $\|x_{n+1} - x_n\| < 10^{-6}$ as stopping criterion. First, we study the behaviour of the sequence generated by Algorithm 3 by choosing $\mu = 0.23, \theta = 0.5, \lambda_0 = 0.1$, $\epsilon_n = \frac{1}{n^s}$ $(s = 0.1, 0.4, 0.8, 0.99)$, $\alpha_n = \epsilon_n^2$, $\eta_n = \frac{2n}{5n+8}, \beta_n = 1 - \alpha_n - \beta_n$. We used the following initial points: Case I: $x_0 = (-5, -5, -5)', x_1 = (2, 2, 2)'$, Case II: $x_0 = (10, 10, 10)', x_1 = (4, 4, 4)'$. The numerical result are shown in Table 1 and Figure 1. We also compare the performance of Algorithm 3 with Algorithm (1.15) of Sahu and Singh [31]. For (1.15), we use $\mu_1 = 0.38, \mu_2 = 0.24, \rho = 1.78, \mu = 0.04, l = 0.28, \alpha_n = \frac{3n}{8(n+1)}, \beta_n = \frac{1}{n+1}$ and $\theta_n = \frac{1}{(n+1)^2}$. More so, the following input values were used for the computation:

Case I: $x_0 = (3, -3, 4)'$, $x_1 = (5, -5, 5)'$;

case II: $x_0 = (7, 9, -10)'$, $x_1 = (2, 5, 8)'$;

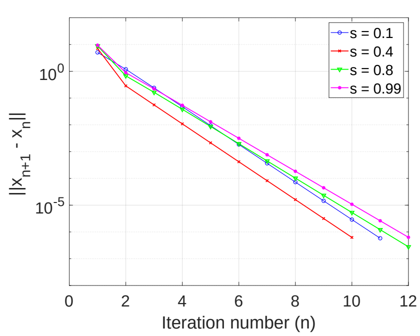
Case III: $x_0 = (1, 10, 5)'$, $x_1 = (-4, 4, 8)'$;

Case IV: $x_0 = (-3, 3, 9)'$, $x_1 = (6, -1, 4)'$.

The computational results are shown in Table 2 and Figure 2.

Table 1. Computation result for Example 5.1.

		$s = 0.1$	$s = 0.4$	$s = 0.8$	$s = 0.99$
Case I	No of Iter.	11	10	12	12
	CPU time (sec)	0.0037	0.0030	0.0046	0.0039
Case II	No of Iter.	11	10	11	12
	CPU time (sec)	0.0033	0.0032	0.0039	0.0038



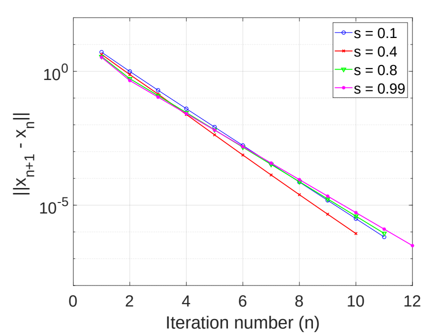


Figure 1. Example 5.1: Performance of Algorithm 3 for different values of s; Left: Case I, Right: Case II.

Next, we give an example in infinite dimensional spaces to support the strong convergence of Theorem 4.1.

Table 2. Comparison of the performance of Algorithm 3 with Algorithm (1.15) for Example 5.1.

		Algorithm 3	Algorithm (1.15)
Case 1	No of Iter.	12	31
	CPU time (sec)	0.0029	0.0065
Case 2	No of Iter.	12	29
	CPU time (sec)	0.0040	0.0076
Case 3	No of Iter.	11	31
	CPU time (sec)	0.0030	0.0073
Case 4	No of Iter.	11	23
	CPU time (sec)	0.0016	0.0043

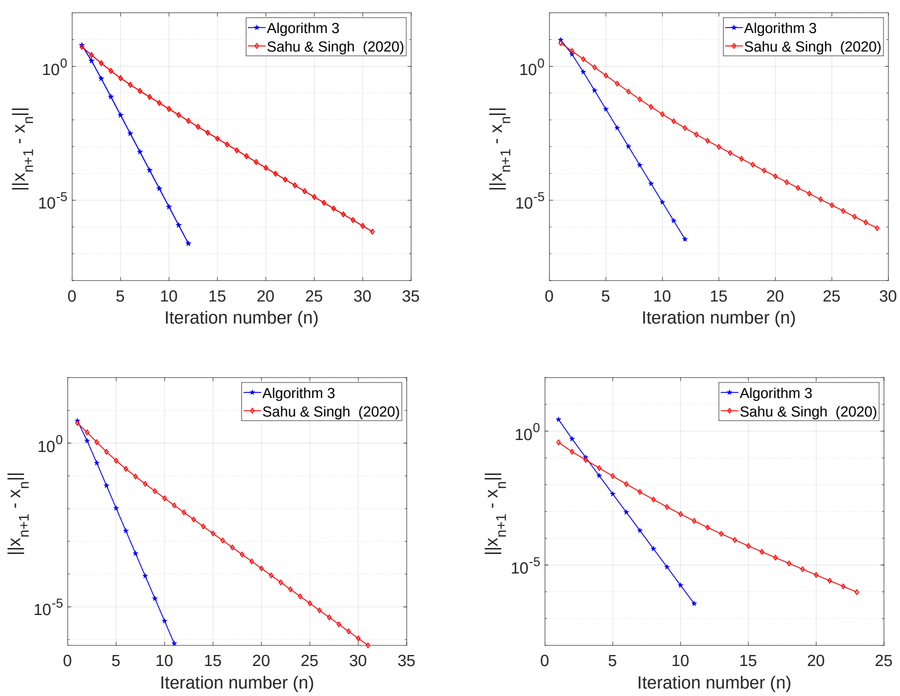


Figure 2. Example 5.1, Top Left: Case 1; Top Right: Case 2; Bottom Left: case 3; Bottom Right: Case 4.

Example 5.2. Let $H_1 = H_2 = \ell_2$ be the linear space whose elements consists of all 2-summable sequence of scalars $(x_1, x_2, \dots, x_j, \dots)$, i.e.,

$$\ell_2 = \left\{ \bar{x} = (x_1, x_2, \dots, x_j, \dots) \quad \text{and} \quad \sum_{j=1}^{\infty} |x_j|^2 < \infty \right\}$$

with inner product $\langle \cdot, \cdot \rangle : \ell_2 \times \ell_2 \to \mathbb{R}$ defined by $\langle \bar{x}, \bar{y} \rangle = \sum_{j=1}^{\infty} x_j y_j$ and norm $\|x\|_2 := \left(\sum_{j=1}^{\infty} |x_j|^2 \right)^{\frac{1}{2}}$, where $\bar{x} = \{x_j\} \in \ell_2$ and $\bar{y} = \{y_j\} \in \ell_2$. Let C be defined by $C = \{x \in \ell_2 : \langle a, x \rangle = b\}$ where $a = (1, 3, 1, 0, \dots, 0, \dots)$ and $b = 2$ and $D_i := \{x \in \ell_2 : \langle c, x \rangle \geq d_i\}$ where $c = (2, -1, 1, 0, \dot{,} 0, \dots)$ and $d_i = -2i$ for $i = 1, 2$. Then, we have

$$P_C(\bar{x}) = \max \left\{ 0, \frac{b - \langle a, \bar{x} \rangle}{\|a\|_2^2} \right\} a + \bar{x},$$

and

$$P_{D_i}(\bar{x}) = \frac{d_i - \langle c, \bar{x} \rangle}{\|c\|_2^2} c + \bar{x}.$$

Let $A : \ell_2 \to \ell_2$ be given by $A\bar{x} = (2x_1, 2x_2, \dots, 2x_j, \dots)$ and $A_1 : D_1 \to \ell_2$ and $A_2 : D_2 \to \ell_2$ are given by $A_1 \bar{x} = \left(\frac{x_1}{6}, \frac{x_2}{6}, \dots, \frac{x_j}{6}, \dots \right)$ and $A_2 \bar{x} = \left(\frac{x_1}{5}, \frac{x_2}{5}, \dots, \frac{x_j}{5} \right)$, respectively. Then A_1 and A_2 are 1-inverse strongly monotone with $\nu_1, \nu_2 \in (0, 2)$. Let $B : \ell_2 \to \ell_2$ be defined by $B\bar{x} = 2\bar{x}$, then B is a bounded linear operator. We choose $\mu = 0.5, \theta = 0.025, \epsilon_n = \frac{1}{(n+1)^s} (s = 0.1, 0.4, 0.6, 0.8, 0.99), \alpha_n = \epsilon_n^2, \theta_n = \frac{1}{n+1}, \beta_n = \frac{9n}{10n+1}, \eta_n = 1 - \alpha_n - \beta_n, u = (1, 1, 1, \dots, 1, \dots)$. One can easily verified that the hypothesis of Theorem 4.1 are satisfied. The numerical result for the performmance of Algorithm 3 for the values of s are shown in Table 3 and Figure 3. We also compare the performance of Algorithm 3 with (1.15). We choose $\rho = 1.99, \mu_1 = 0.58, \mu_2 = 0.24, \mu = 1, \zeta = 0.58, l = 0.55, \alpha_n = \frac{1}{\sqrt{n+1}}$, and $\theta_n = \frac{1}{n+1}$. We test the algorithms using the following initial points and $\|x_{n+1} - x_n\| < 10^{-6}$ as stopping criterion:

Case 1: Take $x_0 = (1, 2, 3, \dots)$, $x_1 = (3.2158, -5.8091, 0, \dots)$.

Case 2: Take $x_0 = (1, 0.5, 0.25, \dots)$, $x_1 = (2.7601, -3.6457, 0, \dots)$.

Case 3: Take $x_0 = (2, 2, 2, \dots)$, $x_1 = (1.8501, -2.7557, 0, \dots)$.

Case 4: Take $x_0 = (1, 3, 1, \dots)$, $x_1 = (1.4501, -2.3457, 0, \dots)$.

The computation results can be seen in Figure 4 and Table 4.

Table 3. Computation result for Example 5.2.

		$s = 0.1$	$s = 0.4$	$s = 0.8$	$s = 0.99$
Case I	No of Iter.	7	7	9	9
	CPU time (sec)	0.0478	0.0441	0.0512	0.0556
Case II	No of Iter.	7	8	10	10
	CPU time (sec)	0.0204	0.0214	0.0228	0.0293

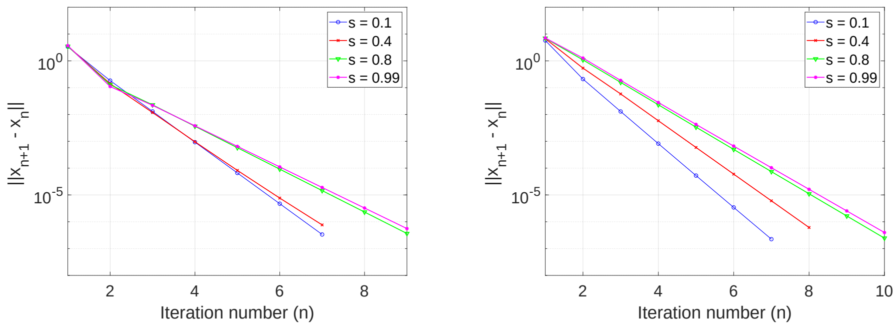


Figure 3. Example 5.2: Performance of Algorithm 3 for different values of s; Left: Case I, Right: Case II.

Table 4. Comparison of the performance of Algorithm 3 with Algorithm (1.15) for Example 5.2.

		Algorithm 3	Algorithm (1.15)
Case 1	No of Iter.	7	38
	CPU time (sec)	0.0103	0.0157
Case 2	No of Iter.	7	37
	CPU time (sec)	0.0046	0.0474
Case 3	No of Iter.	9	36
	CPU time (sec)	0.0109	0.0140
Case 4	No of Iter.	7	8
	CPU time (sec)	0.0051	0.0102

Example 5.3. Let $H_1 = H_2 = L_2([0,1])$ be equipped with the inner product

$$\langle x, y\rangle = \int_0^1 x(t)y(t)dt \quad \forall \quad x, y \in L_2([0,1]) \quad \text{and} \quad \|x\| := \sqrt{\int_0^1 |x(t)|^2 dt}$$

$\forall x, y, \in L_2([0,1])$.

Now, define the operators $A, A_1, A_2, B : L_2([0,1]) \to L_2([0,1])$ by

$$Ax(t) = \int_0^1 \left(x(t) - \left(\frac{2tse^{t+s}}{e\sqrt{e^2-1}} \right) \cos x(s) \right) ds + \frac{2te^t}{e\sqrt{e^2-1}}, \quad x \in L_2([0,1]),$$
$$A_1x(t) = 2x, \quad x \in L_2([0,1]),$$
$$A_2x(t) = \frac{2}{5}x, \quad x \in L_2([0,1]),$$
$$Bx(t) = \max\{0, x(t)\}, \quad t \in [0,1].$$

Then A is Lipschitz continuous and monotone, and B is maximal monotone on $L_2([0,1])$. Let $S : L_2([0,1]) \to L_2([0,1])$ be defined by

$$Sx(t) = \int_0^1 \kappa(s,t)x(t)dt \quad \forall \quad x \in L_2([0,1]),$$

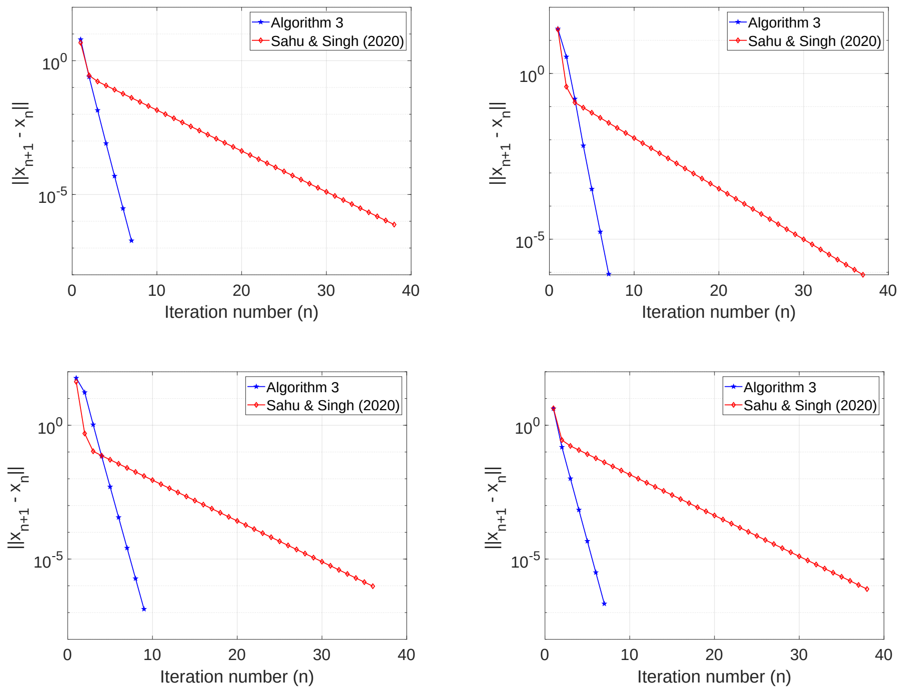


Figure 4. Example 5.2, Top Left: Case 1; Top Right: Case 2; Bottom Left: case 3; Bottom Right: Case 4.

where κ is a continuous real-valued function defined on $[0,1]\times[0,1]$. Then, S is a bounded linear operator with adjoint

$$S^*x(t)=\int_0^1 \kappa(t,s)x(t)dt \quad \forall \quad x\in L_2([0,1]).$$

Let $T_1,T_2 : L_2([0,1])\to L_2([0,1])$ be defined by

$$T_1x(t)=\int_0^1 tx(s)ds, \quad \text{and} \quad T_2x(t)=\frac{x}{2}(t) \quad t\in[0,1].$$

It is easy to see that T_1 and T_2 are nonexpansive mappings. Let $C=\{y\in L_2([0,1]) : \langle a,y\rangle=\alpha\}$, where $a\neq 0$ and $\alpha\in\mathbb{R}$, then C is a nonempty, closed and convex subset of $L_2([0,1])$. Thus, we define the metric projection P_C as:

$$P_C(x)=x-\frac{\langle a,x\rangle-\alpha}{\|a\|^2}a.$$

First, we study the behaviour of the sequence generated by Algorithm 3 by choosing $\mu=0.23,\theta=0.5,\lambda_0=0.1$, $\epsilon_n=\frac{1}{n^s}$ $(s=0.1,0.4,0.8,0.99)$, $\alpha_n=\epsilon_n^2$, $\eta_n=\frac{2n}{5n+8},\beta_n=1-\alpha_n-\beta_n$. The initial points used for the computation are: Case I: $x_1(t)=3e^t,x_0(t)=\cos(2t)$; Case II: $x_1(t)=\sin(2t)$, $x_0(t)=t^3+1$. The numerical results are shown in Table 5 and Figure 5. Also, we compare the performance of

Algorithm 3 with Algorithm (1.15). We take $\theta = 0.36, \lambda_0 = 0.1, , \epsilon_n = \frac{1}{\sqrt{n+1}}, \alpha_n = \frac{1}{n+1}, \eta_n = \frac{3n}{5n+5}, \beta_n = 1 - \eta_n - \alpha_n$. We use $||x_{n+1} - x_n|| < 10^{-4}$ as tolerance level for the computation with different initial values which are given as follows:

Case 1: Take $x_1(t) = t^2 + 2t + 45, \quad x_0(t) = e^{t^2}$.

Case 2: Take $x_1(t) = t^2 + 2, \quad x_0(t) = 3e^t$.

Case 3: Take $x_1(t) = \cos(t) + 2t, \quad x_0(t) = t^2 + e^t$.

Case 4: Take $x_1(t) = \sin(t) + 2t^2 + 5, \quad x_0(t) = t + e^t + 45$.

The numerical results for the comparison of Algorithm 3 with Algorithm (1.15) are shown in Table 6 and Figure 6.

Table 5. Computation result for Example 5.3.

		$s = 0.1$	$s = 0.4$	$s = 0.8$	$s = 0.99$
Case I	No of Iter.	7	6	6	6
	CPU time (sec)	3.5969	6.5984	6.800	7.0366
Case II	No of Iter.	6	6	6	6
	CPU time (sec)	5.6975	8.3069	8.2852	8.553

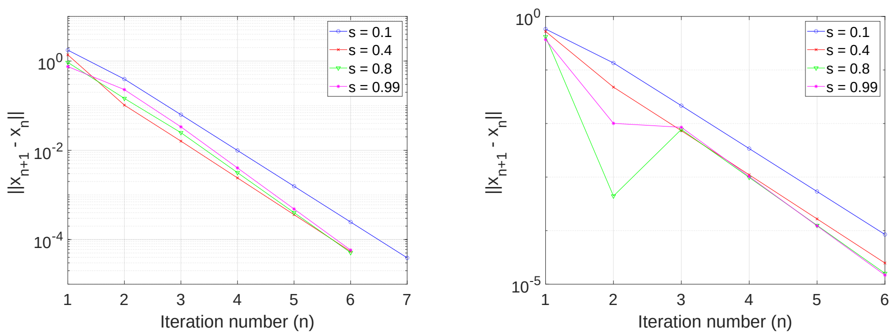


Figure 5. Example 5.3: Performance of Algorithm 3 for different values of s; Left: Case I, Right: Case II.

6. Conclusion

In this paper, we propose a new halpern type inertial extrapolation method for for approximating split system of variational inequalities problems for two inverse strongly monotone operators, variational inequality problem for monotone operator, and the fixed point of two composed mappings and establish that the proposed method converges strongly to a solution set of the aforementioned problems when the underlying operator for the variational inequality problem is monotone which is much more weaker assumptions than the inverse strongly monotonicity assumptions and the monotonicity and Lipschitz continuity assumptions used in the literature. In addition, we present some examples and numerical experiments to show the

Table 6. Comparison of the performance of Algorithm 3 with Algorithm 1.15 for Example 5.3.

		Algorithm 3	Algorithm (1.15)
Case 1	No of Iter.	6	16
	CPU time (sec)	5.7342	7.4992
Case 2	No of Iter.	7	19
	CPU time (sec)	4.1436	7.7330
Case 3	No of Iter.	7	15
	CPU time (sec)	8.3193	11.6183
Case 4	No of Iter.	7	21
	CPU time (sec)	1.7011	3.5331

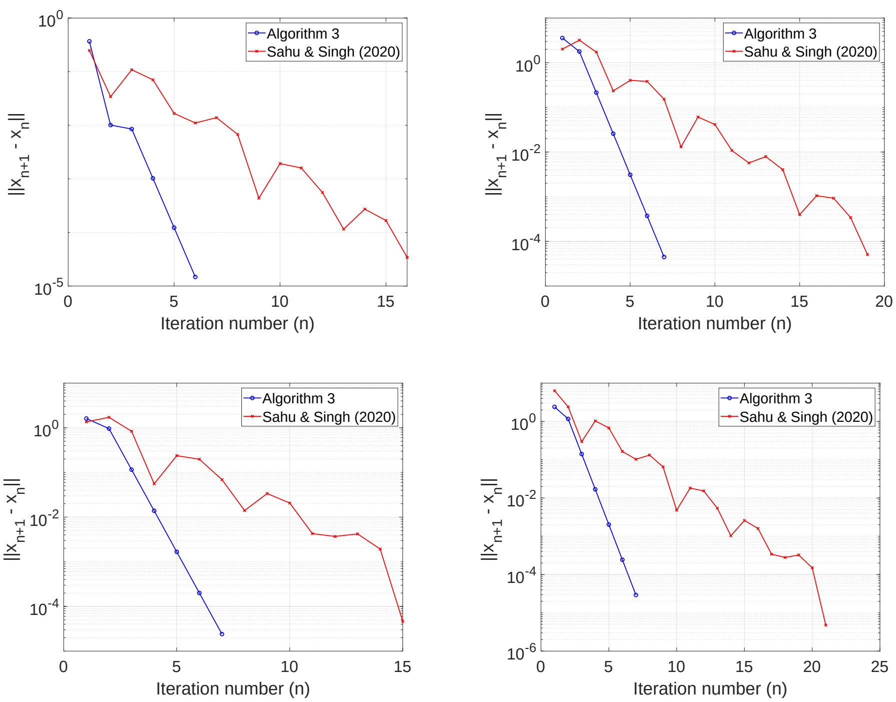


Figure 6. Example 5.3, Top Left: Case 1; Top Right: Case 2; Bottom Left: case 3; Bottom Right: Case 4.

efficiency and applicability of our method in comparison with the iterative algorithm introduced in the framework of infinite and finite dimensional Hilbert spaces.

List of Abbreviations. VIP: Variational Inequality Problem; SFP: Split Feasibility Problem; GSFP: Generalized Split Feasibility Problem; SNVIP: System of Nonlinear Variational Inequality.

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