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OPTIMAL ERROR ANALYSIS OF PARTIALLY-UPDATED PROJECTION FEM SCHEME FOR THE LANDAU-LIFSHITZ EQUATION BASED ON THE CRANK-NICOLSON DISCRETIZATION*

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Abstract This paper presents a Crank-Nicolson partially-updated projection finite element scheme for the numerical approximation of the Landau-Lifshitz equation which describes the dynamics of magnetization in ferromagnetic materials and is a strongly nonlinear parabolic problem with the non-convex constraint. The proposed scheme is a semi-implicit scheme by using the extrapolation technique and the implicit-explicit method to linearize the nonlinear terms. Furthermore, the sphere projection is used to preserve the unit length such that numerical solutions satisfy the non-convex constraint exactly. The optimal second-order convergence rate in time and space is derived under the reasonable time step condition. Finally, the numerical experiment is presented to confirm the theoretical result.

Keywords Landau-Lifshitz equation, Crank-Nicolson scheme, sphere projection method, finite element method, error analysis.

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1. Introduction

The dynamics of the magnetic distribution in a ferromagnetic material is governed by the Landau-Lifshitz (LL) equation (cf. [29]):

$$\mathbf{m}_t = \gamma \mathbf{m} \times \Delta \mathbf{m} - \lambda \gamma \mathbf{m} \times (\mathbf{m} \times \Delta \mathbf{m}). \tag{1.1}$$

where $\mathbf{m} : (0, T] \times \Omega \to \mathbf{R}^3$ denotes the magnetization and Ω is a bounded and convex domain in $\mathbf{R}^d$ with $d = 2, 3$. The constant $\lambda > 0$ denotes the dimensionless damping parameter and $\gamma > 0$ is the exchange constant. Since the exchange constant γ is not critical for the description of the Crank-Nicolson scheme, we set $\gamma = 1$. It is easy to see that $|\mathbf{m}(t)|$ is constant in time. Then we assume that $|\mathbf{m}(t)| = 1$ in the point-wise sense. The LL equation (1.1) is closed by imposing the following initial and boundary conditions:

$$\mathbf{m}(0, \mathbf{x}) = \mathbf{m}_0 \text{ with } |\mathbf{m}_0| = 1 \text{ in } \Omega \quad \text{and} \quad \partial_{\mathbf{n}} \mathbf{m} = 0 \text{ on } (0, T) \times \partial\Omega, \tag{1.2}$$

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where $\mathbf{n}$ is the unit outward normal vector to $\partial\Omega$.

From $|\mathbf{m}(t)| = 1$ for any $t > 0$ and the following vector formula:

$$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w},$$

the Landau-Lifshitz equation (1.1) can be rewritten as (cf. [23])

$$\mathbf{m}_t - \lambda\Delta\mathbf{m} - \mathbf{m} \times \Delta\mathbf{m} = \lambda|\nabla\mathbf{m}|^2\mathbf{m}. \tag{1.3}$$

Since $|\mathbf{m}(t)|$ remains the constant during the evolution process, a key issue in designing numerical algorithms is that one need to take into account the non-convex constraint. In general, there are two methods to deal with this difficulty, i.e., the scheme of approximating the unit length of magnetization and the scheme of preserving the unit length of magnetization exactly.

For the first method, a technique is to relax $|\mathbf{m}| = 1$ by introducing some penalty terms. In [35], based upon different penalty strategies, some penalty schemes were studied. Pistella and Valente [34] used Ginzburg-Landau penalty function $\frac{1}{\varepsilon}(|\mathbf{m}|^2 - 1)\mathbf{m}$ to relax the non-convex constraint and gave an explicit finite difference scheme to solve the penalty problem numerically. They proved the convergence of the numerical solution under the time step restriction $\tau \le C(\varepsilon)h^2$, where τ and h are the time step size and the mesh size, respectively. Usually, the penalty methods result in the use of a very small time step as h and ε tend to zero, and extremely time-consuming in practical computations. Another choice is using the semi-implicit schemes to discrete the LL equation, such as the first-order backward Euler scheme [16, 21] and the second-order Crank-Nicolson scheme [3]. But these semi-implicit schemes fail to preserve the unit length of magnetization exactly.

For the second method, the numerical explicit scheme with a finite element approximation to the weak solution was proposed by Alouges and Jaisson [2], where the orthogonal projection method was used to preserve the unit length of magnetization. However, at each time step, one has to build a new finite element space which is orthogonal point-wisely to the finite element solution at the previous time step. Bartels, Ko and Prohl [8] studied Alouges and Jaisson's scheme and proved the convergence of numerical solutions. Bartles and Prohl [10] studied a fully implicit FEM scheme. Although the fully implicit scheme in [10] was unconditionally stable and can preserve the unit length of magnetization at each time step, but one has to solve a nonlinear problem which was solved by using a fixed-point strategy.

The sphere projection method is a natural numerical method in designing the numerical scheme of preserving the unit length of magnetization. The first sphere projection method was suggested by E and Wang [20] where the term $|\nabla\mathbf{m}|^2$ in the LL equation was treated as the Lagrange multiplier for the constraint $|\mathbf{m}| = 1$. The proposed projection scheme consists of two steps. Firstly, since $|\nabla\mathbf{m}|^2$ was treated as the Lagrange multiplier, a simplified linear system was solved to obtain an intermediate magnetization field $\widetilde{\mathbf{m}}_h^n$. Secondly, the intermediate magnetization field was projected to obtain $\mathbf{m}_h^n = \frac{\widetilde{\mathbf{m}}_h^n}{|\widetilde{\mathbf{m}}_h^n|}$ at the next time step. The idea of treating the term $|\nabla\mathbf{m}|^2$ as a Lagrange multiplier was later discussed in the framework of the weak solution for the harmonic map heat flows [7, 9]. In [35], Prohl proposed a nonlinear Euler projection scheme by using a more general projection $\mathbf{m}_h^n = \frac{\widetilde{\mathbf{m}}_h^n}{|\widetilde{\mathbf{m}}_h^n|^{2-\gamma}}$, which remains the non-convex constraint only when $\gamma = 1$. Moreover, the sub-optimal error estimate in L^2-norm for 2D problem was derived. Recently, a second-order partially-updated projection finite difference scheme was studied by Chen, Wang and Xie in [13], where the following un-updated BDF approximation

$$\mathbf{m}_t^{n+1} \approx \frac{1}{\tau}\left(\widetilde{\mathbf{m}}_h^{n+1} - \frac{3}{2}\widetilde{\mathbf{m}}_h^{n} + \frac{1}{2}\widetilde{\mathbf{m}}_h^{n-1}\right)$$

was used in the time direction. If we consider the updated approximation

$$\mathbf{m}_t^{n+1} \approx \frac{1}{\tau}\left(\widetilde{\mathbf{m}}_h^{n+1} - \mathbf{m}_h^{n}\right),$$

the fully-updated projection Euler and Crank-Nicolson finite difference schemes were studied in [5], where the optimal convergence rates were derived. A modified BDF2 scheme in [13] was proposed for the micromagnetics simulations in [39]. A linear Euler finite element scheme with the fully-updated sphere projection was investigated in [6] and the optimal convergence rate in L^2-norm was derived by using the r-th order element to approximate the magnetization with $r \geq 2$. There have other numerical schemes of preserving the unit length of magnetization in the point-wise sense, such as Gauss-Seidel Projection method [12, 38], the mid-point scheme [18, 33, 36] and the mimetic finite difference method [27]. For a review of numerical methods, we refer to Cimrák [17], Kruzík and Prohl [28] and Prohl [35] and other references cited therein.

In this paper, we study a partially-updated projection scheme for the LL equation (1.3) based upon the second-order Crank-Nicolson scheme. The nonlinear term $\mathbf{m} \times \Delta\mathbf{m}$ is treated by the second-order extrapolation technique [37]. For the nonlinear term $|\nabla\mathbf{m}|^2\mathbf{m}$, we use the implicit-explicit technique. Thus, the proposed projection scheme is a linearized scheme and leads to a linear system at each time step. For the spatial discretization, the piecewise linear finite element is used to approximate the magnetization field $\mathbf{m}$. It is proved that the proposed partially-updated projection scheme is of the optimal second-order convergence rate in time and space under the time step restriction $\tau \leq Ch$.

The rest of the paper is organized as follows. In Section 2, we present the Crank-Nicolson partially-updated projection scheme for the approximation of the LL equation (1.3) and the main result on the optimal $\mathbf{L}^2$ error estimates. Its proof is given in Section 3 by using the method of mathematical induction. In Section 4, numerical results are provided to confirm our theoretical analysis.

2. Crank-Nicolson partially-updated projection scheme

For $k \in \mathbb{N}^+$, $1 \leq p \leq +\infty$, let $W^{k,p}(\Omega)$ denote the standard Sobolev space. For $p = 2$, we use $H^k(\Omega)$ to denote $W^{k,2}(\Omega)$. The boldface Sobolev spaces $\mathbf{H}^k(\Omega), \mathbf{W}^{k,p}(\Omega)$ and $\mathbf{L}^p(\Omega)$ are used to denote the vector Sobolev spaces $H^k(\Omega)^3, W^{k,p}(\Omega)^3$ and $L^p(\Omega)^3$, respectively.

We suppose that Ω is a bounded and convex polygon or polyhedron domain. Let $T_h = \{K\}$ be a shape regular quasi-uniform partition of Ω with mesh size h. For a given partition of Ω, we define

$$\mathbf{V}_h = \{\mathbf{w}_h \in \mathcal{C}(\overline{\Omega})^3, \mid \quad \mathbf{w}_h \in P_1(K)^3, \ \forall\, K \in T_h\},$$

where $P_1(K)$ is space of continuous piecewise linear polynomial on $K \in T_h$.

Let $0 = t_0 < t_1 < \cdots < t_N = T$ be a uniform partition of the time interval $[0, T]$ with time step size $\tau = T/N$ and $t_n = n\tau$. For $0 \le n \le N$, let us denote $\mathbf{m}^n(\mathbf{x}) = \mathbf{m}(t_n, \mathbf{x})$ in Ω. For any sequence of functions $\{f^n\}_{n=0}^N$, we define

$$D_\tau f^{n+1} = \frac{f^{n+1} - f^n}{\tau}, \quad \overline{f}^{n+1/2} = \frac{f^{n+1} + f^n}{2}, \quad \hat{f}^{n+1/2} = \frac{1}{2}(3f^n - f^{n-1}).$$

Start with $\widetilde{\mathbf{m}}_h^0 = \mathbf{m}_h^0 = \Pi_h^0 \mathbf{m}_0 \in \mathbf{V}_h$, where Π_h^0 is a projection operator from $\mathbf{H}^1(\Omega)$ to $\mathbf{V}_h$ and defined by

$$\lambda\left(\nabla(\Pi_h^0\mathbf{u} - \mathbf{u}), \nabla\mathbf{v}_h\right) + \lambda\left(\Pi_h^0\mathbf{u} - \mathbf{u}, \mathbf{v}_h\right) + \left(\mathbf{m}_0 \times \nabla(\Pi_h^0\mathbf{u} - \mathbf{u}), \nabla\mathbf{v}_h\right) = 0.$$

We propose the following extrapolation Crank-Nicolson partially-updated projection finite element scheme for the LL equation (1.3).

Step I: For $n = 0, 1, \cdots, N-1$, we find an intermediate magnetization $\widetilde{\mathbf{m}}_h^{n+1} \in \mathbf{V}_h$ to satisfy

$$\begin{aligned}
&\left(\frac{\widetilde{\mathbf{m}}_h^{n+1} - \widetilde{\mathbf{m}}_h^n}{\tau}, \mathbf{v}_h\right) + \lambda\left(\nabla\overline{\widetilde{\mathbf{m}}}_h^{n+1/2}, \nabla\mathbf{v}_h\right) + \left(\hat{\mathbf{m}}_h^{n+1/2} \times \nabla\overline{\widetilde{\mathbf{m}}}_h^{n+1/2}, \nabla\mathbf{v}_h\right) \\
=&\lambda\left(\nabla\hat{\mathbf{m}}_h^{n+1/2} \cdot \nabla\overline{\widetilde{\mathbf{m}}}_h^{n+1/2}, \hat{\mathbf{m}}_h^{n+1/2} \cdot \mathbf{v}_h\right), \qquad \forall\, \mathbf{v}_h \in \mathbf{V}_h,
\end{aligned} \tag{2.1}$$

where $\hat{\mathbf{m}}_h^{1/2} \in \mathbf{V}_h$ is determined by a semi-implicit Euler scheme:

$$\begin{aligned}
&\left(\frac{\hat{\mathbf{m}}_h^{1/2} - \mathbf{m}_0}{\tau/2}, \mathbf{v}_h\right) + \lambda\left(\nabla\hat{\mathbf{m}}_h^{1/2}, \nabla\mathbf{v}_h\right) + \left(\mathbf{m}_0 \times \nabla\hat{\mathbf{m}}_h^{1/2}, \nabla\mathbf{v}_h\right) \\
=&\lambda\left(|\nabla\mathbf{m}_0|^2\mathbf{m}_0, \mathbf{v}_h\right), \qquad \forall\, \mathbf{v}_h \in \mathbf{V}_h.
\end{aligned} \tag{2.2}$$

Step II: We perform the sphere projection by

$$\mathbf{m}_h^{n+1} = \frac{\widetilde{\mathbf{m}}_h^{n+1}}{|\widetilde{\mathbf{m}}_h^{n+1}|}. \tag{2.3}$$

Remark 2.1. In (2.1), for the diffusion term, we use the standard Crank-Nicolson scheme with the well-known coefficient $1/2$ at time node points t^{n+1} and t^n, which yields the second-order temporal convergence rate. We note that there exists other different second-order time-discrete strategy for the long time numerical simulation. For example, in [14, 24], the authors used the Adams-Moulton interpolation to discrete the diffusion term by taking the coefficients $3/4$ and $1/4$ at time node points t^{n+1} and t^{n-1} for $1 \le n \le N$. The proposed schemes in [14,24] are unconditionally stable and the second-order temporal convergence rate was presented. Such discrete technique has been studied for the Cahn-Hilliard equation in [15,19,25] and can be extended to the LL equation (1.3).

It is clear that the proposed Crank-Nicolson projection scheme (2.1)-(2.3) can preserve the unit length of numerical solutions exactly. The emphasis of this paper is to show the optimal temporal-spatial error estimates for (2.1)-(2.3) under the condition $\tau \le \kappa h$ for some $\kappa > 0$. Thus, we assume that the LL problem (1.3) with (1.2) has a unique smooth solution $\mathbf{m}$ satisfying

$$\begin{aligned}
&\|\mathbf{m}_0\|_{H^2} + \|\mathbf{m}\|_{L^\infty(0,T;W^{2,4})} + \|\mathbf{m}_t\|_{L^\infty(0,T;H^3)} \\
&+ \|\mathbf{m}_{tt}\|_{L^\infty(0,T;H^2)} + \|\mathbf{m}_{ttt}\|_{L^2(0,T;L^2)} \le C.
\end{aligned} \tag{2.4}$$

We present our main result in the following theorem.

Theorem 2.1. *Suppose that the solution* $\boldsymbol{m}$ *to (1.3) with (1.2) satisfies the regularity assumption (2.4). Under the condition* $\tau \leq \kappa h$ *for some* $\kappa > 0$*, there exists some small constant* h_0 *such that when* $h < h_0$*, the discrete system (2.1)-(2.3) admits a unique solution* $\boldsymbol{m}_h^{n+1} \in \boldsymbol{V}_h$ *for* $0 \leq n \leq N-1$*. Moreover, we have the following optimal error estimate:*

$$\max_{1\leq n\leq N} \|\boldsymbol{m}^n - \boldsymbol{m}_h^n\|_{L^2} \leq C_0(\tau^2 + h^2) \tag{2.5}$$

for some $C_0 > 0$ *independent of* h*,* τ *and* κ*.*

The proof of Theorem 2.1 will be given in next section. To prove it, we recall some inequalities frequently used in our proof. The first one is the inverse inequality (cf. [11]):

$$\|\mathbf{v}_h\|_{W^{l,q_1}} \leq Ch^{m-l+d\min\{\frac{1}{q_1}-\frac{1}{q_2},0\}}\|\mathbf{v}_h\|_{W^{m,q_2}} \tag{2.6}$$

for all $\mathbf{v}_h \in \mathbf{V}_h, 1 \leq q_1, q_2 \leq \infty$ and $0 \leq m \leq l$.

The second one is the discrete Gronwall's inequality established in [26].

Lemma 2.1. *Let* a_k, b_k*,* γ_k *and* B *be the nonnegative numbers such that*

$$a_n + \tau\sum_{k=0}^{n} b_k \leq \tau\sum_{k=0}^{n}\gamma_k a_k + B, \quad \textit{for } n \geq 0. \tag{2.7}$$

Suppose that $\tau\gamma_k < 1$ *and set* $\sigma_k = (1-\tau\gamma_k)^{-1}$*. Then*

$$a_n + \tau\sum_{k=0}^{n} b_k \leq B\exp\left(\tau\sum_{k=0}^{n}\gamma_k\sigma_k\right). \tag{2.8}$$

Remark 2.2. If the first sum on the right in (2.7) extends only up to $n-1$, then estimate (2.8) holds for all $\tau > 0$ with $\sigma_k = 1$.

3. Error analysis

Suppose that $\mathbf{m}$ is a solution to (1.3) with $|\mathbf{m}| = 1$ and using

$$\mathbf{m} \times \Delta\mathbf{m} = \mathrm{div}\,(\mathbf{m} \times \nabla\mathbf{m}), \tag{3.1}$$

then the weak formulation of (1.3) with (1.2) reads as

$$(\mathbf{m}_t, \mathbf{v}) + \lambda(\nabla\mathbf{m}, \nabla\mathbf{v}) + (\mathbf{m} \times \nabla\mathbf{m}, \nabla\mathbf{v}) = \lambda(|\nabla\mathbf{m}|^2\mathbf{m}, \mathbf{v}) \quad \forall\ \mathbf{v} \in \mathbf{H}^1(\Omega) \tag{3.2}$$

for $0 < t \leq T$ and $\mathbf{m}(0,\cdot) = \mathbf{m}_0$. Furthermore, the exact solution $\mathbf{m}^{n+1}$ satisfies the following discrete parabolic system:

$$\begin{aligned} &D_\tau\mathbf{m}^{n+1} - \lambda\Delta\overline{\mathbf{m}}^{n+1/2} - \hat{\mathbf{m}}^{n+1/2} \times \Delta\overline{\mathbf{m}}^{n+1/2} - \nabla\hat{\mathbf{m}}^{n+1/2} \times \nabla\overline{\mathbf{m}}^{n+1/2} \\ =&\mathbf{R}^{n+1} + \lambda\left(\nabla\hat{\mathbf{m}}^{n+1/2} \cdot \nabla\overline{\mathbf{m}}^{n+1/2}\right)\hat{\mathbf{m}}^{n+1/2} \end{aligned} \tag{3.3}$$

with Neumann boundary condition $\nabla\mathbf{m}^{n+1} \cdot \mathbf{n} = 0$ on $\partial\Omega$ for $0 \leq n \leq N-1$, and $\hat{\mathbf{m}}^{1/2}$ is defined as the solution to the following linearized problem:

$$\frac{\hat{\mathbf{m}}^{1/2} - \mathbf{m}_0}{\tau/2} - \lambda\Delta\hat{\mathbf{m}}^{1/2} - \mathbf{m}_0 \times \Delta\hat{\mathbf{m}}^{1/2} - \nabla\mathbf{m}_0 \times \nabla\hat{\mathbf{m}}^{1/2} = \mathbf{R}^0 + \lambda|\nabla\mathbf{m}_0|^2\mathbf{m}_0, \tag{3.4}$$

with Neumann boundary condition $\nabla\hat{\mathbf{m}}^{1/2}\cdot\mathbf{n}=0$ on $\partial\Omega$, where the truncation functions $\mathbf{R}^{n+1}$ and $\mathbf{R}^0$ are given by

$$\begin{aligned}&\mathbf{R}^{n+1}\\ =&D_\tau\mathbf{m}^{n+1}-\mathbf{m}_t^{n+1/2}-\lambda\Delta(\overline{\mathbf{m}}^{n+1/2}-\mathbf{m}^{n+1/2})-\hat{\mathbf{m}}^{n+1/2}\times\Delta(\overline{\mathbf{m}}^{n+1/2}-\mathbf{m}^{n+1/2})\\ &-(\hat{\mathbf{m}}^{n+1/2}-\mathbf{m}^{n+1/2})\times\Delta\mathbf{m}^{n+1/2}-\nabla\hat{\mathbf{m}}^{n+1/2}\times\nabla(\overline{\mathbf{m}}^{n+1/2}-\mathbf{m}^{n+1/2})\\ &-\nabla(\hat{\mathbf{m}}^{n+1/2}-\mathbf{m}^{n+1/2})\times\nabla\mathbf{m}^{n+1/2}+\lambda|\nabla\mathbf{m}^{n+1/2}|^2\mathbf{m}^{n+1/2}\\ &-\lambda\left(\nabla\hat{\mathbf{m}}^{n+1/2}\cdot\nabla\overline{\mathbf{m}}^{n+1/2}\right)\hat{\mathbf{m}}^{n+1/2},\end{aligned}$$

and

$$\begin{aligned}\mathbf{R}^0=&\frac{\mathbf{m}^{1/2}-\mathbf{m}_0}{\tau/2}-\mathbf{m}_t^{1/2}-(\mathbf{m}_0-\mathbf{m}^{1/2})\times\Delta\mathbf{m}^{1/2}\\ &-\nabla(\mathbf{m}_0-\mathbf{m}^{1/2})\times\nabla\mathbf{m}^{1/2}+\lambda|\nabla\mathbf{m}^{1/2}|^2\mathbf{m}^{1/2}-\lambda|\nabla\mathbf{m}_0|^2\mathbf{m}_0.\end{aligned}$$

By using the regularity (2.4) and the Taylor's formula, we can derive that

$$\tau\sum_{n=0}^{N-1}\|\mathbf{R}^{n+1}\|_{L^2}^2+\tau^2\|\mathbf{R}^0\|_{L^2}^2\le C\tau^4. \tag{3.5}$$

From (3.1), the variational formulations of (3.3) and (3.4) read, respectively, as

$$\begin{aligned}&\left(D_\tau\mathbf{m}^{n+1},\mathbf{v}\right)+\lambda\left(\nabla\overline{\mathbf{m}}^{n+1/2},\nabla\mathbf{v}\right)+\left(\hat{\mathbf{m}}^{n+1/2}\times\nabla\overline{\mathbf{m}}^{n+1/2},\nabla\mathbf{v}\right)\\ =&\left(\mathbf{R}^{n+1},\mathbf{v}\right)+\lambda\left(\nabla\hat{\mathbf{m}}^{n+1/2}\cdot\nabla\overline{\mathbf{m}}^{n+1/2},\hat{\mathbf{m}}^{n+1/2}\cdot\mathbf{v}\right),\quad\forall\,\mathbf{v}\in\mathbf{H}^1(\Omega),\end{aligned}\tag{3.6}$$

and

$$\begin{aligned}&2\left(\frac{\hat{\mathbf{m}}^{1/2}-\mathbf{m}_0}{\tau},\mathbf{v}\right)+\lambda\left(\nabla\hat{\mathbf{m}}^{1/2},\nabla\mathbf{v}\right)+\left(\mathbf{m}_0\times\nabla\hat{\mathbf{m}}^{1/2},\nabla\mathbf{v}\right)\\ =&\left(\mathbf{R}^0,\mathbf{v}\right)+\lambda\left(|\nabla\mathbf{m}_0|^2\,\mathbf{m}_0,\mathbf{v}\right),\quad\forall\,\mathbf{v}\in\mathbf{H}^1(\Omega).\end{aligned}\tag{3.7}$$

For $1\le n\le N$, we define projection operator Π_h^n from $\mathbf{H}^1(\Omega)$ onto $\mathbf{V}_h$ by

$$\lambda\left(\nabla(\Pi_h^n\mathbf{u}-\mathbf{u}),\nabla\mathbf{v}_h\right)+\lambda\left(\Pi_h^n\mathbf{u}-\mathbf{u},\mathbf{v}_h\right)+\left(\hat{\mathbf{m}}^{n-1/2}\times\nabla(\Pi_h^n\mathbf{u}-\mathbf{u}),\nabla\mathbf{v}_h\right)=0$$

for all $\mathbf{v}_h\in\mathbf{V}_h$. Introduce error functions

$$\mathbf{e}^n=\Pi_h^n\mathbf{m}^n-\mathbf{m}_h^n,\quad\widetilde{\mathbf{e}}^n=\Pi_h^n\mathbf{m}^n-\widetilde{\mathbf{m}}_h^n,\quad\mathbf{E}^n=\Pi_h^n\mathbf{m}^n-\mathbf{m}^n,\quad 0\le n\le N,$$

and

$$\hat{\mathbf{e}}^{1/2}=\Pi_h^0\hat{\mathbf{m}}^{1/2}-\hat{\mathbf{m}}_h^{1/2},\quad\hat{\mathbf{E}}^{1/2}=\Pi_h^0\hat{\mathbf{m}}^{1/2}-\hat{\mathbf{m}}^{1/2}.$$

Then $\mathbf{E}^n$ and $\hat{\mathbf{E}}^{1/2}$ satisfy the following approximation properties (cf. [11,21]):

$$\|\mathbf{E}^n\|_{W^{i,4}}+\|\hat{\mathbf{E}}^{1/2}\|_{W^{i,4}}\le Ch^{2-i}\|\mathbf{m}\|_{L^\infty(0,T;W^{2,4})},\quad i=0,1,\tag{3.8}$$

$$\|\Pi_h^n \mathbf{m}^n\|_{W^{1,p}} \le C, \quad 2 \le p \le +\infty \tag{3.9}$$

for $0 \le n \le N$.

Now, we begin to prove Theorem 2.1. Firstly, $\|\hat{\mathbf{e}}^{1/2}\|_{L^2}$ is estimated in the following lemma.

Lemma 3.1. *Suppose that the solution $\boldsymbol{m}$ to (1.3) with (1.2) satisfies the regularity (2.4). Then there exists some $C > 0$ independent of τ and h such that $\hat{\mathbf{e}}^{1/2}$ satisfies*

$$\|\hat{\mathbf{e}}^{1/2}\|_{L^2} \le C\left(\tau^2 + h^2\right). \tag{3.10}$$

Proof. Taking $\mathbf{v} = \mathbf{v}_h \in \mathbf{V}_h$ in (3.7) and subtracting from (2.2) yields

$$\begin{aligned}
&2\left(\hat{\mathbf{e}}^{1/2}, \mathbf{v}_h\right) + \lambda\tau\left(\nabla\hat{\mathbf{e}}^{1/2}, \nabla\mathbf{v}_h\right) + \tau\left(\mathbf{m}_0 \times \nabla\hat{\mathbf{e}}^{1/2}, \nabla\mathbf{v}_h\right) \\
=&2\left(\hat{\mathbf{E}}^{1/2}, \mathbf{v}_h\right) + \lambda\tau\left(\hat{\mathbf{E}}^{1/2}, \mathbf{v}_h\right) + \tau\left(\mathbf{R}^0, \mathbf{v}_h\right),
\end{aligned}$$

where we use the definition of Π_h^0. Setting $\mathbf{v}_h = \hat{\mathbf{e}}^{1/2}$ leads to

$$\begin{aligned}
&2\|\hat{\mathbf{e}}^{1/2}\|_{L^2}^2 + \lambda\tau\|\nabla\hat{\mathbf{e}}^{1/2}\|_{L^2}^2 \\
\le& 2\|\hat{\mathbf{E}}^{1/2}\|_{L^2}\|\hat{\mathbf{e}}^{1/2}\|_{L^2} + \lambda\tau\|\hat{\mathbf{E}}^{1/2}\|_{L^2}\|\hat{\mathbf{e}}^{1/2}\|_{L^2} + \tau\|\mathbf{R}^0\|_{L^2}\|\hat{\mathbf{e}}^{1/2}\|_{L^2} \\
\le& \|\hat{\mathbf{e}}^{1/2}\|_{L^2}^2 + C\|\hat{\mathbf{E}}^{1/2}\|_{L^2}^2 + C\tau^2\|\hat{\mathbf{E}}^{1/2}\|_{L^2}^2 + C\tau^2\|\mathbf{R}^0\|_{L^2}^2 \\
\le& \|\hat{\mathbf{e}}^{1/2}\|_{L^2}^2 + C(\tau^4 + h^4),
\end{aligned}$$

which completes the proof of (3.10). □

On the other hand, taking $\mathbf{v} = \mathbf{v}_h \in \mathbf{V}_h$ in (3.6) and subtracting the resulting equation from (2.1), we obtain error equations for $n = 0$,

$$\begin{aligned}
&\left(D_\tau\widetilde{\mathbf{e}}^1, \mathbf{v}_h\right) + \lambda\left(\nabla\overline{\widetilde{\mathbf{e}}}^{1/2}, \nabla\mathbf{v}_h\right) \\
=&\left(D_\tau\mathbf{E}^1, \mathbf{v}_h\right) - \frac{\lambda}{2}\left(\mathbf{E}^1 + \mathbf{E}^0, \mathbf{v}_h\right) + \frac{1}{2}\left((\hat{\mathbf{m}}^{1/2} - \mathbf{m}_0) \times \nabla\mathbf{E}^0, \nabla\mathbf{v}_h\right) \\
&+ \left((\hat{\mathbf{m}}^{1/2} - \hat{\mathbf{m}}_h^{1/2}) \times \nabla\overline{\widetilde{\mathbf{e}}}^{1/2}, \nabla\mathbf{v}_h\right) - \left(\hat{\mathbf{m}}^{1/2} \times \nabla\overline{\widetilde{\mathbf{e}}}^{1/2}, \nabla\mathbf{v}_h\right) \\
&- \left((\hat{\mathbf{m}}^{1/2} - \hat{\mathbf{m}}_h^{1/2}) \times \nabla\overline{\mathbf{E}}^{1/2}, \nabla\mathbf{v}_h\right) - \left((\hat{\mathbf{m}}^{1/2} - \hat{\mathbf{m}}_h^{1/2}) \times \nabla\overline{\mathbf{m}}^{1/2}, \nabla\mathbf{v}_h\right) \\
&+ \left(\mathbf{R}^1, \mathbf{v}_h\right) + \lambda\left(\left(\nabla\hat{\mathbf{m}}^{1/2} \cdot \nabla\overline{\mathbf{m}}^{1/2}\right)\hat{\mathbf{m}}^{1/2} - \left(\nabla\hat{\mathbf{m}}_h^{1/2} \cdot \nabla\overline{\widetilde{\mathbf{m}}}_h^{1/2}\right)\hat{\mathbf{m}}_h^{1/2}, \mathbf{v}_h\right),
\end{aligned} \tag{3.11}$$

and

$$\begin{aligned}
&\left(D_\tau\widetilde{\mathbf{e}}^{n+1}, \mathbf{v}_h\right) + \lambda\left(\nabla\overline{\widetilde{\mathbf{e}}}^{n+1/2}, \nabla\mathbf{v}_h\right) \\
=&\left(D_\tau\mathbf{E}^{n+1}, \mathbf{v}_h\right) - \lambda\left(\overline{\mathbf{E}}^{n+1/2}, \mathbf{v}_h\right) + \frac{1}{2}\left((\hat{\mathbf{m}}^{n+1/2} - \hat{\mathbf{m}}^{n-1/2}) \times \nabla\mathbf{E}^n, \nabla\mathbf{v}_h\right) \\
&- \left(\hat{\mathbf{e}}^{n+1/2} \times \nabla\overline{\mathbf{E}}^{n+1/2}, \nabla\mathbf{v}_h\right) + \left(\hat{\mathbf{e}}^{n+1/2} \times \nabla\overline{\widetilde{\mathbf{e}}}^{n+1/2}, \nabla\mathbf{v}_h\right) \\
&+ \left(\hat{\mathbf{E}}^{n+1/2} \times \nabla\overline{\mathbf{E}}^{n+1/2}, \nabla\mathbf{v}_h\right) - \left(\hat{\mathbf{E}}^{n+1/2} \times \nabla\overline{\widetilde{\mathbf{e}}}^{n+1/2}, \nabla\mathbf{v}_h\right)
\end{aligned}$$

$$
\begin{aligned}
&-\left(\hat{\mathbf{m}}^{n+1/2}\times\nabla\overline{\widetilde{\mathbf{e}}}^{n+1/2},\nabla\mathbf{v}_h\right)-\left(\hat{\mathbf{e}}^{n+1/2}\times\nabla\overline{\mathbf{m}}^{n+1/2},\nabla\mathbf{v}_h\right)\\
&+\left(\hat{\mathbf{E}}^{n+1/2}\times\nabla\overline{\mathbf{m}}^{n+1/2},\nabla\mathbf{v}_h\right)+\left(\mathbf{R}^{n+1},\mathbf{v}_h\right)\\
&+\lambda\left(\left(\nabla\hat{\mathbf{m}}^{n+1/2}\cdot\nabla\overline{\mathbf{m}}^{n+1/2}\right)\hat{\mathbf{m}}^{n+1/2}-\left(\nabla\hat{\mathbf{m}}_h^{n+1/2}\cdot\nabla\overline{\widetilde{\mathbf{m}}}_h^{n+1/2}\right)\hat{\mathbf{m}}_h^{n+1/2},\mathbf{v}_h\right)
\end{aligned}\tag{3.12}
$$

for $1\le n\le N-1$. In (3.11) and (3.12), we use

$$
\begin{aligned}
&\lambda\left(\nabla\overline{\mathbf{E}}^{1/2},\nabla\mathbf{v}_h\right)+\left(\hat{\mathbf{m}}^{1/2}\times\nabla\overline{\mathbf{E}}^{1/2},\nabla\mathbf{v}_h\right)\\
=&\frac{1}{2}\left((\hat{\mathbf{m}}^{1/2}-\mathbf{m}_0)\times\nabla\mathbf{E}^0,\nabla\mathbf{v}_h\right)-\frac{\lambda}{2}\left(\mathbf{E}^1+\mathbf{E}^0,\mathbf{v}_h\right)
\end{aligned}
$$

and

$$
\begin{aligned}
&\lambda\left(\nabla\overline{\mathbf{E}}^{n+1/2},\nabla\mathbf{v}_h\right)+\left(\hat{\mathbf{m}}^{n+1/2}\times\nabla\overline{\mathbf{E}}^{n+1/2},\nabla\mathbf{v}_h\right)\\
=&\frac{1}{2}\left((\hat{\mathbf{m}}^{n+1/2}-\hat{\mathbf{m}}^{n-1/2})\times\nabla\mathbf{E}^n,\nabla\mathbf{v}_h\right)-\lambda\left(\overline{\mathbf{E}}^{n+1/2},\mathbf{v}_h\right)
\end{aligned}
$$

thanks to the definitions of Π_h^0 and Π_h^n, respectively.

Lemma 3.2. *Suppose that the solution $\boldsymbol{m}$ to (1.3) with (1.2) satisfies the regularity assumptions (2.4). If $\tau\le\kappa h$, then there exists some $h_1>0$ such that when $h<h_1$, there holds*

$$
\|\widetilde{\mathbf{e}}^1\|_{L^2}=\|\Pi_h^1\boldsymbol{m}^1-\widetilde{\boldsymbol{m}}_h^1\|_{L^2}\le C_1(\tau^2+h^2),\tag{3.13}
$$

where $C_1>0$ is independent of h, τ and κ.

Proof. It follows from (3.8), (3.10) and the inverse inequality that

$$
\|\hat{\mathbf{m}}^{1/2}-\hat{\mathbf{m}}_h^{1/2}\|_{L^2}\le C(1+\kappa^2)h^2,
$$

and

$$
\|\hat{\mathbf{m}}^{1/2}-\hat{\mathbf{m}}_h^{1/2}\|_{H^1}\le\|\hat{\mathbf{m}}^{1/2}-\Pi_h^0\hat{\mathbf{m}}^{1/2}\|_{H^1}+Ch^{-1}\|\hat{\mathbf{e}}^{1/2}\|_{L^2}\le C(1+\kappa^2)h,
$$

under the condition $\tau\le\kappa h$. Due to $\widetilde{\mathbf{e}}^0=0$ and $\overline{\widetilde{\mathbf{e}}}^{1/2}=\frac{1}{2}\widetilde{\mathbf{e}}^1$, we set $\mathbf{v}_h=2\tau\widetilde{\mathbf{e}}^1$ in (3.11) to yield

$$
\begin{aligned}
&2\|\widetilde{\mathbf{e}}^1\|_{L^2}^2+\lambda\tau\|\nabla\widetilde{\mathbf{e}}^1\|_{L^2}^2\\
=&2\left(\mathbf{E}^1-\mathbf{E}^0,\widetilde{\mathbf{e}}^1\right)+2\tau\left(\mathbf{R}^1,\widetilde{\mathbf{e}}^1\right)-\lambda\tau\left(\mathbf{E}^1+\mathbf{E}^0,\widetilde{\mathbf{e}}^1\right)+\tau\left((\hat{\mathbf{m}}^{1/2}-\mathbf{m}_0)\times\nabla\mathbf{E}^0,\nabla\widetilde{\mathbf{e}}^1\right)\\
&-2\tau\left((\hat{\mathbf{m}}^{1/2}-\hat{\mathbf{m}}_h^{1/2})\times\nabla\overline{\mathbf{E}}^{1/2},\nabla\widetilde{\mathbf{e}}^1\right)-2\tau\left((\hat{\mathbf{m}}^{1/2}-\hat{\mathbf{m}}_h^{1/2})\times\nabla\overline{\mathbf{m}}^{1/2},\nabla\widetilde{\mathbf{e}}^1\right)\\
&+2\lambda\tau\left(\left(\nabla\hat{\mathbf{m}}^{1/2}\cdot\nabla\overline{\mathbf{m}}^{1/2}\right)\hat{\mathbf{m}}^{1/2}-\left(\nabla\hat{\mathbf{m}}_h^{1/2}\cdot\nabla\overline{\widetilde{\mathbf{m}}}_h^{1/2}\right)\hat{\mathbf{m}}_h^{1/2},\widetilde{\mathbf{e}}^1\right).
\end{aligned}\tag{3.14}
$$

From the Hölder inequality, the Young inequality, the Sobolev imbedding inequality and (2.4), the right-hand side of (3.14) can be bounded by

$$\begin{aligned}
&2\|\widetilde{\mathbf{e}}^1\|_{L^2}^2 + \lambda\tau\|\nabla\widetilde{\mathbf{e}}^1\|_{L^2}^2 \\
\leq&\frac{1}{2}\|\widetilde{\mathbf{e}}^1\|_{L^2}^2 + \frac{\lambda\tau}{4}\|\widetilde{\mathbf{e}}^1\|_{H^1}^2 + C\left(\|\mathbf{E}^0\|_{L^2}^2 + \|\mathbf{E}^1\|_{L^2}^2 + \tau^2\|\mathbf{R}^1\|_{L^2}^2\right) \\
&+ C\tau^3\|\nabla\mathbf{E}^0\|_{L^2}^2 + C\tau\|\hat{\mathbf{m}}^{1/2} - \hat{\mathbf{m}}_h^{1/2}\|_{H^1}^2\|\nabla\overline{\mathbf{E}}^{1/2}\|_{L^3}^2 + C\tau\|\hat{\mathbf{m}}^{1/2} - \hat{\mathbf{m}}_h^{1/2}\|_{L^2}^2 \\
&+ 2\lambda\tau\left(\left(\nabla\hat{\mathbf{m}}^{1/2}\cdot\nabla\overline{\mathbf{m}}^{1/2}\right)\hat{\mathbf{m}}^{1/2} - \left(\nabla\hat{\mathbf{m}}_h^{1/2}\cdot\nabla\overline{\widetilde{\mathbf{m}}}_h^{1/2}\right)\hat{\mathbf{m}}_h^{1/2}, \widetilde{\mathbf{e}}^1\right) \\
\leq&\frac{1}{2}\|\widetilde{\mathbf{e}}^1\|_{L^2}^2 + \frac{\lambda\tau}{4}\|\widetilde{\mathbf{e}}^1\|_{H^1}^2 + C\left(\tau^4 + h^4 + \tau^3 h^2 + \kappa(1+\kappa^2)^2 h^5\right) \\
&+ 2\lambda\tau\left(\left(\nabla\hat{\mathbf{m}}^{1/2}\cdot\nabla\overline{\mathbf{m}}^{1/2}\right)\hat{\mathbf{m}}^{1/2} - \left(\nabla\hat{\mathbf{m}}_h^{1/2}\cdot\nabla\overline{\widetilde{\mathbf{m}}}_h^{1/2}\right)\hat{\mathbf{m}}_h^{1/2}, \widetilde{\mathbf{e}}^1\right) \\
\leq&\frac{1}{2}\|\widetilde{\mathbf{e}}^1\|_{L^2}^2 + \frac{\lambda\tau}{4}\|\widetilde{\mathbf{e}}^1\|_{H^1}^2 + C\left(\tau^4 + h^4\right) \\
&+ 2\lambda\tau\left(\left(\nabla\hat{\mathbf{m}}^{1/2}\cdot\nabla\overline{\mathbf{m}}^{1/2}\right)\hat{\mathbf{m}}^{1/2} - \left(\nabla\hat{\mathbf{m}}_h^{1/2}\cdot\nabla\overline{\widetilde{\mathbf{m}}}_h^{1/2}\right)\hat{\mathbf{m}}_h^{1/2}, \widetilde{\mathbf{e}}^1\right)
\end{aligned} \tag{3.15}$$

for $h_1 \leq \frac{1}{\kappa(1+\kappa^2)^2}$. Rewrite $\left(\nabla\hat{\mathbf{m}}^{n+1/2}\cdot\nabla\overline{\mathbf{m}}^{n+1/2}\right)\hat{\mathbf{m}}^{n+1/2} - \left(\nabla\hat{\mathbf{m}}_h^{n+1/2}\cdot\nabla\overline{\widetilde{\mathbf{m}}}_h^{n+1/2}\right)\hat{\mathbf{m}}_h^{n+1/2}$ as

$$\begin{aligned}
&\left(\nabla\hat{\mathbf{m}}^{n+1/2}\cdot\nabla\overline{\mathbf{m}}^{n+1/2}\right)\hat{\mathbf{m}}^{n+1/2} - \left(\nabla\hat{\mathbf{m}}_h^{n+1/2}\cdot\nabla\overline{\widetilde{\mathbf{m}}}_h^{n+1/2}\right)\hat{\mathbf{m}}_h^{n+1/2} \\
=&\left(\nabla\hat{\mathbf{m}}^{n+1/2}\cdot\nabla\overline{\mathbf{m}}^{n+1/2}\right)\left(\hat{\mathbf{e}}^{n+1/2} - \hat{\mathbf{E}}^{n+1/2}\right) \\
&+ \nabla(\hat{\mathbf{e}}^{n+1/2} - \hat{\mathbf{E}}^{n+1/2})\cdot\nabla\overline{\widetilde{\mathbf{e}}}^{n+1/2}\left(\hat{\mathbf{e}}^{n+1/2} - \hat{\mathbf{E}}^{n+1/2}\right) \\
&- \nabla(\hat{\mathbf{e}}^{n+1/2} - \hat{\mathbf{E}}^{n+1/2})\cdot\nabla\overline{\mathbf{E}}^{n+1/2}\left(\hat{\mathbf{e}}^{n+1/2} - \hat{\mathbf{E}}^{n+1/2}\right) \\
&- \nabla(\hat{\mathbf{e}}^{n+1/2} - \hat{\mathbf{E}}^{n+1/2})\cdot\nabla\overline{\widetilde{\mathbf{e}}}^{n+1/2}\hat{\mathbf{m}}^{n+1/2} \\
&+ \nabla(\hat{\mathbf{e}}^{n+1/2} - \hat{\mathbf{E}}^{n+1/2})\cdot\nabla\overline{\mathbf{E}}^{n+1/2}\hat{\mathbf{m}}^{n+1/2} \\
&- 2\left(\nabla\hat{\mathbf{m}}^{n+1/2}\cdot\nabla(\overline{\widetilde{\mathbf{e}}}^{n+1/2} - \overline{\mathbf{E}}^{n+1/2})\right)\hat{\mathbf{m}}^{n+1/2} \\
&+ 2\left(\nabla\hat{\mathbf{m}}^{n+1/2}\cdot\nabla(\overline{\widetilde{\mathbf{e}}}^{n+1/2} - \overline{\mathbf{E}}^{n+1/2})\right)\left(\hat{\mathbf{e}}^{n+1/2} - \hat{\mathbf{E}}^{n+1/2}\right) = \sum_{i=1}^{7} I_i.
\end{aligned} \tag{3.16}$$

Taking $n = 0$ in (3.16), by the Hölder inequality, the Young inequality and the inverse inequality (2.6), we can estimate the last term in the right-hand side of (3.15) by

$$\begin{aligned}
2\lambda\tau(I_1, \widetilde{\mathbf{e}}^1) &\leq C\lambda\tau\left(\|\hat{\mathbf{e}}^{1/2}\|_{L^2} + \|\hat{\mathbf{E}}^{1/2}\|_{L^2}\right)\|\widetilde{\mathbf{e}}^1\|_{L^2} \\
&\leq \frac{1}{10}\|\widetilde{\mathbf{e}}^1\|_{L^2}^2 + C\tau^2(\tau^4 + h^4),
\end{aligned}$$

and

$$\begin{aligned}
&2\lambda\tau(I_2, \widetilde{\mathbf{e}}^1) \\
\leq&\, 2\lambda\tau\left(\|\hat{\mathbf{e}}^{1/2}\|_{L^\infty} + \|\hat{\mathbf{E}}^{1/2}\|_{L^\infty}\right)\left(\|\nabla\hat{\mathbf{e}}^{1/2}\|_{L^3} + \|\nabla\hat{\mathbf{E}}^{1/2}\|_{L^3}\right)\|\widetilde{\mathbf{e}}^1\|_{H^1}\|\widetilde{\mathbf{e}}^1\|_{L^6}
\end{aligned}$$

$$\leq C\lambda\tau\left((1+\kappa^2)h^{2-d/2}+h\right)\left((1+\kappa^2)h^{1-d/6}+h\right)\|\widetilde{\mathbf{e}}^1\|_{H^1}^2 \leq \frac{\lambda\tau}{16}\|\widetilde{\mathbf{e}}^1\|_{H^1}^2,$$

for some h_1 with $C\left((1+\kappa^2)h_1^{2-d/2}+h_1\right)\left((1+\kappa^2)h_1^{1-d/6}+h_1\right) \leq \frac{1}{16}$, and

$$\begin{aligned}
&2\lambda\tau(I_3,\widetilde{\mathbf{e}}^1)\\
&\leq 2\lambda\tau\left(\|\hat{\mathbf{e}}^{1/2}\|_{L^2}+\|\hat{\mathbf{E}}^{1/2}\|_{L^2}\right)\left(\|\nabla\hat{\mathbf{e}}^{1/2}\|_{L^3}+\|\nabla\hat{\mathbf{E}}^{1/2}\|_{L^3}\right)\|\nabla\overline{\mathbf{E}}^{1/2}\|_{L^\infty}\|\widetilde{\mathbf{e}}^1\|_{L^6}\\
&\leq C\lambda\tau\left((1+\kappa^2)h^{1-d/6}+h\right)\left(\|\hat{\mathbf{e}}^{1/2}\|_{L^2}+\|\hat{\mathbf{E}}^{1/2}\|_{L^2}\right)\|\widetilde{\mathbf{e}}^1\|_{H^1}\\
&\leq \frac{\lambda\tau}{16}\|\widetilde{\mathbf{e}}^1\|_{H^1}^2+C\tau\left(\|\hat{\mathbf{e}}^{1/2}\|_{L^2}^2+\|\hat{\mathbf{E}}^{1/2}\|_{L^2}^2\right) \leq \frac{\lambda\tau}{16}\|\widetilde{\mathbf{e}}^1\|_{H^1}^2+C\tau\left(\tau^4+h^4\right),
\end{aligned}$$

for some h_1 with $\left((1+\kappa^2)h_1^{1-d/6}+h_1\right) \leq 1$, and

$$\begin{aligned}
2\lambda\tau(I_4,\widetilde{\mathbf{e}}^1) &\leq C\lambda\tau\left(\|\nabla\hat{\mathbf{e}}^{1/2}\|_{L^3}+\|\nabla\hat{\mathbf{E}}^{1/2}\|_{L^3}\right)\|\widetilde{\mathbf{e}}^1\|_{H^1}^2\\
&\leq C\lambda\tau\left((1+\kappa)h^{1-d/6}+h\right)\|\widetilde{\mathbf{e}}^1\|_{H^1}^2 \leq \frac{\lambda\tau}{16}\|\widetilde{\mathbf{e}}^1\|_{H^1}^2,
\end{aligned}$$

for some h_1 with $C\left((1+\kappa^2)h_1^{1-d/6}+h_1\right) \leq \frac{1}{16}$, and, by integration by parts,

$$\begin{aligned}
2\lambda\tau(I_5,\widetilde{\mathbf{e}}^1) \leq& C\lambda\tau\|\overline{\mathbf{E}}^{1/2}\|_{W^{2,4}}\|\hat{\mathbf{m}}^{1/2}\|_{L^\infty}\left(\|\hat{\mathbf{e}}^{1/2}\|_{L^2}+\|\hat{\mathbf{E}}^{1/2}\|_{L^2}\right)\|\widetilde{\mathbf{e}}^1\|_{L^6}\\
&+C\lambda\tau\|\nabla\overline{\mathbf{E}}^{1/2}\|_{L^3}\|\nabla\hat{\mathbf{m}}^{1/2}\|_{L^\infty}\left(\|\hat{\mathbf{e}}^{1/2}\|_{L^2}+\|\hat{\mathbf{E}}^{1/2}\|_{L^2}\right)\|\widetilde{\mathbf{e}}^1\|_{L^6}\\
&+C\lambda\tau\|\nabla\overline{\mathbf{E}}^{1/2}\|_{L^3}\|\hat{\mathbf{m}}^{1/2}\|_{L^\infty}\left(\|\hat{\mathbf{e}}^{1/2}\|_{L^2}+\|\hat{\mathbf{E}}^{1/2}\|_{L^2}\right)\|\nabla\widetilde{\mathbf{e}}^1\|_{L^2}\\
\leq& C\tau\left(\|\hat{\mathbf{e}}^{1/2}\|_{L^2}+\|\hat{\mathbf{E}}^{1/2}\|_{L^2}\right)\|\widetilde{\mathbf{e}}^1\|_{H^1}\\
\leq& \frac{\lambda\tau}{16}\|\widetilde{\mathbf{e}}^1\|_{H^1}^2+C\tau(\tau^4+h^4),
\end{aligned}$$

and

$$\begin{aligned}
2\lambda\tau(I_6,\widetilde{\mathbf{e}}^1) \leq& C\tau\left(\|\widetilde{\mathbf{e}}^1\|_{L^2}+\|\overline{\mathbf{E}}^{1/2}\|_{L^2}\right)\|\widetilde{\mathbf{e}}^1\|_{H^1}\\
\leq& \frac{\lambda\tau}{16}\|\widetilde{\mathbf{e}}^1\|_{H^1}^2+C\tau\left(\|\widetilde{\mathbf{e}}^1\|_{L^2}^2+\|\overline{\mathbf{E}}^{1/2}\|_{L^2}^2\right).
\end{aligned}$$

For last term, we have

$$\begin{aligned}
2\lambda\tau(I_7,\widetilde{\mathbf{e}}^1) \leq& C\tau\left(\|\hat{\mathbf{e}}^{1/2}\|_{L^3}+\|\hat{\mathbf{E}}^{1/2}\|_{L^3}\right)\|\widetilde{\mathbf{e}}^1\|_{H^1}^2\\
&+C\tau\left(\|\hat{\mathbf{e}}^{1/2}\|_{L^2}+\|\hat{\mathbf{E}}^{1/2}\|_{L^2}\right)\|\nabla\overline{\mathbf{E}}^{1/2}\|_{L^3}\|\widetilde{\mathbf{e}}^1\|_{H^1}\\
\leq& \frac{\lambda\tau}{16}\|\widetilde{\mathbf{e}}^1\|_{H^1}^2+C\tau\left((1+\kappa^2)h^{2-d/6}+h^2\right)\|\widetilde{\mathbf{e}}^1\|_{H^1}^2\\
&+C\tau\left(\|\hat{\mathbf{e}}^{1/2}\|_{L^2}^2+\|\hat{\mathbf{E}}^{1/2}\|_{L^2}^2\right)\|\nabla\overline{\mathbf{E}}^{1/2}\|_{L^3}^2\\
\leq& \frac{\lambda\tau}{16}\|\widetilde{\mathbf{e}}^1\|_{H^1}^2+C\tau\left(\|\hat{\mathbf{e}}^{1/2}\|_{L^2}^2+\|\hat{\mathbf{E}}^{1/2}\|_{L^2}^2\right)\|\nabla\overline{\mathbf{E}}^{1/2}\|_{L^3}^2
\end{aligned}$$

$$\leq \frac{\lambda\tau}{16}\|\widetilde{\mathbf{e}}^1\|_{H^1}^2 + Ch\tau\left(\tau^4 + h^4\right),$$

for some h_1 with $C\left((1+\kappa^2)h_1^{2-d/6} + h_1^2\right) \leq \frac{\lambda}{16}$. Substituting these estimates into (3.15), we obtain

$$\frac{3}{2}\|\widetilde{\mathbf{e}}^1\|_{L^2}^2 + \frac{\lambda\tau}{4}\|\widetilde{\mathbf{e}}^1\|_{H^1}^2 \leq C\tau\|\widetilde{\mathbf{e}}^1\|_{L^2}^2 + C(\tau^4 + h^4) \leq \frac{1}{2}\|\widetilde{\mathbf{e}}^1\|_{L^2}^2 + C(\tau^4 + h^4)$$

for some h_1 with $C\tau \leq C\kappa h_1 \leq \frac{1}{2}$. Thus, there exists some small $h_1 > 0$ such that when $h < h_1$, the desired estimate (3.13) holds. □

Lemma 3.3. *Suppose that the solution $\boldsymbol{m}$ to (1.3) with (1.2) satisfies the regularity assumptions (2.4). If $\tau \leq \kappa h$, then there exist some $h_2 > 0$ such that when $h < h_2$, we have*

$$\max_{1\leq k\leq N}\|\widetilde{\boldsymbol{e}}^k\|_{L^2} = \max_{1\leq k\leq N}\|\Pi_h^n \boldsymbol{m}^k - \widetilde{\boldsymbol{m}}_h^k\|_{L^2} \leq C_2(\tau^2 + h^2), \tag{3.17}$$

where $C_2 > 0$ is independent of h, τ and κ.

Proof. We will use the method of mathematical induction to prove this lemma. From Lemma 3.2, the estimate (3.17) is valid for $k = 1$ and $C_2 > C_1$. Now, we assume that (3.17) is valid for $k \leq n$ with $n \leq N-1$, i.e.,

$$\|\widetilde{\mathbf{e}}^n\|_{L^2} \leq C_2(\tau^2 + h^2) \leq C_2(1+\kappa^2)h^2. \tag{3.18}$$

To close the mathematical induction, we need to show that (3.17) is valid for $k \leq n+1$.

By the definition of $\mathbf{m}_h^n$, it is easy to check that

$$|\mathbf{m}_h^n - \widetilde{\mathbf{m}}_h^n| \leq |\mathbf{m}^n - \widetilde{\mathbf{m}}_h^n| \quad \text{in the point-wise sense.} \tag{3.19}$$

Furthermore, under the assumption (3.18), we can prove that there exists some h_2 such that when $h < h_2$, there holds (cf. [4,6]):

$$\|\mathbf{m}_h^n - \widetilde{\mathbf{m}}_h^n\|_{W^{1,i}} \leq C\|\mathbf{m}^n - \widetilde{\mathbf{m}}_h^n\|_{W^{1,i}} \quad \text{for } 1 \leq n \leq N-1 \text{ and } i = 2,3, \tag{3.20}$$

which is a key result in our proof since the inverse inequality is not valid for $\mathbf{m}_h^n$ due to $\mathbf{m}_h^n \notin \mathbf{V}_h$.

For $1 \leq n \leq N-1$, by (3.19), (3.20) and inverse inequality, one has

$$\begin{aligned}
&\|\mathbf{e}^n\|_{L^i} \leq 2\|\widetilde{\mathbf{e}}^n\|_{L^2} + \|\mathbf{E}^n\|_{L^i}, \quad i = 2,3,\\
&\|\mathbf{e}^n\|_{H^1} \leq Ch^{-1}\|\widetilde{\mathbf{e}}^n\|_{L^2} + \|\mathbf{E}^n\|_{H^1},\\
&\|\mathbf{e}^n\|_{W^{1,3}} \leq Ch^{-1-d/6}\|\widetilde{\mathbf{e}}^n\|_{L^2} + \|\mathbf{E}^n\|_{W^{1,3}}.
\end{aligned}$$

Then, from the definition of $\hat{\mathbf{e}}^{n+1/2}$, we have

$$\|\hat{\mathbf{e}}^{n+1/2}\|_{L^2} \leq C\left(\|\widetilde{\mathbf{e}}^n\|_{L^2} + \|\widetilde{\mathbf{e}}^{n-1}\|_{L^2}\right) + Ch^2, \tag{3.21}$$

$$\|\hat{\mathbf{e}}^{n+1/2}\|_{H^1} \leq Ch^{-1}\left(\|\widetilde{\mathbf{e}}^n\|_{L^2} + \|\widetilde{\mathbf{e}}^{n-1}\|_{L^2}\right) + Ch, \tag{3.22}$$

$$\|\hat{\mathbf{e}}^{n+1/2}\|_{W^{1,3}} \leq Ch^{-1-d/6}\left(\|\widetilde{\mathbf{e}}^n\|_{L^2} + \|\widetilde{\mathbf{e}}^{n-1}\|_{L^2}\right) + Ch. \tag{3.23}$$

In (3.12), we take $\mathbf{v}_h = \tau\overline{\widetilde{\mathbf{e}}}^{n+1/2}$ to yield

$$\begin{aligned}
&\frac{1}{2}\|\widetilde{\mathbf{e}}^{n+1}\|_{L^2}^2 - \frac{1}{2}\|\widetilde{\mathbf{e}}^{n}\|_{L^2}^2 + \lambda\tau\|\nabla\overline{\widetilde{\mathbf{e}}}^{n+1/2}\|_{L^2}^2 \\
&= \tau(D_\tau \mathbf{E}^{n+1}, \overline{\widetilde{\mathbf{e}}}^{n+1/2}) - \lambda\tau(\overline{\mathbf{E}}^{n+1/2}, \overline{\widetilde{\mathbf{e}}}^{n+1/2}) + \tau(\mathbf{R}^{n+1}, \overline{\widetilde{\mathbf{e}}}^{n+1/2}) \\
&\quad + \frac{\tau}{2}\left((\hat{\mathbf{m}}^{n+1/2} - \hat{\mathbf{m}}^{n-1/2}) \times \nabla\mathbf{E}^n, \nabla\overline{\widetilde{\mathbf{e}}}^{n+1/2}\right) \\
&\quad - \tau(\hat{\mathbf{e}}^{n+1/2} \times \nabla\overline{\mathbf{E}}^{n+1/2}, \nabla\overline{\widetilde{\mathbf{e}}}^{n+1/2}) + \tau(\hat{\mathbf{E}}^{n+1/2} \times \nabla\overline{\mathbf{E}}^{n+1/2}, \nabla\overline{\widetilde{\mathbf{e}}}^{n+1/2}) \\
&\quad - \tau(\hat{\mathbf{e}}^{n+1/2} \times \nabla\overline{\mathbf{m}}^{n+1/2}, \nabla\overline{\widetilde{\mathbf{e}}}^{n+1/2}) + \tau(\hat{\mathbf{E}}^{n+1/2} \times \nabla\overline{\mathbf{m}}^{n+1/2}, \nabla\overline{\widetilde{\mathbf{e}}}^{n+1/2}) \\
&\quad + \lambda\tau\left(\left(\nabla\hat{\mathbf{m}}^{n+1/2}\cdot\nabla\overline{\mathbf{m}}^{n+1/2}\right)\hat{\mathbf{m}}^{n+1/2} - \left(\nabla\hat{\mathbf{m}}_h^{n+1/2}\cdot\nabla\overline{\widetilde{\mathbf{m}}}_h^{n+1/2}\right)\hat{\mathbf{m}}_h^{n+1/2}, \overline{\widetilde{\mathbf{e}}}^{n+1/2}\right) \\
&:= RHS.
\end{aligned} \tag{3.24}$$

From the Hölder inequality and (2.4), the right-hand side of (3.24) can be bounded by

$$\begin{aligned}
RHS \le & C\tau\left(\|D_\tau\mathbf{E}^{n+1}\|_{L^2} + \|\overline{\mathbf{E}}^{n+1/2}\|_{L^2} + \|\mathbf{R}^{n+1}\|_{L^2}\right)\|\overline{\widetilde{\mathbf{e}}}^{n+1/2}\|_{L^2} \\
& + C\tau^2\|\mathbf{m}_t\|_{L^\infty}\|\nabla\mathbf{E}^n\|_{L^2}\|\nabla\overline{\widetilde{\mathbf{e}}}^{n+1/2}\|_{L^2} \\
& + C\tau\left(\|\hat{\mathbf{e}}^{n+1/2}\|_{L^4} + \|\hat{\mathbf{E}}^{n+1/2}\|_{L^4}\right)\|\nabla\overline{\mathbf{E}}^{n+1/2}\|_{L^4}\|\nabla\overline{\widetilde{\mathbf{e}}}^{n+1/2}\|_{L^2} \\
& + C\tau\left(\|\hat{\mathbf{e}}^{n+1/2}\|_{L^2} + \|\hat{\mathbf{E}}^{n+1/2}\|_{L^2}\right)\|\nabla\overline{\mathbf{m}}^{n+1/2}\|_{L^\infty}\|\nabla\overline{\widetilde{\mathbf{e}}}^{n+1/2}\|_{L^2} \\
& + \lambda\tau\Bigg(\left(\nabla\hat{\mathbf{m}}^{n+1/2}\cdot\nabla\overline{\mathbf{m}}^{n+1/2}\right)\hat{\mathbf{m}}^{n+1/2} \\
& - \left(\nabla\hat{\mathbf{m}}_h^{n+1/2}\cdot\nabla\overline{\widetilde{\mathbf{m}}}_h^{n+1/2}\right)\hat{\mathbf{m}}_h^{n+1/2}, \overline{\widetilde{\mathbf{e}}}^{n+1/2}\Bigg),
\end{aligned}$$

where we use $\hat{\mathbf{m}}^{n+1/2} - \hat{\mathbf{m}}^{n-1/2} = \mathcal{O}(\tau)$. Using the Young inequality, the Sobolev embedding inequality, (2.4) and the inverse inequality (2.6), we obtain

$$\begin{aligned}
RHS \le & \frac{\lambda\tau}{4}\|\overline{\widetilde{\mathbf{e}}}^{n+1/2}\|_{H^1}^2 + C\tau h^4\left(\|D_\tau\mathbf{m}^{n+1}\|_{H^2} + \|\overline{\mathbf{m}}^{n+1/2}\|_{H^2}\right) + C\tau\|\mathbf{R}^{n+1}\|_{L^2}^2 \\
& + C\tau^3\|\nabla\mathbf{E}^n\|_{L^2}^2 + C\tau\left(\|\hat{\mathbf{e}}^{n+1/2}\|_{L^2}^2 + \|\hat{\mathbf{E}}^{n+1/2}\|_{L^2}^2\right) \\
& + Ch^2\tau\left(h^{-d/2}\|\hat{\mathbf{e}}^{n+1/2}\|_{L^2}^2 + \|\hat{\mathbf{E}}^{n+1/2}\|_{L^2}\|\hat{\mathbf{E}}^{n+1/2}\|_{H^1}\right) \\
& + \lambda\tau\Bigg(\left(\nabla\hat{\mathbf{m}}^{n+1/2}\cdot\nabla\overline{\mathbf{m}}^{n+1/2}\right)\hat{\mathbf{m}}^{n+1/2} \\
& - \left(\nabla\hat{\mathbf{m}}_h^{n+1/2}\cdot\nabla\overline{\widetilde{\mathbf{m}}}_h^{n+1/2}\right)\hat{\mathbf{m}}_h^{n+1/2}, \overline{\widetilde{\mathbf{e}}}^{n+1/2}\Bigg),
\end{aligned}$$

which implies that

$$\begin{aligned}
RHS \le & \frac{\lambda\tau}{4}\|\overline{\widetilde{\mathbf{e}}}^{n+1/2}\|_{H^1}^2 + C\tau\left(h^4 + \tau^4 + \|\mathbf{R}^{n+1}\|_{L^2}^2 + \|\widetilde{\mathbf{e}}^{n}\|_{L^2}^2 + \|\widetilde{\mathbf{e}}^{n-1}\|_{L^2}^2\right) \\
& + \lambda\tau\Bigg(\left(\nabla\hat{\mathbf{m}}^{n+1/2}\cdot\nabla\overline{\mathbf{m}}^{n+1/2}\right)\hat{\mathbf{m}}^{n+1/2}
\end{aligned}$$

$$-\left(\nabla\hat{\mathbf{m}}_h^{n+1/2}\cdot\nabla\overline{\widetilde{\mathbf{m}}}_h^{n+1/2}\right)\hat{\mathbf{m}}_h^{n+1/2},\overline{\widetilde{\mathbf{e}}}^{n+1/2}\Big).$$

From (3.16), we rewrite the last term in the right-hand side of of the above inequality as

$$\begin{aligned}
&\lambda\tau\left(\left(\nabla\hat{\mathbf{m}}^{n+1/2}\cdot\nabla\overline{\mathbf{m}}^{n+1/2}\right)\hat{\mathbf{m}}^{n+1/2}-\left(\nabla\hat{\mathbf{m}}_h^{n+1/2}\cdot\nabla\overline{\widetilde{\mathbf{m}}}_h^{n+1/2}\right)\hat{\mathbf{m}}_h^{n+1/2},\overline{\widetilde{\mathbf{e}}}^{n+1/2}\right)\\
=&\lambda\tau\left(\left(\nabla\hat{\mathbf{m}}^{n+1/2}\cdot\nabla\overline{\mathbf{m}}^{n+1/2}\right)\left(\hat{\mathbf{e}}^{n+1/2}-\hat{\mathbf{E}}^{n+1/2}\right),\overline{\widetilde{\mathbf{e}}}^{n+1/2}\right)\\
&+\lambda\tau\left(\nabla(\hat{\mathbf{e}}^{n+1/2}-\hat{\mathbf{E}}^{n+1/2})\cdot\nabla\overline{\widetilde{\mathbf{e}}}^{n+1/2}\left(\hat{\mathbf{e}}^{n+1/2}-\hat{\mathbf{E}}^{n+1/2}\right),\overline{\widetilde{\mathbf{e}}}^{n+1/2}\right)\\
&-\lambda\tau\left(\nabla(\hat{\mathbf{e}}^{n+1/2}-\hat{\mathbf{E}}^{n+1/2})\cdot\nabla\overline{\mathbf{E}}^{n+1/2}\left(\hat{\mathbf{e}}^{n+1/2}-\hat{\mathbf{E}}^{n+1/2}\right),\overline{\widetilde{\mathbf{e}}}^{n+1/2}\right)\\
&-\lambda\tau\left(\nabla(\hat{\mathbf{e}}^{n+1/2}-\hat{\mathbf{E}}^{n+1/2})\cdot\nabla\overline{\widetilde{\mathbf{e}}}^{n+1/2}\hat{\mathbf{m}}^{n+1/2},\overline{\widetilde{\mathbf{e}}}^{n+1/2}\right)\\
&+\lambda\tau\left(\nabla(\hat{\mathbf{e}}^{n+1/2}-\hat{\mathbf{E}}^{n+1/2})\cdot\nabla\overline{\mathbf{E}}^{n+1/2}\hat{\mathbf{m}}^{n+1/2},\overline{\widetilde{\mathbf{e}}}^{n+1/2}\right)\\
&-2\lambda\tau\left(\left(\nabla\hat{\mathbf{m}}^{n+1/2}\cdot\nabla(\overline{\widetilde{\mathbf{e}}}^{n+1/2}-\overline{\mathbf{E}}^{n+1/2})\right)\hat{\mathbf{m}}^{n+1/2},\overline{\widetilde{\mathbf{e}}}^{n+1/2}\right)\\
&+2\lambda\tau\left(\left(\nabla\hat{\mathbf{m}}^{n+1/2}\cdot\nabla(\overline{\widetilde{\mathbf{e}}}^{n+1/2}-\overline{\mathbf{E}}^{n+1/2})\right)\left(\hat{\mathbf{e}}^{n+1/2}-\hat{\mathbf{E}}^{n+1/2}\right),\overline{\widetilde{\mathbf{e}}}^{n+1/2}\right)=\sum_{i=1}^{7}J_i.
\end{aligned}$$

Using (2.4), (3.8), the Hölder inequality and the Young inequality, we can estimate J_1 by

$$\begin{aligned}
J_1&\le C\tau\left(\|\hat{\mathbf{e}}^{n+1/2}\|_{L^2}+\|\hat{\mathbf{E}}^{n+1/2}\|_{L^2}\right)\|\overline{\widetilde{\mathbf{e}}}^{n+1/2}\|_{L^2}\\
&\le\frac{\lambda\tau}{28}\|\overline{\widetilde{\mathbf{e}}}^{n+1/2}\|_{H^1}^2+C\tau\left(\|\hat{\mathbf{e}}^{n+1/2}\|_{L^2}^2+h^4\right).
\end{aligned}$$

For J_2, from (3.8), (3.18) and (3.23), we have

$$\begin{aligned}
J_2&\le C\tau\|\hat{\mathbf{e}}^{n+1/2}-\hat{\mathbf{E}}^{n+1/2}\|_{L^\infty}\left(\|\nabla\hat{\mathbf{e}}^{n+1/2}\|_{L^3}+\|\nabla\hat{\mathbf{E}}^{n+1/2}\|_{L^3}\right)\|\overline{\widetilde{\mathbf{e}}}^{n+1/2}\|_{H^1}^2\\
&\le C\left((1+C_2)(1+\kappa^2)h^{1-\frac{d}{6}}+h\right)\tau\|\overline{\widetilde{\mathbf{e}}}^{n+1/2}\|_{H^1}^2\le\frac{\lambda\tau}{28}\|\overline{\widetilde{\mathbf{e}}}^{n+1/2}\|_{H^1}^2
\end{aligned}$$

for h_2 such that $C\left((1+C_2)(1+\kappa^2)h_2^{1-\frac{d}{6}}+h_2\right)\le\frac{\lambda}{28}$. For J_3, from (3.8), (3.23), we have

$$\begin{aligned}
J_3\le&\, C\tau\left(\|\nabla\hat{\mathbf{e}}^{n+1/2}\|_{L^3}+\|\nabla\hat{\mathbf{E}}^{n+1/2}\|_{L^3}\right)\|\overline{\mathbf{E}}^{n+1/2}\|_{L^\infty}\\
&\times\left(\|\hat{\mathbf{e}}^{n+1/2}\|_{L^2}+\|\hat{\mathbf{E}}^{n+1/2}\|_{L^2}\right)\|\overline{\widetilde{\mathbf{e}}}^{n+1/2}\|_{L^6}\\
\le&\, C\left((1+C_2)(1+\kappa^2)h^{1-\frac{d}{6}}+h\right)\tau\left(\|\hat{\mathbf{e}}^{n+1/2}\|_{L^2}+\|\hat{\mathbf{E}}^{n+1/2}\|_{L^2}\right)\|\overline{\widetilde{\mathbf{e}}}^{n+1/2}\|_{H^1}\\
\le&\,\frac{\lambda\tau}{28}\|\overline{\widetilde{\mathbf{e}}}^{n+1/2}\|_{H^1}^2+C\tau\left(\|\hat{\mathbf{e}}^{n+1/2}\|_{L^2}^2+\|\hat{\mathbf{E}}^{n+1/2}\|_{L^2}^2\right),
\end{aligned}$$

for h_2 such that $(1+C_2)(1+\kappa^2)h_2^{1-\frac{d}{6}}+h_2\le 1$. For J_4, from (3.8), (3.23), we have

$$J_4\le C\tau\left(\|\nabla\hat{\mathbf{e}}^{n+1/2}\|_{L^3}+\|\nabla\hat{\mathbf{E}}^{n+1/2}\|_{L^3}\right)\|\overline{\widetilde{\mathbf{e}}}^{n+1/2}\|_{H^1}^2$$

$$\le C\left((1+C_2)(1+\kappa^2)h^{1-\frac{d}{6}}+h\right)\tau\|\widetilde{\overline{\mathbf{e}}}^{n+1/2}\|_{H^1}^2 \le \frac{\lambda\tau}{28}\|\widetilde{\overline{\mathbf{e}}}^{n+1/2}\|_{H^1}^2$$

for h_2 such that $C\left((1+C_2)(1+\kappa^2)h_2^{1-\frac{d}{6}}+h_2\right) \le \frac{\lambda}{28}$. For J_5, from (3.9) and (3.22), we have

$$\begin{aligned}
J_5 &\le C\tau\|\nabla\hat{\mathbf{e}}^{n+1/2}\|_{L^2}\|\nabla\overline{\mathbf{E}}^{n+1/2}\|_{L^3}\|\widetilde{\overline{\mathbf{e}}}^{n+1/2}\|_{L^6} + C\tau\|\overline{\mathbf{E}}^{n+1/2}\|_{W^{1,3}}^2\|\widetilde{\overline{\mathbf{e}}}^{n+1/2}\|_{L^6}\\
&\le C\tau\left(\|\widetilde{\mathbf{e}}^n\|_{L^2}+\|\widetilde{\mathbf{e}}^{n-1}\|_{L^2}+h^2\right)\|\widetilde{\overline{\mathbf{e}}}^{n+1/2}\|_{H^1}\\
&\le \frac{\lambda\tau}{28}\|\widetilde{\overline{\mathbf{e}}}^{n+1/2}\|_{H^1}^2 + C\tau\left(\|\widetilde{\mathbf{e}}^n\|_{L^2}^2+\|\widetilde{\mathbf{e}}^{n-1}\|_{L^2}^2+h^4\right).
\end{aligned}$$

By integtation by parts, we can estimate J_6 by

$$\begin{aligned}
J_6 \le\; &\lambda\tau\|\hat{\mathbf{m}}^{n+1/2}\|_{W^{2,3}}\|\hat{\mathbf{m}}^{n+1/2}\|_{L^\infty}\left(\|\widetilde{\overline{\mathbf{e}}}^{n+1/2}\|_{L^2}+\|\overline{\mathbf{E}}^{n+1/2}\|_{L^2}\right)\|\widetilde{\overline{\mathbf{e}}}^{n+1/2}\|_{L^6}\\
&+\lambda\tau\|\nabla\hat{\mathbf{m}}^{n+1/2}\|_{L^\infty}^2\left(\|\widetilde{\overline{\mathbf{e}}}^{n+1/2}\|_{L^2}+\|\overline{\mathbf{E}}^{n+1/2}\|_{L^2}\right)\|\widetilde{\overline{\mathbf{e}}}^{n+1/2}\|_{L^6}\\
&+\lambda\tau\|\nabla\hat{\mathbf{m}}^{n+1/2}\|_{L^\infty}\|\hat{\mathbf{m}}^{n+1/2}\|_{L^\infty}\left(\|\widetilde{\overline{\mathbf{e}}}^{n+1/2}\|_{L^2}+\|\overline{\mathbf{E}}^{n+1/2}\|_{L^2}\right)\|\widetilde{\overline{\mathbf{e}}}^{n+1/2}\|_{H^1}\\
\le\; &\frac{\lambda\tau}{28}\|\widetilde{\overline{\mathbf{e}}}^{n+1/2}\|_{H^1}^2 + C\tau\left(\|\widetilde{\overline{\mathbf{e}}}^{n+1/2}\|_{L^2}^2+\|\overline{\mathbf{E}}^{n+1/2}\|_{L^2}^2\right).
\end{aligned}$$

For J_7, from (2.6), (3.21), we have

$$\begin{aligned}
J_7 \le\; &C\tau\left(\|\hat{\mathbf{e}}^{n+1/2}\|_{L^3}+\|\hat{\mathbf{E}}^{n+1/2}\|_{L^3}\right)\|\widetilde{\overline{\mathbf{e}}}^{n+1/2}\|_{L^6}^2\\
&+C\tau\|\nabla\overline{\mathbf{E}}^{n+1/2}\|_{L^3}^2\left(\|\hat{\mathbf{e}}^{n+1/2}\|_{L^2}+\|\hat{\mathbf{E}}^{n+1/2}\|_{L^2}\right)\|\widetilde{\overline{\mathbf{e}}}^{n+1/2}\|_{L^6}\\
\le\; &C\left((1+C_2)(1+\kappa^2)h^{2-\frac{d}{6}}+h^2\right)\tau\|\widetilde{\overline{\mathbf{e}}}^{n+1/2}\|_{H^1}^2\\
&+C\tau h^2\left(\|\hat{\mathbf{e}}^{n+1/2}\|_{L^2}+\|\hat{\mathbf{E}}^{n+1/2}\|_{L^2}\right)\|\widetilde{\overline{\mathbf{e}}}^{n+1/2}\|_{L^6}\\
\le\; &\frac{\lambda\tau}{14}\|\widetilde{\overline{\mathbf{e}}}^{n+1/2}\|_{H^1}^2 + C\tau\left(\|\widetilde{\overline{\mathbf{e}}}^{n+1/2}\|_{L^2}^2+\|\overline{\mathbf{E}}^{n+1/2}\|_{L^2}^2\right),
\end{aligned}$$

for h_2 such that $C\left((1+C_2)(1+\kappa^2)h_2^{2-\frac{d}{6}}+h_2^2\right) \le \frac{\lambda}{28}$. Substituting the above estimates into (3.24) yields

$$\begin{aligned}
&\|\widetilde{\mathbf{e}}^{n+1}\|_{L^2}^2-\|\widetilde{\mathbf{e}}^n\|_{L^2}^2+\lambda\tau\|\widetilde{\overline{\mathbf{e}}}^{n+1/2}\|_{H^1}^2\\
\le\; &C\tau\|\widetilde{\mathbf{e}}^{n+1}\|_{L^2}^2+C\tau\left(h^4+\tau^4+\|\mathbf{R}^{n+1}\|_{L^2}^2+\|\widetilde{\mathbf{e}}^n\|_{L^2}^2+\|\widetilde{\mathbf{e}}^{n-1}\|_{L^2}^2\right).
\end{aligned}$$

For some h_2 such that $C\tau \le C\kappa h_2 < 1/2$, from the discrete Gronwall inequality and (3.5), we have

$$\|\widetilde{\mathbf{e}}^{n+1}\|_{L^2}^2 \le C\exp(TC)(\tau^4+h^4) \le C_2^2(\tau^4+h^4), \qquad \forall\; 1\le n\le N-1$$

for $C_2^2 \ge C\exp(TC)$. Thus, we close the mathematical induction and finish the proof of Lemma 3.3. □

Proof of Theorem 2.1. To prove the existence and uniqueness of solution $\widetilde{\mathbf{m}}_h^n \in \mathbf{V}_h$ to (2.1), we only consider the following homogeneous problem:

$$(\psi_h,\mathbf{v}_h)+\lambda\tau(\nabla\psi_h,\nabla\mathbf{v}_h)+\tau\left(\hat{\mathbf{m}}_h^{n+1/2}\times\nabla\psi_h,\nabla\mathbf{v}_h\right)$$

$$- \lambda\tau \left(\nabla \hat{\mathbf{m}}_h^{n+1/2} \cdot \nabla \psi_h, \hat{\mathbf{m}}_h^{n+1/2} \cdot \mathbf{v}_h\right) = 0, \qquad \forall\ \mathbf{v}_h \in \mathbf{V}_h. \tag{3.25}$$

Taking $\mathbf{v}_h = \psi_h$ in (3.25) leads to

$$\begin{aligned}
0 &= \|\psi_h\|_{L^2}^2 + \lambda\tau\|\nabla\psi_h\|_{L^2}^2 - \lambda\tau\left(\nabla \hat{\mathbf{m}}_h^{n+1/2} \cdot \nabla\psi_h, \hat{\mathbf{m}}_h^{n+1/2} \cdot \psi_h\right) \\
&\geq \|\psi_h\|_{L^2}^2 + \frac{\lambda\tau}{2}\|\nabla\psi_h\|_{L^2}^2 - C\tau\|\nabla\hat{\mathbf{m}}_h^{n+1/2}\|_{L^4}^2\|\psi_h\|_{L^4}^2 \\
&\geq \|\psi_h\|_{L^2}^2 + \frac{\lambda\tau}{4}\|\nabla\psi_h\|_{L^2}^2 - C\tau\|\nabla\hat{\mathbf{m}}_h^{n+1/2}\|_{L^4}^{\frac{8}{4-d}}\|\psi_h\|_{L^2}^2 - C\tau\|\psi_h\|_{L^2}^2 \\
&\geq \|\psi_h\|_{L^2}^2 + \frac{\lambda\tau}{4}\|\nabla\psi_h\|_{L^2}^2 - C\tau\|\psi_h\|_{L^2}^2 \\
&\geq \|\psi_h\|_{L^2}^2 + \frac{\lambda\tau}{4}\|\nabla\psi_h\|_{L^2}^2 - C\kappa h\|\psi_h\|_{L^2}^2 \geq \frac{1}{2}\|\psi_h\|_{L^2}^2 + \frac{\lambda\tau}{4}\|\nabla\psi_h\|_{L^2}^2,
\end{aligned} \tag{3.26}$$

for h_3 with $C\kappa h_3 < 1/2$, where we use the Young inequality, $\|\nabla\hat{\mathbf{m}}_h^{n+1/2}\|_{L^4} \leq C$, and

$$\|\psi_h\|_{L^4} \leq C\|\psi_h\|_{L^2}^{\frac{4-d}{4}}\|\nabla\psi_h\|_{L^2}^{\frac{d}{4}} + C\|\psi_h\|_{L^2}.$$

To prove $\|\nabla\hat{\mathbf{m}}_h^{n+1/2}\|_{L^4} \leq C$, we only need to prove

$$\|\nabla\mathbf{m}_h^n\|_{L^4} = \left\|\nabla\frac{\widetilde{\mathbf{m}}_h^n}{|\widetilde{\mathbf{m}}_h^n|}\right\|_{L^4} \leq C$$

for $1 \leq n \leq N-1$. In fact, from (3.17), we have

$$\|\widetilde{\mathbf{m}}_h^n\|_{L^\infty} \leq \|\Pi_h^n\mathbf{m}^n\|_{L^\infty} + Ch^{-d/2}\|\widetilde{\mathbf{e}}^n\|_{L^2} \leq C + CC_2(1+\kappa^2)h^{2-d/2} \leq C$$

for h_3 with $CC_2(1+\kappa^2)h_3^{2-d/2} < 1$, and

$$\begin{aligned}
\|\widetilde{\mathbf{m}}_h^n\|_{L^\infty} &\geq \|\mathbf{m}^n\|_{L^\infty} - Ch^{-d/2}\|\widetilde{\mathbf{e}}^n\|_{L^2} - C\|\mathbf{E}^n\|_{W^{1,4}} \\
&\geq 1 - CC_2(1+\kappa^2)h^{2-d/2} - Ch \geq 1/2
\end{aligned}$$

for h_3 with $CC_2(1+\kappa^2)h_3^{2-d/2} + Ch_3 < 1/2$, and

$$\begin{aligned}
\|\nabla\widetilde{\mathbf{m}}_h^n\|_{L^4} &\leq \|\nabla\mathbf{m}^n\|_{L^4} + \|\nabla\widetilde{\mathbf{e}}^n\|_{L^4} + \|\nabla\mathbf{E}^n\|_{L^4} \\
&\leq C + Ch + CC_2(1+\kappa^2)h^{1-d/4} \leq C
\end{aligned}$$

for h_3 with $C_2(1+\kappa^2)h_3^{1-d/4} < 1$. The estimate (3.26) implies $\psi \equiv 0$ and we obtain the solvability and uniqueness of the solution to (2.1). Moreover, $\mathbf{m}_h^n$ is uniquely determined by using the projection step (2.3). Next, we prove the optimal error estimate (2.5). It follows from (3.19) and (3.17) that

$$\|\mathbf{m}^n - \mathbf{m}_h^n\|_{L^2} \leq 2\left(\|\widetilde{\mathbf{e}}^n\|_{L^2} + \|\mathbf{E}^n\|_{L^2}\right) \leq C_0(\tau^2 + h^2).$$

We complete the proof of Theorem 2.1 with $h_0 = \min\{h_1, h_2, h_3\}$. □

4. Numerical results

In this section, we present the numerical result to check the optimal error estimate (2.5) derived in Theorem 2.1. We consider the LL problem (1.3) with (1.2) in the unit circle $\Omega = \{(x, y) : \; x^2 + y^2 \leq 1\}$. The initial value $\mathbf{m}_0$ is taken as

$$\mathbf{m}_0 = (\sin(x)\cos(y), \cos(x)\cos(y), \sin(y)).$$

Gilbert damping constant is set as $\lambda = 1$. We take a uniform triangular partition with M nodes on $\partial\Omega$. In addition, we set the final time $T = 1.0$ in this numerical result.

Table 1. Numerical errors and convergence rates for linear FEM

M	$\|\mathbf{m}(1,\cdot) - \widetilde{\mathbf{m}}_h^N\|_{L^2}$	rate	$\|\mathbf{m}(1,\cdot) - \mathbf{m}_h^N\|_{L^2}$	rate	$\|\mathbf{m}(1,\cdot) - \mathbf{m}_h^N\|_{H^1}$	rate
21	4.23322E-03		4.23322E-03		1.05620E-02	
41	1.09936E-03	1.95	1.09936E-03	1.95	3.87279E-03	1.45
61	4.97106E-04	1.96	4.97107E-04	1.96	2.37107E-03	1.21
101	1.76029E-04	2.03	1.76028E-04	2.03	1.33087E-03	1.13

To confirm the optimal convergence rates for the errors $\|\mathbf{m}^n - \widetilde{\mathbf{m}}_h^n\|_{L^2}$, $\|\mathbf{m}^n - \mathbf{m}_h^n\|_{L^2}$ and $\|\mathbf{m}^n - \mathbf{m}_h^n\|_{H^1}$, the time step is taken as $\tau = 0.01/M$. On the other hand, since no analytical solution exists, the reference solution is taken as the numerical solution corresponding to $M = 601$ and $\tau = 0.01/M$. The numerical errors are displayed in Tables 1, from which we can see that the convergence rates are in good agreement with our theoretical analysis in Theorem 2.1.

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