Journal of Applied Analysis and Computation
Volume 11, Number 6, December 2021, 3133–3156
Website:http://www.jaac-online.com
DOI:10.11948/20210321

INVARIANT MANIFOLDS FOR THE NONAUTONOMOUS BOISSONADE SYSTEM IN THREE-DIMENSIONAL TORUS

Na Liu

Abstract We undertake a study of invariant manifolds for the nonautonomous Boissonade system in three-dimensional torus. The system, exhibiting Turing structures, is a activator-inhibitor model for describing the relation between the genuine homogeneous 2D systems and the 3D monolayers. Assuming the diffusivity of the activator be sufficiently large, we prove the existence of a finite-dimensional Lipschitz manifold. The manifold is locally forward invariant and pullback attracts exponentially only those solutions with initial values having a certain regularity. If more assumptions on the external forces are made such that the symbol space is compact, we prove that the manifold is of global type.

Keywords Boissonade system, three-dimensional torus, nonautonomous dynamical system, finite-dimensional manifold of global type.

MSC(2010) 35B42, 37B55, 37D10.

1. Introduction

Invariant manifolds are the most interesting and important objects which arise in the investigation of the long-time behavior of dynamical systems described by nonlinear partial differential equations (see, e.g., [2, 7, 9, 10, 13, 22–24, 27–30, 33]). They can be used to capture complex dynamics of solutions and characterize the qualitative properties of a semiflow nearby invariant sets. What we have to mention is one class of global manifolds, namely inertial manifolds. These manifolds, which are generalizations of the centre-unstable manifolds, are global, exponentially attracting, finite-dimensional. That they are finite-dimensional gives a reasonable and rigorous way to reduce systems to lower-dimensional systems that are more easily analyzed. The concept of inertial manifolds was proposed by Foias etc [9] and after that applied and evolved by many authors; we refer the reader to [1, 3, 6, 7, 9, 10, 12, 18, 20, 24, 25, 30, 33, 36] and the references therein.

The theory of invariant manifolds has been well developed for autonomous dynamical systems. It is known that two alternative methods for constructing invariant manifolds were established by Hadamard [11], Lyapunov [23] and Perron [27–29]. Hadamard's method, also called Hadamard's graph transform method, is a more geometrical in nature than the Lyapunov-Perron method. Zelik [36] studied the problem of finite-dimensional reduction for parabolic partial differential equations.

Email address: naliumath@126.com (N. Liu)
Department of Mathematics, Shanghai Normal University, Guilin Road, 200234 Shanghai, China

He gave a detailed exposition of the classical theory of inertial manifolds as well as various attempts to generalize it based on the so-called Mañé projection theorems. Abu-Hamed etc [10] proved the existence of an inertial manifold for two different sub-grid scale α-models of turbulence: the simplified Bardina model and the modified Leray-α model, in two-dimensional torus. By using the so-called spatial averaging principle, Kostianko [17] proved the existence of an inertial manifold for the modified Leray-α model in three-dimensional space. Lu etc [22] obtained the existence of a finite-dimensional manifold for a generalized phase-field system on the rectangular or cubic spacial domains. This manifold, which has a certain regularity, is locally invariant and attracts exponentially only those solutions with regular enough initial values. Zhao and Wang [37] proved that, in the case of the fast recovery variable, there exists a finite-dimensional global manifold for the FitzHugh-Nagumo system on some two/three-dimensional domains.

There are some works devoted to invariant manifolds for nonautonomous dynamical system by Koksch and Siegmund [16], Latushkin and Layton [19], Wang etc [35]. Koksch and Siegmund's result on the existence of inertial manifolds for nonautonomous dynamical systems was based on two geometrical assumptions, called cone invariance and squeezing property, and two additional technical assumptions, called boundedness and coercivity property. Latushkin and Layton gave optimal gap conditions that imply the existence of infinite-dimensional Lipschitz invariant manifolds for systems of semilinear equations on Banach spaces. The result was used in the proofs of the existence of invariant manifolds for nonautonomous equations and semilinear skew-product flows. Wang etc [35] proved that if a geometrical assumption, called local strong squeezing property and several technical assumptions, called controllability, inverse Lipschitz, and (partial) compactness property are satisfied, then there exists a finite-dimensional Lipschitz invariant manifold for an abstract nonautonomous dynamical system defined on a general Banach space. Then they applied this general framework to scalar reaction-diffusion equations and FitzHugh-Nagumo systems.

In this paper, we shall focus on considering a nonautonomous perturbation of the Boissonade system. This system, exhibiting Turing structures that are found to form one layer after the other so that there is a single layer called "monolayer" beyond the pattern onset, is a simple reaction-diffusion model of coupled type for describing the relation between the genuine homogeneous 2D systems and the 3D monolayers [8,14]. It is noted that this system is a activator-inhibitor model where the activator, u, is responsible for accelerating the reaction, and the inhibitor, v, is responsible for slowing down the reactions caused by the activator [8,15,26,31]. More precisely, the system under consideration is given by

$$\begin{cases} u_t = d\Delta u + u - \alpha v + \gamma uv - u^3 + f(x,t), \\ v_t = \Delta v + u - \beta v + g(x,t) \end{cases} \tag{1.1}$$

with the periodic boundary conditions

$$u(x,t) = u(x+2\pi j,t), \quad v(x,t) = v(x+2\pi j,t), \quad j \in Z^3. \tag{1.2}$$

Here, d is the diffusion coefficient of the activator, α, β, γ are positive parameters, and f, g are the external forces.

We mention two related papers dedicated to the long-time behavior of dynamical system described by the Boissonade system. In [34], the existence and properties of a global attractor for the weak solution semiflow were obtained by a parameter adjusting and grouping estimation method. Moreover, the upper semicontinuity of the global attractors in $H^1 \times H^1$ for the solution semiflow with respect to γ converging to zero was proved. By using a spatial averaging principle of local type and uniform dissipative estimates, Liu [21] proved the existence of a finite-dimensional manifold of global type. The manifold is locally invariant, which attracts uniformly exponentially those solutions with initial values having a certain regularity, but attracts uniformly those solutions starting from the phase space. As far as we all know, the existence of finite-dimensional manifolds of global type for (1.1) has not been studied in the existing literatures. It is in fact this reason that stimulates us to conduct this paper.

It is known that the spectral gap condition is a crucial property to guarantee the existence of inertial manifolds for many evolution equations (see, e.g., Abu-Hamed etc [10], Foias etc [9], Mora [25], Sell and You [32], Zelik [36]). Unfortunately, we notice that the spectral gap condition may fail for the problem under consideration. It is noted that in order to deal with the difficulties caused by the absence of the spectral gap condition, Mallet-Paret and Sell [24] introduced a notion of cone condition for a scalar reaction-diffusion equation and proved the existence of an inertial manifold. They defined the principle of spatial averaging (PSA), which is a property of the Laplace operator and is used to verify the cone condition. This technique was further simplified by Zelik [36], and extended by Kostianko to the three-dimensional modified Leray-α model [17], and by Lu etc to a generalized phase-field system [22]. However, because of the occurrence of the quadratic coupled term uv, it seems rather difficult for us to verify the PSA for the first equation of (1.1) or (1.1) as a whole.

Note that the forcing functions of (1.1) explicitly depend on the time t. At this point, the dynamical system generated by (1.1)–(1.2) differs essentially from the situation in the time-independent case. Thus, the method, e.g. [17, 22, 24, 35, 36], for proving the existence of invariant manifolds in the autonomous case, cannot be extended in a straightforward way to the nonautonomous setting. In order to overcome this problem, we need to obtain the uniform dissipative estimates, which are used to modify the nonlinearities such that the modified functions are independent of the elements from the symbol space driven by the forcing functions.

Our main result states that for every α, β, γ, there exists d such that the dynamical system generated by (1.1)–(1.2) possesses a finite-dimensional Lipschitz manifold $\mathscr{M} = \{\mathscr{M}(\sigma)\}_{\sigma\in\Sigma}$. Here Σ is the symbol space. The manifold is locally forward invariant and pullback attracts exponentially only those solutions with initial values having a certain regularity. If more assumptions on the external forces are made such that the symbol space is compact, we also prove that the manifold is of global type.

2. Preliminaries

Let X be a Banach space with norm $\|\cdot\|_X$. Denote by $\mathrm{dist}_X(\cdot,\cdot)$ the Hausdorff semi-distance on X between B_1 and B_2, i.e.,

$$\mathrm{dist}_X(B_1, B_2) = \sup_{x\in B_1} \inf_{y\in B_2} \|x - y\|_X.$$

We next recall some basic concepts (see [5, Chapter IV]).

A two-parameter family of mappings $\{U(t,\tau)\}$ is said to be a process in X if

$$U(t,s)U(s,\tau) = U(t,\tau), \quad \forall t \geq s \geq \tau, \tau \in \mathbb{R},$$
$$U(\tau,\tau) = I, \quad \tau \in \mathbb{R}.$$

We next consider a family of processes $\{U_\sigma(t,\tau)\}$ depending on a parameter $\sigma \in \Sigma$. The parameter σ is called the symbol of the process $\{U_\sigma(t,\tau)\}$ and the set Σ is called the symbol space.

A bounded set P_1 of X is said to be uniformly (w.r.t. $\sigma \in \Sigma$) absorbing for the family of processes $\{U_\sigma(t,\tau)\}, \sigma \in \Sigma$ if for every bounded set $B \subset X$ there exists $t_0 = t_0(\tau, B) \geq \tau$ such that $\bigcup_{\sigma\in\Sigma} U_\sigma(t,\tau)B \subset P_1$ for all $t \geq t_0$.

A bounded set P_2 of X is said to be uniformly (w.r.t. $\sigma \in \Sigma$) attracting for the family of processes $\{U_\sigma(t,\tau)\}$, $\sigma \in \Sigma$ if for every bounded set $B \subset X$,

$$\lim_{t\to\infty}\left(\sup_{\sigma\in\Sigma} \operatorname{dist}_X(U_\sigma(t,\tau)B, P_2)\right) = 0.$$

A closed set $\mathcal{A}_\Sigma$ of X is said to be the uniform (w.r.t $\sigma \in \Sigma$) attractor of the family of processes $\{U_\sigma(t,\tau)\}, \sigma \in \Sigma$ if it is the minimal uniformly attracting set.

Let $\theta_t, t \in \mathbb{R}$ be a continuous flow defined on a complete metric space Σ such that the map $(t,\sigma) \to \theta_t\sigma$ is continuous in σ and θ_t satisfies

$$\theta_0 = I, \quad \theta_{t_1} \circ \theta_{t_2} = \theta_{t_1+t_2}, \quad t_1, t_2 \in \mathbb{R}.$$

A nonautonomous dynamical system (NDS) defined on the state space X is a cocycle $\phi_{t,\sigma}, t \in \mathbb{R}^+, \sigma \in \Sigma$ over θ_t such that the map $(t,\sigma,u) \to \phi_{t,\sigma}u$ is continuous in (σ, u) and $\phi_{t,\sigma}$ satisfies

$$\phi_{0,\sigma} = I, \quad \phi_{t_1,\theta_{t_2}\sigma} \circ \phi_{t_2,\sigma} = \phi_{t_1+t_2,\sigma}, \quad t_1, t_2 \in \mathbb{R}^+, \sigma \in \Sigma.$$

Let $\Omega = (0, 2\pi)^3$, $H = L_p^2(\Omega) \times L_p^2(\Omega)$ and $E = H_p^1(\Omega) \times H_p^1(\Omega)$. The symbols $\|\cdot\|$ and $(\cdot,\cdot)$ denote the norm and the inner product on H or any component space $L_p^2(\Omega)$, respectively. Let $\|\cdot\|_Y$ denote the norm on any other Banach space Y. We denote the duality product between $H_p^1(\Omega)$ and $H_p^{-1}(\Omega)$ by $\langle\cdot,\cdot\rangle_{H_p^{-1}}$. For $\tau \in \mathbb{R}$, we write $\mathbb{R}_\tau = [\tau, \infty)$ for simplicity. M_i will stand for various positive constants.

Define a linear operator $A : D(A) \to L_p^2(\Omega)$ by

$$A = -\Delta + I, \quad D(A) = H_p^2(\Omega).$$

It is easy to see that $-A$ is the infinitesimal generator of a C_0 semigroup e^{-At}, $t \geq 0$ on $L_p^2(\Omega)$. Furthermore, e^{-At} can be extended to analytic semigroup in some sector around the nonnegative real axis. The operator A possesses the complete orthonormal system of eigenvectors $\{e_j\}$ on $L_p^2(\Omega)$ which correspond to eigenvalues λ_j such that

$$Ae_j = \lambda_j e_j, \quad 0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_j \to \infty,\ j \to \infty.$$

For $N \in \mathbb{N}^+$, we write

$$X_N^u = \operatorname{span}\{e_j : 1 \leq j \leq N\}, \quad X_N^s = \overline{\operatorname{span}}\{e_j : j \geq N+1\}.$$

Let P_N denote the orthogonal projection from $L_p^2(\Omega)$ to X_N^u and $Q_N = I - P_N$.

For $s > 0$, we define the fractional powers of the operator A by

$$A^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-At} \mathrm{d}t, \quad A^s = (A^{-s})^{-1},$$

where $\Gamma(\cdot)$ stands for the Gamma function. It is known that

$$\|A^s e^{-At}\theta\| \le \left[\left(\frac{s}{t}\right)^s + \lambda_1^s\right] e^{-\lambda_1 t}\|\theta\|, \quad t > 0, \theta \in L_p^2(\Omega)$$

and for $t \ge 0$ and $0 < s \le 1$,

$$\|(e^{-At} - I)\theta\| \le \frac{C_{1-s} t^s}{s} \|A^s \theta\|, \quad \theta \in D(A^s),$$

where $C_s > 0$ is bounded for s in any compact interval of $\mathbb{R}^+$. This can be seen in [12, Section 1.4].

We assume that

$$f, g \in C^\delta(\overline{\mathbb{R}}; L_p^2(\Omega)).$$

Here, $C^\delta(\overline{\mathbb{R}}; L_p^2(\Omega))$ is the linear space of Hölder continuous functions with exponent δ that are bounded in $\mathbb{R}$. The space $C_{loc}(\mathbb{R}; L_p^2(\Omega))$ is equipped with the local uniform convergence topology on any bounded interval of the time axis. It easily follows that $C_{loc}(\mathbb{R}; L_p^2(\Omega))$ is metrizable by means of the Fréchet metric

$$\rho(f_1, f_2) = \sum_{n=0}^\infty \frac{1}{2^n} \frac{\|f_1 - f_2\|_n}{1 + \|f_1 - f_2\|_n}, \tag{2.1}$$

where

$$\|f_1 - f_2\|_n = \max_{t \in [-n,n]} \|f_1(t) - f_2(t)\|.$$

Note that $C_{loc}(\mathbb{R}; L_p^2(\Omega))$ with metric (2.1) is complete. Let Σ be the hull of (f, g) in $C_{loc}(\mathbb{R}; L_p^2(\Omega)) \times C_{loc}(\mathbb{R}; L_p^2(\Omega))$, i.e.,

$$\Sigma = \overline{\{(f(\cdot + t), g(\cdot + t)) : t \in \mathbb{R}\}}^{C_{loc}(\mathbb{R}; L_p^2(\Omega)) \times C_{loc}(\mathbb{R}; L_p^2(\Omega))}.$$

It is easy to see that Σ is complete. We also notice that for all $(\sigma_1, \sigma_2) \in \Sigma$,

$$\|\sigma_1\|_{L^\infty(\mathbb{R}; L_p^2(\Omega))} \le \|f\|_{L^\infty(\mathbb{R}; L_p^2(\Omega))},$$
$$\|\sigma_2\|_{L^\infty(\mathbb{R}; L_p^2(\Omega))} \le \|g\|_{L^\infty(\mathbb{R}; L_p^2(\Omega))}.$$

Moreover, for $(\sigma_1, \sigma_2) \in \Sigma$ and $t, s \in \mathbb{R}$, there exist constants $K_1, K_2 > 0$ such that

$$\|\sigma_1(t) - \sigma_1(s)\| \le K_1 |t - s|^\delta,$$
$$\|\sigma_2(t) - \sigma_2(s)\| \le K_2 |t - s|^\delta.$$

Define a translation group $\theta_t : \Sigma \to \Sigma, t \in \mathbb{R}$ by

$$[\theta_t \sigma](s) = (\sigma_1(t + s), \sigma_2(t + s)), \quad \sigma = (\sigma_1, \sigma_2) \in \Sigma, s \in \mathbb{R}.$$

Note that the map $(t, \sigma) \to \theta_t \sigma$ is continuous from $\mathbb{R} \times \Sigma \to \Sigma$.

Let us write

$$h_1(u,v)=(1+d)u-\alpha v+\gamma uv-u^3,\quad h_2(v)=(1-\beta)v.$$

Then (1.1) can be rewritten into the following problem

$$\begin{cases} u'=-dAu+h_1(u,v)+\sigma_1(t),\\ v'=-Av+u+h_2(v)+\sigma_2(t). \end{cases} \tag{2.2}$$

By using standard Galerkin method, for every $\sigma=(\sigma_1,\sigma_2)\in\Sigma$, $\tau\in\mathbb{R}$, $(u_\tau,v_\tau)\in H$, (2.2) has a unique weak solution $(u,v)\in C(\mathbb{R}_\tau;H)\cap L^2_{loc}(\mathbb{R}_\tau;E)$ satisfying $u(\tau)=u_\tau, v(\tau)=v_\tau$ and

$$\begin{cases} \langle u'(t),w\rangle_{H_p^{-1}}+d(\nabla u(t),\nabla w)=(u(t)-\alpha v(t)+\gamma u(t)v(t)-u^3(t),w)+(\sigma_1(t),w),\\ \langle v'(t),\theta\rangle_{H_p^{-1}}+(\nabla v(t),\nabla\theta)=(u(t)-\beta v(t),\theta)+(\sigma_2(t),\theta) \end{cases}$$

for any $w,\theta\in H^1_p(\Omega)$ and a.e. $t\in\mathbb{R}_\tau$.

Hence, we can define a family of processes $\{U_\sigma(t,\tau)\},\sigma\in\Sigma$ by

$$U_\sigma(t,\tau)(u_\tau,v_\tau)=(u(t),v(t)),\quad (u_\tau,v_\tau)\in H,\tau\in\mathbb{R},t\in\mathbb{R}_\tau,$$

where (u,v) is the weak solution of the IVP of (2.2). Because of the uniqueness of weak solution, the following translation identity holds

$$U_\sigma(t+s,\tau+s)=U_{\theta_s\sigma}(t,\tau),\quad \sigma\in\Sigma,s\ge 0,t\in\mathbb{R}_\tau,\tau\in\mathbb{R}.$$

Note that the map $(t,\sigma,(u_\tau,v_\tau))\to(u(t),v(t))$ is continuous from $\mathbb{R}_\tau\times\Sigma\times H\to H$. We then can define a NDS $\psi_{t,\sigma},t\in\mathbb{R}^+,\sigma\in\Sigma$ by

$$\psi_{t,\sigma}(u_0,v_0)=U_\sigma(t,0)(u_0,v_0),\quad (u_0,v_0)\in H.$$

Using the classical global well-posedness theory, for every $\sigma=(\sigma_1,\sigma_2)\in\Sigma$ and $(u_0,v_0)\in E$, the IVP of (2.2) has a global mild solution $(u,v)\in C(\mathbb{R}^+;E)$, which can be defined using the formulas of variations of constants

$$\begin{cases} u(t)=e^{-dAt}u_0+\displaystyle\int_0^t e^{-dA(t-s)}(h_1(u(s),v(s))+\sigma_1(s))\mathrm{d}s,\\ v(t)=e^{-At}v_0+\displaystyle\int_0^t e^{-A(t-s)}(u(s)+h_2(v(s))+\sigma_2(s))\mathrm{d}s. \end{cases}$$

Since the map $(t,\sigma,(u_0,v_0))\to(u(t),v(t))$ is continuous from $\mathbb{R}^+\times\Sigma\times E$ to E, we can define a NDS $\hat\psi_{t,\sigma},t\in\mathbb{R}^+,\sigma\in\Sigma$ by mild solutions of the IVP of (2.2). Note that $h_1(u(\cdot),v(\cdot))+\sigma_1$ and $u+h_2(v(\cdot))+\sigma_2$ belong to $L^2_{loc}(\mathbb{R}^+;L^2_p(\Omega))$. From [4, Proposition 3.6] (see also [35, Section 2]), it follows that (u,v) is also a strong solution. Since a strong solution is also a weak solution, we obtain that for any $t\in\mathbb{R}^+,\sigma\in\Sigma,(u_0,v_0)\in E$,

$$\psi_{t,\sigma}(u_0,v_0)=\hat\psi_{t,\sigma}(u_0,v_0). \tag{2.3}$$

3. Uniform Dissipativity

In this section we are devoted to obtaining the uniform dissipativity and the existence of uniform attractor for (1.1)–(1.2).

We first prove the following

Theorem 3.1. *There exists a constant $R_0 > 0$ satisfying that for every bounded subset $\mathcal{B}$ of E, there is $t_0 = t_0(\mathcal{B}) > 0$ such that for any $\sigma \in \Sigma$ and $(u_0, v_0) \in \mathcal{B}$,*

$$\psi_{t,\sigma}(u_0, v_0) \subset \mathscr{B}_1 = \{\psi_{t,\sigma}(u_0, v_0) \in D(A) \times D(A) : \|\psi_{t,\sigma}(u_0, v_0)\|_{H_p^2 \times H_p^2} \leq R_0\}$$

for $t \geq t_0$, where R_0 is independent of d for $d \geq d_0 > 0$.

Proof. Let $\mathcal{B}$ be a bounded set of E. For $(u_0, v_0) \in \mathcal{B}$ and $\sigma = (\sigma_1, \sigma_2) \in \Sigma$, we write $(u(t), v(t)) = \psi_{t,\sigma}(u_0, v_0), t \in \mathbb{R}^+$. Multiplying the first equation in (2.2) by u and the second one by $2\gamma^2\beta^{-1}v$, we get that

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|u\|^2 + 2\gamma^2\beta^{-1}\|v\|^2) + d\|\nabla u\|^2 + 2\gamma^2\beta^{-1}\|\nabla v\|^2 \\
&\leq \left(\frac{5}{4} + (2\gamma^2\beta^2)^{-1}(2\gamma^2 - \alpha\beta)^2\right)|\Omega| - \frac{1}{2}\int_\Omega u^4 \mathrm{d}x - \frac{\gamma^2}{4}\int_\Omega v^2 \mathrm{d}x \\
&\quad + \int_\Omega \left(\sigma_1^2(t) + \frac{4\gamma^2}{\beta^2}\sigma_2^2(t)\right)\mathrm{d}x.
\end{aligned}$$

Using the condition $d \geq d_0$, we obtain that

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|u\|^2 + 2\gamma^2\beta^{-1}\|v\|^2) \\
&\quad + d_0\|\nabla u\|^2 + 2\gamma^2\beta^{-1}\|\nabla v\|^2 + \frac{1}{2}\|u\|^2 + \frac{\gamma^2}{4}\|v\|^2 \\
&\leq \left(\frac{11}{8} + (2\gamma^2\beta^2)^{-1}(2\gamma^2 - \alpha\beta)^2\right)|\Omega| + \int_\Omega \left(\sigma_1^2(t) + \frac{4\gamma^2}{\beta^2}\sigma_2^2(t)\right)\mathrm{d}x \\
&\leq M_1,
\end{aligned} \tag{3.1}$$

where $M_1 = [\frac{11}{8} + (2\gamma^2\beta^2)^{-1}(2\gamma^2 - \alpha\beta)^2]|\Omega| + \|f\|_{L^\infty(\mathbb{R};L_p^2)} + \frac{4\gamma^2}{\beta^2}\|g\|_{L^\infty(\mathbb{R};L_p^2)}$. Putting $M_2 = \min\{\frac{1}{2}, \frac{\beta}{8}\}$, we end up with

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|u\|^2 + 2\gamma^2\beta^{-1}\|v\|^2) + 2M_2(\|u\|^2 + 2\gamma^2\beta^{-1}\|v\|^2) \leq M_1.$$

Thanks to the Gronwall inequality, we have that

$$\|u(t)\|^2 + 2\gamma^2\beta^{-1}\|v(t)\|^2 \leq e^{-2M_2(t-\tau)}(\|u_\tau\|^2 + 2\gamma^2\beta^{-1}\|v_\tau\|^2) + \frac{M_1}{2M_2}.$$

Putting $M_3 = \min\{1, 2\gamma^2\beta^{-1}\}$ and $M_4 = \max\{1, 2\gamma^2\beta^{-1}\}$, we get that

$$\|u(t)\|^2 + \|v(t)\|^2 \leq \frac{M_4}{M_3}e^{-2M_2(t-\tau)}(\|u_\tau\|^2 + \|v_\tau\|^2) + \frac{M_1}{2M_2M_3}.$$

Hence, there exists a $t_0 = t_0(\mathcal{B})$ such that

$$\|u(t)\|^2 + \|v(t)\|^2 \leq M_5, \quad t \geq t_0. \tag{3.2}$$

Here $M_5 = \frac{M_1}{M_2 M_3}$ is independent of $t, \sigma, \mathcal{B}, d$. By integrating (3.1), we also obtain that

$$\int_t^{t+1} (\|\nabla u(s)\|^2 + \|\nabla v(s)\|^2) \mathrm{d}s \le M_6, \quad t \ge t_0, \tag{3.3}$$

where $M_6 = \frac{2M_1 + M_5 \max\{1, 2\gamma^2\beta^{-1}\}}{2\min\{d_0, 2\gamma^2\beta^{-1}\}}$. As is usual, the formal estimates given below can be rigorously justified by using the Galerkin approximation method. Multiplying the first equation in (2.2) by $-\Delta u$ and the second one by $-\Delta v$, we get that

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|\nabla u\|^2 + \|\nabla v\|^2) + \frac{d}{4}\|\Delta u\|^2 + \frac{1}{4}\|\Delta v\|^2 \\ &\le \|\nabla u\|^2 + \frac{\alpha^2}{d}\|v\|^2 + \frac{\gamma^2}{2d}(\|u\|_{L_p^4}^4 + \|v\|_{L_p^4}^4) + \frac{1}{2}\|u\|^2 + \frac{1}{d}\|\sigma_1(t)\|^2 + \|\sigma_2(t)\|^2. \end{aligned}$$

It follows that

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|\nabla u\|^2 + \|\nabla v\|^2) + \frac{d}{4}\|\Delta u\|^2 + \frac{1}{4}\|\Delta v\|^2 \\ &\le \|\nabla u\|^2 + \frac{\alpha^2}{d}\|v\|^2 + \frac{\gamma^2 M_7^4}{2d}(\|u\|_{H_p^1}^4 + \|v\|_{H_p^1}^4) + \frac{1}{2}\|u\|^2 + \frac{1}{d}\|\sigma_1(t)\|^2 + \|\sigma_2(t)\|^2, \end{aligned}$$

where M_7 is a constant satisfying $\|\theta\|_{L_p^4} \le M_7\|\theta\|_{H_p^1}$ for all $\theta \in H_p^1(\Omega)$. From (3.2), it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|\nabla u\|^2 + \|\nabla v\|^2) \le (\|\nabla u\|^2 + \|\nabla v\|^2)\left(\frac{2M_7^4\gamma^2}{d_0}(\|\nabla u\|^2 + \|\nabla v\|^2) + 2\right) + M_8,$$

where $M_8 = 2M_5(\frac{\alpha^2}{d_0} + \frac{1}{2}) + \frac{2M_5^2 M_7^4 \gamma^2}{d_0} + \frac{2}{d_0}\|f\|_{L^\infty(\mathbb{R}; L_p^2)} + 2\|g\|_{L^\infty(\mathbb{R}; L_p^2)}$. This together with (3.3) and the uniform Gronwall lemma implies that

$$\|\nabla u(t)\|^2 + \|\nabla v(t)\|^2 \le (M_6 + M_8)\exp(\frac{2M_6 M_7^4 \gamma^2}{d_0} + 2), \quad t \ge t_0 + 1.$$

Hence, we get that

$$\|u\|_{H_p^1}, \|v\|_{H_p^1} \le M_9, \quad t \ge t_0 + 1. \tag{3.4}$$

Here M_9 is independent of $t, \sigma, \mathcal{B}, d$. Note that h_1 is locally Lipschitz from E to $L_p^2(\Omega)$. This implies in particular that

$$\|h_1(u, v)\| \le M_{10}, \quad t \ge t_0 + 1. \tag{3.5}$$

Note that

$$u(t) = e^{-dA(t - t_0 - 1)}u(t_0 + 1) + \int_{t_0+1}^t e^{-dA(t-s)}(h_1(u(s), v(s)) + \sigma_1(s))\mathrm{d}s.$$

This together with (3.4) and (3.5) implies that

$$\|u(t+h) - u(t)\|_{H_p^1} \le M_{11}(h^{\frac{1}{2}} + h^{\frac{1}{4}} + h), \quad t \ge t_0 + 1, h \ge 0.$$

A similar calculation further shows that

$$\|v(t+h) - v(t)\|_{H_p^1} \le M_{11}(h^{\frac{1}{2}} + h^{\frac{1}{4}} + h), \quad t \ge t_0 + 1, h \ge 0.$$

Accordingly, $h_1(u,v)$ is Hölder continuous on $[t_0+1,\infty)$. Hence, one has that

$$\begin{aligned}\|v(t)\|_{H_p^2} \le{}& \|Ae^{-A(t-t_0-1)}v(t_0+1)\| + \|A\int_{t_0+1}^t e^{-A(t-s)}\sigma_2(t)\mathrm{d}s\| \\ &+ \int_{t_0+1}^t \|A^{\frac{1}{2}}e^{-A(t-s)}A^{\frac{1}{2}}(u(s)+(1-\beta)v(s))\|\mathrm{d}s \\ &+ \int_{t_0+1}^t \|Ae^{-A(t-s)}(\sigma_2(s)-\sigma_2(t))\|\mathrm{d}s \\ \le{}& M_{12}\end{aligned}$$

and

$$\begin{aligned}\|u(t)\|_{H_p^2} \le{}& \|Ae^{-d_0A(t-t_0-1)}u(t_0+1)\| + \int_{t_0+1}^t \|Ae^{-d_0A(t-s)}(\sigma_1(s)-\sigma_1(t))\|\mathrm{d}s \\ &+ \int_{t_0+1}^t \|Ae^{-d_0A(t-s)}[h_1(u(s),v(s))-h_1(u(t),v(t))]\|\mathrm{d}s \\ &+ \|A\int_{t_0+1}^t e^{-d_0A(t-s)}(h_1(u(t),v(t))+\sigma_1(t))\mathrm{d}s\| \\ \le{}& M_{13}\end{aligned}$$

for $t \ge t_0+2$. Here M_{12}, M_{13} are independent of $t, \sigma, \mathcal{B}, d$. □

Remark 3.1. From Theorem 3.1 and (2.3) we observe that for every bounded subset $\mathcal{B}$ of E, there is $t_0 = t_0(\mathcal{B}) > 0$ such that

$$\|\hat{\psi}_{t,\sigma}(u_0,v_0)\|_{H_p^2\times H_p^2} \le R_0, \quad t \ge t_0, \sigma\in\Sigma, (u_0,v_0)\in\mathcal{B},$$

where R_0 is independent of d for $d \ge d_0 > 0$.

A direct calculation similar to that in Theorem 3.1, we can obtain

Corollary 3.1. *The family of processes* $\{U_\sigma(t,\tau)\}, \sigma\in\Sigma$ *possesses a uniform (w.r.t.* $\sigma\in\Sigma$*) absorbing set.*

A curve $(u(s),v(s)), s\in\mathbb{R}$ is called a complete trajectory of (2.2) if

$$U_\sigma(t,\tau)(u_\tau,v_\tau) = (u(t),v(t)), \quad \forall t\ge\tau, \tau\in\mathbb{R}. \tag{3.6}$$

A set of all bounded complete trajectories of the process $\{U_\sigma(t,\tau)\}$ is called the kernel $\mathcal{K}_\sigma$ of the process $\{U_\sigma(t,\tau)\}$, that is

$\mathcal{K}_\sigma = \{(u(\cdot),v(\cdot)) : (u(\cdot),v(\cdot))$ satisfies (3.6) and $\|(u(s),v(s))\|$ is bounded for $s\in\mathbb{R}\}$.

The set

$$\mathcal{K}_\sigma(0) = \{(u(0),v(0)) : (u(\cdot),v(\cdot))\in\mathcal{K}_\sigma\}$$

is called the kernel section at $t=0$.

Thanks to Corollary 3.1, an application of [5, Theorem IV 5.1] obtains the following result.

Theorem 3.2. *The family of processes* $\{U_\sigma(t,\tau)\}, \sigma\in\Sigma$ *possesses a compact uniform (w.r.t.* $\sigma\in\Sigma$*) attractor* $\mathcal{A}_\Sigma$*. Moreover, if* Σ *is compact, then* $\mathcal{A}_\Sigma$ *can be represented as*

$$\mathcal{A}_\Sigma = \bigcup_{\sigma\in\Sigma}\mathcal{K}_\sigma(0).$$

4. A modification scheme

In this section, our main purpose is to establish a modification scheme for (2.2). This is mainly based on the uniform dissipative estimates.

We first introduce a smooth cut-off function φ_1 which is given by

$$\varphi_1(s) = 1, \quad s \in [0, (cR_0)^2], \qquad \varphi_1(s) = 0, \quad s \in [4(cR_0)^2, \infty),$$

where c is a positive constant satisfying $\|\theta\|_{L_p^\infty} \leq c\|\theta\|_{H_p^2}$ for $\theta \in H_p^2(\Omega)$ and R_0 is given in Theorem 3.1. We modify h_1 by setting

$$\begin{aligned}\hat{h}_1(u,v)(x) &= (1+d)\varphi_1(u^2(x))u(x) - \alpha\varphi_1(v^2(x))v(x)\\ &\quad + \gamma\varphi_1(u^2(x))u(x)\varphi_1(v^2(x))v(x) - \varphi_1(u^2(x))u^3(x).\end{aligned}$$

It is easy to see that $\hat{h}_1 : H \to L_p^2(\Omega)$ is globally Lipschitz with Lipschitz constant $L_{\hat{h}_1}$ and globally bounded. At the same time, we also modify h_2 by setting

$$\hat{h}_2(v)(x) = (1-\beta)v(x)\varphi_1(v^2(x)).$$

Clearly, $\hat{h}_2$ is globally Lipschitz from $L_p^2(\Omega)$ to $L_p^2(\Omega)$ and globally bounded. Denote by $L_{\hat{h}_2}$ the Lipschitz constant of $\hat{h}_2$. Moreover, $\hat{h}_2$ is Gâteaux differentiable. The derivative is denoted by $d\hat{h}_2(v)$ which can be represented as

$$[d\hat{h}_2(v)l](x) = h_2'(v(x))l(x), \quad v \in L_p^2(\Omega), l \in L_p^2(\Omega).$$

For $\sigma = (\sigma_1, \sigma_2) \in \Sigma$, we consider the following coupled system

$$\begin{cases} u_t = -dAu + \hat{h}_1(u,v) + \sigma_1(t), & t > 0,\\ v_t = -Av + u + \hat{h}_2(v) + \sigma_2(t), & t > 0,\\ u(0) = u_0, v(0) = v_0. \end{cases} \tag{4.1}$$

Clearly, for every $(u_0, v_0) \in H$, (4.1) has a global mild solution $(u,v) \in C([0,\infty); H)$. Hence, we can define a NDS $\tilde{\psi}_{t,\sigma}$ by

$$\tilde{\psi}_{t,\sigma}(u_0, v_0) = (u(t), v(t)), \quad t > 0, \sigma \in \Sigma, (u_0, v_0) \in H.$$

Notice that for every $\sigma \in \Sigma$ and $(u_0, v_0) \in E$ satisfying $\|\hat{\psi}_{t,\sigma}(u_0, v_0)\|_{H_p^2 \times H_p^2} \leq R_0$ for t in some interval $[t_0, \infty)$,

$$\hat{\psi}_{t,\sigma}(u_0, v_0) = \tilde{\psi}_{t-t_0,\theta_{t_0}\sigma}(\hat{\psi}_{t_0,\sigma}(u_0, v_0)), \quad t \geq t_0. \tag{4.2}$$

For $k > 0$ and $\lambda_N > k$, we define the projection operators from $L_p^2(\Omega)$ to $L_p^2(\Omega)$ by the formulas

$$\begin{aligned}\mathcal{P}_{N,k}u &= \sum_{\lambda_j < \lambda_N - k} (u, e_j)e_j,\\ \mathcal{R}_{N,k}u &= \sum_{\lambda_N - k \leq \lambda_j \leq \lambda_N + k} (u, e_j)e_j,\\ \mathcal{Q}_{N,k}u &= \sum_{\lambda_j > \lambda_N + k} (u, e_j)e_j.\end{aligned}$$

The following proposition can be seen from [36, Example 2.38].

Proposition 4.1. *There exists $\varrho \in (0,2)$ such that for every $\varepsilon, k > 0$ and bounded subset $\mathcal{B}$ of $H_p^{2-\varrho}$, there exist infinitely many values of $N \in \mathbb{N}^+$ satisfying*

$$\lambda_{N+1} - \lambda_N \geq 1 \tag{4.3}$$

and for any $v \in \mathcal{B}$,

$$\|\mathcal{R}_{N,k} \circ (d\hat{h}_2(v) - a(v)I) \circ \mathcal{R}_{N,k} l\| \leq \varepsilon \|l\|, \quad l \in L_p^2(\Omega), \tag{4.4}$$

where $a(v) = \frac{1}{|\Omega|} \int_\Omega h_2'(v(x))\mathrm{d}x$.

As in [22], we introduce a smooth cut-off function φ_2 defined as

$$\begin{aligned} &\varphi_2(s) = 1, && s \in [0, R_0^2], \\ &\varphi_2(s) = \frac{1}{2}, && s \in [4R_0^2, \infty), \\ &\varphi_2'(s) \leq 0, && s \in [0, \infty), \\ &s\varphi_2'(s) + \rho\varphi_2(s) \geq 0, && s \in [0, \infty). \end{aligned}$$

Here, $\rho \in (0,1)$. Clearly, the existence of φ_2 is not standard. The specific example of φ_2 can be seen from [24]. Modify the nonlinearity $\hat{h}_2(v)$ by

$$H_N(v) = AP_{N} v - \varphi_2(\|P_{N} v\|^2_{H_p^2}) AP_{N} v + \hat{h}_2(v), \quad v \in L_p^2(\Omega). \tag{4.5}$$

It is easy to see that H_N is globally Lipschitz from $L_p^2(\Omega)$ to $L_p^2(\Omega)$. Moreover, H_N is Gâteaux differentiable. The Gâteaux derivative is denoted by $dH_N(v)$.

For $\sigma = (\sigma_1, \sigma_2) \in \Sigma$, we consider the following modified system

$$\begin{cases} u_t = -dAu + \hat{h}_1(u,v) + \sigma_1(t), & t > 0, \\ v_t = -Av + u + H_N(v) + \sigma_2(t), & t > 0, \\ u(0) = u_0, v(0) = v_0. \end{cases} \tag{4.6}$$

Clearly, we can define a NDS $\overline{\psi}_{t,\sigma}$ generated by mild solutions of (4.6). Moreover, for every $\sigma \in \Sigma$ and $(u_0, v_0) \in E$ satisfying $\|\tilde{\psi}_{t,\sigma}(u_0, v_0)\|_{H_p^2 \times H_p^2} \leq R_0$ for t in some interval $[t_0, \infty)$,

$$\tilde{\psi}_{t,\sigma}(u_0, v_0) = \overline{\psi}_{t-t_0, \theta_{t_0}\sigma}(\tilde{\psi}_{t_0,\sigma}(u_0, v_0)), \quad t \geq t_0. \tag{4.7}$$

For $N \in \mathbb{N}^+$ and $r > 0$, we define a set $D_{N,r}$ by

$$D_{N,r} = \{(u,v) \in D(A) \times H_p^{2-\varrho} : \|u\|^2 + \|Q_{N} v\|^2_{H_p^{2-\varrho}} \leq r\}.$$

Lemma 4.1. *There exists $R \geq R_0$ such that for every $N \in \mathbb{N}^+$, $\overline{\psi}_{t,\sigma}(0,p) \in D_{N,R}$ for all $t \in \mathbb{R}^+, \sigma \in \Sigma, p \in X_{N}^u$.*

Proof. Let $t \in \mathbb{R}^+, \sigma = (\sigma_1, \sigma_2) \in \Sigma, p \in X_{N}^u$. One readily sees that $\overline{\psi}_{t,\sigma}(0,p) \in D(A) \times D(A)$. For simplicity, we write

$$(u(t;\sigma,(0,p)), v(t;\sigma,(0,p))) = \overline{\psi}_{t,\sigma}(0,p).$$

Note that

$$u(t;\sigma,(0,p)) = \int_0^t e^{-dA(t-s)}(\hat{h}_1(u(s;\sigma,(0,p)),v(s;\sigma,(0,p))) + \sigma_1(s))\mathrm{d}s.$$

Then there exists M_1 such that $\|u(t;\sigma,(0,p))\| \leq M_1$ for all $t \in \mathbb{R}^+$. This together with (4.5) implies that

$$\begin{aligned}
&\|Q_N v(t;\sigma,(0,p))\|_{H_p^{2-\varrho}} \\
&\leq \int_0^t \|A^{1-\frac{\varrho}{2}} e^{-A(t-s)}(u(s;\sigma,(0,p)) + Q_N H_N(v(s;\sigma,(0,p))) + \sigma_2(s))\|\mathrm{d}s \\
&\leq M_2 \int_0^t [(t-s)^{1-\frac{\varrho}{2}} + \lambda_{N+1}^{1-\frac{\varrho}{2}}]e^{-\lambda_{N+1}(t-s)}(\|u(s)\| + \|\hat{h}_2(v(s))\| + \|\sigma_2(s)\|)\mathrm{d}s \\
&\leq M_3
\end{aligned}$$

for all $t \in \mathbb{R}^+$. By defining $R = \max\{M_1 + M_3, R_0\}$, we complete the proof. □

For $N \in \mathbb{N}^+$, let us introduce the following indefinite quadratic form $W_N : L_p^2(\Omega) \to \mathbb{R}$ by

$$W_N(w) = \|Q_N w\|^2 - \|P_N w\|^2, \quad w \in L_p^2(\Omega).$$

We also define a functional $V_N : L_p^2(\Omega) \times D(A) \to \mathbb{R}$ by

$$V_N(v,l) = (-Al + dH_N(v)l, Q_N l - P_N l), \quad v \in L_p^2(\Omega), l \in D(A).$$

Let $J_N(v) = \frac{\lambda_{N+1}}{4} + \frac{\lambda_N}{2}$ if $\|P_N v\|_{H_p^2} \geq 2R_0$ and $J_N(v) = \frac{\lambda_{N+1}}{2} + \frac{\lambda_N}{2} - L_{\hat{h}_2}$ if $\|P_N v\|_{H_p^2} < 2R_0$.

Lemma 4.2. *There exists $N_0 \in \mathbb{N}^+$ such that*

$$V_{N_0}(v,l) + J_{N_0}(v)W_{N_0}(l) \leq -\frac{\rho}{4}\|l\|^2$$

for every $v \in H_p^{2-\varrho}$ with $\|Q_{N_0} v\|^2_{H_p^{2-\varrho}} \leq R$ and $l \in D(A)$.

Proof. Choose ε, k such that

$$(1-\rho)k > L_{\hat{h}_2}, \quad \varepsilon + \frac{L_{\hat{h}_2}^2}{1-\rho}\left(k - \frac{L_{\hat{h}_2}}{1-\rho}\right)^{-1} < \frac{\rho}{4}. \tag{4.8}$$

Let $\mathcal{B}$ be a bounded subset satisfying $\|v\|_{H_p^{2-\varrho}} \leq 4R_0^2 + R^2$. It follows from Proposition 4.1 that there exists $N_0 \in \mathbb{N}^+$ such that

$$4L_{\hat{h}_2} - \lambda_{N_0+1} \leq -\rho \tag{4.9}$$

and (4.3)–(4.4) are satisfied with $N = N_0$.

For $v \in L_p^2(\Omega)$,

$$\begin{aligned}
dH_{N_0}(v)l = AP_{N_0}l - 2\varphi_2'(\|P_{N_0}v\|^2_{H_p^2})(AP_{N_0}v, AP_{N_0}l)AP_{N_0}v \\
- \varphi_2(\|P_{N_0}v\|^2_{H_p^2})AP_{N_0}l + d\hat{h}_2(v)l,
\end{aligned}$$

where $l \in L^2_p(\Omega)$. By virtue of [24, Lemma 3.10], one has that

$$\begin{aligned}&V_{N_0}(v,l) + \mu W_{N_0}(l)\\ &\le ((\mu - A)Q_{N_0}l, Q_{N_0}l) - \mu\|P_{N_0}l\| + (\hat{h}_2(v)l, Q_{N_0}l - P_{N_0}l)\\ &\quad + \varphi_2(\|P_{N_0}v\|^2_{H^2_p})\|AP_{N_0}l\|\|P_{N_0}l\|,\end{aligned}$$

where $\mu = \frac{\lambda_{N_0}+\lambda_{N_0+1}}{2}$.

If $\|P_{N_0}v\|_{H^2_p} \ge 2R_0$, then $\varphi_2(\|P_{N_0}v\|^2_{H^2_p}) = \frac{1}{2}$. We hence have that

$$V_{N_0}(v,l) + \mu W_{N_0}(l) \le -\frac{\lambda_{N_0+1}}{2}\|P_{N_0}l\|^2 + L_{\hat{h}_2}\|l\|^2.$$

This together with the condition (4.9) implies that

$$V_{N_0}(v,l) + J_{N_0}(v)W_{N_0}(l) \le \left(-\frac{\lambda_{N_0+1}}{4} + L_{\hat{h}_2}\right)\|l\|^2 \le -\frac{\rho}{4}\|l\|^2.$$

If $\|P_{N_0}v\|_{H^2_p} < 2R_0$, then $\varphi_2(\|P_{N_0}v\|^2_{H^2_p}) = 1$. Thus, one has that

$$\begin{aligned}&V_{N_0}(v,l) + \mu W_{N_0}(l)\\ &\le \rho((\mu - A)Q_{N_0}l, Q_{N_0}l) - \mu\rho\|P_{N_0}l\|^2 + \rho\lambda_{N_0}\|P_{N_0}l\|^2 + (d\hat{h}_2(v)l, Q_{N_0}l - P_{N_0}l)\\ &\quad + (1-\rho)(((A-\mu)P_{N_0}l, P_{N_0}l) + ((\mu - A)Q_{N_0}l, Q_{N_0}l))\\ &\le -\frac{\rho}{2}\|l\|^2 + (d\hat{h}_2(v)l, Q_{N_0}l - P_{N_0}l)\\ &\quad + (1-\rho)(((A-\mu)P_{N_0}l, P_{N_0}l) + ((\mu - A)Q_{N_0}l, Q_{N_0}l)).\end{aligned}$$

Here, we have used the condition (4.3). Following [36], we can obtain that

$$\begin{aligned}((\mu - A)Q_{N_0}l, Q_{N_0}l) &\le -k\|\mathcal{Q}_{N_0,k}l\|^2,\\ ((A-\mu)P_{N_0}l, P_{N_0}l) &\le -k\|\mathcal{P}_{N_0,k}l\|^2.\end{aligned}$$

Since $l = \mathcal{P}_{N_0,k}l + \mathcal{R}_{N_0,k}l + \mathcal{Q}_{N_0,k}l$, one finds that

$$\begin{aligned}&(d\hat{h}_2(v)l, Q_{N_0}l - P_{N_0}l)\\ &\le (\mathcal{R}_{N_0,k} \circ d\hat{h}_2(v) \circ \mathcal{R}_{N_0,k}l, Q_{N_0}l - P_{N_0}l)\\ &\quad + (\mathcal{R}_{N_0,k} \circ d\hat{h}_2(v) \circ (\mathcal{P}_{N_0,k} + \mathcal{Q}_{N_0,k})l, Q_{N_0}l - P_{N_0}l)\\ &\quad + (d\hat{h}_2(v)l, \mathcal{Q}_{N_0,k}l - \mathcal{P}_{N_0,k}l).\end{aligned}$$

Thanks to $|a(v)| \le L_{\hat{h}_2}$ and (4.4), we get that

$$\begin{aligned}&(\mathcal{R}_{N_0,k} \circ d\hat{h}_2(v) \circ \mathcal{R}_{N_0,k}l, Q_{N_0}l - P_{N_0}l)\\ &\le (\mathcal{R}_{N_0,k} \circ (d\hat{h}_2(v) - a(v)) \circ \mathcal{R}_{N_0,k}l, Q_{N_0}l - P_{N_0}l) + |a(v)|(\mathcal{R}_{N_0,k}l, Q_{N_0}l - P_{N_0}l)\\ &\le \varepsilon\|l\|^2 + L_{\hat{h}_2}(\|\mathcal{P}_{N_0,k}l\|^2 + \|\mathcal{Q}_{N_0,k}l\|^2) + L_{\hat{h}_2}(\|Q_{N_0}l\|^2 - \|P_{N_0}l\|^2).\end{aligned}$$

Furthermore, using the Hölder inequality and the Cauchy-Schwarz inequality, one has that

$$\begin{aligned}&(\mathcal{R}_{N_0,k} \circ d\hat{h}_2(v) \circ (\mathcal{P}_{N_0,k} + \mathcal{Q}_{N_0,k})l, Q_{N_0}l - P_{N_0}l) + (d\hat{h}_2(v)l, \mathcal{Q}_{N_0,k}l - \mathcal{P}_{N_0,k}l)\\ &\le 2L_{\hat{h}_2}\|l\|(\|\mathcal{P}_{N_0,k}l\|^2 + \|\mathcal{Q}_{N_0,k}l\|^2)^{\frac{1}{2}}\\ &\le \frac{L^2_{\hat{h}_2}}{1-\rho}\left(k - \frac{L_{\hat{h}_2}}{1-\rho}\right)^{-1}\|l\|^2 + (1-\rho)\left(k - \frac{L_{\hat{h}_2}}{1-\rho}\right)(\|\mathcal{P}_{N_0,k}l\|^2 + \|\mathcal{Q}_{N_0,k}l\|^2).\end{aligned}$$

We hence conclude, due to condition (4.8), that

$$V_{N_0}(v,l)+J_{N_0}(v)W_{N_0}(l)\le\left[-\frac{\rho}{2}+\varepsilon+\frac{L_{\hat{h}_2}^2}{1-\rho}\left(k-\frac{L_{\hat{h}_2}}{1-\rho}\right)^{-1}\right]\|l\|^2\le-\frac{\rho}{4}\|l\|^2.$$

□

To continue, for $N\in\mathbb{N}^+$, let us define

$$Y_N(w_1,w_2)=\|w_1\|^2+\|Q_Nw_2\|^2-\|P_Nw_2\|^2,\quad (w_1,w_2)\in H.$$

For $\sigma\in\Sigma$, we also define

$$D(\sigma)=\{(u,v)\in H:\overline{\psi}_{t,\sigma}(u,v)\in D_{N_0,R}\text{ for all }t\in\mathbb{R}^+\}.$$

With the preparation above at hand, we can prove the following

Theorem 4.1. *There exist $R\ge R_0$ and $N_0\in\mathbb{N}^+$ such that if*

$$d\ge\max\left\{\frac{\rho}{16\lambda_1}+\frac{4L_{\hat{h}_1}^2}{\rho\lambda_1}+\frac{4}{\rho\lambda_1}+\frac{\lambda_{N_0}+\lambda_{N_0+1}}{2\lambda_1},d_0\right\},\tag{4.10}$$

then for every $\sigma\in\Sigma$ and $(u_{0i},v_{0i})\in D(\sigma)$, $i=1,2$, there exists a function $y:\mathbb{R}^+\to[\frac{\lambda_{N_0+1}}{2}+\lambda_{N_0},\lambda_{N_0+1}+\lambda_{N_0}]$ satisfying

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}Y_{N_0}(\overline{\psi}_{t,\sigma}(u_{01},v_{01})-\overline{\psi}_{t,\sigma}(u_{02},v_{02}))\\&\le-y(t)Y_{N_0}(\overline{\psi}_{t,\sigma}(u_{01},v_{01})-\overline{\psi}_{t,\sigma}(u_{02},v_{02}))\end{aligned}\tag{4.11}$$

for a.e. $t\in\mathbb{R}^+$, where $y\in L^1(0,T)$ for every $T>0$.

Proof. Let $\sigma\in\Sigma$ and $(u_{0i},v_{0i})\in D(\sigma)$, $i=1,2$. For simplicity, we write

$$\begin{aligned}&(u(t;\sigma,(u_{0i},v_{0i})),v(t;\sigma,(u_{0i},v_{0i})))=\overline{\psi}_{t,\sigma}(u_{0i},v_{0i}),\ i=1,2,\\&w_1(t)=u(t;\sigma,(u_{01},v_{01}))-u(t;\sigma,(u_{02},v_{02})),\\&w_2(t)=v(t;\sigma,(u_{01},v_{01}))-v(t;\sigma,(u_{02},v_{02})).\end{aligned}$$

It is easy to see that $(u(t;\sigma,(u_{0i},v_{0i})),v(t;\sigma,(u_{0i},v_{0i}))),i=1,2$ are strong solutions. Hence, $Y_{N_0}(\overline{\psi}_{t,\sigma}(u_{01},v_{01})-\overline{\psi}_{t,\sigma}(u_{02},v_{02}))$ is absolutely continuous on any closed subset of $\mathbb{R}^+$. Thanks to Lemma 4.2, we conclude that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}Y_{N_0}(w_1,w_2)\\&=(-dAw_1+\hat{h}_1(\overline{\psi}_{t,\sigma}(u_{01},v_{01}))-\hat{h}_1(\overline{\psi}_{t,\sigma}(u_{02},v_{02})),w_1)\\&\quad+(-Aw_2+H_{N_0}(v_1)+H_{N_0}(v_2),Q_{N_0}w_2-P_{N_0}w_2)+(w_1,Q_{N_0}w_2-P_{N_0}w_2)\\&\le-d\lambda_1\|w_1\|^2+\frac{\rho}{8}(\|w_1\|^2+\|w_2\|^2)+\frac{2L_{\hat{h}_1}^2}{\rho}\|w_1\|^2\\&\quad-\int_0^1J_{N_0}(sv(t;\sigma,(u_{01},v_{01}))+(1-s)v(t;\sigma,(u_{02},v_{02})))\mathrm{d}sW_{N_0}(w_2)\\&\quad-\frac{\rho}{4}\|w_2\|^2+\frac{\rho}{8}\|w_2\|^2+\frac{2}{\rho}\|w_1\|^2.\end{aligned}$$

Here, we have used the Hölder inequality and Young inequality. Putting

$$y(t) = 2\int_0^1 J_{N_0}(sv(t;\sigma,(u_{01},v_{01})) + (1-s)v(t;\sigma,(u_{02},v_{02})))\mathrm{d}s.$$

Clearly, $y \in L^1(0,T)$ for any $T > 0$ and $y(t) \in [\frac{\lambda_{N_0+1}}{2} + \lambda_{N_0}, \lambda_{N_0+1} + \lambda_{N_0}]$ for all $t \in \mathbb{R}^+$. Then we obtain that

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t} Y_{N_0}(w_1(t), w_2(t)) + y(t) Y_{N_0}(w_1(t), w_2(t)) \\ &\le 2\left(-d\lambda_1 + \frac{\rho}{16} + \frac{4L_{\hat{h}_1}^2}{\rho} + \frac{4}{\rho} + \frac{\lambda_{N_0} + \lambda_{N_0+1}}{2}\right)\|w_1(t)\|^2 - \frac{\rho}{8}\|w_2(t)\|^2 \end{aligned} \tag{4.12}$$

and the theorem is proved. □

Define a cone K by

$$K = \{(u,v) \in H : \|u\|^2 + \|Q_{N_0} v\|^2 \le \|P_{N_0} v\|^2\}.$$

In order to show the existence of invariant manifolds of global type for (1.1)–(1.2), we will need the following theorem.

Theorem 4.2. *Assume the condition of Theorem* 4.1. *Let* $\sigma \in \Sigma$ *and* $(u_i, v_i) \in D(\sigma), i = 1,2$. *Then the following properties hold.*

(i) If $(u_1,v_1) - (u_2,v_2) \in K$, *then* $\overline{\psi}_{t,\sigma}(u_1,v_1) - \overline{\psi}_{t,\sigma}(u_2,v_2) \in K$ *for all* $t \in \mathbb{R}^+$.

(ii) There exist positive constants M *and* ν *such that if*

$$P_{N_0} v(r;\sigma,(u_1,v_1)) = P_{N_0} v(r;\sigma,(u_2,v_2)) \tag{4.13}$$

for $r > 0$, *then*

$$\|\overline{\psi}_{t,\sigma}(u_1,v_1) - \overline{\psi}_{t,\sigma}(u_2,v_2)\| \le M\|(u_1, Q_{N_0} v_1) - (u_2, Q_{N_0} v_2)\| e^{-\nu t}$$

for any $t \in [0,r]$, *where* $(u(t;\sigma,(u_i,v_i)), v(t;\sigma,(u_i,v_i))) = \overline{\psi}_{t,\sigma}(u_i,v_i)$, $i = 1,2$.

Proof. Let $\sigma \in \Sigma$ and $(u_i,v_i) \in D(\sigma), i = 1,2$. For simplicity, we write

$$\begin{aligned} w_1(t) &= u(t;\sigma,(u_1,v_1)) - u(t;\sigma,(u_2,v_2)), \\ w_2(t) &= v(t;\sigma,(u_1,v_1)) - v(t;\sigma,(u_2,v_2)). \end{aligned}$$

From (4.11), it follows that

$$\begin{aligned} &\|w_1(t)\|^2 + \|Q_{N_0} w_2(t)\|^2 - \|P_{N_0} w_2(t)\|^2 \\ &\le e^{-\int_0^t y(s)\mathrm{d}s}(\|w_1(0)\|^2 + \|Q_{N_0} w_2(0)\|^2 - \|P_{N_0} w_2(0)\|^2). \end{aligned}$$

Using the condition $(u_1,v_1) - (u_2,v_2) \in K$, we get that for all $t \in \mathbb{R}^+$,

$$\|w_1(t)\|^2 + \|Q_{N_0} w_2(t)\|^2 - \|P_{N_0} w_2(t)\|^2 \le 0.$$

As a consequence, we obtain the conclusion (i).

Again by using (4.11), it therefore follows that

$$\begin{aligned} &\|w_1(r)\|^2 + \|Q_{N_0} w_2(r)\|^2 - \|P_{N_0} w_2(r)\|^2 \\ &\le e^{-\int_t^r y(s)\mathrm{d}s}(\|w_1(t)\|^2 + \|Q_{N_0} w_2(t)\|^2 - \|P_{N_0} w_2(t)\|^2) \end{aligned}$$

for all $t \in [0, r]$. We obtain, due to (4.13), that for $t \in [0, r]$,

$$\|w_1(t)\|^2 + \|Q_{N_0} w_2(t)\|^2 \geq \|P_{N_0} w_2(t)\|^2. \tag{4.14}$$

It is not difficult to see, using the Hölder inequality and Young inequality, that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|w_1\|^2 \leq (1 + L_{\hat{h}_1})\|w_1\|^2 + L_{\hat{h}_1}\|w_2\|^2 \tag{4.15}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\|w_2\|^2 \leq (1 + L_{H_{N_0}})\|w_2\|^2 + \|w_1\|^2. \tag{4.16}$$

Let us define

$$Y^a_{N_0}(w_1, w_2) = a\|w_1\|^2 + a\|w_2\|^2 + Y_{N_0}(w_1, w_2).$$

Put

$$ay(t) + a(2 + L_{\hat{h}_1}) \leq 2\left(d\lambda_1 - \frac{\rho}{16} - \frac{4L^2_{\hat{h}_1} + 4}{\rho} - \frac{\lambda_{N_0} + \lambda_{N_0+1}}{2}\right)$$

and

$$ay(t) + a(1 + L_{\hat{h}_1} + 2L_{H_{N_0}}) \leq \frac{\rho}{8}.$$

Making use of (4.12), (4.15), (4.16), we obtain that

$$\frac{\mathrm{d}}{\mathrm{d}t}Y^a_{N_0}(w_1(t), w_2(t)) \leq -y(t)Y^a_{N_0}(w_1(t), w_2(t)).$$

This together with (4.14) implies that

$$\begin{aligned}
\|w_1(t)\|^2 + \|w_2(t)\|^2 &\leq a^{-1}Y^a_{N_0}(w_1(t), w_2(t)) \\
&\leq a^{-1}e^{-\int_0^t y(s)\mathrm{d}s}Y^a_{N_0}(w_1(0), w_2(0)) \\
&\leq a^{-1}(1 + 2a)(\|w_1(0)\|^2 + \|Q_{N_0}w_2(0)\|^2)e^{-\left(\frac{\lambda_{N_0+1}}{2} + \lambda_{N_0}\right)t}
\end{aligned}$$

for $t \in [0, r]$. Thus, the theorem is proved. □

5. Existence of An Invariant Manifold of Global Type

The present section is concerned with the existence of an invariant manifold of global type for (1.1)–(1.2).

We first show the following properties of $\overline{\psi}_{t,\sigma}$.

Lemma 5.1. *For every $r > 0$, there exists $M_0 = M_0(r) > 0$ such that for every $\sigma \in \Sigma$ and $(u_i, v_i) \in D(\sigma), i = 1, 2$ satisfying $(u_1, v_1) - (u_2, v_2) \in K$,*

$$\|P_{N_0}v_1 - P_{N_0}v_2\| \leq M_0\|P_{N_0}v(t; \sigma, (u_1, v_1)) - P_{N_0}v(t; \sigma, (u_2, v_2))\|, \quad t \in [0, r], \tag{5.1}$$

where $v(\cdot; \sigma, (u_i, v_i))$, $i = 1, 2$ satisfy (4.6) with initial values (u_i, v_i).

Proof. Let $\sigma \in \Sigma$ and $(u_i, v_i) \in D(\sigma), i = 1, 2$ satisfy $(u_1, v_1) - (u_2, v_2) \in K$. For simplicity, we keep here the notations of the proof of Theorem 4.2. From Theorem 4.2(i), it follows that $(w_1(t), w_2(t)) \in K$ for all $t \in \mathbb{R}^+$. We thus conclude that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|P_{N_0} w_2\|^2 &\geq -M_1\|P_{N_0} w_2\|^2 - 2\|P_{N_0} w_2\|\|w_1\| - M_2\|P_{N_0} w_2\|\|w_2\| \\ &\geq -M_3\|P_{N_0} w_2\|^2 - M_4(\|w_1\|^2 + \|Q_{N_0} w_2\|^2) \\ &\geq -M_5\|P_{N_0} w_2\|^2.\end{aligned}$$

Then, integrating the above inequality on $[0, r]$ enables us to obtain (5.1). □

Lemma 5.2. *For every $\sigma \in \Sigma, p \in X^u_{N_0}, T > 0$, the set*

$$\{P_{N_0} v(\cdot; \sigma, (u_0, p + q))|_{[0,T]} : (u_0, q) \in Q_{N_0} D_{N_0,R}\}$$

is relatively compactly in $C([0, T]; X^u_{N_0})$, where $v(\cdot; \sigma, (u_0, p+q))$ satisfies (4.6) *with initial value $(u_0, p + q)$.*

Proof. Let $\sigma \in \Sigma, p \in X^u_{N_0}, T > 0$. Clearly, for $(u_0, q) \in Q_{N_0} D_{N_0,R}$,

$$(u(t; \sigma, (u_0, p + q)), v(t; \sigma, (u_0, p + q))) \in D(A) \times D(A), \quad t \in [0, T].$$

By the Gronwall inequality, we get that the set $\{\overline{\psi}_{\cdot,\sigma}(u_0, p + q)|_{[0,T]} : (u_0, q) \in Q_{N_0} D_{N_0,R}\}$ is bounded in $C([0, T]; H)$. Then for every $t \in [0, T]$, the set

$$\{P_{N_0} v(t; \sigma, (u_0, p + q)) : (u_0, q) \in Q_{N_0} D_{N_0,R}\}$$

is bounded in $X^u_{N_0}$. Moreover, for every $(u_0, q) \in Q_{N_0} D_{N_0,R}$ and $t \in [0, T - l]$ with $l \in [0, T]$,

$$\begin{aligned}&\|P_{N_0} v(t + l; \sigma, (u_0, p + q)) - P_{N_0} v(t; \sigma, (u_0, p + q))\| \\ &\leq \int_t^{t+l} \| - AP_{N_0} v(s) + u(s) + H_{N_0}(v(s)\|\mathrm{d}s + l\|h\|_{L^\infty(\mathbb{R};L^2_p(\Omega))},\end{aligned}$$

which implies that the set $\{P_{N_0} v(\cdot; \sigma, (u_0, p + q))|_{[0,T]} : (u_0, q) \in Q_{N_0} D_{N_0,R}\}$ is equicontinuous on $[0, T]$. Thanks to Ascoli-Arzelà theorem, we obtain the conclusion of the lemma. □

We next prove the following

Theorem 5.1. *There exist $R > 0, N_0 \in \mathbb{N}^+$ such that for every d satisfying* (4.10), *there exists a map $\Phi(\sigma, \cdot) : X^u_{N_0} \to L^2_p(\Omega) \times X^s_{N_0}, \sigma \in \Sigma$ such that*

(i) for $p_1, p_2 \in X^u_{N_0}$,
$$\|\Phi(\sigma, p_1) - \Phi(\sigma, p_2)\| \leq \|p_1 - p_2\|;$$
(ii) $\mathscr{M} = \{\mathscr{M}(\sigma)\}_{\sigma\in\Sigma}$ with $\mathscr{M}(\sigma) = \mathrm{graph}\Phi(\sigma, \cdot)$ is locally forward invariant under the NDS $\psi_{t,\sigma}$, i.e., for every $\sigma \in \Sigma$ and $z_0 \in \mathscr{M}(\sigma)$ satisfying $\psi_{t,\sigma} z_0 \in \mathscr{B}_1$ for $t \in [0, \varepsilon)$, one has $\psi_{t,\sigma} z_0 \in \mathscr{M}(\theta_t\sigma)$ for any $t \in (0, \varepsilon)$;

(iii) $\mathscr{M}(\sigma) \subset D_{N_0,R}$ for all $\sigma \in \Sigma$.

Proof. Let $r > 0, \sigma \in \Sigma$ and

$$\mathscr{M}_r(\sigma) = (u(r; \theta_{-r}\sigma, (0, X^u_{N_0})), v(r; \theta_{-r}\sigma, (0, X^u_{N_0}))).$$

From Lemma 4.1, it follows that

$$\mathscr{M}_r(\sigma) \subset D_{N_0,R}. \tag{5.2}$$

Define a map $G_r(\sigma,\cdot): X^u_{N_0} \to X^u_{N_0}$ by

$$G_r(\sigma,p) = P_{N_0} v(r;\theta_{-r}\sigma,(0,p)), \quad p \in X^u_{N_0}.$$

It is easy to see that $G_r(\sigma,\cdot)$ is continuous. We shall prove that $G_r(\sigma,\cdot)$ is injective. In fact, let $p_1, p_2 \in X^u_{N_0}$ and $p_1 \neq p_2$. Then we obtain that, due to Lemma 4.1, $\overline{\psi}_{r,\theta_{-r}\sigma}(0,p_i) \in D_{N_0,R}$, $i = 1,2$. Note that $(0,p_1)-(0,p_2) \in K$. By Lemma 5.1, one has that

$$\|p_1 - p_2\| \le M_0 \|G_r(\sigma,p_1) - G_r(\sigma,p_2)\|,$$

which implies that $G_r(\sigma,\cdot)$ is injective. Following a similar proof as in [22, Theorem 7.3], we can obtain that $G_r(\sigma,\cdot)$ is surjective. Hence, $G_r(\sigma,\cdot)$ is a homeomorphism on $X^u_{N_0}$.

For $p \in X^u_{N_0}$, define $\Phi_r(\sigma,p) = (\Phi^1_r(\sigma,p), \Phi^2_r(\sigma,p))$, where

$$\Phi^1_r(\sigma,p) = u(r;\theta_{-r}\sigma,(0,G_r(\sigma,\cdot)^{-1}(p))),$$
$$\Phi^2_r(\sigma,p) = Q_{N_0} v(r;\theta_{-r}\sigma,(0,G_r(\sigma,\cdot)^{-1}(p))).$$

We next claim that

$$\mathscr{M}_r(\sigma) = \mathrm{graph}(\Phi_r(\sigma,\cdot)). \tag{5.3}$$

In fact, if $(u_0,v_0) \in \mathrm{graph}(\Phi_r(\sigma,\cdot))$, then

$$\begin{aligned}
&(u_0,v_0)\\
&= (\Phi^1_r(\sigma,P_{N_0}v_0), P_{N_0}v_0 + \Phi^2_r(\sigma,P_{N_0}v_0))\\
&= (u(r;\theta_{-r}\sigma,(0,G_r(\sigma,\cdot)^{-1}(P_{N_0}v_0))), v(r;\theta_{-r}\sigma,(0,G_r(\sigma,\cdot)^{-1}(P_{N_0}v_0))))\\
&\in \mathscr{M}_r(\sigma).
\end{aligned}$$

Here, we have used the fact $P_{N_0}v_0 = G_r(\sigma,\cdot)[G_r(\sigma,\cdot)^{-1}(P_{N_0}v_0)]$. If $(u_0,v_0) \in \mathscr{M}_r(\sigma)$, then there exists $p_0 \in X^u_{N_0}$ such that

$$(u_0,v_0) = (u(r;\theta_{-r}\sigma,(0,p_0)), v(r;\theta_{-r}\sigma,(0,p_0))).$$

This implies that $P_{N_0}v_0 = P_{N_0}v(r;\theta_{-r}\sigma,(0,p_0)) = G_r(\sigma,p_0)$. Hence,

$$u_0 = \Phi^1_r(\sigma,P_{N_0}v_0), \quad Q_{N_0}v_0 = \Phi^2_r(\sigma,P_{N_0}v_0),$$

i.e., $(u_0,v_0) \in \mathrm{graph}(\Phi_r(\sigma,\cdot))$. Conversely, if $p_1, p_2 \in X^u_{N_0}$ and $p_1 \neq p_2$, then $(0,G_r(\sigma,\cdot)^{-1}(p_1)) - (0,G_r(\sigma,\cdot)^{-1}(p_2)) \in K$ and

$$\overline{\psi}_{r,\theta_{-r}\sigma}(0,G_r(\sigma,\cdot)^{-1}(p_i)) \in D_{N_0,R}, \quad i = 1,2.$$

Using Theorem 4.2(i), we obtain that

$$\overline{\psi}_{r,\theta_{-r}\sigma}(0,G_r(\sigma,\cdot)^{-1}(p_1)) - \overline{\psi}_{r,\theta_{-r}\sigma}(0,G_r(\sigma,\cdot)^{-1}(p_2)) \in K.$$

Then one has that

$$\|\Phi_r(\sigma,p_1) - \Phi_r(\sigma,p_2)\| \le \|p_1 - p_2\|. \tag{5.4}$$

On the other hand, from (5.3), it follows that

$$(\Phi_r^1(\sigma, p), \Phi_r^2(\sigma, p)) \in Q_{N_0} D_{N_0,R}. \tag{5.5}$$

Let $r_1 > r_2 > 0$ and $p \in X_{N_0}^u$. Note that

$$\begin{aligned} &P_{N_0} v(r_1; \theta_{-r_1}\sigma, (0, G_{r_1}(\sigma, \cdot)^{-1}(p))) = P_{N_0} v(r_2; \theta_{-r_2}\sigma, (0, G_{r_2}(\sigma, \cdot)^{-1}(p))), \\ &\overline{\psi}_{t,\theta_{-r_i}\sigma}(0, G_{r_i}(\sigma, \cdot)^{-1}(p)) \in D_{N_0,R}, \quad t \in \mathbb{R}^+, i = 1, 2. \end{aligned} \tag{5.6}$$

Thanks to the cocycle property of $\overline{\psi}_{t,\sigma}$, we have that

$$\overline{\psi}_{r_1,\theta_{-r_1}\sigma}(0, G_{r_1}(\sigma, \cdot)^{-1}(p)) = \overline{\psi}_{r_2,\theta_{-r_2}\sigma}(\overline{\psi}_{r_1-r_2,\theta_{-r_1}\sigma}(0, G_{r_1}(\sigma, \cdot)^{-1}(p))). \tag{5.7}$$

This together with (5.6) implies that

$$\begin{aligned} &P_{N_0} v(r_2; \theta_{-r_2}\sigma, (0, G_{r_2}(\sigma, \cdot)^{-1}(p))) \\ &= P_{N_0} v(r_2; \theta_{-r_2}\sigma, \overline{\psi}_{r_1-r_2,\theta_{-r_1}\sigma}(0, G_{r_1}(\sigma, \cdot)^{-1}(p))). \end{aligned}$$

By Theorem 4.2(*ii*) and (5.7), one has that

$$\begin{aligned} &\|\Phi_{r_1}(\sigma, p) - \Phi_{r_2}(\sigma, p)\| \\ &= \|\overline{\psi}_{r_1,\theta_{-r_1}\sigma}(0, G_{r_1}(\sigma, \cdot)^{-1}(p)) - \overline{\psi}_{r_2,\theta_{-r_2}\sigma}(0, G_{r_2}(\sigma, \cdot)^{-1}(p))\| \\ &\le M_6 e^{-\nu r_2}. \end{aligned} \tag{5.8}$$

This implies that $\Phi_r(\sigma, p)$ satisfies a Cauchy condition as $r \to \infty$.

Let

$$\lim_{r\to\infty} \Phi_r^1(\sigma, p) = \Phi^1(\sigma, p), \quad \lim_{r\to\infty} \Phi_r^2(\sigma, p) = \Phi^2(\sigma, p). \tag{5.9}$$

This together with (5.4) yields conclusion (*i*).

We write $\mathscr{M}(\sigma) = \text{graph}\Phi(\sigma, \cdot)$. Let $p \in X_{N_0}^u$. For $r > 0$, define

$$\begin{aligned} &\psi_r(t, \sigma) = \overline{\psi}_{t+r,\theta_{-r}\sigma}(0, G_r(\sigma, \cdot)^{-1}(p)), \\ &\hat{\psi}_r(t, \sigma) = (u(t + r; \theta_{-r}\sigma, (0, G_r(\sigma, \cdot)^{-1}(p))), Q_{N_0} v(t + r; \theta_{-r}\sigma, (0, G_r(\sigma, \cdot)^{-1}(p)))), \\ &\tilde{\psi}_r(t, \sigma) = P_{N_0} v(t + r; \theta_{-r}\sigma, (0, G_r(\sigma, \cdot)^{-1}(p))), \quad t \ge -r, \sigma \in \Sigma. \end{aligned}$$

By the cocycle property of $\overline{\psi}_{t,\sigma}$, we get that

$$\begin{aligned} \tilde{\psi}_r(t, \sigma) &= P_{N_0} v(t; \sigma, (u(r; \theta_{-r}\sigma, (0, G_r(\sigma, \cdot)^{-1}(p))), v(r; \theta_{-r}\sigma, (0, G_r(\sigma, \cdot)^{-1}(p))))) \\ &= P_{N_0} v(t; \sigma, (\Phi_r^1(\sigma, p), p + \Phi_r^2(\sigma, p))). \end{aligned}$$

This together with Lemma 5.2 and (5.5) yields that the set $\{\tilde{\psi}_r(\cdot, \sigma)|_{[0,T]} : r > 0\}$ is relatively compact in $C([0, T]; X_{N_0}^u)$. Hence, there exists a sequence r_n such that $r_n \to \infty$ and $\{\tilde{\psi}_{r_n}(\cdot, \sigma)\}_n$ converges uniformly on any compact subset of $\mathbb{R}^+$. We then define a function on $X_{N_0}^u$ by

$$\tilde{\psi}(t, \sigma) = \lim_{n\to\infty} \tilde{\psi}_{r_n}(t, \sigma). \tag{5.10}$$

From (5.4) and (5.8), it follows that for $t \geq -d_n$ and $r_k \geq r_n$,

$$\begin{aligned} &\|\hat{\psi}_{r_k}(t,\sigma) - \hat{\psi}_{r_n}(t,\sigma)\| \\ &\leq \|\Phi_{t+r_k}(\theta_t\sigma, \tilde{\psi}_{r_k}(t,\sigma)) - \Phi_{t+r_n}(\theta_t\sigma, \tilde{\psi}_{r_k}(t,\sigma))\| \\ &\quad + \|\Phi_{t+r_n}(\theta_t\sigma, \tilde{\psi}_{r_k}(t,\sigma)) - \Phi_{t+r_n}(\theta_t\sigma, \tilde{\psi}_{r_n}(t,\sigma))\| \\ &\leq M_7 e^{-\nu(r_n+t)} + \|\tilde{\psi}_{r_k}(t,\sigma) - \tilde{\psi}_{r_n}(t,\sigma)\|, \end{aligned}$$

which implies that $\{\hat{\psi}_{r_n}(\cdot,\sigma)\}_n$ converges uniformly on any compact subset of $\mathbb{R}^+$. We hence conclude that $\{\psi_{r_n}(\cdot,\sigma)\}_n$ converges uniformly on any compact subset of $\mathbb{R}^+$. Let

$$(\psi_u(t,\sigma), \psi_v(t,\sigma)) = \lim_{n\to\infty} \psi_{r_n}(t,\sigma).$$

Note that $\tilde{\psi}(t,\sigma) = P_{N_0}\psi_v(t,\sigma)$. For $t \geq -r_n$, using (5.4) again, we can obtain that

$$\begin{aligned} &\|\Phi_{t+r_n}(\theta_t\sigma, \tilde{\psi}_{r_n}(t,\sigma)) - \Phi(\theta_t\sigma, \tilde{\psi}(t,\sigma))\| \\ &\leq \|\tilde{\psi}_{r_n}(t,\sigma) - \tilde{\psi}(t,\sigma)\| + \|\Phi_{t+r_n}(\theta_t\sigma, \tilde{\psi}(t,\sigma)) - \Phi(\theta_t\sigma, \tilde{\psi}(t,\sigma))\|. \end{aligned}$$

This together with (5.9) and (5.10) yields that

$$\Phi_{t+r_n}(\theta_t\sigma, \tilde{\psi}_{r_n}(t,\sigma)) \to \Phi(\theta_t\sigma, \tilde{\psi}(t,\sigma))$$

as $n \to \infty$ for any $t \in \mathbb{R}^+$. By the uniqueness of limit, one has that

$$(\psi_u(t,\sigma), Q_{N_0}\psi_v(t,\sigma)) = \Phi(\theta_t\sigma, P_{N_0}\psi_v(t,\sigma)).$$

Hence, $(\psi_u(t,\sigma), \psi_v(t,\sigma)) \in \mathscr{M}(\theta_t\sigma)$.

On the other hand, fix $n \in \mathbb{N}^+$. Using the cocycle property of $\overline{\psi}_{t,\sigma}$, we have that

$$\begin{aligned} \psi_{r_n}(t,\sigma) &= \overline{\psi}_{t,\sigma}\psi_{r_n}(0,\sigma) \\ &= \overline{\psi}_{t,\sigma}(\Phi^1_{r_n}(\sigma,p), p + \Phi^2_{r_n}(\sigma,p)). \end{aligned}$$

Then we obtain that

$$(\psi_u(t,\sigma), \psi_v(t,\sigma)) = \overline{\psi}_{t,\sigma}(\Phi^1(\sigma,p), p + \Phi^2(\sigma,p)),$$

which implies that $\overline{\psi}_{t,\sigma}(\mathscr{M}(\sigma)) \subset \mathscr{M}(\theta_t\sigma)$.

Let $\sigma \in \Sigma$ and $z_0 \in \mathscr{M}(\sigma)$ satisfy $\psi_{t,\sigma}z_0 \in \mathscr{B}_1$ for $t \in [0,\varepsilon)$. Using (2.3), (4.2) and (4.7), one has $\psi_{t,\sigma}z_0 = \overline{\psi}_{t,\sigma}z_0$ for $t \in [0,\varepsilon)$. Hence, conclusion (*ii*) holds. Thanks to (5.2) and (5.3), we obtain conclusion (*iii*). □

The following lemma is considered as another important property of $\overline{\psi}_{t,\sigma}$.

Lemma 5.3. *For every $\sigma \in \Sigma$ and $(u_1,v_1) \in D(\sigma)$, there exists $(u_2,v_2) \in \mathscr{M}(\sigma)$ such that for all $t \in \mathbb{R}^+$,*

$$\|\overline{\psi}_{t,\sigma}(u_1,v_1) - \overline{\psi}_{t,\sigma}(u_2,v_2)\| \leq M_8 e^{-\nu t}.$$

Proof. Let $\sigma \in \Sigma$, $(u_1,v_1) \in D(\sigma)$ and t_n be a monotonically increasing sequence satisfying $t_n \to \infty$ as $n \to \infty$. We claim that following problem

$$P_{N_0}v(t_n;\sigma,(\Phi^1(\sigma,p), p + \Phi^2(\sigma,p))) = P_{N_0}v(t_n;\sigma,(u_1,v_1)), \quad p \in X^u_{N_0} \tag{5.11}$$

has a unique solution in $X^u_{N_0}$. To illustrate this, we define a map $F : X^u_{N_0} \to X^u_{N_0}$ by

$$F(p) = P_{N_0} v(t_n; \sigma, (\Phi^1(\sigma, p), p + \Phi^2(\sigma, p))).$$

Clearly, F is continuous. Let $p_1, p_2 \in X^u_{N_0}$ and $p_1 \neq p_2$. From Theorem 5.1(i),(iii), it follows that

$$\begin{aligned}&(\Phi^1(\sigma, p_i), p_i + \Phi^2(\sigma, p_i)) \in D_{N_0,R}, \quad i = 1, 2,\\&(\Phi^1(\sigma, p_1), p_1 + \Phi^2(\sigma, p_1)) - (\Phi^1(\sigma, p_2), p_2 + \Phi^2(\sigma, p_2)) \in K.\end{aligned}$$

Using Lemma 5.1, we obtain that

$$\|p_1 - p_2\| \leq M_0 \|F(p_1) - F(p_2)\|,$$

which implies that F is injective. Following a similar proof as in [22, Theorem 7.3], we get that F is surjective. Thus, F is a homeomorphism on $X^u_{N_0}$. Hence, (5.11) has a unique solution $p_n = F^{-1}(P_{N_0} v(t_n; \sigma, (u_1, v_1)))$.

Thanks to (5.11) and Theorem 4.2(ii), there exists M_9 such that

$$\begin{aligned}&\|\overline{\psi}_{t,\sigma}(\Phi^1(\sigma, p_n), p_n + \Phi^2(\sigma, p_n)) - \overline{\psi}_{t,\sigma}(u_1, v_1)\|\\&\leq M \|(\Phi^1(\sigma, p_n), \Phi^2(\sigma, p_n)) - (u_1, Q_{N_0} v_1)\| e^{-\nu t_n}\\&\leq M_9 e^{-\nu t_n}\end{aligned}$$

for all $t \in [0, t_n]$. In particular, for $t = 0$, one has that

$$\|(\Phi^1(\sigma, p_n), p_n + \Phi^2(\sigma, p_n)) - (u_1, v_1)\| \leq M_9 \| e^{-\nu t_n}.$$

We hence obtain that p_n is bounded. Then there is a convergent subsequence of p_n, denoted again by p_n, such that $p_n \to \tilde{p}$ as $n \to \infty$. Therefore,

$$\|\overline{\psi}_{t,\sigma}(\Phi^1(\sigma, \tilde{p}), \tilde{p} + \Phi^2(\sigma, \tilde{p})) - \overline{\psi}_{t,\sigma}(u_1, v_1)\| \leq M_9 e^{-\nu t_n}.$$

By putting $(u_2, v_2) = (\Phi^1(\sigma, \tilde{p}), \tilde{p} + \Phi^2(\sigma, \tilde{p})) \in \mathscr{M}(\sigma)$, we complete the proof. □

Thanks to Lemma 5.3, we obtain that $\mathscr{M}$ is pullback exponentially attracting for NDS $\psi_{t,\sigma}$. More precisely, we have the following theorem.

Theorem 5.2. *There exists M_{10} such that for every bounded subset $\mathcal{B}$ of E, there is $t_0 = t_0(\mathcal{B}) > 0$ satisfying*

$$\mathrm{dist}_H(\psi_{t,\theta_{-t}\sigma}\mathcal{B}, \mathscr{M}(\sigma)) \leq M_{10} e^{-\nu(t-t_0)}, \quad t \geq t_0.$$

Proof. Let $\mathcal{B}$ be bounded subset of E. From Theorem 3.1, it follows that there is $t_0 = t_0(\mathcal{B}) > 0$ such that for any $\sigma \in \Sigma$, $(u_0, v_0) \in \mathcal{B}$, $t \geq t_0$,

$$\|\psi_{t,\sigma}(u_0, v_0)\|_{H^2_p \times H^2_p} \leq R_0.$$

This implies that $\psi_{t,\sigma}(u_0, v_0) \in D_{N_0,R}$. Applying Lemma 5.3, we obtain that there exists $(u, v) \in \mathscr{M}(\theta_{t_0}\sigma)$ such that for all $t \geq t_0$,

$$\|\overline{\psi}_{t-t_0,\theta_{t_0}\sigma}\psi_{t,\sigma}(u_0, v_0) - \overline{\psi}_{t-t_0,\theta_{t_0}\sigma}(u, v)\| \leq M_8 e^{-\nu(t-t_0)}. \tag{5.12}$$

On the other hand, we note that

$$\begin{aligned}\psi_{t,\sigma}(u_0, v_0) &= \psi_{t-t_0,\theta_{t_0}\sigma}\psi_{t_0,\sigma}(u_0, v_0)\\&= \overline{\psi}_{t-t_0,\theta_{t_0}\sigma}\psi_{t_0,\sigma}(u_0, v_0).\end{aligned}$$

This together with (5.12) yields that

$$\operatorname{dist}_H(\psi_{t,\sigma}\mathcal{B}, \mathscr{M}(\theta_t\sigma)) \leq M_{10}e^{-\nu(t-t_0)}, \quad t \geq t_0.$$

Replacing σ by $\theta_t\sigma$, we complete the proof of the theorem. □

In the remainder of this paper, we assume that $f, g \in W^{1,\infty}(\mathbb{R}; H_p^s)$, $s > 0$. From [5, Theorem V 1.1], it follows that Σ is compact.

With the help of Corollary 3.1 and Theorem 5.2, one can prove the following theorem.

Theorem 5.3. *$\mathcal{K}_\sigma(0) \subset \mathscr{M}(\sigma)$ for all $\sigma \in \Sigma$ and hence $\mathcal{A}_\Sigma \subset \bigcup_{\sigma\in\Sigma} \mathscr{M}(\sigma)$.*

Proof. From [5, Proposition 5.1], it follows that there exists the semiflow $\{S(t), t \geq 0\}$ acting in the extended phase space $H \times \Sigma$ that corresponds to the family of processes $\{U_\sigma(t,\tau)\}, \sigma \in \Sigma$. More precisely,

$$S(t)((u_0, v_0), \sigma) = (\psi_{t,\sigma}(u_0, v_0), \theta_t\sigma), \quad t \in \mathbb{R}^+, ((u_0, v_0), \sigma) \in H \times \Sigma.$$

Noticing Corollary 3.1, an application of [5, IV Theorem 5.1] yields that the semiflow $S(t)$ possesses a global attractor $\mathcal{A}$ satisfying

$$\mathcal{A} = \bigcup_{\sigma\in\Sigma} \mathcal{K}_\sigma(0) \times \{\sigma\}.$$

Thanks to Theorems 3.2 and 5.2, an argument similar to that in [35, Theorem 4.4] yields that for $\sigma \in \Sigma$ and $(u_0, v_0) \in \mathcal{K}_\sigma(0)$,

$$\operatorname{dist}_{H\times\Sigma}((S(t)\mathcal{A}, \mathcal{M}) \leq M_{10}e^{-\nu(t-t_0)}, \quad t \geq t_0,$$

where

$$\mathcal{M} = \bigcup_{\sigma\in\Sigma} \mathscr{M}(\sigma) \times \{\sigma\}.$$

Hence, $\operatorname{dist}_{H\times\Sigma}((S(t)\mathcal{A}, \mathcal{M}) \to 0$ as $t \to \infty$. Moreover, it is easy to see that $\mathcal{M}$ is closed $H \times \Sigma$. From [22, Lemma 8.1], it follows that $\mathcal{A} \subset \mathcal{M}$. This implies that $\mathcal{K}_\sigma(0) \subset \mathscr{M}(\sigma)$ for all $\sigma \in \Sigma$. Hence, $\mathcal{A}_\Sigma \subset \bigcup_{\sigma\in\Sigma} \mathscr{M}(\sigma)$. □

Acknowledgements

The author takes this opportunity to express her appreciation and gratitude to Professor Rong-Nian Wang for suggesting this problem, and for his advice and guidance.

References

[1] A. V. Babin, *Inertial manifolds for travelling-wave solutions of reaction-diffusion systems*, Comm. Pure Appl. Math., 1995, 48(2), 167–198.

[2] P. W. Bates, K. Lu and C. Zeng, *Existence and persistence of invariant manifolds for semiflows in Banach space*, Mem. Amer. Math. Soc., 1998, 135(645), 1–132.

[3] A. Bonfoh, *Existence and continuity of inertial manifolds for the hyperbolic relaxation of the viscous Cahn-Hilliard equation*, Appl. Math. Optim., 2021, 84(3), 3339–3416.

[4] H. Brezis, *Opérateurs Maximaux Monotones et Semigroupes de Contractions dans un Espace de Hilbert*, North-Holland Pub. Co., Amsterdam, 1973.

[5] V. V. Chepyzhov and M. I. Vishik, *Attractors for Equations of Mathematical Physics*, American Mathematical Society, Providence, RI, 2002.

[6] V. V. Chepyzhov, A. Kostianko and S. Zelik, *Inertial manifolds for the hyperbolic relaxation of semilinear parabolic equations*, Discrete Contin. Dyn. Syst. Ser. B, 2019, 24(3), 1115–1142.

[7] S. N. Chow and K. Lu, *Invariant manifolds for flows in Banach spaces*, J. Differential Equations, 1988, 74(2), 285–317.

[8] V. Dufiet and J. Boissonade, *Dynamics of Turing pattern monolayers close to onset*, Phys. Rev. E, 1996, 53(5), 4883–4892.

[9] C. Foias, G. R. Sell and R. Temam, *Inertial manifolds for nonlinear evolutionary equations*, J. Differential Equations, 1988, 73(2), 309–353.

[10] M. Abu Hamed, Y. Guo and E. S. Titi, *Inertial manifolds for certain subgrid-scale α-models of turbulence*, SIAM J. Appl. Dyn. Syst., 2015, 14(3), 1308–1325.

[11] J. Hadamard, *Sur l'iteration et les solutions asymptotiques des equations differentielles*, Bull. Soc. Math. France, 1901, 29, 224–228.

[12] D. Henry, *Geometric Theory of Semilinear Parabolic Equations*, Springer, Berlin, 1981.

[13] J. Jin, Z. Lin and C. Zeng, *Invariant manifolds of traveling waves of the 3D Gross-Pitaevskii equation in the energy space*, Comm. Math. Phys., 2018, 364(3), 981–1039.

[14] R. Kapral and K. Showalter, *Chemical Waves and Patterns*, Springer Netherlands, Heidelberg, 1995.

[15] T. Kojima and Y. Oshita, *Linear stability of radially symmetric equilibrium solutions to the singular limit problem of three-component activator-inhibitor model*, Math. J. Okayama Univ., 2021, 63, 201–217.

[16] N. Koksch and S. Siegmund, *Pullback attracting inertial manifolds for nonautonomous dynamical systems*, J. Dynam. Differential Equations, 2002, 14(4), 889–941.

[17] A. Kostianko, *Inertial manifolds for the 3D modified-Leray-α model with periodic boundary conditions*, J. Dynam. Differential Equations, 2018, 30(1), 1–24.

[18] M. Kwak and X. X. Sun, *Remarks on the existence of an inertial manifold*, J. Korean Math. Soc., 2021, 58(5), 1261–1277.

[19] Y. Latushkin and B. Layton, *The optimal gap condition for invariant manifolds*, Discrete Contin. Dynam. Systems, 1999, 5(2), 233–268.

[20] A. M. Le, *Inertial manifolds for neutral functional differential equations with infinite delay and applications*, Ann. Polon. Math., 2020, 125(3), 255–271.

[21] N. Liu, *Global finite-dimensional invariant manifolds of the Boissonade system*, submitted.

[22] K. Lu, R. N. Wang and J. C. Zhao, *Invariant manifolds of the generalized phase-field systems*, submitted.

[23] A. M. Lyapunov, *Problème Général de la Stabilité du Mouvement*, Oxford University Press, London, 1947.

[24] J. Mallet-Paret and G. R. Sell, *Inertial manifolds for reaction diffusion equations in higher space dimensions*, J. Amer. Math. Soc., 1988, 1(4), 805–866.

[25] X. Mora, *Finite-dimensional attracting invariant manifolds for damped semilinear wave equations*, Contributions to nonlinear partial differential equations, Vol. II (Paris, 1985), 172–183, Pitman Res. Notes Math. Ser., 155, Longman Sci. Tech., Harlow, 1987.

[26] J. D. Murry, *Mathematical Biology. I. An introduction*, Springer-Verlag, New York, 2002.

[27] O. Perron, *Die Stabilitätsfrage bei Differentialgleichungen*, Math. Z., 1930, 32(1), 703–728.

[28] O. Perron, *Über Stabilität und asymptotisches verhalten der integrale von differentialgleichungssystemen*, Math. Z., 1929, 29, 129–160.

[29] O. Perron, *Über Stabilität und asymptotisches verhalten der lösungen eines systems endlicher differenzengleichungen*, J. Reine Angew. Math., 1929, 161, 41–64.

[30] J. C. Robinson, *Infinite-dimensional Dynamical Systems. An Introduction to Dissipative Parabolic PDEs and the Theory of Global Attractors*, Cambridge University Press, Cambridge, 2001.

[31] S. I. Rubinow, *Introduction to Mathematical Biology*, John Wiley, New York, 1975.

[32] G. R. Sell and Y. You, *Inertial manifolds: the non-selfadjoint case*, J. Differential Equations, 1992, 96(2), 203–255.

[33] R. Temam, *Infinite-dimensional Dynamical Systems in Mechanics and Physics*, Springer-Verlag, New York, 1997.

[34] J. Tu, *Global attractors and robustness of the Boissonade system*, J. Dynam. Differential Equations, 2015, 27(1), 187–211.

[35] R. N. Wang, J. H. Wu and J. C. Zhao, *Theory of invariant manifolds for infinite dimensional nonautonomous dynamical systems and applications*, submitted.

[36] S. Zelik, *Inertial manifolds and finite-dimensional reduction for dissipative PDEs*, Proc. Roy. Soc. Edinburgh Sect. A, 2014, 144(6), 1245–1327.

[37] J. C. Zhao and R. N. Wang, *The invariant manifold approach applied to global long-time dynamics of Fitzhugh-Nagumo systems*, submitted.