Journal of Applied Analysis and Computation
Volume 12, Number 1, February 2022, 1–16
Website:http://www.jaac-online.com
DOI:10.11948/20200406

ANALYTIC INTEGRABILITY AROUND THE ORIGIN OF CERTAIN DIFFERENTIAL SYSTEM*

Jaume Giné[1,†] and Claudia Valls[2]

Abstract In this work we consider the polynomial differential system $\dot{x} = -y + xy^{n-1}$, $\dot{y} = x + ayx^{n-1}$, where $a \in \mathbb{R}$ and $n \geq 2$ with $n \in \mathbb{N}$. This system is a certain generalization of the classical Liénard system. We study the center problem and consequently the analytic integrability problem for such family around the origin for any value of n.

Keywords Analytic first integral, center problem, order of a focus.

MSC(2010) 34C05, 34A05, 37G05, 34C20.

1. Introduction and statement of the main results

Two of the main problems in the qualitative theory of differential systems are the center/focus problem and the integrability problem that are equivalent for systems with a linear part of center type. For other singularities the existence of analytic invariant curves is also connected with the analytic integrability and the existence of a explicit first integral, see for instance [1] and references therein. Here we consider systems of the form

$$\dot{x} = P(x, y), \quad \dot{y} = Q(x, y) \tag{1.1}$$

where P and Q are polynomials in the variables x and y. Despite the intense activity on the well-known center/focus problem, there are very few satisfactory results on characterizing whether a given finite singular point is a center or a weak focus for any polynomial system in function of its degree, see for instance [4–6, 26]. This is mainly due to the fact that most of the results on the center/focus problem has been done considering particular differential systems because the computations of the focal values (see below for a definition) are very involved needing a computer algebra assistance, in the majority of the cases. Moreover when the degree of the polynomial system is arbitrary the problem becomes extremely difficult.

Assume that the origin is a singular point and that the system can be written into the form

$$\dot{x} = \lambda x + y + p(x, y), \quad \dot{y} = -x + \lambda y + q(x, y) \tag{1.2}$$

†The corresponding author. Email address: jaume.gine@udl.cat (J. Giné)
[1]Departament de Matemàtica, Universitat de Lleida, Avda. Jaume II, 69; 25001 Lleida, Catalonia, Spain
[2]Departamento de Matemática, Instituto Superior Técnico, Universidade de Lisboa, Av. Rovisco Pais 1049-001, Lisboa, Portugal
*The first author is partially supported by a MINECO/ FEDER grant number PID2020-113758GB-I00 and an AGAUR (Generalitat de Catalunya) grant number 2017SGR 1276. The second author is supported by FCT/Portugal through UID/MAT/04459/2013.

where p and q are polynomials that do not have neither linear nor constant terms. When $\lambda = 0$, the origin is said to be a *weak focus*. One of the existing methods to distinguish if we have a center or a weak focus is to propose a power series H such that, its derivative with respect to the time, $\dot{H}$, which is computed as

$$\dot{H} = \frac{\partial H}{\partial x}\dot{x} + \frac{\partial H}{\partial y}\dot{y},$$

that is, is the rate of change of H along the solutions of the system, is of the form

$$\dot{H} = v_2 r^2 + v_4 r^4 + \ldots, \quad \text{where } r^2 = x^2 + y^2,$$

where the quantities $v_{2\ell}$ are the so-called *Poincaré-Liapunov constants* or simply *focal values*. It is known that they are the coefficients of the even terms of the development of $\dot{H}$, see for instance [8,13]. If $v_{2\ell} = 0$ for all ℓ then we have a *center*, otherwise we have a weak focus and we say that the *order* of the weak focus is ℓ if

$$v_{2k} = 0 \quad \text{for } k \leq \ell \text{ and } v_{2k+2} \neq 0. \tag{1.3}$$

In other words, ℓ is in fact the multiplicity of the origin as a fixed point of the Poincaré map and at most ℓ limit cycles bifurcate from this weak focus of order ℓ. Moreover these focal values are polynomials in the parameters of system (1.1). If system (1.1) is polynomial then the Hilbert basis theorem assures that there exists $m \in \mathbb{N}$ such that $v_{2\ell} = 0$ for all ℓ if $v_{2\ell} = 0$ for $\ell \leq m$, see for instance [6,8,14,20] and references therein. These method can be generalized through the search for an inverse integrating factor [13], method that can be used also for some nilpotent centers, see [25]. Moreover, system (1.2) can be complexified obtaining a differential system with a resonant saddle at the origin and we can find, in this context, the integrability conditions for such singular point, see [9, 10] where is presented a method to find formal integrability.

One of the most famous second order differential equation is the well-known Liénard equation given by

$$\ddot{x} + f(x)\dot{x} + g(x) = 0, \tag{1.4}$$

mainly due to the fact that they appear in many applications and also that many other systems can be transformed to it, see for instance [11,29]. Equation (1.4) can be written as

$$\dot{x} = y, \qquad \dot{y} = -g(x) - yf(x), \tag{1.5}$$

where $f(x)$ and $g(x)$ are analytic functions with $f(0) = g(0) = 0$ and $g'(0) > 0$. For these equations there are many results for the existence, nonexistence and uniqueness of periodic orbits (see [3,29,30]) and if one fixes some class of functions of f and g some lower bounds on the maximum number of bifurcating limit cycles has been found, see [22].

In the last decades several generalizations of the Liénard equations have been proposed, see for instance [2, 5, 7, 16–18, 23, 24, 27] where the authors studied the center problem and the number of limit cycles that bifurcate from the singular point at the origin.

Recently in [21] is studied the polynomial differential system

$$\dot{x} = -y + xf(y), \quad \dot{y} = x + yf(x), \tag{1.6}$$

where f is a polynomial. This system is a certain generalization of the classical Liénard system which has $g(x) = x$. For such system, the center and ciclicity problems of the origin are solved in [21]. More specifically the system has a center if and only if all the even terms of the development of f are zero. Some particular systems which are possible generalizations of system (1.6) have been studied recently, see for instance [28, 31, 32].

In this paper we propose the simple case generalizing (1.6) which is when f is different in both equations but a unique monomial

$$\dot{x} = -y + xy^{n-1}, \quad \dot{y} = x + ayx^{n-1}, \tag{1.7}$$

where $a \in \mathbb{R}$ and $n \geq 2$ with $n \in \mathbb{N}$. We will see that although is one of the simplest systems that we can propose as generalization of system (1.6) the center problem for such system becomes very involved and in fact we will see that we cannot completely solve. The paper shows how difficult is to solve the center problem for a polynomial system of arbitrary degree.

The following is the main result of the paper.

Theorem 1.1. *The following statements hold for system* (1.7).

1. *For n odd it has a center at the origin if and only if $a = -1$.*
2. *For $n = 2$ it has a center at the origin for all $a \in \mathbb{R}$.*
3. *For n even with $n \geq 4$ it has a center at the origin if $a \in \{-1, 0, 1\}$.*

From the previous results and the computations given in the last part of the paper in order to compute the first non zero focal value of system (1.7) we can establish the following conjecture.

Conjecture 1.1. *For n even with $n \geq 4$ it has a center at the origin if and only if $a \in \{-1, 0, 1\}$.*

The proof of Theorem 1.1 is given in the following sections. In order to proof Theorem 1 we need to prove the necessity and sufficiency. The necessity is proved computing the focal values and establishing the decomposition in primary ideals of the ideal generated by the focal values. The sufficiency is proved looking for a first integral or the reversibility that is approached by several methods, see for instance [15, 19, 26] and references therein. However there is no a unified method.

In the proof of Theorem 1.1 for n odd it is used the technique of the wedge product to compare the vector fields defined by system (1.7) and the one with a center. The idea of this technique was introduced for the first time in [2]. This method to study the centers in the Liénard systems was used lately in [4, 12, 17, 21].

The proof of Theorem 1.1 for n odd is given in Section 2 and for n even in Section 3.

2. Proof of Theorem 1.1 for n odd

First we prove the sufficiency. We apply the linear change of variables

$$x = \frac{u}{\sqrt{2}} + \frac{v}{\sqrt{2}}, \qquad y = -\frac{u}{\sqrt{2}} + \frac{v}{\sqrt{2}}, \tag{2.1}$$

and system (1.7) with $a=-1$ takes the form

$$\begin{aligned} \dot u &= -v + \tfrac{1}{2}u\left(-\tfrac{u}{\sqrt2}+\tfrac{v}{\sqrt2}\right)^{n-1} + \tfrac12 v\left(-\tfrac{u}{\sqrt2}+\tfrac{v}{\sqrt2}\right)^{n-1} \\ &\quad -\tfrac12 u\left(\tfrac{u}{\sqrt2}+\tfrac{v}{\sqrt2}\right)^{n-1} + \tfrac12 v\left(\tfrac{u}{\sqrt2}+\tfrac{v}{\sqrt2}\right)^{n-1}, \\ \dot v &= u + \tfrac12 u\left(-\tfrac{u}{\sqrt2}+\tfrac{v}{\sqrt2}\right)^{n-1} + \tfrac12 v\left(-\tfrac{u}{\sqrt2}+\tfrac{v}{\sqrt2}\right)^{n-1} \\ &\quad +\tfrac12 u\left(\tfrac{u}{\sqrt2}+\tfrac{v}{\sqrt2}\right)^{n-1} - \tfrac12 v\left(\tfrac{u}{\sqrt2}+\tfrac{v}{\sqrt2}\right)^{n-1}. \end{aligned} \tag{2.2}$$

System (2.2) is time-reversible because has the symmetry $(u,v,t)\to(-u,v,-t)$. This can be seen verifying that $\dot u - \dot u_{u\to -u} = 0$ and $\dot v + \dot v_{u\to-u} = 0$ for all $n\in\mathbb{N}$. As the system is time-reversible, the system has a center at the origin, see [26].

To prove necessity we will proceed by contradiction using the method of the wedge product of an arbitrary system with the system with a center. Hence we consider the wedge product of system (1.7) with n odd, that in the vector field form we call $\mathcal{X}$, and system

$$\dot x = -y + xy^{n-1}, \quad \dot y = x - yx^{n-1}, \tag{2.3}$$

that in the vector field form we call $\mathcal{X}_c$. This wedge product gives

$$\mathcal{X}\wedge\mathcal{X}_c = -(1+a)x^{n-1}y^2 + O(n+1). \tag{2.4}$$

If $a\neq -1$, from equation (2.4) we deduce that, in a neighborhood of the origin, the level curves of the solutions of $\mathcal{X}_c$ do not have contact with the flow of $\mathcal{X}$ giving the impossibility of having a center for $\mathcal{X}$. This contradiction shows that $a=-1$ is necessary for having a center. Note that this argument is only valid if n is odd and so it concludes the proof of statement (1) of Theorem 1.1.

3. Proof of Theorem 1.1 for n even

System (1.7) for $n=2$ has the first integral

$$H(x,y) = e^{a(ax-y)}(x-1)^{a^2}(1+ay),$$

and the inverse integrating factor $V=(x-1)(1+ay)$. Taking into account that $V(0,0)\neq 0$ and that the origin is a nondegenerate singular point by the Reeb's criterion we have that the origin of system (1.7) is a center, see [13]. Hence the system has a center for any value of a. This prove statement (2) of Theorem 1.1.

Now we can assume that $n\geq 4$ with n even. For $a=0$ system (1.7) takes the form

$$\dot x = -y + xy^{n-1}, \quad \dot y = x, \tag{3.1}$$

which is a time reversible system because has the symmetry $(x,y,t)\to(x,-y,-t)$. Hence, system (3.1) has a center at the origin and consequently an analytic first integral around it.

For $a\neq 0$ we apply the linear change of variables (2.1) and system (1.7) can be written as

$$\begin{aligned} \dot u &= \frac{(\sqrt2\, a(u-v)^2(u+v)^n - \sqrt2(v-u)^n(u+v)^2 + 2^{\frac{n+2}{2}}v(v^2-u^2)}{2^{\frac{n+2}{2}}(u^2-v^2)}, \\ \dot v &= \frac{(-\sqrt2 a(u-v)^2(u+v)^n - \sqrt2(v-u)^n(u+v)^2 + 2^{\frac{n+2}{2}}u(u^2-v^2)}{2^{\frac{n+2}{2}}(u^2-v^2)}. \end{aligned} \tag{3.2}$$

Now it is easy to see that for the case $a = 1$ with n even and $n \geq 4$ system (3.2) is time-reversible because has the symmetry $(u, v, t) \to (u, -v, -t)$. This can be seen verifying that the first equation of system (3.2) satisfies $\dot{u} + \dot{u}_{v\to -v} = 0$ and the second equation of system (3.2) verifies $\dot{v} - \dot{v}_{u\to -u} = 0$ for all $n \in \mathbb{N}$. For $a = -1$ happens the same that in the proof of statement (1) of Theorem 1.1. This completes the proof of statement (3) of Theorem 1.1.

4. About the proof of the conjecture

The study of the necessary conditions needs many computations which have been made proposing a power series of the form

$$\begin{aligned} H(x,y) &= x^2 + y^2 + H_{n+1}(x,y) + H_{2n}(x,y) + H_{3n-1}(x,y) + H_{4n-2}(x,y) + \cdots \\ &= x^2 + y^2 + \sum_{j=0}^{n+1} b_j x^{n+1-j} y^j + \sum_{j=0}^{2n} c_j x^{2n-j} y^j \\ &\quad + \sum_{j=0}^{3n-1} d_j x^{3n-1-j} y^j + \sum_{j=0}^{4n-2} f_j x^{4n-2-j} y^j + \cdots \end{aligned}$$

as a first integral. The main strategy of the proof is as follows: we will compute $H_{n+1}(x,y)$, $H_{2n}(x,y)$, $H_{3n-1}(x,y)$, and $H_{4n-2}(x,y)$ and we will show that focal value $V_{4n-2}(x,y)$ is a polynomial in a of the form $V_{4n-2}(x,y) = C_{4n-2}a(a^2-1)$ with some constant C_{4n-2} that depends on n. We will see that we are not able to prove that $C_{4n-2} \neq 0$ for all $n \geq 4$ and we only can provide a table with the values of C_{4n-2} for $n = 4, \ldots, 10$ where it is shown that indeed this constant is different from zero.

Computation of H_{n+1}

Computing the terms of order $n+1$ for H we need to solve

$$\begin{aligned} &2x^2y^{n-1} + 2ay^2x^{n-1} - \sum_{j=0}^{n+1} (n+1-j) b_j x^{n-j} y^{j+1} + j b_j x^{n+2-j} y^{j-1} \\ =&2x^2y^{n-1} + 2ay^2x^{n-1} - b_n y^{n+1} + b_1 x^{n+1} \\ &- \sum_{j=1}^{n} [(n+2-j) b_{j-1} - (j+1) b_{j+1}] x^{n+1-j} y^j. \end{aligned}$$

Hence,

$$b_1 = b_n = 0, \quad b_3 = -\frac{2a}{3}, \quad b_{n-2} = \frac{2}{3}$$

and proceeding recursively we get

$$b_{2l} = \frac{2}{3} \prod_{k=l}^{n/2-2} \frac{2k+2}{n-2k+1}, \quad l = 0, \ldots, \frac{n}{2} - 1,$$

and

$$b_{2l+1} = -\frac{2a}{3} \prod_{k=2}^{l} \frac{n+2-2k}{2k+1}, \quad l = 1, \ldots, \frac{n}{2},$$

where we have used the convention that $\prod_{k=2}^{1} \cdot_k = 1$. In particular taking the change $l = n/2 - p$ and $k = n/2 - j$ (and then renaming p by l and j by k) we can see that

$$ab_{2l} = -b_{n-2l+1}, \quad l = 0, \ldots, n/2. \tag{4.1}$$

Computation of H_{2n} and some useful relations

Computing the terms of order $2n$ for H and proceeding as above we need to solve

$$\begin{aligned}
&\sum_{j=1}^{n+1}(n+1-j)b_j x^{n-j+1}y^{n-1+j} + \sum_{j=1}^{n+1} ajb_j x^{2n-j}y^j \\
&-\sum_{j=0}^{2n}(2n-j)c_j x^{2n-1-j}y^{j+1} + jc_j x^{2n+1-j}y^{j-1} \\
=&\sum_{j=n-1}^{2n-1}(2n-j)b_{j-n+1}x^{2n-j}y^j + \sum_{j=1}^{n+1} ajb_j x^{2n-j}y^j - c_{2n-1}y^{2n} \\
&+ c_1 x^{2n} - \sum_{j=1}^{2n-1}[(2n+1-j)c_{j-1} - (j+1)c_{j+1}]x^{2n-j}y^j.
\end{aligned} \tag{4.2}$$

Let

$$T_j = \begin{cases} 0, & j = 0, \\ ajb_j, & j = 1, \ldots, n-2 \\ ajb_j + b_{j-n+1}(2n-j), & j = n-1; n+1 \\ 0, & j = n \\ (2n-j), & j = n+1 \\ b_{j-n+1}(2n-j), & j = n+2, \ldots, 2n-1, \\ 0 & j = 2n. \end{cases} \tag{4.3}$$

Then relation (4.2) becomes

$$(2n+1-j)c_{j-1} - (j+1)c_{j+1} = T_j, \quad c_1 = 0, \quad c_{2n-1} = 0. \tag{4.4}$$

First note that the solution of (4.4) is $c_1 = 0$ and

$$c_{2p+1} = -\frac{1}{2p+1}\sum_{j=1}^{p}\prod_{k=j}^{p-1}\frac{2n-1-2k}{2k+1}T_{2j} \tag{4.5}$$

for $p = 1, \ldots, n/2 - 1$ and

$$c_{2p} = \prod_{k=1}^{p}\frac{2n+2-2k}{2k}c_0 - \frac{1}{2p}\sum_{j=2}^{p}\prod_{k=j}^{p-1}\frac{n-k}{k}T_{2j-1}, \tag{4.6}$$

for $p = 1, \ldots, n/2$ (note that since $T_1 = 0$, the index in j starts with $j = 2$).

Observe that in view of (4.1) we have

$$T_{2j} = -T_{2n-2j} \quad \text{for } j = 0, \ldots, n/2, \quad (\text{note that } T_n = 0)$$

and

$$T_{2j-1} = -a^2 T_{2n-2j+1} \quad \text{for } j = 0, \dots, n/2-1 \tag{4.7}$$

and

$$T_{n-1} = -T_{n+1} + (1-a^2)(n+1)b_2. \tag{4.8}$$

Indeed, it follows from (4.3) that $T_n = 0$ and for $j = 1, \dots, n/2-1$ we get

$$\begin{aligned} T_{2j} + T_{2n-2j} &= 2jab_{2j} + (2n-(2n-2j))b_{2n-2j-n+1} \\ &= 2j(ab_{2j} + b_{n-2j+1}) = 0. \end{aligned}$$

Moreover, for $j = 1, \dots, n/2-1$

$$\begin{aligned} T_{2j-1} + a^2 T_{2n-2j+1} =& (2j-1)ab_{2j-1} \\ &+ a^2(2n-(2n-2j+1))b_{2n-2j+1-n+1} \\ =& (2j-1)a(b_{2j-1} + ab_{n-2j+2}) = 0 \end{aligned}$$

and

$$\begin{aligned} T_{n-1} + T_{n+1} &= a(n-1)b_{n-1} + (n+1)b_0 + a(n+1)b_{n+1} + (n-1)b_2 \\ &= -a^2(n-1)b_2 + (n+1)b_0 - a^2(n+1)b_0 + (n-1)b_2 \\ &= (1-a^2)((n+1)b_0 + (n-1)b_2) = (1-a^2)(n+1)b_2. \end{aligned}$$

Using (4.5) and (4.6) we get

$$c_{2p+1} = c_{2n-2p-1}, \quad p = 0, \dots, n/2-1, \tag{4.9}$$

with

$$\begin{aligned} c_{2p+1} = -\frac{2a}{3}\Big(&\frac{1}{2p+1}\sum_{j=2}^{p} 2j \prod_{k=j}^{p-1} \frac{2n-1-2k}{2k+1} \prod_{k=j}^{n/2-2} \frac{2k+2}{n-2k+1} \\ &+ \frac{2}{3}\prod_{k=1}^{n/2-2}\frac{2k+2}{n-2k+1}\prod_{k=2}^{p}\frac{2n+1-2k}{2k+1}\Big) \end{aligned} \tag{4.10}$$

for $p = 1, \dots, n/2-1$ with $c_1 = c_{2n-1} = 0$ (here we have used the convention that $\sum_{j=2}^{1} \cdot_j = 0$). Moreover,

$$c_{2p} = \alpha_{2p} - \beta_{2p} := c_0 \prod_{k=1}^{p} \frac{2n+2-2k}{2k} - \frac{1}{2p}\sum_{j=2}^{p}\prod_{k=j}^{p-1}\frac{n-k}{k} T_{2j-1}$$

where $\alpha_{2n} = 1$, and

$$\alpha_{2p} = \alpha_{2n-2p}, \quad p = 1, \dots, n/2-1. \tag{4.11}$$

Note that $c_0 = c_{2n}$. Finally, we recall a relation that will be used later on. We claim that for $1 \le j < n/2$ we obtain

$$c_{2j} - c_{2n-2j} = (a^2-1)K_j \tag{4.12}$$

where

$$K_j = -\frac{1}{2(n-j)}\left((n+1)b_2\prod_{k=n/2}^{n-j-1}\frac{n-k}{k}+\sum_{i=j+1}^{n/2-1}T_{2n-2i+1}\prod_{k=i}^{n-j-1}\frac{n-k}{k}\right) \tag{4.13}$$

and taking into account that $b_n = 0$ we get that $K_0 = \frac{1}{n}K_1$.

For $j = n/2+1,\dots,n$ we write $j = n-p$ with $p = 1,\dots,n/2-1$ and then we have

$$\begin{aligned} c_{2j} - c_{2n-2j} &= c_{2n-2p} - c_{2p} = -(c_{2p} - c_{2n-2p}) = -(a^2-1)K_p \\ &= -(a^2-1)K_{n-j}. \end{aligned} \tag{4.14}$$

Now we shall prove the claim. By definition and in view of (4.11) we get

$$\begin{aligned} c_{2p} - c_{2n-2p} &= \alpha_{2p} - \alpha_{2n-2p} - \beta_{2p} + \beta_{2n-2p} = -\beta_{2p} + \beta_{2n-2p} \\ &= -\frac{1}{2p}\sum_{j=2}^{p}\prod_{k=j}^{p-1}\frac{n-k}{k}T_{2j-1} + \frac{1}{2(n-p)}\sum_{j=2}^{n-p}\prod_{k=j}^{n-p-1}\frac{n-k}{k}T_{2j-1} \\ &= -\frac{1}{2p}\sum_{j=2}^{p}\prod_{k=j}^{p-1}\frac{n-k}{k}T_{2j-1} + \frac{1}{2(n-p)}\sum_{j=2}^{p}\prod_{k=j}^{n-p-1}\frac{n-k}{k}T_{2j-1} \\ &\quad + \frac{1}{2(n-p)}\sum_{j=p+1}^{n-p}\prod_{k=j}^{n-p-1}\frac{n-k}{k}T_{2j-1}. \end{aligned} \tag{4.15}$$

Note that

$$\frac{1}{2p}\sum_{j=2}^{p}\prod_{k=j}^{p-1}\frac{n-k}{k}T_{2j-1} - \frac{1}{2(n-p)}\sum_{j=2}^{p}\prod_{k=j}^{n-p-1}\frac{n-k}{k}T_{2j-1} = 0$$

because

$$\frac{1}{n-p}\prod_{k=j}^{n-p-1}\frac{n-k}{k} = \frac{1}{n-p}\prod_{k=j}^{p-1}\frac{n-k}{k}\prod_{k=p}^{n-p-1}\frac{n-k}{k} = \frac{1}{p}\prod_{k=j}^{p-1}\frac{n-k}{k}.$$

Moreover,

$$\begin{aligned} &\sum_{j=p+1}^{n-p}\prod_{k=j}^{n-p-1}\frac{n-k}{k}T_{2j-1} = \sum_{j=p+1}^{n/2}\prod_{k=j}^{n-p-1}\frac{n-k}{k}T_{2j-1} \\ &\quad + \sum_{j=n/2+1}^{n-p}\prod_{k=j}^{n-p-1}\frac{n-k}{k}T_{2j-1} \\ &= \prod_{k=n/2}^{n-p-1}\frac{n-k}{k}T_{n-1} + \sum_{j=p+1}^{n/2-1}\prod_{k=j}^{n-p-1}\frac{n-k}{k}T_{2j-1} \\ &\quad + \prod_{k=n/2+1}^{n-p-1}\frac{n-k}{k}T_{n+1} + \sum_{j=n/2+2}^{n-p}\prod_{k=j}^{n-p-1}\frac{n-k}{k}T_{2j-1} \end{aligned}$$

and

$$\sum_{j=n/2+2}^{n-p}\prod_{k=j}^{n-p-1}\frac{n-k}{k}T_{2j-1}=\sum_{m=n/2-1}^{p+1}\prod_{k=n+1-m}^{n-p-1}\frac{n-k}{k}T_{2n-2m+1}$$
$$=\sum_{m=p+1}^{n/2-1}\prod_{k=m}^{n-p-1}\frac{n-k}{k}T_{2n-2m+1},$$

where we have used that (recall that $m\geq n/2+1$)

$$\prod_{k=n+1-m}^{n-p-1}\frac{n-k}{k}=\prod_{k=m}^{n-p-1}\frac{n-k}{k}\prod_{k=n+1-m}^{m-1}\frac{n-k}{k}\prod_{k=m+1}^{n+1-m}\frac{n-k}{k}$$
$$=\prod_{k=m}^{n-p-1}\frac{n-k}{k}.$$

Now using (4.7) and (4.8) we conclude that

$$\sum_{j=p+1}^{n-p}\prod_{k=j}^{n-p-1}\frac{n-k}{k}T_{2j-1}=\prod_{k=n/2}^{n-p-1}\frac{n-k}{k}T_{n-1}+\prod_{k=n/2+1}^{n-p-1}\frac{n-k}{k}T_{n+1}$$
$$-a^2\sum_{j=p+1}^{n/2-1}\prod_{k=j}^{n-p-1}\frac{n-k}{k}T_{2n-2j+1}+\sum_{j=p+1}^{n/2-1}\prod_{k=j}^{n-p-1}\frac{n-k}{k}T_{2n-2j+1}$$
$$=(1-a^2)\left((n+1)b_2\prod_{k=n/2}^{n-p-1}\frac{n-k}{k}+\sum_{j=p+1}^{n/2-1}\prod_{k=j}^{n-p-1}\frac{n-k}{k}T_{2n-2j+1}\right).$$

This last relation together with (4.15) yields the claim provided by relations (4.12) and (4.13). Note that in view of (4.3) and the definition of b_j the constant K_j in (4.13) is in fact

$$K_j=-\frac{n+1}{3(n-j)}\prod_{k=1}^{n/2-2}\frac{2k+2}{n-2k+1}\prod_{k=n/2}^{n-j-1}\frac{n-k}{k}$$
$$-\frac{1}{3(n-j)}\sum_{i=j+1}^{n/2-1}(2i-1)\prod_{k=n/2-i+1}^{n/2-2}\frac{2k+2}{n-2k+1}\prod_{k=i}^{n-j-1}\frac{n-k}{k}.$$

Note that $K_j<0$. Moreover, for $n/2\leq j\leq n-1$ it follows from equation (4.10) that

$$a^2c_{3n-1-2j}-c_{2j-n+1}=(a^2-1)c_{2j-n+1}=-\frac{2}{3}a(a^2-1)D_j$$

with

$$D_j=\frac{2}{2j-n+1}\left(\sum_{i=2}^{j-n/2}i\prod_{k=i}^{j-n/2-1}\frac{2n-1-2k}{2k+1}\prod_{k=i}^{n/2-2}\frac{2k+2}{n-2k+1}\right.$$
$$\left.+\prod_{k=1}^{n/2-2}\frac{2k+2}{n-2k+1}\prod_{k=1}^{j-n/2-1}\frac{2n-1-2k}{2k+1}\right).$$

Note that $D_j > 0$ and $D_{n/2} = 0$ because $c_1 = 0$. Moreover, when $j = n, \dots, 3n/2-1$ in view of (4.9) we have that

$$\begin{aligned} a^2 c_{3n-1-2j} - c_{2j-n+1} &= (a^2-1)c_{2j-n+1} = (a^2-1)c_{3n-2j-1} \\ &= -\frac{2}{3} a(a^2-1) D_{2n-1-j} \end{aligned}$$

(for the last relation we just need to set $3n-2j-1 = 2p-n+1$ and we get $p = 2n-1-j$).

Computation of H_{3n-1} and some useful relations

Computing the terms of order $3n-1$ for H and proceeding as above we need to solve

$$\begin{aligned} &\sum_{j=1}^{2n-1} (2n-j) c_j x^{2n-j} y^{n-1+j} + \sum_{j=1}^{2n} a j c_j x^{3n-1-j} y^j \\ &- \sum_{j=0}^{3n-1} (3n-1-j) d_j x^{3n-2-j} y^{j+1} + j d_j x^{3n-j} y^{j-1} \\ = &\sum_{j=n-1}^{3n-2} (3n-1-j) c_{j-n+1} x^{3n-1-j} y^j + \sum_{j=1}^{2n} a j c_j x^{3n-1-j} y^j \\ &- d_{3n-2} y^{3n-1} + d_1 x^{3n-1} - \sum_{j=1}^{3n-2} [(3n-j) d_{j-1} - (j+1) d_{j+1}] x^{3n-1-j} y^j . \end{aligned}$$

Let

$$R_j = \begin{cases} 0, & j = 0, \\ a j c_j, & j = 1, \dots, n-2 \\ a j c_j + c_{j-n+1}(3n-1-j), & j = n-1, \dots, 2n \\ c_{j-n+1}(3n-1-j), & j = 2n+1, \dots, 3n-2, \\ 0 & j = 3n-1 \end{cases}$$

We have that the solution is given recursively by the relation

$$(3n-j) d_{j-1} - (j+1) d_{j+1} = R_j, \quad d_1 = d_{3n-2} = 0.$$

Note that again by recursivity the general solution for j odd is

$$d_{2p+1} = -\frac{1}{2p+1} \sum_{j=1}^{p} \prod_{k=j}^{p-1} \frac{3n-2-2k}{2k+1} R_{2j} \tag{4.16}$$

(using that $d_1 = 0$) and for j even is

$$d_{2p} = \frac{1}{3n-2p-1} \sum_{j=p}^{3n/2-2} \prod_{k=p}^{j-1} \frac{2k+2}{3n-3-2k} R_{2j+1} \tag{4.17}$$

(using that $d_{3n-2} = 0$ and so the recursion have to be made isolating d_{j-1} in terms of d_{j+1} instead of the other way around as we do for the case in which j is odd). Note that here there are no restrictions in the computations of the constants d_j.

Now we compute some relation that will be used later on. For $1 \le p \le \frac{3n}{2} - 2$ we claim that

$$\begin{aligned} &ad_{2p} + d_{3n-1-2p} \\ =&\frac{1}{3n-2p-1} \sum_{j=1}^{3n/2-1-p} \prod_{k=p+1}^{3n/2-1-j} \frac{2k}{3n-2k-1}(aR_{3n-1-2j} - R_{2j}). \end{aligned} \tag{4.18}$$

We prove the claim. We note that it follows from (4.16) with the change $m = 3n/2 - 1 - j$ that

$$\begin{aligned} d_{2p} &= \frac{1}{3n-2p-1} \sum_{m=1}^{3n/2-1-p} \prod_{k=p}^{3n/2-2-m} \frac{2k+2}{3n-3-2k} R_{3n-1-2m} \\ &= \frac{1}{3n-2p-1} \sum_{m=1}^{3n/2-1-p} \prod_{k=p+1}^{3n/2-1-m} \frac{2k}{3n-1-2k} R_{3n-1-2m}, \end{aligned} \tag{4.19}$$

where in the last identity we have changed k by $k+1$.

On the other hand, it follows from (4.16) (with p interchanged by $3n/2 - 1 - p$) that

$$d_{3n-1-2p} = -\frac{1}{3n-2p-1} \sum_{j=1}^{3n/2-1-p} \prod_{k=j}^{3n/2-2-p} \frac{3n-2-2k}{2k+1} R_{2j}.$$

Now making the change $\ell = 3n/2 - 1 - k$ we get

$$d_{3n-1-2p} = -\frac{1}{3n-2p-1} \sum_{j=1}^{3n/2-1-p} \prod_{\ell=p+1}^{3n/2-1-j} \frac{2\ell}{3n-1-2\ell} R_{2j}. \tag{4.20}$$

Therefore, the relation in (4.18) follows directly from (4.19) and (4.20) and so the claim follows.

Note that

$$ad_{2p} + d_{3n-1-2p} = 0 \quad \text{for } p = \frac{3n}{2} - 1.$$

Now we want to compute the quantity $aR_{3n-1-2j} - R_{2j}$ for $1 \le j \le n-1$. To do so, we note that for $1 \le j \le n/2 - 1$ we have

$$aR_{3n-1-2j} - R_{2j} = 2ja(c_{2n-2j} - c_{2j})$$

and for $n/2 \le j \le n-1$ we have

$$aR_{3n-1-2j} - R_{2j} = 2ja(c_{2n-2j} - c_{2j}) + (3n-1-2j)(a^2 c_{3n-1-2j} - c_{2j-n+1}).$$

On the other hand, for $j = n, \ldots, 3n/2 - 2$, taking into account that $c_0 = c_{2n}$ we get

$$aR_{3n-1-2j} - R_{2j} = (3n-1-2j)(a^2 c_{3n-1-2j} - c_{2j-n+1})$$

and for $j = \frac{3n}{2} - 1$ we have

$$aR_{3n-1-2j} - R_{2j} = 0.$$

From (4.12), (4.14) and the fact that $K_n = -K_0 = -K_1/n$ we get that

$$
\begin{aligned}
&aR_{3n-1-2j} - R_{2j} \\
&= a(a^2-1)\begin{cases} -2jK_j & \text{for } 1 \le j \le n/2-1\ , \\ 0 & \text{for } j = n/2\ , \\ 2jK_{n-j} - \frac{2}{3}(3n-1-2j)D_j & \text{for } n/2+1 \le j \le n-1, \\ 2K_1 - \frac{2}{3}(n-1)D_{n-1} & \text{for } j = n\ , \\ -\frac{2}{3}(3n-1-2j)D_{2n-1-j} & \text{for } n+1 \le j \le 3n/2-2, \\ 0 & \text{for } j = 3n/2-1. \end{cases}
\end{aligned} \tag{4.21}
$$

Therefore,

$$
ad_{2p} + d_{3n-1-2p} = a(a^2-1)F_p \tag{4.22}
$$

where in view of (4.21) we get for $1 \le p \le n/2-1$,

$$
\begin{aligned}
F_p =& \frac{1}{3n-2p-1}\Bigg(-\sum_{j=1}^{n/2-1} \prod_{k=p+1}^{3n/2-1-j} \frac{2k}{3n-2k-1} 2jK_j \\
&+ \sum_{j=n/2+1}^{n-1} \prod_{k=p+1}^{3n/2-1-j} \frac{2k}{3n-2k-1}(2jK_{n-j} - \frac{2}{3}(3n-1-2j)D_j) \\
&+ \prod_{k=p+1}^{n/2-1} \frac{2k}{3n-2k-1}(2K_1 - \frac{2}{3}(n-1)D_{n-1}) \\
&- \sum_{j=n+1}^{3n/2-1-p} \prod_{k=p+1}^{3n/2-1-j} \frac{4k}{3(3n-2k-1)}(3n-1-2j)D_{2n-1-j} \Bigg),
\end{aligned}
$$

for $p = n/2, \dots, n-1$ we get

$$
\begin{aligned}
F_p =& \frac{1}{3n-2p-1}\Bigg(-\sum_{j=1}^{n/2-1} \prod_{k=p+1}^{3n/2-1-j} \frac{2k}{3n-2k-1} 2jK_j \\
&+ \sum_{j=n/2+1}^{3n/2-1-p} \prod_{k=p+1}^{3n/2-1-j} \frac{2k}{3n-2k-1}(2jK_{n-j} - \frac{2}{3}(3n-1-2j)D_j) \Bigg),
\end{aligned}
$$

for $p = n, \dots, 3n/2-2$ we have

$$
F_p = -\frac{1}{3n-2p-1} \sum_{j=1}^{3n/2-1-j} \prod_{k=p+1}^{3n/2-1-j} \frac{2k}{3n-2k-1} 2jK_j,
$$

and for $p = 3n/2-1$,

$$
F_p = 0.
$$

Computation of H_{4n-2} and some useful relations

Computing the terms of order $4n-2$ for H and proceeding as above we need to solve

$$\begin{aligned}
&\sum_{j=0}^{3n-2}(3n-1-j)d_j x^{3n-1-j}y^{n-1+j} + \sum_{j=1}^{3n-1} ajd_j x^{4n-2-j}y^j \\
&-\sum_{j=0}^{4n-2}(4n-2-j)f_j x^{4n-3-j}y^{j+1} + jf_j x^{4n-1-j}y^{j-1} \\
=&\sum_{j=n-1}^{4n-3}(4n-2-j)d_{j-n+1}x^{4n-2-j}y^j + \sum_{j=1}^{3n-1} ajd_j x^{4n-1-j}y^j \\
&-f_{4n-3}y^{4n-2} + f_1 x^{4n-2} - \sum_{j=1}^{4n-3}[(4n-1-j)f_{j-1}-(j+1)f_{j+1}]x^{4n-2-j}y^j.
\end{aligned}$$

Let

$$S_j = \begin{cases} 0, & j=0, \\ ajd_j, & j=1,\dots,n-2 \\ ajd_j + d_{j-n+1}(4n-2-j), & j=n-1,\dots,3n-1 \\ d_{j-n+1}(4n-2-j), & j=3n,\dots,4n-3, \\ 0 & j=4n-2. \end{cases}$$

We have that the solution is given recursively by the relation

$$(4n-1-j)f_{j-1}-(j+1)f_{j+1} = S_j, \quad f_1 = f_{4n-3} = 0.$$

Note that for j even there are no conditions on f_j (and so thye can be obtained in function of a new parameter f_0) and the condition will be given in the case of j odd. In this case the general solution is

$$f_{2p+1} = -\frac{1}{2p+1}\sum_{j=1}^{p}\prod_{k=j}^{p-1}\frac{4n-3-2k}{2k+1}S_{2j}$$

(using that $d_1 = 0$). Then, we need to impose the condition $f_{4n-3}=0$ which will lead to the Liapunov constant. Note that

$$\begin{aligned}
0 = -(4n-3)f_{4n-3} &= \sum_{j=1}^{2n-2}\prod_{k=j}^{2n-3}\frac{4n-3-2k}{2k+1}S_{2j} \\
&= \sum_{j=1}^{n-1}\prod_{k=j}^{2n-3}\frac{4n-3-2k}{2k+1}S_{2j} + \sum_{j=n}^{2n-2}\prod_{k=j}^{2n-3}\frac{4n-3-2k}{2k+1}S_{2j} \\
&= \sum_{j=1}^{n-1}\prod_{k=j}^{2n-3}\frac{4n-3-2k}{2k+1}S_{2j} + \sum_{m=1}^{n-1}\prod_{k=2n-1-m}^{2n-3}\frac{4n-3-2k}{2k+1}S_{4n-2-2m} \\
&= \sum_{j=1}^{n-1}\prod_{k=j}^{2n-3}\frac{4n-3-2k}{2k+1}(S_{2j}+S_{4n-2-2j}),
\end{aligned}$$

where we have used that

$$\prod_{k=2n-1-m}^{2n-3} \frac{4n-3-2k}{2k+1} = \prod_{k=m}^{2n-3} \frac{4n-3-2k}{2k+1} \left(\prod_{k=m}^{2n-2-m} \frac{4n-3-2k}{2k+1} \right)^{-1} = \prod_{k=m}^{2n-3} \frac{4n-3-2k}{2k+1}.$$

Moreover, we have that for $j = 1, \ldots, n/2-1$,

$$S_{2j} + S_{4n-2-2j} = 2j(ad_{2j} + d_{3n-1-2j})$$

and for $j = n/2, \ldots, n-1$ then

$$S_{2j} + S_{4n-2-2j} = 2j(ad_{2j} + d_{3n-1-2j}) + (4n-2-2j)(ad_{4n-2-2j} + d_{2j-n+1}).$$

Applying identities (4.22) we conclude that

$$S_{2j} + S_{4n-2-2j} = a(a^2-1) \begin{cases} 2jF_j, & \text{for } j = 1, \ldots, n/2-1 \\ 2jF_j + (2n-1-j)F_{2n-1-j} & \text{for } j = n/2, \ldots, n-1, \end{cases}$$

where we have used that

$$(4n-2-2j)(ad_{4n-2-2j} + d_{2j-n+1}) = ad_{2p} + d_{3n-1-2p} = F_{2n-1-p}$$

with $p = 2n-1-j$. So,

$$0 = \sum_{j=1}^{n-1} \prod_{k=j}^{2n-3} \frac{4n-3-2k}{2k+1} (S_{2j} + S_{4n-2-2j}) = C_{4n-2} a(a^2-1),$$

where

$$C_{4n-2} = \sum_{j=1}^{n/2-1} 2jF_j \prod_{k=j}^{2n-3} \frac{4n-3-2k}{2k+1} + \sum_{j=n/2}^{n-1} (2jF_j + (2n-1-j)F_{2n-1-j}) \prod_{k=j}^{2n-3} \frac{4n-3-2k}{2k+1}.$$

We claim that $C_{4n-2} \neq 0$ for all $n \geq 4$ but we are not able to prove this statement. In the following we add a table of the values of C_{4n-2}.

Table 1. Values of C_{4n-2} for n even and $n \geq 4$.

n	4	6	8	10	12	14
C_{4n-2}	$\frac{4}{429}$	$\frac{2552}{793611}$	$\frac{7556}{581690713}$	$\frac{490921616}{773177645625}$	$\frac{3415220}{9653339997}$	$\frac{9525314385967112}{44271577283526521775}$

Acknowledgements

The first author is partially supported by a MINECO/ FEDER grant number PID2020-113758GB-I00 and an AGAUR (Generalitat de Catalunya) grant number 2017SGR 1276. The second author is supported by FCT/Portugal through UID/MAT/04459/2013.

References

[1] A. Algaba, C. García and M. Reyes, *Invariant curves and analytic integrability of a planar vector field*, J. Differential Equations, 2019, 266(2–3), 1357–1376.

[2] L. A. Cherkas, *On the conditions for a center for certain equations of the form* $yy' = P(x) + Q(x)y + R(x)y^2$, Differ. Uravn., 1974, 8, 1435–1439; Differ. Equ., 1974, 8, 1104–1107.

[3] L. A. Cherkas, *Estimation of the number of limit cycles of autonomous systems*, Differential Equations, 1977, 13, 529–547.

[4] C. J. Christopher, *An algebraic approach to the classification of centers in polynomial Liénard systems*, J. Math. Anal. Appl., 1999, 229, 319–329.

[5] C. J. Christopher and C. Li, *Limit cycles of differential equations*, Advanced Courses in Mathematics. CRM Barcelona. Birkhäuser-Verlag, Basel, 2007.

[6] C. J. Christopher and N. G. Lloyd, *Small-amplitude limit cycles in polynomial Liénard systems*, NoDEA Nonlinear Differential Equations Appl., 1996, 3, 183–190.

[7] C. Christopher and D. Schlomiuk, *On general algebraic mechanisms for producing centers in polynomial differential systems*, J. Fixed Point Theory Appl., 2008, 3(2), 331–351.

[8] A. Cima, A. Gasull and F. Mañosas, *A Note on the Lyapunov and Period Constants*, Qual. Theory Dyn. Syst., 2020, 19(1), 44.

[9] B. Fercec and J. Giné, *A blow-up method to prove formal integrability for some planar differential systems*, J. Appl. Anal. Comput., 2018, 8(6), 1833–1850.

[10] B. Fercec and J. Giné, *Blow-up method to compute necessary conditions of integrability for planar differential systems*, Appl. Math. Comput., 2019, 358, 16–24.

[11] A. Gasull, *Differential equations that can be transformed into equations of Liénard type*, in "17 Colóquio Brasileiro de Matemática, 1989."

[12] A. Gasull, J. Giné and J. Torregrosa, *Center problem for systems with two monomial nonlinearities*, Commun. Pure Appl. Anal., 2016, 15(2), 577–598.

[13] J. Giné, *The nondegenerate center problem and the inverse integrating factor*, Bull. Sci. Math., 2006, 130(2), 152–161.

[14] J. Giné, *The center problem for a linear center perturbed by homogeneous polynomials*, Acta Math. Sin. (Engl. Ser.), 2006, 22(6), 1613–1620.

[15] J. Giné, *Singularity analysis in planar vector fields*, J. Math. Phys., 2014, 55, Article ID 112703.

[16] J. Giné, *Center conditions for generalized polynomial Kukles systems*, Commun. Pure Appl. Anal., 2017, 16(2), 417–425.

[17] J. Giné, *Center conditions for polynomial Liénard systems*, Qual. Theory Dyn. Syst., 2017, 16(1), 119–126.

[18] J. Giné and J. Llibre, *Analytic reducibility of nondegenerate centers: Cherkas systems*, Electron. J. Qual. Theory Differ. Equ., 2016, 49, 1–10.

[19] J. Giné and S. Maza, *The reversibility and the center problem*, Nonlinear Anal., 2011, 74(2), 695–704.

[20] J. Giné and X. Santallusia, *Implementation of a new algorithm of computation of the Poincaré-Liapunov constants*, J. Comput. Appl. Math., 2004, 166(2), 465–476.

[21] J. Giné and C. Valls, *Small-amplitude limit cycles of certain planar differential systems* Qual. Theory Dyn. Syst., 2020, 19(2), 53.

[22] M. Han, Y. Tian and P. Yu, *Small-amplitude limit cycles of polynomial Liénard systems*, Sci. China Math., 2013, 56(8), 1543–1556.

[23] M. Hu, T. Li and X. Chen, *Bi-center problem and Hopf cyclicity of a cubic Liénard system*, Discrete Contin. Dyn. Syst. Ser. B, 2020, 25(1), 401–414.

[24] J. Jiang and M. Han, *Small-amplitude limit cycles of some Liénard-type systems*, Nonlinear Anal., 2009, 71(12), 6373–6377.

[25] F. Li, P. Yu and Y. Liu, *Analytic integrability of two lopsided systems*, Internat. J. Bifur. Chaos Appl. Sci. Engrg., 2016, 26(2), 14.

[26] V. G. Romanovski and D. S. Shafer, *The center and cyclicity problems: acomputational algebra approach*, Birkhäuser, Boston, 2009.

[27] Y. Wu, H. Li and A. Alsaedi, *Center conditions and bifurcation of limit cycles created from a class of second-order ODEs*, Internat. J. Bifur. Chaos Appl. Sci. Engrg., 2019, 29(1), 12.

[28] Y. Yan and Z. Zhou, *Research on the composition center of a class of rigid differential systems*, J. Appl. Anal. Comput., 2020, 10(6), 2506–2520.

[29] Y. Ye, S. Cai, L. Chen, K. Huang, D. Luo, Z. Ma, E. Wang, M. Wang and X. Yang, *Theory of Limit Cycles*, 2nd ed., Transl. Math. Monogr., Amer. Math. Soc., Providence, 1986, 66.

[30] Z. Zhang, T. Ding, W. Huang and Z. Dong, *Qualitative Theory of Differential Equations*, Transl. Math. Monogr., Amer. Math. Soc., Providence, 1992, 101.

[31] Z. Zhou, *On the center-focus problem for a family of high-order polynomial differential systems*, J. Dynam. Differential Equations, 2020, 32(1), 393–418.

[32] Z. Zhou and V.G. Romanovski, *The center problem and the composition condition for a family of quartic differential systems*, Electron. J. Qual. Theory Differ. Equ., 2018, 15, 17.