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ORBITAL STABILITY OF SOLITARY WAVES FOR THE NONLINEAR SCHRÖDINGER-KDV SYSTEM*

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Abstract This paper investigates the stability of solitary waves for the nonlinear Schrödinger-KdV system. We establish the existence and orbital stability of solitary waves solutions by applying the abstract results and detailed spectral analysis, and this result improves the previous one by Chen (1999).

Keywords Solitary waves, orbital stability, nonlinear Schrödinger-KdV system.

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1. Introduction

This paper is concerned with the orbital stability of solitary waves for the nonlinear Schrödinger-KdV system

$$\begin{cases} i\varepsilon_t + \varepsilon_{xx} + \lambda n\varepsilon = \alpha|\varepsilon|^2\varepsilon, \\ n_t + nn_x + n_{xxx} + \nu(|\varepsilon|^2)_x = 0, \end{cases} x \in \mathbb{R}, \tag{1.1}$$

where $\lambda, \nu, \alpha \in \mathbb{R}$, $\varepsilon(t,x)$ is a complex function, $n(t,x)$ is a real function. The system (1.1) arises in fluid mechanics as well as plasma physics, which describes the interactions between short-wave ε and long-wave n.

Corcho and Linares [5] established the global well-posedness of strong solution to the system (1.1) on the assumption that the initial data $(\varepsilon(0,x), n(0,x)) \in H^1(\mathbb{R})$ with $\lambda\nu > 0$. By virtue of an appropriate change of both independent and dependent variables, one can take $\lambda = \nu = 1$ to get

$$\begin{cases} i\varepsilon_t + \varepsilon_{xx} + n\varepsilon = \alpha|\varepsilon|^2\varepsilon, \\ n_t + nn_x + n_{xxx} + (|\varepsilon|^2)_x = 0, \end{cases} x \in \mathbb{R}, \tag{1.2}$$

where $\alpha \in \mathbb{R}$. When $\alpha = 0$, it studies the resonant interaction between short and long capillary-gravity waves on water of uniform finite depth, in plasma physics and

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in a diatomic lattice system. The well-posedness of Cauchy problem was considered in [8, 10, 11]. Chen [3] studied the stability of solitary waves for the system (1.2) with $\alpha = 0$. In 1998, Guo and Chen [9] considered the orbital stability of solitary waves for the Long-short wave resonance system

$$\begin{cases} i\varepsilon_t + \varepsilon_{xx} = n\varepsilon + \alpha|\varepsilon|^2\varepsilon, \\ n_t = (|\varepsilon|^2)_x, \end{cases} x \in \mathbb{R}, \tag{1.3}$$

and recently, Zheng, Di and Peng [20] solved the orbital stability to the Long-short wave resonance system with a cubic-quintic strong nonlinear terms

$$\begin{cases} i\varepsilon_t + \varepsilon_{xx} = \gamma n\varepsilon + \alpha|\varepsilon|^2\varepsilon + \delta|\varepsilon|^4\varepsilon, \\ n_t = \beta(|\varepsilon|^2)_x. \end{cases} x \in \mathbb{R}, \tag{1.4}$$

Zhang, Li, Li and Chen [19] proved the orbital stability for a generalized Boussinesq system with two nonlinear terms. Lu, Chen and Deng [17] considered the orbital stability of peakons for a generalized Camassa-Holm system. One can also see [2,4,12,15,16] for some results on a single nonlinear Schödinger system and [13,14,18] on a single KdV-type system.

In this paper, we consider the existence and orbital stability of solitary waves to system (1.2). We study the solitary waves solutions of the following forms

$$\varepsilon(t,x) = \mathrm{e}^{-i\omega t}\mathrm{e}^{iq(x-vt)}\hat{\varepsilon}_{\omega,v}(x-vt) \quad \text{and} \quad n(t,x) = n_{\omega,v}(x-vt), \tag{1.5}$$

where $q, \omega, v \in \mathbb{R}$. Our results cover the orbital stability of solitary waves for the nonlinear Schrödinger-KdV system in [3] with $\alpha = 0$. Let T_1, T_2 be one-parameter groups of the unitary operator on the function space X

$$T_1(s_1)\overrightarrow{u}(\cdot) = \overrightarrow{u}(\cdot - s_1), \; T_2(s_2)\overrightarrow{u}(\cdot) = (\mathrm{e}^{-is_2}\varepsilon(\cdot), n(\cdot)), \tag{1.6}$$

for $\overrightarrow{u}(\cdot) = (\varepsilon(\cdot), n(\cdot)) \in X$, $s_1, s_2 \in \mathbb{R}$. Denote

$$\overrightarrow{\Phi}_{\omega,v}(x) = \left(\varepsilon_{\omega,v}(x), n_{\omega,v}(x)\right), \tag{1.7}$$

where $\varepsilon_{\omega,v}(x) = \mathrm{e}^{iqx}\hat{\varepsilon}_{\omega,v}(x)$. The orbital stability of solitary waves $T_1(vt)T_2(\omega t)$ $\overrightarrow{\Phi}_{\omega,v}(x)$ can be defined as follows.

Definition 1.1 ([6]). The solitary wave $T_1(vt)T_2(\omega t)\overrightarrow{\Phi}_{\omega,v}(x)$ is orbitally stable if for all $\epsilon > 0$, there exists $\delta > 0$ with the following property: If $\|\overrightarrow{u}_0 - \overrightarrow{\Phi}_{\omega,v}\|_X < \delta$ and $\overrightarrow{u}(t)$ is a solution of (1.2) in some interval $[0, t_0)$ with $\overrightarrow{u}(0) = \overrightarrow{u}_0$, then $\overrightarrow{u}(t)$ can be continued to a solution in $0 \le t < +\infty$, and

$$\sup_{0<t<\infty} \inf_{s_1\in\mathbb{R}} \inf_{s_2\in\mathbb{R}} \|\overrightarrow{u}(t) - T_1(s_1)T_2(s_2)\overrightarrow{\Phi}_{\omega,v}\|_X < \epsilon. \tag{1.8}$$

Otherwise, $T_1(vt)T_2(\omega t)\overrightarrow{\Phi}_{\omega,v}$ is called orbitally unstable.

The paper is organized as follows. In section 2, we first introduce the existence of solitary waves and in section 3, we present the spectral analysis of some self-adjoint operators necessary to obtain our stability result and we prove the Theorem 3.1 under the sufficient conditions.

2. The existence of solitary waves

In this section, we consider the exact solitary waves of the following nonlinear Schrödinger-KdV system

$$\begin{cases} i\varepsilon_t + \varepsilon_{xx} + n\varepsilon = \alpha|\varepsilon|^2\varepsilon, \\ n_t + nn_x + n_{xxx} = -(|\varepsilon|^2)_x. \end{cases} \tag{2.1}$$

Let

$$\varepsilon(t,x) = \mathrm{e}^{-i\omega t}\varepsilon_{\omega,v}(x-vt) = \mathrm{e}^{-i\omega t}\mathrm{e}^{iq(x-vt)}\hat{\varepsilon}_{\omega,v}(x-vt) \text{ and } n(t,x) = n_{\omega,v}(x-vt) \tag{2.2}$$

be the solitary waves of system (2.1), where ω, v, q are constants, $\hat{\varepsilon}_{\omega,v}$ and $n_{\omega,v}$ are real function. Assume $\hat{\varepsilon}_{\omega,v}(x), \hat{\varepsilon}'_{\omega,v}(x), \hat{\varepsilon}''_{\omega,v}(x), n_{\omega,v}(x), n''_{\omega,v}(x) \to 0$, as $x \to \infty$, then substituting the waveform (2.2) into the system (2.1), we can obtain that $\varepsilon_{\omega,v}$ satisfies the equation

$$\varepsilon''_{\omega,v} + n_{\omega,v}\varepsilon_{\omega,v} - \alpha|\varepsilon_{\omega,v}|^2\varepsilon_{\omega,v} + \omega\varepsilon_{\omega,v} - iv\varepsilon'_{\omega,v} = 0, \tag{2.3}$$

$\hat{\varepsilon}_{\omega,v}$ satisfies

$$\hat{\varepsilon}''_{\omega,v} + i(2q-v)\hat{\varepsilon}'_{\omega,v} + (\omega + qv - q^2 + n_{\omega,v} - \alpha\hat{\varepsilon}^2_{\omega,v})\hat{\varepsilon}_{\omega,v} = 0, \tag{2.4}$$

and $n_{\omega,v}$ satisfies

$$n''_{\omega,v} - vn_{\omega,v} + \frac{1}{2}n^2_{\omega,v} + \hat{\varepsilon}^2_{\omega,v} = 0. \tag{2.5}$$

By (2.4) we have $q = \frac{v}{2}$. Let $\hat{\varepsilon}_{\omega,v} = c_1\mathrm{sech}c_2x$ satisfy (2.4) with constants c_1, c_2 to be determined later, then we deduce

$$\hat{\varepsilon}''_{\omega,v} = (c_2^2 - 2c_2^2\mathrm{sech}^2c_2x)\hat{\varepsilon}_{\omega,v} = (-w - \frac{v^2}{4} - n_{\omega,v} + \alpha c_1^2\mathrm{sech}^2c_2x)\hat{\varepsilon}_{\omega,v}. \tag{2.6}$$

Moreover, by (2.6) one has

$$n_{\omega,v} = (2c_2^2 + \alpha c_1^2)\mathrm{sech}^2c_2x - c_2^2 - \omega - \frac{v^2}{4} = (2c_2^2 + \alpha c_1^2)\mathrm{sech}^2c_2x, \tag{2.7}$$

and

$$c_2^2 = -\omega - \frac{v^2}{4}. \tag{2.8}$$

Insetting (2.7)-(2.8) into (2.5), we arrive at

$$\begin{aligned} &-v(2c_2^2 + \alpha c_1^2)\mathrm{sech}^2c_2x + \frac{1}{2}(2c_2^2 + \alpha c_1^2)^2\mathrm{sech}^4c_2x \\ =&2c_2^2(2c_2^2 + \alpha c_1^2)(3\mathrm{sech}^4c_2x - 2\mathrm{sech}^2c_2x) - c_1^2\mathrm{sech}^2c_2x. \end{aligned} \tag{2.9}$$

Thus, by (2.9), it holds that

$$2c_2^2 + \alpha c_1^2 = 12c_2^2,\ c_2 = \sqrt{-\omega - \frac{v^2}{4}},\ c_1 = \sqrt{12(-\omega - \frac{v^2}{4})(v + 4\omega + v^2)}. \tag{2.10}$$

From (2.4), (2.7), (2.8) and (2.10), it yields

$$\begin{cases} \hat{\varepsilon}_{\omega,v} = \sqrt{12(-\omega - \frac{v^2}{4})(v + 4\omega + v^2)}\mathrm{sech}(\sqrt{-\omega - \frac{v^2}{4}}x), \\ n_{\omega,v} = 12(-\omega - \frac{v^2}{4})\mathrm{sech}^2(\sqrt{-\omega - \frac{v^2}{4}}x),\ q = \frac{v}{2}. \end{cases} \tag{2.11}$$

Therefore, we have the following theorem

Theorem 2.1. *For any real constants ω, v satisfying*

$$-\frac{v^2}{4} - \frac{v}{4} < \omega < -\frac{v^2}{4}, \quad v > 0, \tag{2.12}$$

there exists solitary waves of system (2.1) in the form of (2.2) with $\hat{\varepsilon}_{\omega,v}$ and $n_{\omega,v}$ satisfying (2.11).

3. Orbital stability of solitary waves

In this section, we prove that the orbital stability of solitary waves solution for the nonlinear Schröinger-KdV system (2.1). First, let $\overrightarrow{u} = (\varepsilon, n)$, system (2.1) can be rewritten in the Hamiltonian form

$$\frac{\mathrm{d}\overrightarrow{u}}{\mathrm{d}t} = JE'(\overrightarrow{u}), \tag{3.1}$$

where

$$J = \begin{bmatrix} -\frac{i}{2} & \\ & \frac{\partial}{\partial x} \end{bmatrix}, \tag{3.2}$$

$$E(\overrightarrow{u}) = \int \left(|\varepsilon_x|^2 - n|\varepsilon|^2 + \frac{\alpha}{2}|\varepsilon|^4 - \frac{1}{6}n^3 + \frac{1}{2}(n_x)^2 \right) \mathrm{d}x \tag{3.3}$$

and the Frechet derivatives of $E(\overrightarrow{u})$

$$E'(\overrightarrow{u}) = \begin{bmatrix} -2\varepsilon_{xx} - 2n\varepsilon + 2\alpha|\varepsilon|^2\varepsilon \\ -|\varepsilon|^2 - \frac{1}{2}n^2 - n_{xx} \end{bmatrix}. \tag{3.4}$$

In order to present our main results, we state the function space which will be used in this paper. Define $X = H^1_{complex}(\mathbb{R}) \times H^1_{real}(\mathbb{R})$ with the inner product

$$(\overrightarrow{u}_1, \overrightarrow{u}_2) = \mathrm{Re} \int_{\mathbb{R}} (\varepsilon_1\overline{\varepsilon}_2 + \varepsilon_{1x}\overline{\varepsilon}_{2x} + n_1n_2 + n_{1x}n_{2x})\mathrm{d}x, \tag{3.5}$$

and the dual space of X is $X^* = H^{-1}_{complex}(\mathbb{R}) \times H^{-1}_{real}(\mathbb{R})$. There exists a natural isomorphism $I : X \to X^*$ defined by $\langle I\overrightarrow{u}_1, \overrightarrow{u}_2\rangle = (\overrightarrow{u}_1, \overrightarrow{u}_2)$, where $\langle\cdot,\cdot\rangle$ denotes the pairing between X and X^*, and

$$\langle \overrightarrow{f}, \overrightarrow{u} \rangle = \mathrm{Re} \int_{\mathbb{R}} (f_1\overline{\varepsilon} + f_2 n)\mathrm{d}x. \tag{3.6}$$

By (3.5)-(3.6), we have $I = \begin{bmatrix} 1 - \frac{\partial^2}{\partial x^2} & \\ & 1 - \frac{\partial^2}{\partial x^2} \end{bmatrix}$ and it is easy to verify that the operator J of (3.2) is a skew symmetric operator, namely, $\langle J\overrightarrow{u_1}, \overrightarrow{u_2}\rangle = -\langle \overrightarrow{u_1}, J\overrightarrow{u_2}\rangle$. In addition, it yields from (1.6)

$$T_1'(0) = \begin{bmatrix} -\frac{\partial}{\partial x} & \\ & -\frac{\partial}{\partial x} \end{bmatrix} \quad \text{and} \quad T_2'(0) = \begin{bmatrix} -i & \\ & 0 \end{bmatrix}. \tag{3.7}$$

As in [6,7], by $T_1'(0) = JB_1$, $T_2'(0) = JB_2$, one gets

$$B_1 = \begin{bmatrix} -2i\frac{\partial}{\partial x} & \\ & -1 \end{bmatrix} \text{ and } B_2 = \begin{bmatrix} 2 & \\ & 0 \end{bmatrix}.$$

Define

$$Q_1(\overrightarrow{u}) = \frac{1}{2}\langle B_1\overrightarrow{u}, \overrightarrow{u}\rangle = -\frac{1}{2}\int n^2\mathrm{d}x - i\int \varepsilon_x\varepsilon\mathrm{d}x = -\frac{1}{2}\int n^2\mathrm{d}x + \mathrm{Im}\int \varepsilon_x\bar{\varepsilon}\mathrm{d}x, \quad (3.8)$$

$$Q_2(\overrightarrow{u}) = \frac{1}{2}\langle B_2\overrightarrow{u}, \overrightarrow{u}\rangle = \int |\varepsilon|^2\mathrm{d}x. \quad (3.9)$$

Combining (1.6), (3.3), (3.8) and (3.9), we can prove that $E(\overrightarrow{u})$, $Q_1(\overrightarrow{u})$ and $Q_2(\overrightarrow{u})$ are invariant under T_1 and T_2, namely

$$\begin{aligned} &E(T_1(s_1)T_2(s_2)\overrightarrow{u}) = E(\overrightarrow{u}),\\ &Q_1(T_1(s_1)T_2(s_2)\overrightarrow{u}) = Q_1(\overrightarrow{u}),\\ &Q_2(T_1(s_1)T_2(s_2)\overrightarrow{u}) = Q_2(\overrightarrow{u}) \end{aligned} \quad (3.10)$$

for any $s_1, s_2 \in \mathbb{R}$. Moreover, $E(\overrightarrow{u})$ is formally conserved under the flow of (3.1). Indeed, by the skew symmetry of operator J, we can get

$$\frac{\mathrm{d}E(\overrightarrow{u})}{\mathrm{d}t} = \langle E'(\overrightarrow{u}), \frac{\mathrm{d}\overrightarrow{u}}{\mathrm{d}t}\rangle = \langle E'(\overrightarrow{u}), JE'(\overrightarrow{u})\rangle = 0, \quad (3.11)$$

that is $E(\overrightarrow{u}(t)) = E(\overrightarrow{u}(0))$. Similarly, we have

$$Q_1(\overrightarrow{u}(t)) = Q_1(\overrightarrow{u}(0)),\ Q_2(\overrightarrow{u}(t)) = Q_2(\overrightarrow{u}(0)) \quad (3.12)$$

for any $t \in \mathbb{R}$.

By (2.3) and (2.5) we deduce that $\overrightarrow{\Phi}_{\omega,v}$ is a critical point of function $E - vQ_1 - \omega Q_2$, that is

$$E'(\overrightarrow{\Phi}_{\omega,v}) - vQ_1'(\overrightarrow{\Phi}_{\omega,v}) - \omega Q_2'(\overrightarrow{\Phi}_{\omega,v}) = 0, \quad (3.13)$$

where $Q_1'(\overrightarrow{u}) = \begin{bmatrix} -2i\varepsilon_x \\ -n \end{bmatrix}$ and $Q_2'(\overrightarrow{u}) = \begin{bmatrix} 2\varepsilon \\ 0 \end{bmatrix}$ are the Frechet derivatives of Q_1 and Q_2, respectively.

Now, we consider the operator $H_{\omega,v} : X \to X^*$,

$$H_{\omega,v} = E''(\overrightarrow{\Phi}_{\omega,v}) - vQ_1''(\overrightarrow{\Phi}_{\omega,v}) - \omega Q_2''(\overrightarrow{\Phi}_{\omega,v}), \quad (3.14)$$

where

$$E''(\overrightarrow{u}) = \begin{bmatrix} -2\partial_{xx} - 2n + 2\alpha(|\varepsilon|^2 + 2\varepsilon^2) & -2\varepsilon \\ -2\varepsilon & -n - \partial_{xx} \end{bmatrix}, \quad (3.15)$$

$$Q_1''(\overrightarrow{u}) = B_1 = \begin{bmatrix} -2i\partial_x & 0 \\ 0 & -1 \end{bmatrix} \quad (3.16)$$

and

$$Q_2''(\overrightarrow{u}) = B_2 = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}. \tag{3.17}$$

By (3.14)-(3.17), we obtain that

$$H_{\omega,v}\overrightarrow{\psi} = \begin{bmatrix} -2\psi_{1xx} - 2n\psi_1 + 2\alpha(|\varepsilon|^2\psi_1 + \overline{\varepsilon}\psi_1\varepsilon + \varepsilon\overline{\psi}_1\varepsilon) + 2iv\psi_{1x} - 2\omega\psi_1 - 2\varepsilon\psi_2 \\ -\overline{\varepsilon}\psi_1 - \varepsilon\overline{\psi}_1 - n\psi_2 - \psi_{2xx} + v\psi_2 \end{bmatrix} \tag{3.18}$$

with $\overrightarrow{\psi} = (\psi_1, \psi_2) \in X$. Observe that $H_{\omega,v}$ is self-adjoint operator, i.e., $H^*_{\omega,v} = H_{\omega,v}$, and $I^{-1}H_{\omega,v}$ is a bounded self-adjoint operator. The spectrum of $H_{\omega,v}$ consists of the real numbers λ such that $H_{\omega,v} - \lambda I$ is not invertible.
By (1.7), (2.3)-(2.5), (3.7) and (3.18) it holds that

$$H_{\omega,v}T_1'(0)\overrightarrow{\Phi}_{\omega,v}(x) = 0 \quad \text{and} \quad H_{\omega,v}T_2'(0)\overrightarrow{\Phi}_{\omega,v}(x) = 0. \tag{3.19}$$

Let

$$Z = \{k_1T_1'(0)\overrightarrow{\Phi}_{\omega,v}(x) + k_2T_2'(0)\overrightarrow{\Phi}_{\omega,v}(x) \big| k_1, k_2 \in R\}, \tag{3.20}$$

then by (3.19), Z is contained in the kernel of $H_{\omega,v}$.

We shall apply the abstract stability theory of Grillakis, Shatah and Strauss [6,7] to prove the orbital stability of solitary waves $T_1(vt)T_2(\omega t)\overrightarrow{\Phi}_{\omega,v}(x)$ for system (2.1). First, we need to study the operator $H_{\omega,v}$ and state the assumption on $H_{\omega,v}$ which will be used to prove the orbital stability of solitary waves.

Assumption 3.1 ([6] **Spectral decomposition of** $H_{\omega,v}$**).** *The space X is decomposed as a direct sum*

$$X = N + Z + P, \tag{3.21}$$

where Z is defined in (3.20)*, N is a finite-dimensional subspace such that*

$$\langle H_{\omega,v}\overrightarrow{u}, \overrightarrow{u}\rangle < 0, \ \text{for } 0 \neq \overrightarrow{u} \in N, \tag{3.22}$$

and P is a closed subspace such that

$$\langle H_{\omega,v}\overrightarrow{u}, \overrightarrow{u}\rangle \geq \delta\|\overrightarrow{u}\|^2_X, \ \text{for } \overrightarrow{u} \in P \tag{3.23}$$

with some constant $\delta > 0$ independent of $\overrightarrow{u}$.

For any $\overrightarrow{\psi} \in X$, rewrite it as

$$\overrightarrow{\psi} = (\mathrm{e}^{i\frac{v}{2}x}z_1, z_2) \tag{3.24}$$

with $z_1 = y_1 + iy_2$, $y_1 = \mathrm{Re}z_1$, $y_2 = \mathrm{Im}z_1$, then by (3.18) we have

$$\langle H_{\omega,v}\overrightarrow{\psi}, \overrightarrow{\psi}\rangle = \mathrm{Re}\int_{\mathbb{R}} \Big[2\left(-\frac{\partial^2}{\partial x^2} - \omega - \frac{v^2}{4} - n\right) z_1\overline{z}_1$$

$$
\begin{aligned}
&+2\alpha\left(|\hat{\varepsilon}|^2 z_1\overline{z}_1+\overline{\varepsilon}z_1\varepsilon\overline{z}_1+\varepsilon e^{-i\frac{v}{2}x}\overline{z}_1\varepsilon e^{-i\frac{v}{2}x}\overline{z}_1\right)-2\varepsilon z_2 e^{-i\frac{v}{2}x}\overline{z}_1\Big]\mathrm{d}x\\
&+\mathrm{Re}\int_{\mathbb{R}}\Big[-\overline{\varepsilon}e^{i\frac{v}{2}x}z_1z_2-\varepsilon e^{-i\frac{v}{2}x}\overline{z}_1z_2+\left(-\frac{\partial^2}{\partial x^2}+v-n\right)z_2^2\Big]\mathrm{d}x\\
=&\langle\overline{L}_1y_1,y_1\rangle+\langle L_2y_2,y_2\rangle+\langle L_4z_2,z_2\rangle+4\mathrm{Re}\int_{\mathbb{R}}-\hat{\varepsilon}z_2\overline{z}_1\mathrm{d}x \qquad (3.25)\\
=&\langle L_1y_1,y_1\rangle+\langle L_2y_2,y_2\rangle+\langle L_3z_2,z_2\rangle\\
&+\int_{\mathbb{R}}\left(\frac{2\hat{\varepsilon}y_1}{\sqrt{v+4\omega+v^2}}-\sqrt{v+4\omega+v^2}z_2\right)^2\mathrm{d}x,
\end{aligned}
$$

where

$$
\begin{aligned}
\overline{L}_1=&2\left(-\frac{\partial^2}{\partial x^2}-\omega-\frac{v^2}{4}-n+3\alpha\hat{\varepsilon}^2\right),\\
L_1=&\overline{L}_1-\frac{4\hat{\varepsilon}^2}{v+4\omega+v^2}=-2\frac{\partial^2}{\partial x^2}+2\left(-\omega-\frac{v^2}{4}\right)-n,\\
L_2=&2\left(-\frac{\partial^2}{\partial x^2}-\omega-\frac{v^2}{4}-n+\alpha\hat{\varepsilon}^2\right)=-2\frac{\partial^2}{\partial x^2}+2\left(-\omega-\frac{v^2}{4}\right)-\frac{1}{3}n, \qquad (3.26)\\
L_4=&-\frac{\partial^2}{\partial x^2}+v-n,\\
L_3=&L_4-\left(v+4\omega+v^2\right)=-\frac{\partial^2}{\partial x^2}+4\left(-\omega-\frac{v^2}{4}\right)-n.
\end{aligned}
$$

Next, we focus on the spectrum structure of the linear operators L_1, L_2 and L_3. Note that $L_1=-2\frac{\partial^2}{\partial x^2}+2\left(-\omega-\frac{v^2}{4}\right)+M_1(x)$, $L_2=-2\frac{\partial^2}{\partial x^2}+2\left(-\omega-\frac{v^2}{4}\right)+M_2(x)$ and $L_3=-\frac{\partial^2}{\partial x^2}+4\left(-\omega-\frac{v^2}{4}\right)+M_3(x)$ with $M_i(x)\to 0$ as $|x|\to+\infty$, where $i=1,2,3$. Therefore, by Wely's theorem on the essential spectrum, it is easy to see that $\sigma_{ess}(L_1)=\sigma_{ess}(L_2)=\left[2\left(-\omega-\frac{v^2}{4}\right),+\infty\right)$ and $\sigma_{ess}(L_3)=\left[4\left(-\omega-\frac{v^2}{4}\right),+\infty\right)$. Moreover, combining (2.4), (2.5), (2.11) and (3.26), one gets

$$
L_1\hat{\varepsilon}_x=0,\quad L_2\hat{\varepsilon}=0,\quad L_3n_x=0, \qquad (3.27)
$$

and $\hat{\varepsilon}_x$ and n_x have a simple zero point at $x=0$, respectively. Thanks to the Sturm-Liouville theorem, we know that 0 is the second eigenvalue of L_1 and L_3. Thus, L_1 and L_3 only have one strictly negative eigenvalue, respectively. In fact, by the simple calculation, we have

$$
L_1(\hat{\varepsilon}^2)=-6\left(-\omega-\frac{v^2}{4}\right)(\hat{\varepsilon}^2)\quad\text{and}\quad L_3(2n\hat{\varepsilon})=-5\left(-\omega-\frac{v^2}{4}\right)(2n\hat{\varepsilon}). \qquad (3.28)
$$

Thus, the first negative eigenvalue of L_1 is $-6\left(-\omega-\frac{v^2}{4}\right)$, with corresponding eigenfunction is $\hat{\varepsilon}^2$ and the first negative eigenvalue of L_3 is $-5\left(-\omega-\frac{v^2}{4}\right)$, with corresponding eigenfunction is $2n\hat{\varepsilon}$. Since $\hat{\varepsilon}$ has no zero point, 0 is the first simple eigenvalue of L_2 by the Sturm-Liouville theorem.

According to [1, 3], we have the following lemmas.

Lemma 3.1. *For any real functions $y_1 \in H^1(\mathbb{R})$ satisfying*

$$\langle y_1, \hat{\varepsilon}^2\rangle = \langle y_1, \hat{\varepsilon}_x\rangle = 0, \tag{3.29}$$

there exists a positive number $\overline{\delta}_1, \delta_1 > 0$ such that $\langle L_1 y_1, y_1\rangle \geq \overline{\delta}_1 \|y_1\|^2_{L^2}$. Moreover, we have

$$\langle L_1 y_1, y_1\rangle \geq \delta_1 \|y_1\|^2_{H^1}. \tag{3.30}$$

Lemma 3.2. *For any real functions $y_2 \in H^1(\mathbb{R})$ satisfying*

$$\langle y_2, \hat{\varepsilon}\rangle = 0, \tag{3.31}$$

there exists a positive number $\delta_2 > 0$ such that

$$\langle L_2 y_2, y_2\rangle \geq \delta_2 \|y_2\|^2_{H^1}. \tag{3.32}$$

Lemma 3.3. *For any real functions $z_2 \in H^1(\mathbb{R})$ satisfying*

$$\langle z_2, 2n\hat{\varepsilon}\rangle = \langle z_2, n_x\rangle = 0, \tag{3.33}$$

there exists a positive number $\overline{\delta}_3, \delta_3 > 0$ such that $\langle L_3 z_2, z_2\rangle \geq \overline{\delta}_3 \|z_2\|^2_{L^2}$. Moreover, we have

$$\langle L_3 z_2, z_2\rangle \geq \delta_3 \|z_2\|^2_{H^1}. \tag{3.34}$$

From (3.24), we denote $\overrightarrow{\psi}^-$ by

$$\overrightarrow{\psi}^- = (\mathrm{e}^{i\frac{v}{2}x}(y_1^- + iy_2^-), z_2^-), \tag{3.35}$$

where $y_1^- = \hat{\varepsilon}^2$, $y_2^- = 0$ and $z_2^- = 2n\hat{\varepsilon}$. Then by (3.25), (3.26) and (3.28) we obtain

$$\langle H_{\omega,v}\overrightarrow{\psi}^-, \overrightarrow{\psi}^-\rangle = -6\left(-\omega - \frac{v^2}{4}\right)\langle \hat{\varepsilon}^2, \hat{\varepsilon}^2\rangle - 5\left(-\omega - \frac{v^2}{4}\right)\langle 2n\hat{\varepsilon}, 2n\hat{\varepsilon}\rangle < 0. \tag{3.36}$$

Moreover, one also gets that the kernel of $H_{\omega,v}$ is spanned by the following two vectors

$$\overrightarrow{\psi}_{0,1} = (\hat{\varepsilon}_x, 0, n_x) \quad \text{and} \quad \overrightarrow{\psi}_{0,2} = (0, \hat{\varepsilon}, 0). \tag{3.37}$$

Let

$$Z = \{k_1\overrightarrow{\psi}_{0,1} + k_2\overrightarrow{\psi}_{0,2}/k_1, k_2 \in \mathbb{R}\}, \tag{3.38}$$

$$P = \big\{\overrightarrow{p} \in X/\overrightarrow{p} = (p_1, p_2, p_3), \langle p_1, \hat{\varepsilon}^2\rangle + \langle p_3, 2n\hat{\varepsilon}\rangle = 0,$$
$$\langle p_1, \hat{\varepsilon}_x\rangle + \langle p_3, n_x\rangle = 0, \langle p_2, \hat{\varepsilon}\rangle = 0\big\}, \tag{3.39}$$

$$N = \{k\overrightarrow{\psi}^-/k \in \mathbb{R}\}. \tag{3.40}$$

For any $\overrightarrow{u} = (\mathrm{e}^{i\frac{v}{2}x}(y_1 + iy_2), z_2) \in X$, we choose $a_1 = \frac{\langle y_1, \hat{\varepsilon}^2\rangle + \langle z_2, 2n\hat{\varepsilon}\rangle}{\langle \hat{\varepsilon}^2, \hat{\varepsilon}^2\rangle + \langle 2n\hat{\varepsilon}, 2n\hat{\varepsilon}\rangle}$, $b_1 = \frac{\langle y_1, \hat{\varepsilon}_x\rangle + \langle z_2, n_x\rangle}{\langle \hat{\varepsilon}_x, \hat{\varepsilon}_x\rangle + \langle n_x, n_x\rangle}$, $b_2 = \frac{\langle y_2, \hat{\varepsilon}\rangle}{\langle \hat{\varepsilon}, \hat{\varepsilon}\rangle}$, then $\overrightarrow{u}$ can be uniquely represented by

$$\overrightarrow{u} = a_1\overrightarrow{\psi}^- + b_1\overrightarrow{\psi}_{0,1} + b_2\overrightarrow{\psi}_{0,2} + \overrightarrow{p} \tag{3.41}$$

with $\overrightarrow{p} \in P$, which implies (3.21) and (3.22) hold. For subspace P, we have the following lemma.

Lemma 3.4. *For any $\overrightarrow{p} \in P$ defined by* (3.39), *there exists a constant $\delta > 0$ such that*

$$\langle H_{\omega,v}\overrightarrow{p}, \overrightarrow{p}\rangle \geq \delta\|\overrightarrow{p}\|_X \tag{3.42}$$

with δ independent of $\overrightarrow{p}$.

Proof. For any $\overrightarrow{p} \in P$, by (3.25) and Lemmas 3.1-3.3, we have

$$\begin{aligned}\langle H_{\omega,v}\overrightarrow{p}, \overrightarrow{p}\rangle \geq&\delta_1\|p_1\|^2_{H^1} + \delta_2\|p_2\|^2_{H^1} + \delta_3\|p_3\|^2_{H^1}\\ &+ \int_{\mathbb{R}}\left(\frac{2\hat{\varepsilon}p_1}{\sqrt{v+4\omega+v^2}} - \sqrt{v+4\omega+v^2}p_3\right)^2 \mathrm{d}x.\end{aligned} \tag{3.43}$$

(1) If $\|p_3\|_{L^2} \geq 2M\|p_1\|_{L^2}$, $M = \frac{2\|\hat{\varepsilon}\|_{L^\infty}}{v+4\omega+v^2}$, then

$$\int_{\mathbb{R}}\left(\frac{2\hat{\varepsilon}p_1}{\sqrt{v+4\omega+v^2}} - \sqrt{v+4\omega+v^2}p_3\right)^2 \mathrm{d}x \geq \frac{v+4\omega+v^2}{4}\|p_3\|_{L^2}. \tag{3.44}$$

(2) If $\|p_3\|_{L^2} \leq 2M\|p_1\|_{L^2}$, then

$$\delta_1\|p_1\|^2_{H^1} \geq \frac{\delta_1}{2}\|p_1\|^2_{H^1} + \frac{\delta_1}{4M}\|p_3\|^2_{L^2}. \tag{3.45}$$

Thus, for any $\overrightarrow{p} \in P$, by (3.42)-(3.44), we arrive at

$$\langle H_{\omega,v}\overrightarrow{p}, \overrightarrow{p}\rangle \geq\delta\|\overrightarrow{p}\|_X, \tag{3.46}$$

and the proof of Lemma 3.4 is finished. □

Therefore, we prove that Assumption 3.1 holds and $n(H_{\omega,v}) = 1$, where $n(H_{\omega,v})$ denotes the number of negative eigenvalues of $H_{\omega,v}$. Finally, we define $d(\omega, v)$: $R \times R \to R$ by

$$d(\omega, v) = E(\overrightarrow{\Phi}_{\omega,v}) - vQ_1(\overrightarrow{\Phi}_{\omega,v}) - \omega Q_2(\overrightarrow{\Phi}_{\omega,v}), \tag{3.47}$$

and define $d''(\omega, v)$ to be a Hessian matrix of the function $d(\omega, v)$. Now, let us state our main results about stability of solitary wave $T_1(vt)T_2(\omega t)\overrightarrow{\Phi}_{\omega,v}(x)$.

Theorem 3.1. *Under the condition of the Theorem 2.1, if*

$$-\frac{v^2}{4} - \frac{3}{2} < \omega < -\frac{v^2}{4}, \tag{3.48}$$

then the solitary waves $T_1(vt)T_2(\omega t)\overrightarrow{\Phi}_{\omega,v}(x)$ of system (2.1) *are orbital stable.*

Proof. Let $p(d'')$ be the number of positive eigenvalues of its Hessian at (ω, v). To prove Theorem 3.1, we first need to prove $n(H_{\omega,v}) = p(d'') = 1$. By (3.13) and (3.47), we deduce that

$$\begin{aligned}d_\omega(\omega, v) =& - Q_2(\Phi_{\omega,v}) = -\int_{\mathbb{R}}\varepsilon\bar{\varepsilon}\mathrm{d}x\\ =& - \int_{\mathbb{R}} 12\left(-\omega - \frac{v^2}{4}\right)(v+4\omega+v^2)\mathrm{sech}^2\left(\sqrt{-\omega-\frac{v^2}{4}}x\right)\mathrm{d}x\\ =& - 24\sqrt{-\omega-\frac{v^2}{4}}(v+4\omega+v^2),\end{aligned} \tag{3.49}$$

$$
\begin{aligned}
d_v(\omega,v) =& -Q_1(\Phi_{\omega,v}) = \frac{1}{2}\int_{\mathbb{R}} n^2 \mathrm{d}x - \operatorname{Im}\int_{\mathbb{R}} \varepsilon_x \bar{\varepsilon} \mathrm{d}x \\
=& 96\left(-\omega-\frac{v^2}{4}\right)^{\frac{3}{2}} - 12v\sqrt{-\omega-\frac{v^2}{4}}(v+4\omega+v^2),
\end{aligned} \tag{3.50}
$$

$$
d_{\omega\omega}(\omega,v) = \frac{12(v+12\omega+3v^2)}{\sqrt{-\omega-\frac{v^2}{4}}}, \tag{3.51}
$$

$$
d_{\omega v}(\omega,v) = \frac{6v(v+4\omega+v^2)}{\sqrt{-\omega-\frac{v^2}{4}}} - 24(1+2v)\sqrt{-\omega-\frac{v^2}{4}}, \tag{3.52}
$$

$$
d_{v\omega}(\omega,v) = \frac{6v(v+4\omega+v^2)}{\sqrt{-\omega-\frac{v^2}{4}}} - 24(6+2v)\sqrt{-\omega-\frac{v^2}{4}}, \tag{3.53}
$$

and

$$
d_{vv}(\omega,v) = -12v\sqrt{-\omega-\frac{v^2}{4}}(7+2v) + \frac{3v^2-12\left(-\omega-\frac{v^2}{4}\right)}{\sqrt{-\omega-\frac{v^2}{4}}}(v+4\omega+v^2). \tag{3.54}
$$

Hence

$$
\begin{aligned}
\det(d'') =& d_{\omega\omega}d_{vv} - d_{\omega v}d_{v\omega} \\
=& \frac{1}{\sigma}\left(-12\cdot 12v\sigma(y-8\sigma)(7+2v) + 12y(y-8\sigma)(3v^2+12\sigma)\right) - \frac{1}{\sigma}\big(36v^2y^2 \\
& - 6\cdot 24vy\sigma(1+2v) - 6\cdot 24vy\sigma(6+2v) + 24\cdot 24\sigma^2(1+2v)(6+2v)\big) \\
=& -144\left[(y-4\sigma)^2 + 8\sigma(3-2\sigma)\right] < 0
\end{aligned}
$$

provided $-\frac{v^2}{4}-\frac{3}{2}<\omega$, where $y=v+4\omega+v^2>0$, $\sigma=-\omega-\frac{v^2}{4}>0$. It follows that d'' has exactly one positive and one negative eigenvalue, thus $p(d'')=1$ and we complete the proof of Theorem 3.1. □

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