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LACUNARY STATISTICAL HARMONIC SUMMABILITY

Fatih Nuray[1,†]

Abstract In this paper, the concepts of lacunary (H,1) summability, lacunary strongly harmonically summability, lacunary statistical (H,1) summability, lacunary statistical logarithmic convergence of sequences of real numbers are introduced and relations between these concepts are investigated.

Keywords Harmonic convergence, statistical convergence, lacunary sequence.

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1. Introduction and Background

The natural density of a set K of positive integers is defined by

$$\delta(K) := \lim_{n\to\infty} \frac{1}{n} |\{i \le n : \quad i \in K\}|,$$

where $|\{i \le n : \quad i \in K\}|$ denotes the number of elements of K not exceeding n.

A sequence (x_i) of real or complex numbers is said to be statistically convergent to the number a if for every $\epsilon > 0$,

$$\delta(\{i : \quad |x_i - a| \ge \epsilon\}) = 0.$$

A statistically convergent sequence may be bounded or unbounded. For example, define $x_i = 1$ if i is a square and $x_i = 0$ otherwise. Then

$$|\{i \le n : \quad |x_i - 0| \ge \epsilon\}| \le \sqrt{n},$$

so $st - \lim x_i = 0$.

Statistical convergence of sequences of real or complex numbers was introduced by Fast [5]. In [17] Schoenberg established some basic properties of statistical convergence and also studied the concept as a summability method. The reader can refer to the articles [3, 7, 9] for more information on this topic.

There is a natural relationship [2] between statistical convergence and strong Cesàro summability:

$$[C, 1] := \{x : \quad \text{for some} \quad a, \quad \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} |x_i - a| = 0\}.$$

By a lacunary sequence we mean an increasing integer sequence $\theta = (k_r)$ of nonnegative integers such that $k_0 = 0$ and $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$. The

[†]The corresponding author. Email:fnuray@aku.edu.tr(F. Nuray)
[1]Department of Mathematics, Afyon Kocatepe University, Afyonkarahisar, Turkey

intervals determined by $\theta = (k_r)$ is denoted by $I_r := (k_{r-1}, k_r]$, and ratio $\frac{k_r}{k_{r-1}}$ is abbreviated by q_r and $q_1 = k_1$ for convenience. In recent years, lacunary sequences have been studied in ([11, 12]).

There is a strong connection [6] between $[C, 1]$ and the sequence space N_θ, which is defined by

$$N_\theta := \{x : \quad \text{for some} \quad a, \quad \lim_{r\to\infty} \frac{1}{h_r} \sum_{i\in I_r} |x_i - a| = 0\}.$$

Definition of lacunary statistical convergent sequence was given in [8] as follows: Let $\theta = (k_r)$ be a lacunary sequence. A sequence (x_i) of real or complex numbers is said to be lacunary statistically convergent to the number a if for every $\epsilon > 0$,

$$\lim_{r\to\infty} \frac{1}{h_r} |\{i \in I_r : \quad |x_i - a| \geq \epsilon\}| = 0$$

and the authors of [8] established some inclusion relations between the set of lacunary statistically convergent sequences and N_θ and also between the set of lacunary statistically convergent sequences and the set of statistically convergent sequences.

The harmonic means of the sequence (x_i) defined by

$$\tau_n := \frac{1}{\ell_n} \sum_{i=1}^{n} \frac{x_i}{i}, \text{ where } \ell_n := \sum_{i=1}^{n} \frac{1}{i} \sim \log n \text{ for } n = 1, 2, \ldots.$$

A sequence (x_i) is said to be (H,1) summable to a if

$$\lim_{n\to\infty} \frac{1}{\ell_n} \sum_{i=1}^{n} \frac{x_i}{i} = a.$$

The set of all (H,1) summable real sequences will be denoted H. H^0 will denote the set of all real sequences (H,1) summable to 0. It is well known that ordinary convergence always implies harmonic summability, and that the converse implication holds only under additional conditions.

A sequence (x_i) is said to be strongly harmonically summable to a if

$$\lim_{n\to\infty} \frac{1}{\ell_n} \sum_{i=1}^{n} \frac{|x_i - a|}{i} = 0.$$

The set of all strongly harmonically summable real sequences will be denoted $[H]$. $[H^0]$ will denote the set of all real sequences strongly harmonically summable to 0. The articles [4, 10, 13, 15, 16, 18] are written on these topics.

In this paper, we will introduce lacunary (H,1) summable sequence and establish some inclusion relations.

2. Lacunary (H,1) Summability

In this paper, log means the natural logarithm. Furthermore, given two sequences (x_i) and (y_i) of positive numbers (except possibly a finite number of terms), we write $x_i \sim y_i$ if $\lim_{i\to\infty} x_i/y_i = 1$.

Let define lacunary harmonic means of the sequence (x_i) by

$$T_r := \frac{1}{L_r}\sum_{i\in I_r}\frac{x_i}{i}, \text{ where } L_r := \sum_{i\in I_r}\frac{1}{i} \qquad r = 1, 2, ...$$

Definition 2.1. Let $\theta = (k_r)$ be a lacunary sequence. A sequence (x_i) of real numbers is said to be lacunary (H,1) summable to the number a if

$$\lim_{r\to\infty}\frac{1}{L_r}\sum_{i\in I_r}\frac{x_i}{i} = a.$$

The set of all lacunary (H,1) summable real sequences will be denoted H_θ.

Definition 2.2. Let $\theta = (k_r)$ be a lacunary sequence. A sequence (x_i) of real numbers is said to be lacunary strongly harmonically summable to the number a if

$$\lim_{r\to\infty}\frac{1}{L_r}\sum_{i\in I_r}\frac{|x_i - a|}{i} = 0.$$

The set of all lacunary strongly harmonically summable real sequences will be denoted $[H_\theta]$.

Theorem 2.1. *Let $\theta = (k_r)$ be lacunary sequence. Then*

(i) $[H] \subseteq [H_\theta]$ if and only if $\liminf_r \frac{\ell_{k_r}}{\ell_{k_{r-1}}} > 1$;

(ii) $[H_\theta] \subseteq [H]$ if and only if $\limsup_r \frac{\ell_{k_r}}{\ell_{k_{r-1}}} < \infty$;

(iii) $[H_\theta] = [H]$ if and only if $1 < \liminf_r \frac{\ell_{k_r}}{\ell_{k_{r-1}}} \leq \limsup_r \frac{\ell_{k_r}}{\ell_{k_{r-1}}} < \infty$.

Proof. (i)If $\liminf_r \frac{\ell_{k_r}}{\ell_{k_{r-1}}} > 1$ then there exits $\alpha > 0$ such that $1 + \alpha \leq \frac{\ell_{k_r}}{\ell_{k_{r-1}}}$ for all $r \geq 1$. For $x \in [H^0]$, we write

$$\begin{aligned} T_r = \frac{1}{L_r}\sum_{i\in I_r}\frac{|x_i|}{i} &= \frac{1}{L_r}\sum_{i=1}^{k_r}\frac{|x_i|}{i} - \frac{1}{L_r}\sum_{i=1}^{k_{r-}}\frac{|x_i|}{i} \\ &= \frac{\ell_{k_r}}{L_r}\left(\frac{1}{\ell_{k_r}}\sum_{i=1}^{k_r}\frac{|x_i|}{i}\right) - \frac{\ell_{k_{r-1}}}{L_r}\left(\frac{1}{\ell_{k_{r-1}}}\sum_{i=1}^{k_{r-1}}\frac{|x_i|}{i}\right). \end{aligned}$$

Since $L_r = \ell_{k_r} - \ell_{k_{r-1}}$, we have

$$\frac{\ell_{k_r}}{L_r} \leq \frac{1+\alpha}{\alpha} \text{ and } \frac{\ell_{k_{r-1}}}{L_r} \leq \frac{1}{\alpha}.$$

The terms in the brackets both converge to 0, and hence we get $T_r \to 0$, that x is lacunary strongly harmonically summable to 0. The general inclusion $[H] \subseteq [H_\theta]$ follows by the linearity.

Now assume that $\liminf_r \frac{\ell_{k_r}}{\ell_{k_{r-1}}} = 1$. Since θ is a lacunary sequence, we can select a subsequence (k_{r_j}) of θ satisfying

$$\frac{\ell_{k_{r_j}}}{\ell_{k_{r_j-1}}} < 1 + \frac{1}{j} \text{ and } \frac{\ell_{k_{r_j-1}}}{\ell_{k_{r_{j-1}}}} < 1 + \frac{1}{j}$$

where $r_j \geq r_{j-1}+2$.

Define $x=(x_i)$ by

$$x_i = \begin{cases} 1, & \text{if } i \in I_{r_j}, \text{ for some } j = 1,2,\dots, \\ 0, & \text{otherwise.} \end{cases}$$

Then, for any real a,

$$\frac{1}{L_{r_j}} \sum_{i \in I_{r_j}} \frac{|x_i - a|}{i} = |1-a| \text{ for } j = 1,2,3,...,$$

and

$$\frac{1}{L_r} \sum_{i \in I_r} \frac{|x_i - a|}{i} = |a| \text{ for } r \neq r_j.$$

It follows that $x \notin [H_\theta]$. However, $x \in [H]$ since t ia any sufficiently large integer we can find the unique j for which $\ell_{k_{r_j}-1} < \ell_t \leq \ell_{k_{r_{j+1}}-1}$ and write

$$\frac{1}{\ell_t} \sum_{i=1}^{t} \frac{|x_i|}{i} \leq \frac{\ell_{k_{r_{j-1}}} + L_{r_j}}{\ell_{k_{r_j}-1}} \leq \frac{2}{j}.$$

As $t \to \infty$ it follows that also $j \to \infty$. Hence $x \in [H^0]$.

(ii) If $\limsup_r \frac{\ell_{k_r}}{\ell_{k_{r-1}}} < \infty$ there exists $P > 0$ such that $\frac{\ell_{k_r}}{\ell_{k_{r-1}}} < P$ for alll $r \geq 1$. Let $x \in [H_\theta^0]$ and $\epsilon > 0$, we can find $Q > 0$ and $S > 0$ such that

$$\sup_{i \geq Q} T_i < \epsilon \text{ and } T_i < P \text{ for all } i = 1,2,....$$

Then if t is any integer with $\ell_{k_{r-1}} < \ell_t \leq \ell_{k_r}$ where $r > Q$, we can write

$$\begin{aligned}
\frac{1}{\ell_t} \sum_{i=1}^{t} \frac{|x_i|}{i} &\leq \frac{1}{\ell_{k_{r-1}}} \sum_{i=1}^{k_r} \frac{|x_i|}{i} \\
&= \frac{1}{\ell_{k_{r-1}}} \left(\sum_{I_1} \frac{|x_i|}{i} + \sum_{I_2} \frac{x_i}{i} + ... + \sum_{I_r} \frac{|x_i|}{i} \right) \\
&= \frac{\ell_{k_1}}{\ell_{k_{r-1}}} T_1 + \frac{\ell_{k_2} - \ell_{k_1}}{\ell_{k_{r-1}}} T_2 + ... + \frac{\ell_{k_Q} - \ell_{k_{Q-1}}}{\ell_{k_{r-1}}} T_Q \\
&\quad + ... + \frac{\ell_{k_{Q+1}} - \ell_{k_Q}}{\ell_{k_{r-1}}} T_{Q_{R+1}} + ... + \frac{\ell_{k_r} - \ell_{k_{r-1}}}{\ell_{k_{r-1}}} T_r \\
&\leq \sup_{i \geq 1} T_i \frac{\ell_{k_Q}}{\ell_{k_{r-1}}} + \sup_{i \geq Q} T_i \frac{\ell_{k_r} - \ell_{k_Q}}{\ell_{k_{r-1}}} \\
&< S \frac{\ell_{k_Q}}{\ell_{k_{r-1}}} + \epsilon P.
\end{aligned}$$

Since $\ell_{k_{r-1}} \to \infty$ as $t \to \infty$, it follows that $\frac{1}{\ell_t} \sum_{i=1}^{t} \frac{|x_i|}{i} \to 0$ and, therefore, we have $x \in [H^0]$.

Now suppose that $\limsup_r \frac{\ell_{k_r}}{\ell_{k_{r-1}}} = \infty$ and construct a sequence in $[H_\theta]$ such that not in [H]. Select a subsequence (k_{r_j}) of θ with $\frac{\ell_{k_r}}{\ell_{k_{r-1}}} > j$ and define $x = (x_i)$ by

$$x_i = \begin{cases} 1, & \text{if } k_{r_j - 1} < i \leq 2k_{r_j - 1}, \text{ for some } i = 1, 2, \ldots, \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$T_r = \frac{\ell_{k_{r_j-1}}}{\ell_{k_{r_j}} - \ell_{k_{r_j-1}}} < \frac{1}{j-1}$$

and, if $r \neq r_j$, $T_r = 0$. Thus $x \in [H_\theta^0]$. Any sequence in [H] consisting of 0's and 1's has an associated [H] limit which is 0 or 1. For the sequence $x = (x_i)$ above, and $i = 1, 2, ..., k_{r_j}$

$$\frac{1}{\ell_{k_{r_j}}} \sum_i \frac{|x_i|}{i} \geq \frac{1}{\ell_{k_{r_j}}} (\ell_{k_{r_j}} - 2\ell_{k_{r_j-1}}) = 1 - \frac{2\ell_{k_{r_j}}}{\ell_{k_{r_j}}} > 1 - \frac{2}{j}$$

which converges to 1, and for $i = 1, 2, ..., 2\ell_{k_{r_j-1}}$,

$$\frac{1}{2\ell_{k_{r_j-1}}} \sum_i \frac{|x_i|}{i} \geq \frac{\ell_{k_{r_j-1}}}{2\ell_{k_{r_j-1}}} = \frac{1}{2},$$

and it follows that $x \notin H$.

(iii) Combining (i) and (ii) we have (iii). □

3. Lacunary statistical (H,1) Summability

The concept of statistical summability (H,1), which is a generalization of statistical convergence, has recently been introduced by Móricz. By using harmonic means, Móricz [14] defined statistically (H,1) summable sequence as follows:

A sequence (x_i) of real numbers is said to be statistically (H,1) summable to the number a if for every $\epsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{n} \left| \left\{ j \leq n : \quad \left| \frac{1}{\ell_j} \sum_{i=1}^{j} \frac{x_i}{i} - a \right| \geq \epsilon \right\} \right| = 0.$$

The set of all statistically (H,1) summable real sequences will be denoted SH.

In this section, we will introduce lacunary statistically (H,1) summable sequence and establish some inclusion relations.

Definition 3.1. Let $\theta = (k_r)$ be lacunary sequence. A sequence (x_i) of real numbers is said to be lacunary statistically (H,1) summable to the number a if for every $\epsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{n} \left| \left\{ j \leq n : \quad \left| \frac{1}{L_j} \sum_{i \in I_j} \frac{x_i}{i} - a \right| \geq \epsilon \right\} \right| = 0.$$

The set of all lacunary statistically (H,1) summable real sequences will be denoted SH_θ.

The proof of the following theorem similar to the proof of Theorem 3, we will not give the proof to avoid repeating it.

Theorem 3.1. *Let $\theta = (k_r)$ be lacunary sequence. Then*

(i) $SH \subseteq SH_\theta$ *if and only if* $\liminf_r \frac{\ell_{k_r}}{\ell_{k_{r-1}}} > 1$;

(ii) $SH_\theta \subseteq SH$ *if and only if* $\limsup_r \frac{\ell_{k_r}}{\ell_{k_{r-1}}} < \infty$.

(iii) $SH_\theta = SH$ *if and only if* $1 < \liminf_r \frac{\ell_{k_r}}{\ell_{k_{r-1}}} \leq \limsup_r \frac{\ell_{k_r}}{\ell_{k_{r-1}}} < \infty$.

4. Lacunary statistical logarithmic convergence

In [1] the concept of statistical logarithmic convergence was defined as follows:

Definition 4.1 ([1]). A sequence (x_i) of real numbers is said to be statistically logarithmic convergent to the number a if for every $\epsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{\ell_n} \left| \left\{ i \leq n : \quad \frac{1}{i} |x_i - a| \geq \epsilon \right\} \right| = 0.$$

Definition 4.2. Let $\theta = (k_r)$ be lacunary sequence. A sequence (x_i) of real numbers is said to be lacunary statistically logarithmic convergent to the number a if for every $\epsilon > 0$,

$$\lim_{r\to\infty} \frac{1}{L_r} \left| \left\{ i \in I_r : \quad \frac{1}{i} |x_i - a| \geq \epsilon \right\} \right| = 0.$$

In the next theorem we establish the inclusion relation between lacunary logarithmic statistical convergence and $[H_\theta]$-summability.

Theorem 4.1. *(i) If a sequence $x = (x_i)$ is $[H_\theta]$-summable to the number a, then it is lacunary logarithmic statistically convergent to a;*

(ii) If a sequence $x = (x_{i)})$ is bounded and lacunary logarithmic statistically convergent to a, then $x_i \to a[H_\theta]$.

Proof. (i) Let $x_i \to a[H_\theta]$, then

$$\frac{1}{L_r} \sum_{i\in I_r} \frac{|x_i - a|}{i} \geq \frac{1}{L_r} \sum_{\substack{i\in I_r \\ \frac{|x_i-a|}{i} \geq \epsilon}} \frac{|x_i - a|}{i} \geq \frac{\epsilon}{L_r} \left| \left\{ i \in I_r : \quad \frac{1}{i} |x_i - a| \geq \epsilon \right\} \right|$$

and as $r \to \infty$ we have

$$\lim_{r\to\infty} \frac{1}{L_r} \left| \left\{ i \in I_r : \quad \frac{1}{i} |x_i - a| \geq \epsilon \right\} \right| = 0,$$

that is $x = (x_i)$ is lacunary logarithmic statistically convergent to a.

(ii) Assume that $x = (x_i)$ is bounded and lacunary logarithmic statistically convergent to a. Then,

$$\lim_{r\to\infty} \frac{1}{L_r} \left| \left\{ i \in I_r : \quad \frac{1}{i} |x_i - a| \geq \epsilon \right\} \right| = 0.$$

Since $x \in \ell_\infty$, there exists $M > 0$ such that $|x_i - a| \leq M$ (i=1,2, . . .). We have

$$\begin{aligned}\frac{1}{L_r}\sum_{i\in I_r}\frac{|x_i-a|}{i} &= \frac{1}{L_r}\sum_{\substack{i\in I_r\\ \frac{|x_i-a|}{i}\geq\epsilon}}\frac{1}{i}|x_i-a| + \frac{1}{L_r}\sum_{\substack{i\in I_r\\ \frac{|x_i-a|}{i}<\epsilon}}\frac{1}{i}|x_i-a| \\ &\leq \sup|x_i-a|\frac{\left|\left\{i\in I_r: \quad \frac{1}{i}|x_i-a|\geq\epsilon\right\}\right|}{L_r}+\epsilon \\ &\leq M\frac{\left|\left\{i\in I_r: \quad \frac{1}{i}|x_i-a|\geq\epsilon\right\}\right|}{L_r}+\epsilon\to 0\end{aligned}$$

as $r \to \infty$. Hence $x_i \to a[H_\theta]$. □

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