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REALIZATION OF NEURAL NETWORK FOR GAIT CHARACTERIZATION OF QUADRUPED LOCOMOTION*

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Abstract Animal central pattern generator (CPG) is a device used to simulate the nervous system of animals. It is widely used in the design and control of four-legged robots. The objective of this paper is to establish a CPG network which is formed by a set of mutually symmetric neural networks combined with time delay to generate rhythmic motion patterns. Firstly, a symmetric delayed neural network consists of two loops and composed of eight neurons that can produce multi-phase locked oscillation patterns corresponding to the quadruped gaits. Then the primary gaits of all six types can be produced,and gait transitions between the different gaits are generated by altering the delay as the parameter, i.e., different ranges of delay correspond to different patterns of neural neurons. At last£¬ the simulation results show that the delayed neural network can generate multiple periodic oscillations corresponding to the gait of quadruped locomotion.

Keywords Hopfield neural network, symmetry, delay, bifurcating periodic solutions, gait.

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1. Introduction

CPG exists in the spinal cord or thoracic and abdominal ganglion of animals. It is a distributed oscillatory network composed of intermediate neurons, which can produce and possess stable phase interlock relationship of rhythm signals. Synaptic connections between neurons in CPG are plastic and exhibit multiple out put lines, see [3,10,12,13,15,17]. This can be used to control the animals to achieve a variety of movement patterns. The application of CPG to the gait regulation of quadruped mobile robot is a research hotspot. In the view of mathematics, the nonlinear neural network can be used to model CPG, and the gait characteristics can be changed by adjusting the parameters of CPG model. Golubitsky's work, published in Nature in 1999, can be considered groundbreaking in this field [6].

When constructing mathematical models of the gait of quadrupeds, a network of four symmetrical cells is generally considered sufficiently. However, Golubitsky

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indicated that some gait characteristics obtained in the four symmetric coupling units, such as trot and pace, do not exist simultaneously. Therefore, the eight-element symmetric coupling unit is sufficient, see [2].

Consider the output signal $x(t) = (x_1(t), x_2(t), \cdots, x_8(t))$ sent from the CPG to the legs, where $x_1(t), x_7(t)$ send to the left hind leg, $x_3(t), x_5(t)$ send to the left fore leg, $x_2(t), x_8(t)$ send to the right hind leg, and $x_4(t), x_6(t)$ send to the right fore leg. These connections are represented in Figure 1.

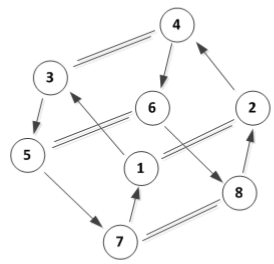

Figure 1. Schematic diagram of the system (1.1).

In Figure 1, the arrows and double lines determine the symmetry of the network as well as possible coupling between the neurons. The symmetry ρ cyclicly permutes the neurons connected by unidirectional arrows. The κ symmetry interchanges neurons connected by double lines.

In [2], the authors show that these CPG networks model primary gaits pronk, pace, bound, trot, walk, and jump. Primary gaits of the CPG network are periodic solutions with symmetry. Let $x_i \in \mathrm{R}^s$ be the state variable of neuron i. Then the dynamics of the quadruped CPG network can be given by a coupled differential system with eight state variables $x(t) = (x_1(t), x_2(t), \cdots, x_8(t))$. The authors point out that the dimension s of state variable x_i of neuron i must satisfy $s \geq 2$.

In this paper, we concentrate on the delayed Hopfield network. The dimension s of state variable x_i of neuron i can be $s = 1$. Suppose that there exist time delays in signal transmission between the two hind legs and between the two front legs. This situation leads to the following delayed differential equation system:

$$
\begin{cases}
\dot{x}_1(t) = ax_1(t) + b\tanh(x_1(t)) + d\tanh(x_7(t)) + c\tanh(x_2(t-\tau)), \\
\dot{x}_2(t) = ax_2(t) + b\tanh(x_2(t)) + d\tanh(x_8(t)) + c\tanh(x_1(t-\tau)), \\
\dot{x}_3(t) = ax_3(t) + b\tanh(x_3(t)) + d\tanh(x_1(t)) + c\tanh(x_4(t-\tau)), \\
\dot{x}_4(t) = ax_4(t) + b\tanh(x_4(t)) + d\tanh(x_2(t)) + c\tanh(x_3(t-\tau)), \\
\dot{x}_5(t) = ax_5(t) + b\tanh(x_5(t)) + d\tanh(x_3(t)) + c\tanh(x_6(t-\tau)), \\
\dot{x}_6(t) = ax_6(t) + b\tanh(x_6(t)) + d\tanh(x_4(t)) + c\tanh(x_5(t-\tau)), \\
\dot{x}_7(t) = ax_7(t) + b\tanh(x_7(t)) + d\tanh(x_5(t)) + c\tanh(x_8(t-\tau)), \\
\dot{x}_8(t) = ax_8(t) + b\tanh(x_8(t)) + d\tanh(x_6(t)) + c\tanh(x_7(t-\tau)),
\end{cases}
\tag{1.1}
$$

where $\tau \geq 0$ is the time delay.

Choosing two permutations $\sigma^1 = \begin{pmatrix} 12345678 \\ 78123456 \end{pmatrix}$, $\sigma^2 = \begin{pmatrix} 12345678 \\ 21436587 \end{pmatrix}$. Then (1.1) can be rewritten as

$$\dot{x}_i(t) = ax_i(t) + b\tanh(x_i(t)) + d\tanh(x_{\sigma^1(i)}(t)) + c\tanh(x_{\sigma^2(i)}(t-\tau)), i = 1, 2, \cdots 8. \tag{1.2}$$

The Hopfield neural network of this ring topology consists of two coupled one-way rings, each ring having four neurons. The rings have symmetric group $\Gamma = Z_4 \times Z_2$, that means the global symmetry Z_2 and internal symmetry Z_4. See Figure 1. In this paper, we will show that this neural network can also be a CPG model of a quadrupedal locomotor.

In this model, time delay is incorporated into the coupled system. The time delay enables the system to produce periodic solutions at lower dimensions. From the mathematical point of view, time delay makes the problem more difficult to deal with. In fact, the state vector describing the nonlinear system with time delay evolves in the infinite functional space [1,4,5,8,9,11,14,18].

Coupled networks have discrete spatial structure and continuous temporal structure and can describe the actual model with geometry and symmetry. The spatiotemporal pattern of this symmetric structure has received great attention in recent years. The bifurcation theory of symmetric differential equations is the theoretical basis for studying this kind of problems. See the works of M. Golubitsky and I. Stewart [7].

In this paper, the dynamic properties of a time-delay neural network model (1.2) with global and internal symmetry are studied by using equivariant bifurcation theory of functional differential equations, and the gaits of quadruped animals are depicted by using the periodic solutions obtained. In the next section, we will concentrate on linear stability analysis of trivial equilibrium. We use the equivariant bifurcation theory to describe the distribution of eigenvalues. Use the symmetric theorem, we find that the system(1.2) is equivariant under the group $Z_4 \times Z_2$. In section 3, we obtain some important results about the multiple bifurcation of periodic solutions, and we give the consistency of the several kinds spatio-temporal patterns with the gaits of quadrupedal locomotor. Numerical simulations are given in section 4.

2. Symmetry and Eigenvalue Distribution

It is clear that $(x_1, x_2, \cdots, x_8) = (0, 0, \cdots, 0)$ is an equilibrium point of Eq.(1.2). The linearization of Eq.(1.2) at the origin leads to

$$\dot{x}_i(t) = (a+b)x_i(t) + dx_{\sigma^1(i)}(t) + cx_{\sigma^2(i)}(t-\tau), i = 1, 2, \cdots 8. \tag{2.1}$$

Rewriting (1.2) and (2.1) as (2.2) and (2.3), respectively

$$\dot{U}(t) = fU(t) \tag{2.2}$$

with

$$f\varphi_i = a\varphi_i(0) + b\tanh\varphi_i(0) + d\tanh\varphi_{\sigma^1(i)}(0) + c\tanh\varphi_{\sigma^2(i)}(-\tau), i = 1, 2, \cdots 8.$$

$$\dot{U}(t) = LU(t) \tag{2.3}$$

with

$$L\varphi_i = (a+b)\varphi_i(0) + d\varphi_{\sigma^1(i)}(0) + c\varphi_{\sigma^2(i)}(-\tau), i = 1, 2, \cdots 8.$$

It is clear that both the system (2.2) and (2.3) are all $Z_4 \times Z_2$ equivariant. In fact, for state variables $X = (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8)$, the symmetric group is $\Gamma = Z_4 \times Z_2$, where Z_4 and Z_2 be the cycle group, the action on X follows: $\rho \in Z_4$, $\kappa \in Z_2$:

$$\rho(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) = (x_7, x_8, x_1, x_2, x_3, x_4, x_5, x_6),$$
$$\kappa(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) = (x_2, x_1, x_4, x_3, x_6, x_5, x_8, x_7).$$

The infinitesimal generator of the C_0-semigroup generated by the linear system (2.3) is $\mathcal{A}(\tau)$ with

$$\mathcal{A}(\tau)\phi = \dot{\phi}, \phi \in Dom(\mathcal{A}(\tau)), Dom(\mathcal{A}(\tau)) = \{\phi \in C, \dot{\phi} \in C, \dot{\phi}(0) = L(\tau)\phi\}.$$

Regarding τ as the parameter, we determine when the infinitesimal generator $A(\tau)$ of the C^0-semigroup generated by the linear system (2.3) has a pair of pure imaginary eigenvalues. Using the equivariant bifurcation theory of functional differential equations, the associated characteristic equation of Eq.(2.3) takes the form

$$\det(\Delta(\lambda, \tau)) = 0,$$

where

$$\Delta(\lambda, \tau) = \lambda E_{8\times 8} - Circ(A, O_{2\times 2}, O_{2\times 2}, D),$$

and

$$A = \begin{pmatrix} a+b & ce^{-\lambda\tau} \\ ce^{-\lambda\tau} & a+b \end{pmatrix}, D = \begin{pmatrix} d & 0 \\ 0 & d \end{pmatrix}.$$

Using the lemma 2.1 in Ref. [19], the product factors of the characteristic equation are

$$\begin{aligned} & [\lambda - a - b - d - ce^{-\lambda\tau}] \times [\lambda - a - b - d + ce^{-\lambda\tau}] \times [\lambda - a - b + \mathrm{i}d - ce^{-\lambda\tau}] \\ & \times [\lambda - a - b + \mathrm{i}d + ce^{-\lambda\tau}] \times [\lambda - a - b + d - ce^{-\lambda\tau}] \times [\lambda - a - b + d + ce^{-\lambda\tau}] \\ & \times [\lambda - a - b - \mathrm{i}d - ce^{-\lambda\tau}] \times [\lambda - a - b - \mathrm{i}d + ce^{-\lambda\tau}] \\ = & \Delta_1\Delta_2\Delta_3\Delta_4\Delta_5\Delta_6\Delta_7\Delta_8 = 0. \end{aligned} \tag{2.4}$$

To discuss the distribution of eigenvalues, we make the following assumption:

$(H_1) : |c| > |a + b + d|$ if $(a + b)d > 0$.

$(H_2) : |c| > |a + b - d|$ if $(a + b)d < 0$.

In the following, we consider the distribution of pure imaginary roots of $\Delta_i = 0 (i = 1, 2, 3, 4, 5, 6, 7, 8)$. The distribution of the pure imaginary roots of the characteristic equation is listed in Table 1.

In order to study the structure of multiple periodic solutions, we consider the generalized eigenspace corresponding to the pure virtual eigenvalues of $\mathcal{A}(\tau)$.

Assume (H_1)-(H_2) hold, Eq.(2.4) has roots $\pm i\omega_1$ when $\tau = \tau_1^k$. Using the Lemma 2.1 in [16], we know that the generalized eigenspace U_1 consisting of eigenvectors of $\mathcal{A}(\tau_1^k)$ corresponding to $\pm i\omega_1$ is

$$U_1 = \{x_1\phi_1 + x_2\varepsilon_1, x_r \in \mathrm{R},\ r = 1, 2\},$$

Table 1. Analysis of characteristic equation (2.4)

Δ_i	analysis of characteristic equation	transversality conditions
Δ_1	$\begin{cases} \omega_1 = \sqrt{c^2-(a+b+d)^2}; \\ \cos(\omega_1\tau_1^k) = \frac{a+b+d}{-c}; \\ \sin(\omega_1\tau_1^k) = \frac{\omega_1}{-c}; k=0,1,2,\cdots. \end{cases}$	$\mathrm{Re}(\frac{d\lambda}{d\tau}\|_{\tau=\tau_1^k})^{-1} = \frac{1}{(\omega_1)^2+(a+b+d)^2} > 0$
Δ_2	$\begin{cases} \omega_1 = \sqrt{c^2-(a+b+d)^2}; \\ \cos(\omega_1\tau_2^k) = \frac{a+b+d}{c}; \\ \sin(\omega_1\tau_2^k) = \frac{\omega_1}{c}; k=0,1,2,\cdots. \end{cases}$	$\mathrm{Re}(\frac{d\lambda}{d\tau}\|_{\tau=\tau_2^k})^{-1} = \frac{1}{(\omega_1)^2+(a+b+d)^2} > 0$
Δ_3	$\begin{cases} \omega_2 = -d+\sqrt{c^2-(a+b)^2}; \\ \cos(\omega_2\tau_3^k) = \frac{a+b}{-c}; \\ \sin(\omega_2\tau_3^k) = \frac{\omega_2+d}{-c}; k=0,1,2,\cdots. \end{cases}$	$\mathrm{Re}(\frac{d\lambda}{d\tau}\|_{\tau=\tau_3^k})^{-1} = \frac{\omega_2+d}{\omega_2(\omega_2+d)^2+\omega_2(a+b)^2} > 0$
Δ_4	$\begin{cases} \omega_2 = -d+\sqrt{c^2-(a+b)^2}; \\ \cos(\omega_2\tau_4^k) = \frac{a+b}{c}; \\ \sin(\omega_2\tau_4^k) = \frac{\omega_2+d}{c}; k=0,1,2,\cdots. \end{cases}$	$\mathrm{Re}(\frac{d\lambda}{d\tau}\|_{\tau=\tau_4^k})^{-1} = \frac{\omega_2+d}{\omega_2(\omega_2+d)^2+\omega_2(a+b)^2} > 0$
Δ_5	$\begin{cases} \omega_3 = \sqrt{c^2-(a+b-d)^2}; \\ \cos(\omega_3\tau_5^k) = \frac{a+b-d}{-c}; \\ \sin(\omega_3\tau_5^k) = \frac{\omega_3}{-c}; k=0,1,2,\cdots. \end{cases}$	$\mathrm{Re}(\frac{d\lambda}{d\tau}\|_{\tau=\tau_5^k})^{-1} = \frac{1}{(\omega_3)^2+(a+b-d)^2} > 0$
Δ_6	$\begin{cases} \omega_3 = \sqrt{c^2-(a+b-d)^2}; \\ \cos(\omega_3\tau_6^k) = \frac{a+b-d}{c}; \\ \sin(\omega_3\tau_6^k) = \frac{\omega_3}{c}; k=0,1,2,\cdots. \end{cases}$	$\mathrm{Re}(\frac{d\lambda}{d\tau}\|_{\tau=\tau_6^k})^{-1} = \frac{1}{(\omega_3)^2+(a+b-d)^2} > 0$
Δ_7	$\begin{cases} \omega_4 = d+\sqrt{c^2-(a+b)^2}; \\ \cos(\omega_4\tau_7^k) = \frac{a+b}{-c}; \\ \sin(\omega_4\tau_7^k) = \frac{\omega_4-d}{-c}; k=0,1,2,\cdots. \end{cases}$	$\mathrm{Re}(\frac{d\lambda}{d\tau}\|_{\tau=\tau_7^k})^{-1} = \frac{\omega_4-d}{\omega_4(\omega_4-d)^2+\omega_4(a+b)^2} > 0$
Δ_8	$\begin{cases} \omega_4 = d+\sqrt{c^2-(a+b)^2}; \\ \cos(\omega_4\tau_8^k) = \frac{a+b}{c}; \\ \sin(\omega_4\tau_8^k) = \frac{\omega_4-d}{c}; k=0,1,2,\cdots. \end{cases}$	$\mathrm{Re}(\frac{d\lambda}{d\tau}\|_{\tau=\tau_8^k})^{-1} = \frac{\omega_4-d}{\omega_4(\omega_4-d)^2+\omega_4(a+b)^2} > 0$

where

$$\phi_1(\theta) = \cos(\omega_1\theta)\mathrm{Re}\{V_1\} - \sin(\omega_1\theta)\mathrm{Im}\{V_1\},$$

$$\varepsilon_1(\theta) = \sin(\omega_1\theta)\mathrm{Re}\{V_1\} + \cos(\omega_1\theta)\mathrm{Im}\{V_1\},$$

and

$$V_1 = (1,1,1,1,1,1,1,1)^T, \quad \theta \in [-1,0].$$

The analysis of other generalized feature spaces is similar to the above analysis, hence they are omitted here, see the following Table 2:

3. Multiple Bifurcating Periodic Solutions and Gaits

Since (1.2) has internal and global symmetries, in the following, we will determine the effects of the time delay and symmetric coupling on the system. Let $C([-\tau,0],\mathrm{R}^8)$ denote the Banach space of continuous mapping from $[-\tau,0]$ into R^8 equipped with the supermum norm $\parallel \varphi \parallel = \sup\limits_{-\tau\le\theta\le 0} \mid \varphi(\theta) \mid$ for $\varphi \in C([-\tau,0],\mathrm{R}^8)$.

Let $\sigma \in \mathrm{R}, A \ge 0$, $X : [\sigma-\tau, \sigma+A] \to \mathrm{R}^8, t \in [\sigma, \sigma+A]$ be defined by $X_t(\theta) = X(t+\theta)$ for $-\tau \le \theta \le 0$.

We study the multiple Hopf bifurcation of the origin under the action of a group Γ, where $\Gamma = Z_4 \times Z_2$. It is clear that

$$\Gamma = Z_4 \times Z_2 = \{1, \rho, \rho^2, \rho^3, \kappa, \kappa\rho, \kappa\rho^2, \kappa\rho^3\}.$$

Table 2. Generalized eigenspace

U_j	eigenvectors of $\mathcal{A}(\tau)$
$U_1=\{x_1\phi_1+x_2\varepsilon_1, x_r\in \mathrm{R},\ r=1,2\}$	$\begin{cases}\phi_1(\theta)=\cos(\omega_1\theta)\mathrm{Re}\{V_1\}-\sin(\omega_1\theta)\mathrm{Im}\{V_1\};\\ \varepsilon_1(\theta)=\sin(\omega_1\theta)\mathrm{Re}\{V_1\}+\cos(\omega_1\theta)\mathrm{Im}\{V_1\};\ \theta\in[-1,0].\\ V_1=(1,1,1,1,1,1,1,1)^T;\end{cases}$
$U_2=\{x_1\phi_2+x_2\varepsilon_2, x_r\in \mathrm{R},\ r=1,2\}$	$\begin{cases}\phi_2(\theta)=\cos(\omega_1\theta)\mathrm{Re}\{V_2\}-\sin(\omega_1\theta)\mathrm{Im}\{V_2\};\\ \varepsilon_2(\theta)=\sin(\omega_1\theta)\mathrm{Re}\{V_2\}+\cos(\omega_1\theta)\mathrm{Im}\{V_2\};\ \theta\in[-1,0].\\ V_2=(1,-1,1,-1,1,-1,1,-1)^T;\end{cases}$
$U_3=\{x_1\phi_3+x_2\varepsilon_3, x_r\in \mathrm{R},\ r=1,2\}$	$\begin{cases}\phi_3(\theta)=\cos(\omega_2\theta)\mathrm{Re}\{V_3\}-\sin(\omega_2\theta)\mathrm{Im}\{V_3\};\\ \varepsilon_3(\theta)=\sin(\omega_2\theta)\mathrm{Re}\{V_3\}+\cos(\omega_2\theta)\mathrm{Im}\{V_3\};\ \theta\in[-1,0].\\ V_3=(1,1,\mathrm{i},\mathrm{i},-1,-1,-\mathrm{i},-\mathrm{i})^T;\end{cases}$
$U_4=\{x_1\phi_4+x_2\varepsilon_4, x_r\in \mathrm{R},\ r=1,2\}$	$\begin{cases}\phi_4(\theta)=\cos(\omega_2\theta)\mathrm{Re}\{V_4\}-\sin(\omega_2\theta)\mathrm{Im}\{V_4\};\\ \varepsilon_4(\theta)=\sin(\omega_2\theta)\mathrm{Re}\{V_4\}+\cos(\omega_2\theta)\mathrm{Im}\{V_4\};\ \theta\in[-1,0].\\ V_4=(1,-1,\mathrm{i},-\mathrm{i},-1,1,-\mathrm{i},\mathrm{i})^T;\end{cases}$
$U_5=\{x_1\phi_5+x_2\varepsilon_5, x_r\in \mathrm{R},\ r=1,2\}$	$\begin{cases}\phi_5(\theta)=\cos(\omega_3\theta)\mathrm{Re}\{V_5\}-\sin(\omega_3\theta)\mathrm{Im}\{V_5\};\\ \varepsilon_5(\theta)=\sin(\omega_3\theta)\mathrm{Re}\{V_5\}+\cos(\omega_3\theta)\mathrm{Im}\{V_5\};\ \theta\in[-1,0].\\ V_5=(1,1,-1,-1,1,1,-1,-1)^T;\end{cases}$
$U_6=\{x_1\phi_6+x_2\varepsilon_6, x_r\in \mathrm{R},\ r=1,2\}$	$\begin{cases}\phi_6(\theta)=\cos(\omega_3\theta)\mathrm{Re}\{V_6\}-\sin(\omega_3\theta)\mathrm{Im}\{V_6\};\\ \varepsilon_6(\theta)=\sin(\omega_3\theta)\mathrm{Re}\{V_6\}+\cos(\omega_3\theta)\mathrm{Im}\{V_6\};\ \theta\in[-1,0].\\ V_6=(1,-1,-1,1,1,-1,-1,1)^T;\end{cases}$
$U_7=\{x_1\phi_7+x_2\varepsilon_7, x_r\in \mathrm{R},\ r=1,2\}$	$\begin{cases}\phi_7(\theta)=\cos(\omega_4\theta)\mathrm{Re}\{V_7\}-\sin(\omega_4\theta)\mathrm{Im}\{V_7\};\\ \varepsilon_7(\theta)=\sin(\omega_4\theta)\mathrm{Re}\{V_7\}+\cos(\omega_4\theta)\mathrm{Im}\{V_7\};\ \theta\in[-1,0].\\ V_7=(1,1,-\mathrm{i},-\mathrm{i},-1,-1,\mathrm{i},\mathrm{i})^T;\end{cases}$
$U_8=\{x_1\phi_8+x_2\varepsilon_8, x_r\in \mathrm{R},\ r=1,2\}$	$\begin{cases}\phi_8(\theta)=\cos(\omega_4\theta)\mathrm{Re}\{V_8\}-\sin(\omega_4\theta)\mathrm{Im}\{V_8\};\\ \varepsilon_8(\theta)=\sin(\omega_4\theta)\mathrm{Re}\{V_8\}+\cos(\omega_4\theta)\mathrm{Im}\{V_8\};\ \theta\in[-1,0].\\ V_8=(1,-1,-\mathrm{i},\mathrm{i},-1,1,\mathrm{i},-\mathrm{i})^T;\end{cases}$

The possible bifurcating solutions can be obtained by giving the isotropy subgroup and fixed-point subspaces. From Section 2, we have obtained the generalized eigenspace corresponding to pure imaginary eigenvalues of $\mathcal{A}(\tau_j^k)(j=1,2,\cdots,8;\ k=0,1,\cdots.)$. Hence, we know their corresponding isotropy subgroup, see Table 3.

Table 3. The twisted isotropy subgroups for Γ-equivariant system (1.2)

τ	group action	twisted isotropy subgroups	bases of Fix $\sum$
τ_1^k	$\mathbf{1}z=z;\mathbf{1}z=z$	$\sum<\mathbf{1},\mathbf{1}>$	$\phi_1(\theta),\varepsilon_1(\theta)$
τ_2^k	$\mathbf{1}z=z;\kappa z=-z$	$\sum<\mathbf{1},\kappa>$	$\phi_2(\theta),\varepsilon_2(\theta)$
τ_3^k	$\rho z=e^{\frac{\pi}{2}\mathrm{i}}z;\mathbf{1}z=z$	$\sum<\rho,\mathbf{1}>$	$\phi_3(\theta),\varepsilon_3(\theta)$
τ_4^k	$\rho z=e^{\frac{\pi}{2}\mathrm{i}}z;\kappa z=-z$	$\sum<\rho,\kappa>$	$\phi_4(\theta),\varepsilon_4(\theta)$
τ_5^k	$\rho^2z=-z;\mathbf{1}z=z$	$\sum<\rho^2,\mathbf{1}>$	$\phi_5(\theta),\varepsilon_5(\theta)$
τ_6^k	$\rho^2z=-z;\kappa z=-z$	$\sum<\rho^2,\kappa>$	$\phi_6(\theta),\varepsilon_6(\theta)$
τ_7^k	$\rho^3z=e^{\frac{3\pi}{2}\mathrm{i}}z;\mathbf{1}z=z$	$\sum<\rho^3,\mathbf{1}>$	$\phi_7(\theta),\varepsilon_7(\theta)$
τ_8^k	$\rho^3z=e^{\frac{3\pi}{2}\mathrm{i}}z;\kappa z=-z$	$\sum<\rho^3,\kappa>$	$\phi_8(\theta),\varepsilon_8(\theta)$

By using the equivariant bifurcation theory, we can obtain the existence of small amplitude periodic solutions of the system (1.2), and further, we know that the spatiotemporal symmetries of the solutions can be completely characterized by the property of the corresponding isotropic subgroup.

In case one, $\Delta_1 = 0$ implies that the action of $Z_4 \times Z_2$ is given by $\rho z = z; \kappa z = z$. Obviously, all neurons in two rings are synchronous:

$$(x(t), x(t), x(t), x(t), x(t), x(t), x(t), x(t)).$$

Similar to the analysis in $\Delta_1 = 0$, $\Delta_2 = 0$ implies that the action of $Z_4 \times Z_2$ is given by $\rho z = z; \kappa z = -z$. The all neurons in two rings are anti-synchronous:

$$(x(t), -x(t), x(t), -x(t), x(t), -x(t), x(t), -x(t)).$$

From the Table 3, in summary, we write the results in the following Table 4.

By using irreducible representations of the bases of U_j and the relationships between these representations and the basic gait, we can get the correspondence between the basic gait and the periodic solution. See Table 5.

Table 4. Bifurcating periodic solutions

twisted isotropy subgroups $\sum$	periodic solutions
$\sum < \mathbf{1}, \mathbf{1} >$	$(x(t), x(t), x(t), x(t), x(t), x(t), x(t), x(t))$
$\sum < \mathbf{1}, \kappa >$	$(x(t), x(t+\frac{T}{2}), x(t), x(t+\frac{T}{2}), x(t), x(t+\frac{T}{2}), x(t), x(t+\frac{T}{2}))$
$\sum < \rho, \mathbf{1} >$	$(x(t), x(t), x(t+\frac{T}{4}), x(t+\frac{T}{4}), x(t+\frac{T}{2}), x(t+\frac{T}{2}), x(t+\frac{3T}{4}), x(t+\frac{3T}{4}))$
$\sum < \rho, \kappa >$	$(x(t), x(t+\frac{T}{2}), x(t+\frac{T}{4}), x(t+\frac{3T}{4}), x(t+\frac{T}{2}), x(t), x(t+\frac{3T}{4}), x(t+\frac{T}{4}))$
$\sum < \rho^2, \mathbf{1} >$	$(x(t), x(t), x(t+\frac{T}{2}), x(t+\frac{T}{2}), x(t), x(t), x(t+\frac{T}{2}), x(t+\frac{T}{2}))$
$\sum < \rho^2, \kappa >$	$(x(t), x(t+\frac{T}{2}), x(t+\frac{T}{2}), x(t), x(t), x(t+\frac{T}{2}), x(t+\frac{T}{2}), x(t))$
$\sum < \rho^3, \mathbf{1} >$	$(x(t), x(t), x(t+\frac{3T}{4}), x(t+\frac{3T}{4}), x(t+\frac{T}{2}), x(t+\frac{T}{2}), x(t+\frac{T}{4}), x(t+\frac{T}{4}))$
$\sum < \rho^3, \kappa >$	$(x(t), x(t+\frac{T}{2}), x(t+\frac{3T}{4}), x(t+\frac{T}{4}), x(t+\frac{T}{2}), x(t), x(t+\frac{T}{4}), x(t+\frac{3T}{4}))$

Table 5. quadrupeds gait and periodic solutions of $x_1(t), x_2(t), x_3(t), x_4(t)$

quadruped locomotion gait	periodic solutions of Legs $x_1(t), x_2(t), x_3(t), x_4(t)$
Pronk	$x(t), x(t), x(t), x(t),$
Pace	$x(t), x(t+\frac{T}{2}), x(t), x(t+\frac{T}{2})$
Walk	$\begin{cases} x(t), x(t+\frac{T}{2}), x(t+\frac{T}{4}), x(t+\frac{3T}{4}) \\ x(t), x(t+\frac{T}{2}), x(t+\frac{3T}{4}), x(t+\frac{T}{4}) \end{cases}$
Bound	$x(t), x(t), x(t+\frac{T}{2}), x(t+\frac{T}{2})$
Trot	$x(t), x(t+\frac{T}{2}), x(t+\frac{T}{2}), x(t)$
Jump	$\begin{cases} x(t), x(t), x(t+\frac{3T}{4}), x(t+\frac{3T}{4}) \\ x(t), x(t), x(t+\frac{T}{4}), x(t+\frac{T}{4}) \end{cases}$

4. Computer simulation

In this section, we perform simulation for special cases based on the analysis results:

Let $a = -4.5,\ b = 1,\ c = 6.5,\ d = -2$. Using the results Tables 1-4, the spatiotemporal patterns of periodic bifurcation oscillations vary with the propagation time delay.

If the case 6 occurs, then we have $\omega_3 = 6.325$ and $\tau_6^0 = 0.2852$.

If the case 8 occurs, then we have $\omega_4 = 3.477$ and $\tau_8^0 = 0.6153$.

Let $\tau = 0.4 > \tau_6^0 = 0.2852$, then the case 6 in Table 4 appears:

$$(x(t), x(t+\frac{T}{2}), x(t+\frac{T}{2}), x(t)).$$

Consistent with this periodic solution, the quadruped has the gait trot. See Figure 2.

Let $\tau = 0.7 > \tau_8^0 = 0.6153$, then the case 8 in Table 4 appears:

$$(x(t), x(t+\frac{T}{2}), x(t+\frac{3T}{4}), x(t+\frac{T}{4})).$$

Consistent with this periodic solution, the quadruped has the gait walk. See Figure 3.

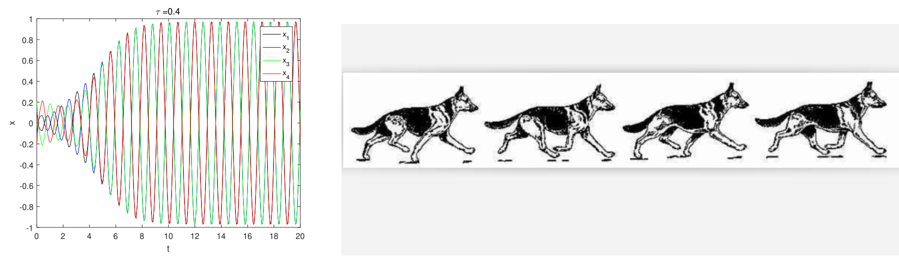

Figure 2. $x_1(t) = x_4(t)$ and $x_2(t) = x_3(t)$; and they are $\frac{T}{2}$ out of phase with $\tau = 0.4$ and initial condition (0.1,-0.1,0.2, -0.2, 0.1, -0.1,0.2, -0.2).

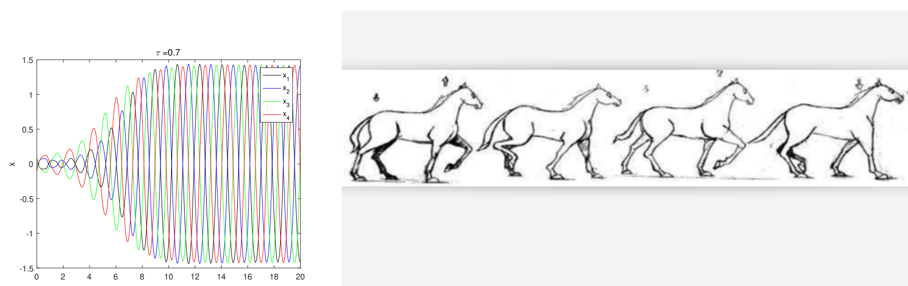

Figure 3. $x_1(t), x_4(t), x_2(t)$ and $x_3(t)$ are $\frac{T}{4}$ out of phase with $\tau = 0.7$ and initial condition (0.1,- 0.1, 0.1, -0.1, 0.2,- 0.2, 0.2, -0.2).

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