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ON A CLASS OF CHOQUARD-TYPE EQUATION WITH UPPER CRITICAL EXPONENT AND INDEFINITE LINEAR PART*

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Abstract The existence of ground states to the strongly indefinite Choquard type equation

$$\begin{cases} -\Delta u + (V(x) - W(x))u = a(x)(I_\alpha * |u|^{\frac{2^*_\alpha}{2}})|u|^{\frac{2^*_\alpha}{2}-2}u + b(x)f(u), \quad x \in \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \end{cases}$$

is proved, where $N \geq 3$, $\alpha \in (0, N)$, $2^*_\alpha = \frac{2(N+\alpha)}{N-2}$, I_α denotes the Riesz potential, and 0 belongs to the gap of the spectrum of $-\Delta + V$. We consider the asymptotically periodic case, i.e., V is periodic in x, $\lim_{|x|\to+\infty} W(x) = 0$, and a, b are asymptotically periodic in x. Some results in the literature are completed.

Keywords Critical exponent, Choquard equation, ground state, strongly indefinite functional.

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1. Introduction

We study the indefinite Choquard type equation with upper critical exponent

$$\begin{cases} -\Delta u + (V(x) - W(x))u = a(x)(I_\alpha * |u|^{\frac{2^*_\alpha}{2}})|u|^{\frac{2^*_\alpha}{2}-2}u + b(x)f(u), \quad x \in \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \end{cases} \tag{1.1}$$

where $N \geq 3$, $\alpha \in (0, N)$, $2^*_\alpha = \frac{2(N+\alpha)}{N-2}$, I_α denotes the Riesz potential on $\mathbb{R}^N \setminus \{0\}$ defined by

$$I_\alpha(x) = \frac{\Gamma(\frac{N-\alpha}{2})}{2^\alpha \pi^{\frac{N}{2}} \Gamma(\frac{\alpha}{2})|x|^{N-\alpha}},$$

V, W, a, $b \in C(\mathbb{R}^N, \mathbb{R})$, and $f : \mathbb{R} \mapsto \mathbb{R}$ has a subcritical growth.

Problem (1.1) is closely related to the classical Choquard-Pekar equation

$$-\Delta u + Vu = (I_\alpha * |u|^p)|u|^{p-2}u, u \in H^1(\mathbb{R}^N), \tag{1.2}$$

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which has application in a variety of physical situations, for example, it was introduced by Fröhlich [7] and Pekar [22] in 1954 to model the quantum mechanics of a polaron at rest, and by Choquard [16] in 1976 to characterize an electron trapped in its hole. The readers can refer to [4,23] for more details of physical background related to equation (1.2).

In the last decade, problem (1.2) has been studied extensively by using variational methods (see [6,16,18,30] and the survey [21] for example). In the case when $V = 1$, Moroz and Van Schaftingen [19] explored the existence, regularity, symmetry and decay of positive ground states to (1.2) for $p \in (\frac{N+\alpha}{N}, \frac{N+\alpha}{N-2})$. They also showed the nonexistence of nontrivial solutions to (1.2) for $p \leq \frac{N+\alpha}{N}$ or $p \geq \frac{N+\alpha}{N-2}$. The power $p = \frac{N+\alpha}{N}$ and $p = \frac{N+\alpha}{N-2}$ are called lower and upper critical exponents respectively owing to the Hardy-Littlewood-Sobolev inequality [17, Theorem 4.3]. Recently, increasing attention has been paid to the critical case of equation (1.2) with variable potential or subcritical perturbation. See [15, 20, 25–27, 29] for the lower critical case and [5,8,9,11,13–15] for the upper critical case.

For the study of strongly indefinite problems, as we know, a considerable amount of effort has been devoted to the Schrödinger equation

$$-\Delta u + V(x)u = f(x, u),\ u \in H^1(\mathbb{R}^N), \tag{1.3}$$

where 0 is in a spectral gap of $-\Delta + V$ (see [2,12,24,31] for example), while limited study has been done in the Choquard type equations. In [1], Ackermann obtained infinitely many solutions of the strongly indefinite equation

$$-\Delta u + V(x)u = (W * |u|^2)u,\ u \in H^1(\mathbb{R}^3).$$

Gao and Yang [8] explored the existence of nontrivial solutions to the upper critical problem with indefinite linear part

$$-\Delta u + V(x)u = (I_\alpha * G(x, u))g(x, u),\ u \in H^1(\mathbb{R}^N), \tag{1.4}$$

where $g(x, u) = k(x)|u|^{\frac{2^*_\alpha}{2}-2}u + f(x, u)$, $G(x, u) = \int_0^u g(x, t)dt$, and $f(x, u)$ has a subcritical growth. Guo and Tang [11] proved the existence of nontrivial solutions to the strongly indefinite problem with upper critical exponent and local perturbation

$$-\Delta u + V(x)u = (I_\alpha * |u|^{\frac{2^*_\alpha}{2}})|u|^{\frac{2^*_\alpha}{2}-2}u + g(u),\ u \in H^1(\mathbb{R}^N). \tag{1.5}$$

The $V(x)$, $k(x)$, $f(x, u)$ in equations (1.4) and (1.5) are periodic in x.

Motivated by the work mentioned previously, we are interested in the strongly indefinite problem (1.1) with asymptotically periodic coefficients. Our goal is to show the existence of ground states (nontrivial solutions with minimal energy) to this problem. For this purpose, we make assumptions as follows.

(V1) $V \in C(\mathbb{R}^N, \mathbb{R})$, V is 1-periodic in each x_i, $1 \leq i \leq N$, and

$$\underline{\lambda} := \sup(\sigma(-\Delta + V) \cap (-\infty, 0]) < 0 < \overline{\lambda} := \inf(\sigma(-\Delta + V) \cap [0, +\infty)).$$

(V2) $W \in C(\mathbb{R}^N, \mathbb{R})$, $\lim_{|x|\to+\infty} W(x) = 0$, and

$$0 \leq W(x) \leq \overline{W} = \sup_{x\in\mathbb{R}^N} W(x) < \overline{\lambda},\ \forall x \in \mathbb{R}^N.$$

(A1) $a(x) = a_0(x) + a_*(x)$, $b(x) = b_0(x) + b_*(x)$, where a_*, b_*, a_0, $b_0 \in C(\mathbb{R}^N, \mathbb{R})$, a_0 is positive, a_*, b_*, b_0 are nonnegative, a_0, b_0 are 1-periodic in each x_i, $1 \leq i \leq N$, and

$$\lim_{|x|\to+\infty} a_*(x) = \lim_{|x|\to+\infty} b_*(x) = 0.$$

(A2) There exists $x_0 \in \mathbb{R}^N$ such that $a(x_0) = \max_{x\in\mathbb{R}^N} a(x)$, $a(x) - a(x_0) = o(|x - x_0|^2)$ as $x \to x_0$, and $V(x_0) < 0$.

(F1) $f \in C(\mathbb{R}, \mathbb{R})$, and $f(u) = o(u)$ as $u \to 0$.

(F2) $f(u) \leq C(1 + |u|^{p-1})$ for some $2 < p < 2^*$ and $C > 0$. Additionally, for the case $N = 3$, there is $\lambda > 0$ and $2 < q < 2^*$ such that $|f(s)| \geq \lambda |s|^{q-1}$ if $s \neq 0$.

(F3) $u \mapsto \dfrac{f(u)}{|u|}$ is nondecreasing on $(-\infty, 0)$ and $(0, +\infty)$.

The main results we obtained can be stated as follow.

Theorem 1.1. *Assume that $(V1)$, $(V2)$, $(A1)$, $(A2)$, $(F1) - (F3)$ and $\alpha \in ((N-4)^+, N)$ are satisfied. Then equation (1.1) possesses a ground state for $N \geq 4$. For $N = 3$, there exists $\tilde{b} > 0$ such that equation (1.1) possesses a ground state if $\inf_{x\in\mathbb{R}^N} b(x) \geq \tilde{b}$.*

Remark 1.1. Note that the existence of ground states to the equation without subcritical perturbation

$$\begin{cases} -\Delta u + (V(x) - W(x))u = a(x)(I_\alpha * |u|^{\frac{2^*_\alpha}{2}})|u|^{\frac{2^*_\alpha}{2}-2}u, & x \in \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \end{cases} \tag{1.6}$$

for $N \geq 4$ and to the equation with periodic coefficients

$$\begin{cases} -\Delta u + V(x)u = a_0(x)(I_\alpha * |u|^{\frac{2^*_\alpha}{2}})|u|^{\frac{2^*_\alpha}{2}-2}u + b_0(x)f(u), & x \in \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \end{cases} \tag{1.7}$$

is included in Theorem 1.1.

Theorem 1.1 completes the results in [8,11] in view of the following three reasons.

(1) Neither the case $N = 3$ nor the asymptotically periodic case was analysed in [8,11].

(2) We obtain a ground state while [8,11] found a nontrivial solution.

(3) The existence of ground states to (1.6) was excluded in the studies of [8,11].

To prove Theorem 1.1, we mainly employ the generalized Nehari manifold method originated from [24] (see also [2,31]). However, the arguments in [2,24,31] can not be used directly since our equation has a convolution term and a general subcritical perturbation. We follow some arguments in [11]. Moreover, we have to cope with the difficulty caused by the non-periodic coefficients. By a concentration-compactness type Lemma [9, Lemma 2.5], and the periodicity of the limit equation (1.7), we are able to overcome it.

2. Notations

In this paper, $|\cdot|_p$ and $\|\cdot\|_H$ denote the usual norms in $L^p(\mathbb{R}^N)$ and $H^1(\mathbb{R}^N)$ respectively, $B_r(y)$ represents the open ball on $\mathbb{R}^N$ of center y and radius r, $\int \cdot$ is the simplified form of the integral $\int_{\mathbb{R}^N} \cdot dx$, and C_1, C_2, C_3 are positive constants.

A weak solution to problem (1.1) is a critical point to the energy functional defined on $H^1(\mathbb{R}^N)$ by

$$E_W(u) := \frac{1}{2}\int (|\nabla u|^2 + (V(x) - W(x))u^2) - \frac{1}{2_\alpha^*}\int a(x)(I_\alpha * |u|^{\frac{2_\alpha^*}{2}})|u|^{\frac{2_\alpha^*}{2}} - \int b(x)F(u), \tag{2.1}$$

where $F(s) = \int_0^s f(t)dt$. As in [2, 24], there exists an equivalent norm $\|\cdot\|$ in $X := H^1(\mathbb{R}^N)$ such that

$$E_W(u) = \frac{1}{2}(\|u^+\|^2 - \|u^-\|^2) - \frac{1}{2}\int W(x)u^2 - \frac{1}{2_\alpha^*}\int a(x)(I_\alpha * |u|^{\frac{2_\alpha^*}{2}})|u|^{\frac{2_\alpha^*}{2}} - \int b(x)F(u),$$

where $u = u^+ + u^- \in X^+ \oplus X^-$, and $X = X^+ \oplus X^-$ corresponds to the positive and negative subspace of the spectral decomposition of $-\Delta + V$. Then the energy functional on $H^1(\mathbb{R}^N)$ associated with (1.7) is

$$E(u) := \frac{1}{2}(\|u^+\|^2 - \|u^-\|^2) - \frac{1}{2_\alpha^*}\int a_0(x)(I_\alpha * |u|^{\frac{2_\alpha^*}{2}})|u|^{\frac{2_\alpha^*}{2}} - \int b_0(x)F(u). \tag{2.2}$$

In the sequel, we denote

$$\mathcal{M} := \{u \in H^1(\mathbb{R}^N) \setminus X^- \mid E_W'(u)u = 0 \text{ and } E_W'(u)v = 0,\ \forall\, v \in X^-\},$$
$$X(u) := X^- \oplus \mathbb{R}u,\ \ \hat{X}(u) := X^- \oplus [0, +\infty)u,$$

and set

$$A_W := \inf_{\mathcal{M}} E_W(u) \text{ and } A := \inf_{\mathcal{M}} E(u).$$

3. Technical lemmas

In this section, we assume that all the conditions in Theorem 1.1 are satisfied, and we will provide some technical lemmas.

To begin with, we set

$$H(x,t) := \frac{1}{2}W(x)t^2 + \frac{1}{2_\alpha^*}a(x)(I_\alpha * |t|^{\frac{2_\alpha^*}{2}})|t|^{\frac{2_\alpha^*}{2}} + b(x)F(t),$$

and

$$h(x,t) := W(x)t + a(x)(I_\alpha * |t|^{\frac{2_\alpha^*}{2}})|t|^{\frac{2_\alpha^*}{2}-2}t + b(x)f(t).$$

Lemma 3.1. *$H(x,u) > 0$ and $\frac{1}{2}h(x,u)u > H(x,u)$ if $u \neq 0$.*

Proof. From (F1), (F3) and $\alpha \in ((N-4)^+, N)$, we see that

$$u \mapsto \frac{h(x,u)}{|u|} \text{ is increasing strictly on } (-\infty, 0) \text{ and } (0, +\infty),$$

and

$$h(x,u) = o(u) \text{ uniformly in } x \text{ as } |u| \to 0,$$

which lead to $h(x,u) > 0$, $\forall u > 0$, and $h(x,u) < 0$, $\forall u < 0$. Then it follows $H(x,u) = \int_0^u h(x,t)dt > 0$, and

$$H(x,u) = \int_0^u \frac{h(x,t)}{t}tdt < \int_0^u \frac{h(x,u)}{u}tdt = \frac{1}{2}h(x,u)u,\ \forall u \neq 0.$$

□

Lemma 3.2. *Let $u, v, s \in \mathbb{R}$ satisfy $su + v \neq 0$ and $s \geq -1$. Then*

$$\left(\frac{s^2}{2} + s\right) h(x,u)u + (s+1)h(x,u)v + H(x,u) - H(x,(1+s)u+v) < 0,\ \forall x \in \mathbb{R}^N.$$

Proof. The conclusion follows by Lemma 3.1 and an argument similar to the one adopted in [24, lemma 2.2]. □

Lemma 3.3. *If $u \in \mathcal{M}$, then*

$$E_W(u+z) < E_W(u),\ \forall z \in \mathcal{B} := \{su + v |\, s \geq -1,\ v \in X^-\},\ z \neq 0.$$

Proof. Direct calculation yields

$$\begin{aligned}&E_W(u+z) - E_W(u)\\ &= -\frac{1}{2}\|v\|^2 + \int \left(\left(\frac{s^2}{2} + s\right) h(x,u)u + (s+1)h(x,u)v + H(x,u) - H(x,u+z)\right).\end{aligned}$$

Then the desired results follows from Lemma 3.2. □

Lemma 3.4. *If $\mathcal{Q} \subset X^+ \setminus \{0\}$ is a compact subset, then for every $u \in \mathcal{Q}$, there exists $R > 0$ such that*

$$E_W(z) \leq 0,\ \forall z \in X(u) \setminus B_R(0).$$

Proof. Set

$$\Phi(u) := \frac{1}{2}(\|u^+\|^2 - \|u^-\|^2) - \int a(x)(I_\alpha * |u|^{\frac{2_\alpha^*}{2}})|u|^{\frac{2_\alpha^*}{2}}.$$

It is obvious that

$$E_W(u) \leq \Phi(u),\ \forall u \in H^1(\mathbb{R}^N).$$

Then it suffices to show that

$$\Phi(z) \leq 0,\ \forall z \in X(u) \setminus B_R(0),\ u \in \mathcal{Q}, \tag{3.1}$$

for some $R > 0$. Without loss of generality, we may suppose that $\|u\| = 1$ for each $u \in \mathcal{Q}$. If (3.1) would not hold, then there is $u_n \in \mathcal{Q}$ and $z_n \in X(u_n)$ satisfying $\|z_n\| \to \infty$ as $n \to \infty$, and $\Phi(z_n) > 0$ for all n. Taking a subsequence, we may suppose that $u_n \to u \in X^+$ with $\|u\| = 1$. Let $w_n := \frac{z_n}{\|z_n\|} = s_n u_n + w_n^-$, then

$$s_n^2 = 1 - \|w_n^-\|^2, \tag{3.2}$$

and

$$0 \leq \frac{\Phi(z_n)}{\|z_n\|^2} = \frac{1}{2}(s_n^2 - \|w_n^-\|^2) - \int a(x)(I_\alpha * |z_n|^{\frac{2_\alpha^*}{2}})|z_n|^{\frac{2_\alpha^*}{2}-2} w_n^2. \tag{3.3}$$

Combining (3.2) and (3.3) suggests that $s_n \in [\frac{1}{\sqrt{2}}, 1]$. Then we assume that $s_n \to s > 0$ (up to a subsequence), and $w_n \rightharpoonup w = su + w^- \neq 0$. Since $\alpha \in ((N-4)^+, N)$, we have $\frac{2_\alpha^*}{2} - 2 > 0$. Applying the Fatou's Lemma, we see that

$$\begin{aligned}&\liminf_{n\to\infty} \int a(x)(I_\alpha * |z_n|^{\frac{2_\alpha^*}{2}})|z_n|^{\frac{2_\alpha^*}{2}-2} w_n^2\\ &\geq \int \liminf_{n\to\infty} a(x)(I_\alpha * |z_n|^{\frac{2_\alpha^*}{2}})|z_n|^{\frac{2_\alpha^*}{2}-2} w_n^2 \to +\infty,\end{aligned}$$

which contradicts (3.3). We have thus prove the lemma. □

Lemma 3.5. *(i) There is $\beta > 0$ such that $A_W \geq \inf_{S_\beta} E_W(u) > 0$, where $S_\beta := \{u \in X^+ : \|u\| = \beta\}$.*

(ii) $\|u^+\| \geq \max\{\|u^-\|, \sqrt{2A_W}\}$ for all $u \in \mathcal{M}$.

Proof. (i) From (F1) and (F2), we know that for any $\epsilon > 0$, there is $C_\epsilon > 0$ such that $F(u) \leq \epsilon|u|^2 + C_\epsilon|u|^p$ for some $2 < p < 2^*$. Then for $u \in X^+$,

$$\begin{aligned}
E_W(u) &= \frac{1}{2}\|u\|^2 - \frac{1}{2}\int W(x)u^2 - \frac{1}{2^*_\alpha}\int a(x)(I_\alpha * |u|^{\frac{2^*_\alpha}{2}})|u|^{\frac{2^*_\alpha}{2}} - \int b(x)F(u) \\
&\geq \frac{1}{2}\|u\|^2 - \frac{\overline{W}}{2}\int u^2 - \frac{\overline{a}}{2^*_\alpha}|u|_{2^*}^{\frac{2(N+\alpha)}{N-2}} - \overline{b}\int(\epsilon|u|^2 + C_\epsilon|u|^p) \\
&\geq \frac{1}{2}\left(1 - \frac{\overline{W}}{\overline{\lambda}} - C_1\epsilon\right)\|u\|^2 - C_2\|u\|^{\frac{2(N+\alpha)}{N-2}} - C_3C_\epsilon\|u\|^p,
\end{aligned}$$

where $\overline{a} = \sup_{x\in\mathbb{R}^N} a(x)$, $\overline{b} = \sup_{x\in\mathbb{R}^N} b(x)$. By (V2), for $\epsilon > 0$ sufficiently small, there exists $\beta > 0$ such that

$$\frac{1}{2}\left(1 - \frac{\overline{W}}{\overline{\lambda}} - C_1\epsilon\right)\beta^2 - C_2\beta^{\frac{2(N+\alpha)}{N-2}} - C_3C_\epsilon\beta^p > 0,$$

which implies

$$\inf_{u\in S_\beta} E_W(u) > 0.$$

Now denote

$$l := \inf_{S_\beta\cap X^+} E_W(u) > 0.$$

According to Lemma 3.3, we obtain

$$E_W(u) \geq E_W\left(\frac{\beta u^+}{\|u\|}\right) \geq l, \ \forall u \in \mathcal{M}.$$

Thus,

$$A_W = \inf_{\mathcal{M}} E_W \geq l > 0.$$

(ii) For each $u \in \mathcal{M}$, since

$$A_W \leq E_W(u) \leq \frac{1}{2}(\|u^+\|^2 - \|u^-\|^2),$$

we conclude that (ii) holds. □

Lemma 3.6. *For each $u \in H^1(\mathbb{R}^N) \setminus X^-$, there exists precisely one element $\hat{m}(u)$ in $\mathcal{M} \cap \hat{X}(u)$. Moreover, $\hat{m}(u)$ is the unique global maximum of $E_W|_{\hat{X}(u)}$.*

Proof. According to Lemma 3.3, we need only show that $\mathcal{M} \cap \hat{X}(u) \neq \emptyset$. By the fact $\hat{X}(u) = \hat{X}(u^+)$, we may suppose that $u \in X^+$ and $\|u\| = 1$. Combining Lemma 3.4 and Lemma 3.5 (a) leads to $0 < \sup_{\hat{X}(u)} E_W < \infty$. Noting the weakly upper semi-continuity of E_W on $\hat{X}(u)$, we derive that there exists $z_0 \in \hat{X}(u) \setminus \{0\}$ such that $E_W(z_0) = \sup_{\hat{X}(u)} E_W$. Then it follows that $E'_W(z_0)z_0 = E'_W(z_0)v = 0$ for all $v \in \hat{X}(u)$, implying $z_0 \in \mathcal{M} \cap \hat{X}(u)$. □

Lemma 3.7. *E_W is coercive on $\mathcal{M}$, that is, $E_W(u)\to\infty$ as $\|u\|\to\infty$, $\forall\, u\in\mathcal{M}$.*

Proof. If the statement is false, then there is a sequence $\{u_n\}\subset\mathcal{M}$ with $E_W(u_n)\leq c$ and $\|u_n\|\to\infty$ for some $c>0$. Then we have for n large enough,

$$\begin{aligned} c+o(1) &\geq E_W(u_n)-\frac{1}{2}E'_W(u_n)u_n \\ &= \frac{2^*_\alpha-2}{2\cdot 2^*_\alpha}\int (I_\alpha * |u_n|^{\frac{2^*_\alpha}{2}})|u_n|^{\frac{2^*_\alpha}{2}} + \int b(x)(\frac{1}{2}f(u_n)u_n - F(u_n)). \end{aligned} \tag{3.4}$$

From (F1), (F3) and similar arguments as in Lemma 3.1, we have $F(s)\geq 0$ and $\frac{1}{2}f(s)s-F(s)\geq 0$ for all $s\in\mathbb{R}$, which combined (3.4) gives

$$\int (I_\alpha * |u_n|^{\frac{2^*_\alpha}{2}})|u_n|^{\frac{2^*_\alpha}{2}} \leq C_1 \tag{3.5}$$

for some $C_1>0$. Set $v_n=\frac{u_n}{\|u_n\|}$. We may assume that $v_n\rightharpoonup v$ in $H^1(\mathbb{R}^N)$, $v_n\to v$ in $L^p_{loc}(\mathbb{R}^N)(2\leq p<2^*)$, and $v_n\to v$ a.e on $\mathbb{R}^N$. We claim that there is $\{y_n\}\subset\mathbb{R}^N$ such that $\int_{B(y_n,1)}|v_n^+|^2\geq\delta>0$. In fact, by Lemma 3.5(ii), we have

$$\|v_n^+\|\geq\frac{1}{2}. \tag{3.6}$$

From (3.5), we obtain

$$\int (I_\alpha * |v_n^+|^{\frac{2^*_\alpha}{2}})|v_n^+|^{\frac{2^*_\alpha}{2}} \leq \int (I_\alpha * |v_n|^{\frac{2^*_\alpha}{2}})|v_n|^{\frac{2^*_\alpha}{2}} = \frac{\int (I_\alpha * |u_n|^{\frac{2^*_\alpha}{2}})|u_n|^{\frac{2^*_\alpha}{2}}}{\|u_n\|^{2^*_\alpha}}\to 0. \tag{3.7}$$

Assume by contradiction that

$$\sup_{y\in\mathbb{R}^N}\int_{B(y,1)}|v_n^+|^2dx\to 0 \text{ as } n\to\infty.$$

Then by Lion's Lemma [28][Lemma 1.21], we have $v_n^+\to 0$ in $L^p(\mathbb{R}^N)(2<p<2^*)$, which combined (A1), (F1) and (F2) yields

$$\int b(x)F(sv_n^+)\to 0,\ \forall\, s\in\mathbb{R}. \tag{3.8}$$

Moreover, we see from (V2) there exists $R>0$ sufficiently large such that

$$\int W(x)(v_n^+)^2 = \int_{\mathbb{R}^N\setminus B(0,R)} W(x)(v_n^+)^2dx + \int_{B(0,R)} W(x)(v_n^+)^2dx \to 0. \tag{3.9}$$

Since $sv_n^+\in\hat{X}(u_n)$ for $s\geq 0$, we see from Lemma 3.3, (3.6), (3.7), (3.8) and (3.9) that

$$\begin{aligned} c &\geq E_W(u_n)\geq E_W(sv_n^+) \\ &= \frac{s^2}{2}\left(\|v_n^+\|^2-\int W(x)|v_n^+|^2\right) - \int b(x)F(sv_n^+) - \frac{1}{2^*_\alpha}\int (I_\alpha * |sv_n^+|^{\frac{2^*_\alpha}{2}})|sv_n^+|^{\frac{2^*_\alpha}{2}} \\ &\geq \frac{s^2}{4} - \frac{s^2}{2}\int W(x)|v_n^+|^2 - \int b(x)F(sv_n^+) - \frac{1}{2^*_\alpha}\int (I_\alpha * |sv_n^+|^{\frac{2^*_\alpha}{2}})|sv_n^+|^{\frac{2^*_\alpha}{2}} \\ &\to \frac{s^2}{4}, \end{aligned}$$

which is a contradiction if $s \geq \sqrt{4c}$, so the claim holds. By the invariance of $\mathcal{M}$ and E_W under the transformation $u \mapsto u(\cdot - k)$ with $k \in \mathbb{Z}^N$, we may assume that $\{y_n\}$ is bounded. Then $v_n^+ \to v^+$ in $L^2_{loc}(\mathbb{R}^N)$ with $v^+ > 0$. Applying Fatou's Lemma, we obtain

$$\frac{\int (I_\alpha * |u_n|^{\frac{2^*_\alpha}{2}})|u_n|^{\frac{2^*_\alpha}{2}}}{\|u_n\|^2} = \int (I_\alpha * |u_n|^{\frac{2^*_\alpha}{2}})|u_n|^{\frac{2^*_\alpha}{2}-2} v_n^2 \to \infty \text{ as } n \to \infty,$$

which contradicts (3.5), showing the desired result. □

According to Lemma 3.6, we consider the map

$$\hat{m}(u): \ X^+ \setminus \{0\} \to \mathcal{M}, \text{ where } \hat{m}(u) \in \hat{X}(u) \cap \mathcal{M}.$$

Now we establish the continuity of the $\hat{m}$.

Lemma 3.8. *The function $\hat{m}: X^+ \setminus \{0\} \mapsto \mathcal{M}$ is continuous.*

Proof. The proof of Lemma 3.8 will not be included since it is analogous to that in [24, Lemma 2.8]. □

By Lemma 3.8, we define a continuous functional $\hat{J}: \ X^+ \setminus \{0\} \mapsto \mathbb{R}$ by

$$\hat{J}(u) = E_W(\hat{m}(u)),$$

and denote the restriction of the functional $\hat{J}$ to set S^+ by $J: S^+ \mapsto \mathbb{R}$, where $S^+ := \{z \in X^+ : \|z\| = 1\}$. Then as in [24], we obtain the following properties related to $\hat{J}$ and J.

Lemma 3.9. *$\hat{J} \in C^1(X^+ \setminus \{0\}, \mathbb{R})$ and*

$$\hat{J}'(z)w = \frac{\|\hat{m}(z)^+\|}{\|z\|} E'_W(\hat{m}(z))w, \ \forall\, z, w \in X^+, \ z \neq 0.$$

Corollary 3.1. *(i) $J \in C^1(S^+)$, and*

$$J'(z)w = \|\hat{m}(z)^+\| E'_W(\hat{m}(z))w, \text{ for } w \in T_z S^+ := \{y \in X^+ |\, \langle z, y\rangle = 0\}.$$

(ii) $\{z_n\}$ is a $(PS)_c$ sequence of J if and only if $\{\hat{m}(z_n)\}$ is a $(PS)_c$ sequence of E_W.

(iii) If there exists $u \in \mathcal{M}$ such that $E_W(u) = A_W$, then $E'_W(u) = 0$.

The proofs of Lemma 3.9 and Corollary 3.1 can be found in [24]. By Corollary 3.1 and the Ekeland's variational principle, we obtain the existence of $(PS)_{A_W}$ sequence for E_W.

Corollary 3.2. *There exists a sequence $\{u_n\} \subset \mathcal{M}$ satisfying*

$$E_W(u_n) \to A_W, \ E'_W(u_n) \to 0.$$

Next we will give an estimate on the upper bound of A_W, which is crucial in establishing the compactness of $(PS)_{A_W}$ sequence. We begin with two technical lemmas.

Lemma 3.10. *Fix $u \in X^+ \setminus \{0\}$. Then there exist s_0, ρ such that*

$$\sup_{z \in \hat{X}(u)} E_W(z) = \sup_{\substack{\|su+v\| \leq \rho, \\ s \geq s_0, v \in X^-}} E_W(su + v).$$

Proof. By Lemma 3.4 and 3.5, there is $\rho > 0$ such that

$$\sup_{z\in\hat{X}(u)} E_W(z) = \sup_{z\in\hat{X}(u)\cap B(0,\rho)} E_W(z).$$

Let $\{v_n\} \in X^-$ and $\{s_n\} \subset [0, +\infty)$ be any sequences satisfying $\|s_n u + v_n\| \le \rho$ and

$$E_W(s_n u + v_n) \to \sup_{z\in\hat{X}(u)} E_W(z).$$

If the conclusion is false, then $s_n \to 0$. It follows that

$$0 < A_W \le \sup_{z\in\hat{X}(u)} E_W(z) = E_W(s_n u + v_n) + o(1) \le \frac{1}{2} s_n^2 \|u\|^2 + o(1) \to 0,$$

which is absurd. The lemma is proved. □

Lemma 3.11. *Let $N = 3$, $u \in X^+ \setminus \{0\}$ and s_0, $\rho > 0$. Then for all $v \in X^-$ and $s \ge s_0$ satisfying $\|su + v\| \le \rho$, there is $q \in (2, 2^*)$ and $\eta > 0$ such that*

$$\eta|su|^q \le F(su + v).$$

Proof. From (F3), we know that there exist $\lambda > 0$ and $q \in (2, 2^*)$ such that

$$F(su + v) \ge \frac{\lambda}{q}|su + v|^q.$$

Then the desired result follows from [2, Lemma 2.7]. □

By (A2), we may suppose that $V(0) < 0$. Denote

$$U_\epsilon := \frac{C\epsilon^{\frac{N-2}{4}}}{(\epsilon + |x|^2)^{\frac{N-2}{2}}}, \tag{3.10}$$

where C is a given constant, and ϵ is a positive parameter. According to [10], U_ϵ is a minimizer of S_* defined by

$$S_* = \inf_{u\in D^{1,2}(\mathbb{R}^N)} \left\{ \int |\nabla u|^2 \;\; \Big| \;\; \int (I_\alpha * |u|^{\frac{2^*_\alpha}{2}})|u|^{\frac{2^*_\alpha}{2}} = 1 \right\}.$$

Choose r sufficiently small such that $V(x) \le -\delta$ for some $\delta > 0$ and $|x| < 2r$, and let $\psi \in C_0^\infty(\mathbb{R}^N)$ be a cut-off function such that $\psi(x) = 1, |x| \le r$, $\psi(x) = 0$, $|x| \ge 2r$, and $0 \le \psi \le 1$. Set

$$\psi_\epsilon(x) = \frac{\psi U_\epsilon}{\left(\int (I_\alpha * |\psi U_\epsilon|^{\frac{2^*_\alpha}{2}})|\psi U_\epsilon|^{\frac{2^*_\alpha}{2}}\right)^{\frac{1}{2^*_\alpha}}}. \tag{3.11}$$

Then we know from [3, 10, 28] that

$$\begin{aligned} &\int (I_\alpha * |\psi_\epsilon|^{\frac{2^*_\alpha}{2}})|\psi_\epsilon|^{\frac{2^*_\alpha}{2}} = 1, \quad \int |\nabla \psi_\epsilon|^2 = S_* + O(\epsilon^{\frac{N-2}{2}}), \\ &\int |\psi_\epsilon| = O(\epsilon^{\frac{N-2}{4}}), \quad \int |\nabla \psi_\epsilon| = O(\epsilon^{\frac{N-2}{4}}), \end{aligned} \tag{3.12}$$

$$\int |\psi_\epsilon|^2 = \begin{cases} O(\epsilon), \ N \geq 5, \\ O(\epsilon ln|\epsilon|), \ N = 4, \end{cases} \tag{3.13}$$

$$\int |\psi_\epsilon|^{2^*-s} = \begin{cases} O(\epsilon^{\frac{(N-2)s}{4}}), \ 0 < s < \frac{2^*}{2}, \\ O(\epsilon^{\frac{N}{4}}|ln\epsilon|), \ s = \frac{2^*}{2}, \\ O(\epsilon^{\frac{N}{2}-\frac{(N-2)s}{4}}), \ \frac{2^*}{2} < s < 2^*. \end{cases}$$

Lemma 3.12. *For $N \geq 4$,*

$$A_W < \frac{\alpha+2}{2(N+\alpha)|a|_\infty^{\frac{N-2}{\alpha+2}}} S_*^{\frac{N+\alpha}{\alpha+2}}. \tag{3.14}$$

For $N = 3$, there exists $\tilde{b} > 0$ such that (3.14) *holds if* $\inf_{\mathbb{R}^N} b(x) > \tilde{b}$.

Proof. For the case $N \geq 4$, it is obvious that

$$E_W(u) \leq \frac{1}{2}\int (|\nabla u|^2 + V(x)u^2) - \frac{1}{2^*_\alpha}\int a(x)(I_\alpha * |u|^{\frac{2^*_\alpha}{2}})|u|^{\frac{2^*_\alpha}{2}} := E_*(u).$$

By [11, Lemma 3.5], we know that

$$\sup_{\hat{X}(\psi_\epsilon)} E_* < \frac{\alpha+2}{2(N+\alpha)|a|_\infty^{\frac{N-2}{\alpha+2}}} S_*^{\frac{N+\alpha}{\alpha+2}}.$$

Then by the definition of A_W, we obtain that (3.14) holds.

For $N = 3$, denote $\underline{b} := \inf_{\mathbb{R}^N} b(x)$. Then by Lemma 3.10, Lemma 3.11, we have

$$\begin{aligned} \sup_{\hat{X}(u)} E_W &= \sup_{\substack{\|su+v\|\leq\rho, \\ s\geq s_0, v\in X^-}} E_W(su+v). \\ &\leq \sup_{\substack{\|su+v\|\leq\rho, \\ s\geq s_0, v\in X^-}} \left(\frac{s^2\|u\|^2}{2} - \underline{b}\int F(su+v)\right) \\ &\leq \sup_{\substack{\|su+v\|\leq\rho, \\ s\geq s_0, v\in X^-}} \left(\frac{s^2\|u\|^2}{2} - \underline{b}\eta\int |su|^q)\right) \\ &\leq \max_{s\geq 0}\left(s^2\frac{\|u\|^2}{2} - s^q\underline{b}\eta\int |u|^q\right). \end{aligned}$$

Denote

$$l(s) := s^2\frac{\|u\|^2}{2} - s^q\underline{b}\eta\int |u|^q.$$

Since $\max_{s\geq 0} l(s) \to 0$, as $\underline{b} \to +\infty$, we deduce that there exists $\tilde{b} > 0$ such that

$$\sup_{\hat{X}(u)} E_W < \frac{\alpha+2}{2(N+\alpha)|a|_\infty^{\frac{N-2}{\alpha+2}}} S_*^{\frac{N+\alpha}{\alpha+2}}, \ \forall \underline{b} \geq \tilde{b}.$$

The conclusion follows. □

The next Lemma will be used in proving the existence of ground states of the periodic equation (1.7). Since the proof can be found in the [11], it will not be included.

Lemma 3.13. *If* $\{u_n\} \subset H^1(\mathbb{R}^N)$ *is a* $(PS)_d$ *sequence of* E *with*

$$d \in \left(0, \frac{\alpha+2}{2(N+\alpha)|a_0|_\infty^{\frac{N-2}{\alpha+2}}} S_*^{\frac{N+\alpha}{\alpha+2}}\right),$$

then there exist $\{y_n\} \subset \mathbb{Z}^N$ *and* r, $\delta_0 > 0$ *such that*

$$\limsup_{n\to\infty} \int_{B_r(y_n)} u_n^2 dx \geq \delta_0.$$

Finally, we give a lemma which is important in dealing with the difficulty caused by non-periodic coefficients.

Lemma 3.14. *If* $\{u_n\} \subset H^1(\mathbb{R}^N)$ *is a* $(PS)_d$ *sequence of* E_W *with*

$$d < \frac{\alpha+2}{2(N+\alpha)|a|_\infty^{\frac{N-2}{\alpha+2}}} S_*^{\frac{N+\alpha}{\alpha+2}}, \tag{3.15}$$

and $u_n \rightharpoonup 0$ *in* $H^1(\mathbb{R}^N)$, *then*

$$\int a_*(x)(I_\alpha * |u_n|^{\frac{2_\alpha^*}{2}})|u_n|^{\frac{2_\alpha^*}{2}} \to 0.$$

Proof. By condition (A1) and Lemma 3.7, $\forall \epsilon > 0$, there exists $R > 0$ sufficiently large such that

$$\int_{\mathbb{R}^N \setminus B_R(0)} a_*(x)(I_\alpha * |u_n|^{\frac{2_\alpha^*}{2}})|u_n|^{\frac{2_\alpha^*}{2}} dx < \epsilon.$$

Then it is sufficient to show

$$\int_{B_R(0)} (I_\alpha * |u_n|^{\frac{2_\alpha^*}{2}})|u_n|^{\frac{2_\alpha^*}{2}} dx \to 0. \tag{3.16}$$

According to a Concentration Compactness Lemma [9, Lemma 2.5], we may assume that

$$|\nabla u_n|^2 \rightharpoonup \mu \geq \sum_{j\in\mathcal{J}} \mu_j \delta_{x_j}, \quad (I_\alpha * |u_n|^{\frac{2_\alpha^*}{2}})|u_n|^{\frac{2_\alpha^*}{2}} \rightharpoonup \nu = \sum_{j\in\mathcal{J}} \nu_j \delta_{x_j},$$

where $\{x_j\} \subset \mathbb{R}^N$ is a countable sequence, $\mathcal{J}$ is a countable set and μ_j, $\nu_j > 0, j \in \mathcal{J}$. In order to obtain (3.16), we need only prove

$$\nu_j = 0, \forall j \in \mathcal{J}. \tag{3.17}$$

Observe that

$$d = \lim_{n\to\infty} \left(E_W(u_n) - \frac{1}{2}E'_W(u_n)u_n\right) \geq \frac{\alpha+2}{2(N+\alpha)} \sum_{j\in J} a(x_j)\nu_j. \tag{3.18}$$

Set $\phi_\epsilon(x) := \phi((x-x_j)/\epsilon)$ with $\phi = 1$ on $B_1(0)$, $\phi = 0$ on $\mathbb{R}^N \setminus B_2(0)$, $0 \leq \phi \leq 1$ and $|\nabla\phi| \leq 2$. Then

$$E'_W(u_n)(\phi_\epsilon u_n) \to 0,$$

that is,

$$\int \nabla u_n \nabla(\phi_\epsilon u_n) + \int (V(x) - W(x))\phi_\epsilon u_n^2 - \int a(x)(I_\alpha * |u_n|^{\frac{2^*_\alpha}{2}})|u_n|^{\frac{2^*_\alpha}{2}}\phi_\epsilon$$
$$- \int b(x)f(u_n)u_n\phi_\epsilon \to 0.$$

By (F1), (F2), the fact that $u_n \rightharpoonup 0$ and letting $\epsilon \to 0$, we have

$$\mu(x_j) = a(x_j)\nu_j.$$

Since $\mu_j \leq \mu(x_j)$, it follows

$$S_* \nu_j^{\frac{N-2}{N+\alpha}} \leq \mu_j \leq \mu(x_j) = a(x_j)\nu_j.$$

Thus, we obtain

$$\nu_j \geq \left(\frac{S_*}{a(x_j)}\right)^{\frac{N+\alpha}{\alpha+2}}. \tag{3.19}$$

Combining (3.18) and (3.19) yields

$$d \geq \frac{\alpha+2}{2(N+\alpha)|a|_\infty^{\frac{N-2}{\alpha+2}}} S_*^{\frac{N+\alpha}{\alpha+2}},$$

which contradicts (3.15). □

4. Proof of Theorem 1.1

We divide the proof into two cases: the periodic case and the non-periodic case.

First, We proof the periodic case, that is, the case when $a_* = b_* = W \equiv 0$. By Corollary 3.2, there exists $\{u_n\} \subset \mathcal{M}$ such that

$$E(u_n) \to A,\ E'(u_n) \to 0.$$

Then from Lemma 3.7, $\{u_n\}$ is bounded. By Lemma 3.13, we may assume that $u_n \rightharpoonup u \neq 0$. Then u is a solution of equation (1.7). Moreover,

$$\begin{aligned}
A &= \lim_{n\to\infty}\left(E(u_n) - \frac{1}{2}E'(u_n)u_n\right)\\
&= \liminf_{n\to\infty}\left(\frac{1}{2} - \frac{1}{2^*_\alpha}\right)\int a(x)(I_\alpha * |u_n|^{\frac{2^*_\alpha}{2}})|u_n|^{\frac{2^*_\alpha}{2}} + \int b(x)(\frac{1}{2}f(u_n)u_n - F(u_n))\\
&\geq \left(\frac{1}{2} - \frac{1}{2^*_\alpha}\right)\int a(x)(I_\alpha * |u|^{\frac{2^*_\alpha}{2}})|u|^{\frac{2^*_\alpha}{2}} + \int b(x)(\frac{1}{2}f(u)u - F(u))\\
&= E(u),
\end{aligned} \tag{4.1}$$

that is, u is a ground state of (1.7). The proof of the periodic case is finished.

Second, we study the asymptotically periodic case. From the definition, it is obvious that

$$A_W \leq A.$$

Then we consider two cases:
(a) $A_W = A$. Denote a ground state of the periodic problem (1.7) by u, and let $v \in \hat{X}(u)$ satisfy

$$E_W(v) = \sup_{\hat{X}(u)} E_W.$$

Then

$$A_W \le E_W(v) \le E(v) \le E(u) = A = A_W. \tag{4.2}$$

Combining (4.2) and Corollary 3.1, we obtain that $E_W'(v) = 0$, that is, v is a ground state of equation (1.1).
(b) $A_W < A$. Let $\{v_n\} \subset \mathcal{M}$ be a $(PS)_{A_W}$ sequence, that is,

$$E_W(v_n) \to A_W,\ E_W'(v_n) \to 0. \tag{4.3}$$

Then $\{v_n\}$ is bounded. Taking a subsequence, we may assume that $v_n \rightharpoonup v \in H^1(\mathbb{R}^N)$. We assert that $v \ne 0$. Actually, if $v = 0$, then from assumptions (V2), (A1), (F1),(F2) and Lemma 3.14, we have

$$\begin{aligned} &\int W(x)v_n^2 \to 0, && \int W(x)v_n h \to 0,\\ &\int b_*(x)F(v_n) \to 0, && \int b_*(x)f(v_n)h \to 0,\\ &\int a_*(x)(I_\alpha * |v_n|^{\frac{2_\alpha^*}{2}})|v_n|^{\frac{2_\alpha^*}{2}} \to 0, && \int a_*(x)(I_\alpha * |v_n|^{\frac{2_\alpha^*}{2}})|v_n|^{\frac{2_\alpha^*}{2}-1}h \to 0, \end{aligned} \tag{4.4}$$

for any $h \in H^1(\mathbb{R}^N)$. Combining (4.3), (4.4) and the definition of E_W and E leads to

$$E(v_n) \to A_W,\ \ E'(v_n) \to 0.$$

Applying Lemma 3.13, we obtain that there exist $\{y_n\} \subset \mathbb{Z}^N$ and r, $\delta_0 > 0$ such that

$$\limsup_{n\to\infty} \int_{B_r(y_n)} v_n^2 dx \ge \delta_0. \tag{4.5}$$

Denote $w_n := v_n(x - y_n)$. Then it follows from (4.5) that $w_n \rightharpoonup w \ne 0$ and $E'(w) = 0$. Thus,

$$\begin{aligned} A_W &= \lim_{n\to\infty} E_W(v_n) = \lim_{n\to\infty} E(v_n)\\ &= \lim_{n\to\infty} E(w_n) = \lim_{n\to\infty}\left(E(w_n) - \frac{1}{2}E'(w_n)w_n\right)\\ &\ge E(w) - \frac{1}{2}E'(w)w = E(w) \ge A, \end{aligned}$$

which contradicts $A_W < A$. Thus, $v \ne 0$. Then as (4.1), we conclude that v is a ground state of (1.1).

References

[1] N. Ackermann, *On a periodic Schrödinger equation with nonlocal superlinear part*, Math. Z, 2004, 248(2), 423–443.

[2] C. O. Alves and G. F. Germano, *Ground state solution for a class of indefinite variational problems with critical growth*, J. Differ. Equa., 2018, 265(1), 444–477.

[3] H. Breźis and L. Nirenberg, *Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents*, Commun. Pur. Appl. Math., 1983, 36(4), 437–477.

[4] L. Diósi, *Gravitation and quantum-mechanical localization of macro-objects*, Phys. Lett. A, 1984, 105(4–5), 199–202.

[5] Y. Ding, F. Gao and M. Yang, *Semiclassical states for Choquard type equations with critical growth: Critical frequency case*, Nonlinearity, 2020, 33, 6695–6728.

[6] L. Du and M. Yang, *Uniqueness and nondegeneracy of solutions for a critical nonlocal equation*, Discrete Cont. Dyn. Syst., 2019, 39(10), 5847–5866.

[7] H. Fröhlich, *Electrons in lattice fields*, Adv. Phys., 1954, 3(11), 325–361.

[8] F. Gao and M. Yang, *A strongly indefinite Choquard equation with critical exponent due to the Hardy-Littlewood-Sobolev inequality*, Commun. Contemp. Math., 2018, 20(4), 1750037.

[9] F. Gao, E. D. da Silva, M. Yang and J. Zhou, *Existence of solutions for critical Choquard equations via the concentration-compactness method*, P. Roy. Soc. A: Math. Phy., 2020, 150(2), 921–954.

[10] F. Gao and M. Yang, *The Brezis-Nirenberg type critical problem for the nonlinear Choquard equation*, Sci. China Math., 2018, 61, 1219–1242.

[11] T. Guo, X. Tang, Q. Zhang and Z. Gao, *Nontrivial solutions for the choquard equation with indefinite linear part and upper critical exponent*, Commun. Pur. Appl. Anal., 2020, 19(3), 1563–1579.

[12] W. Kryszewski and A. Szulkin, *Generalized linking theorem with an application to semilinear Schrödinger equation*, Adv. Differ. equa., 1998, 3(3), 441–472.

[13] Y. Li, G. Li and C. Tang, *Ground state solutions for Choquard equations with Hardy-Littlewood-Sobolev upper critical growth and potential vanishing at infinity*, J. Math. Anal. Appl., 2020, 484(2), 123733.

[14] G. Li and C. Tang, *Existence of a ground state solution for Choquard equation with the upper critical exponent*, Comput. Math. Appl., 2018, 76(11–12), 2635–2647.

[15] X. Li and S. Ma, *Choquard equations with critical nonlinearities*, Commun. Contemp. Math., 2020, 22(4), 1950023.

[16] E. H. Lieb, *Existence and uniqueness of the minimizing solution of Choquard's nonlinear equation*, Stud. Appl. Math., 1977, 57(2), 93–105.

[17] E. H. Lieb and M. Loss, *Analysis, 2nd ed., Graduate studies in mathematics*, vol. 14, American Mathematical Society, Providence, RI, 2001.

[18] P. L. Lions, *The Choquard equation and related questions*, Nonlinear Anal., 1980, 4(6), 1063–1072.

[19] V. Moroz and J. Van Schaftingen, *Groundstates of nonlinear Choquard equations: existence, qualitative properties and decay asymptotics*, J. Funct. Anal., 2013, 265(2), 153–184.

[20] V. Moroz and J. Van Schaftingen, *Groundstates of nonlinear Choquard equations: Hardy-Littlewood-Sobolev critical exponent*, Commun. Contemp. Math., 2015, 17(5) , 1550005.

[21] V. Moroz and J. Van Schaftingen, *A guide to the Choquard equation*, J. Fixed Point Theory Appl., 2017, 19(1), 773–813.

[22] S. I. Pekar, *Untersuchungen über die Elektronentheorie der Kristalle*, Akademie Verlag, Berlin, 1954.

[23] R. Penrose, *On gravity's role in quantum state reduction*, Gen. Relat. Gravit., 1996, 28(5), 581–600.

[24] A. Szulkin and T. Weth, *Ground state solutions for some indefinite variational problems*, J. Funct. Anal., 2009, 257(12), 3802–3822.

[25] J. Van Schaftingen and J. Xia, *Groundstates for a local nonlinear perturbation of the Choquard equations with lower critical exponent*, J. Math. Anal. Appl., 2018, 464(2), 1184–1202.

[26] X. Tang, J. Wei and S. Chen, *Nehari-type ground state solutions for a Choquard equation with lower critical exponent and local nonlinear perturbation*, Math. Meth. Appl. Sci., 2020, 43(10), 6627–6638.

[27] X. Wang and F. Liao, *Ground state solutions for a Choquard equation with lower critical exponent and local nonlinear perturbation*, Nonlinear Anal., 2020, 196, 111831.

[28] M. Willem, *Minimax Theorems*, Birkhäuser, Boston, 1996.

[29] Q. Wu, D. Qin and J. Chen, *Ground states and non-existence results for Choquard type equations with lower critical exponent and indefinite potentials*, Nonlinear Anal., 2020, 197, 111863.

[30] M. Yang, F. Zhao and S. Zhao, *Classification of solutions to a nonlocal equation with doubly Hardy-Littlewood-Sobolev critical exponents*, Discrete Cont. Dyn. Syst., 2021, 41, 5209–5241.

[31] H. Zhang, J. Xu and F. Zhang, *On a class of semilinear Schrödinger equations with indefinite linear part*, J. Math. Anal. Appl., 2014, 414(2), 710–724.