Journal of Applied Analysis and Computation
Volume 12, Number 2, April 2022, 736–753
Website:http://www.jaac-online.com
DOI:10.11948/20210305

ON THE SOLUTIONS OF THREE-DIMENSIONAL SYSTEM OF DIFFERENCE EQUATIONS VIA RECURSIVE RELATIONS OF ORDER TWO AND APPLICATIONS

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Abstract In this paper, we show that the following three-dimensional system of difference equations

$$x_{n+1}=\frac{y_n y_{n-2}}{bx_{n-1}+ay_{n-2}},\ y_{n+1}=\frac{z_n z_{n-2}}{dy_{n-1}+cz_{n-2}},\ z_{n+1}=\frac{x_n x_{n-2}}{fz_{n-1}+ex_{n-2}},$$

for $n\in\mathbb{N}_0$, where the parameters a,b,c,d,e,f and the initial values x_{-i}, y_{-i}, z_{-i}, $i\in\{0,1,2\}$, are real numbers, can be solved, extending further some results in literature. Also, we determine the forbidden set of the initial values by using obtained formulas. Finally, some applications concerning aforementioned system of difference equations are given.

Keywords System of difference equations, explicit solution, forbidden set.

MSC(2010) 39A10, 39A20, 39A23.

1. Introduction and preliminaries

First, remind that $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$, stand for natural, non-negative integer, integer, real and complex numbers, respectively. If $m,n\in\mathbb{Z}$, $m\le n$ the notation $i=\overline{m,n}$ stands for $\{i\in\mathbb{Z}: m\le i\le n\}$. The notation of $\mathbb{R}^n$ is a set of $n-$dimensional Cartesian Products defined in the form $\underbrace{\mathbb{R}\times\mathbb{R}\times\cdots\times\mathbb{R}}_{n\text{ times}}$.

The difference equations and systems of difference equations has been attracted by many authors in recent years [7, 8, 14–16, 19, 20, 24, 28–33, 36].

Firstly, De Moivre solved the following homogeneous linear difference equation

$$x_{n+2}=\alpha x_{n+1}+\beta x_n,\ n\in\mathbb{N}_0, \tag{1.1}$$

when $\beta\ne 0$ and $\alpha^2\ne -4\beta$. He found the general solution for equation (1.1) as follows:

$$x_n=\frac{(x_1-\lambda_2 x_0)\,\lambda_1^n+(\lambda_1 x_0-x_1)\,\lambda_2^n}{\lambda_1-\lambda_2},\ n\in\mathbb{N}_0, \tag{1.2}$$

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where $\lambda_{1,2} = \frac{\alpha \pm \sqrt{\alpha^2+4\beta}}{2}$, for $\alpha^2 + 4\beta \neq 0$, are roots of the polynomial $P(\lambda) = \lambda^2 - \alpha\lambda - \beta = 0$. The equation (1.2) is called the De Moivre formula and also the polynomial P is called the characteristic polynomial associated to the linear equation (1.1) in [4].

It is clear that the solutions of the difference equation with the same characteristic equation as equation (1.1), with initial conditions $s_{-1} = 0$, $s_0 = 1$, are called Binet formula for generalized Fibonacci sequences

$$s_n = \frac{\lambda_1^{n+1} - \lambda_2^{n+1}}{\lambda_1 - \lambda_2},\ n \geq -1, \tag{1.3}$$

where λ_1 and λ_2 are the roots of characteristic equation of equation (1.1). The sequence $(s_n)_{n\geq -1}$ is called the generalized Fibonacci sequence in the literature. As seen in many papers, it is easy to obtain the solution (1.2) in terms of solution (1.3). So solution (1.2) can be written as

$$x_n = \beta x_0 s_{n-2} + x_1 s_{n-1},\ n \in \mathbb{N}_0, \tag{1.4}$$

s_{-2} is calculated by using the following relations $s_n = (s_{n+2} - \alpha s_{n+1})/\beta$ for $n = -2$. By taking $\alpha = 1$, $\beta = 1$ and $\alpha = 2$, $\beta = 1$ in equation (1.1), with $s_{-1} = 0$, $s_0 = 1$, then the sequence $(s_n)_{n\geq -1}$ reduce to the well known Fibonacci sequence and the well known Pell sequence respectively. Such as $(s_n)_{n\geq -1}$ sequence there are a lot of generalization of Fibonacci and Pell sequences in the literature [3, 9–13, 18, 21–23, 25, 34, 35].

One of the most well-known difference equations that can be reduced to equation (1.1) under convenient transformations in the literature, is Riccati difference equation.

The Riccati difference equation is as follows

$$x_{n+1} = \frac{ax_n + b}{cx_n + d},\ n \in \mathbb{N}_0,$$

for $c \neq 0$, $ad \neq bc$, where parameters a, b, c, d and the initial value x_0 are real numbers.

Similarly, there are some papers that can be reduced to the Riccati difference equation under convenient transformations in the literature [17, 26, 27]. The Riccati difference equation is important for those papers that have been made.

One of the following difference equations that reduced to the Riccati difference equation under appropriate transformations,

$$x_{n+1} = \frac{x_n x_{n-2}}{x_{n-1} - x_{n-2}} \quad \text{and} \quad x_{n+1} = \frac{x_n x_{n-2}}{-x_{n-1} + x_{n-2}},\ n \in \mathbb{N}_0, \tag{1.5}$$

was first presented, among other things, by Abo-Zeid et al. in [1]. Then, in [5, 6], equations in (1.5) were generalized to the following equations

$$x_{n+1} = \frac{ax_{n-l}x_{n-k}}{bx_{n-p} + cx_{n-q}} \quad \text{and} \quad x_{n+1} = \frac{ax_{n-l}x_{n-k}}{bx_{n-p} - cx_{n-q}},\ n \in \mathbb{N}_0,$$

where $r := \max\{l, k, p, q\}$ is nonnegative integer, a, b, c are positive constants.

Further, the equations in (1.5) were extended to the following two-dimensional four systems of difference equations

$$x_{n+1} = \frac{y_n y_{n-2}}{x_{n-1} + y_{n-2}},\ y_{n+1} = \frac{x_n x_{n-2}}{\pm y_{n-1} \pm x_{n-2}},\ n \in \mathbb{N}_0 \tag{1.6}$$

in [2]. The solutions of systems in given (1.6) were found by using induction. Induction method didn't give much detail on how solutions were obtained.

But, two-dimensional system of difference equations in (1.6) was extended to the following two-dimensional system of difference equations with constant coefficients

$$x_{n+1} = \frac{y_n y_{n-2}}{b x_{n-1} + a y_{n-2}}, \; y_{n+1} = \frac{x_n x_{n-2}}{d y_{n-1} + c x_{n-2}}, \; n \in \mathbb{N}_0, \tag{1.7}$$

and system (1.7) was solved using convenient transformations in [22].

A natural question is to study both three-dimensional form of equations in (1.5), systems (1.6) and more general system of (1.7) solvable in explicit-form. Here we study such a system. That is, we deal with the following system of difference equations

$$x_{n+1} = \frac{y_n y_{n-2}}{b x_{n-1} + a y_{n-2}}, \; y_{n+1} = \frac{z_n z_{n-2}}{d y_{n-1} + c z_{n-2}}, \; z_{n+1} = \frac{x_n x_{n-2}}{f z_{n-1} + e x_{n-2}}, \; n \in \mathbb{N}_0, \tag{1.8}$$

where the parameters a, b, c, d, e, f and the initial values x_{-i}, y_{-i}, z_{-i}, $i \in \{0, 1, 2\}$, are real numbers. We solve system (1.8) in explicit form. Note that system (1.8) is a natural extension of both equations in (1.5) and systems (1.6), (1.7).

Our paper is organized as follows: In the next section we solve system (1.8) in explicit form by using convenient transformation and determine the forbidden set of the initial values by using the obtained formulas. In the final section we obtain well-known Fibonacci numbers and Pell numbers in the solutions of aforementioned system when $a = b = c = d = e = f = 1$; $a = d = e = f = -1$, $b = c = 1$ and $a = 3$, $b = -1$, $c = 2$, $d = f = 1$, $e = 4$.

2. Explicit solutions of the system (1.8)

Suppose that $x_{n_0} = 0$ for some $n_0 \geq -2$. Then from the third equation in (1.8) it follows that $z_{n_0+1} = 0$. If $z_{n_0+1} = 0$, then from the second equation in (1.8) it follows that $y_{n_0+2} = 0$, and consequently $d y_{n_0+2} + c z_{n_0+1} = 0$, from which it follows that $y_{n_0+4} = 0$ is not defined. Assume that $y_{n_1} = 0$ for some $n_1 \geq -2$. Then from the first equation in (1.8) it follows that $x_{n_1+1} = 0$. If $x_{n_1+1} = 0$, then from the third equation in (1.8) it follows that $z_{n_1+2} = 0$, and consequently $f z_{n_1+2} + e x_{n_1+1} = 0$, from which it follows that $z_{n_1+4} = 0$ is not defined. Suppose that $z_{n_2} = 0$ for some $n_2 \geq -2$. Then from the second equation in (1.8) it follows that $y_{n_2+1} = 0$. If $y_{n_2+1} = 0$, then from the first equation in (1.8) it follows that $x_{n_2+2} = 0$, and consequently $b x_{n_2+2} + a y_{n_2+1} = 0$, from which it follows that $x_{n_2+4} = 0$ is not defined. This means that the set

$$\bigcup_{j=0}^{2} \Big\{ \left(\vec{x}_{-(2,0)}, \vec{y}_{-(2,0)}, \vec{z}_{-(2,0)}\right) \in \mathbb{R}^9 : \; x_{-j} = 0 \text{ or } y_{-j} = 0 \text{ or } z_{-j} = 0 \Big\},$$

where $\vec{x}_{-(2,0)} = (x_{-2}, x_{-1}, x_0)$, $\vec{y}_{-(2,0)} = (y_{-2}, y_{-1}, y_0)$, $\vec{z}_{-(2,0)} = (z_{-2}, z_{-1}, z_0)$, is a subset of the forbidden set of solutions to system (1.8).

Hence, from now on we will assume that $x_n y_n z_n \neq 0$, $n \geq -2$. Note that the system (1.8) can be written in the form

$$\frac{y_n}{x_{n+1}} = b\frac{x_{n-1}}{y_{n-2}} + a, \; \frac{z_n}{y_{n+1}} = d\frac{y_{n-1}}{z_{n-2}} + c, \; \frac{x_n}{z_{n+1}} = f\frac{z_{n-1}}{x_{n-2}} + e, \; n \in \mathbb{N}_0. \tag{2.1}$$

Next, by employing the change of variables

$$u_{n+1}=\frac{y_n}{x_{n+1}},\ v_{n+1}=\frac{z_n}{y_{n+1}},\ t_{n+1}=\frac{x_n}{z_{n+1}},\ n\geq -2, \tag{2.2}$$

system (2.1) can be written as

$$u_{n+1}=\frac{b}{u_{n-1}}+a,\ v_{n+1}=\frac{d}{v_{n-1}}+c,\ t_{n+1}=\frac{f}{t_{n-1}}+e,\ n\in\mathbb{N}_0. \tag{2.3}$$

Let $u_m^{(k)}=u_{2m+k},\ v_m^{(k)}=v_{2m+k},\ t_m^{(k)}=t_{2m+k}$, for $m\geq -1,\ k\in\{1,2\}$. Then, from (2.3) we see that $\left(u_m^{(k)}\right)_{m\geq -1}$, $\left(v_m^{(k)}\right)_{m\geq -1}$, $\left(t_m^{(k)}\right)_{m\geq -1}$, $k\in\{1,2\}$, are three solutions to the following difference equations

$$q_m=\frac{b}{q_{m-1}}+a,\ \widehat{q}_m=\frac{d}{\widehat{q}_{m-1}}+c,\ \widetilde{q}_m=\frac{f}{\widetilde{q}_{m-1}}+e,\ m\in\mathbb{N}_0. \tag{2.4}$$

Equations in (2.4) are solvable. Let

$$q_m=\frac{w_{m+1}}{w_m},\ \widehat{q}_m=\frac{\widehat{w}_{m+1}}{\widehat{w}_m},\ \widetilde{q}_m=\frac{\widetilde{w}_{m+1}}{\widetilde{w}_m},\ m\geq -1, \tag{2.5}$$

where $w_{-1}=1$, $w_0=q_{-1}$, $\widehat{w}_{-1}=1$, $\widehat{w}_0=\widehat{q}_{-1}$, $\widetilde{w}_{-1}=1$, $\widetilde{w}_0=\widehat{q}_{-1}$. From now on, we assume that the sequences of q_m, $\widehat{q}_m$ and $\widetilde{q}_m$ are well defined. By using (2.5) in (2.4), we obtain following equations

$$w_{m+1}=aw_m+bw_{m-1},\ m\in\mathbb{N}_0, \tag{2.6}$$
$$\widehat{w}_{m+1}=c\widehat{w}_m+d\widehat{w}_{m-1},\ m\in\mathbb{N}_0, \tag{2.7}$$
$$\widetilde{w}_{m+1}=e\widetilde{w}_m+f\widetilde{w}_{m-1},\ m\in\mathbb{N}_0. \tag{2.8}$$

Let $(s_m)_{m\geq -1}$, $(\widehat{s}_m)_{m\geq -1}$, $(\widetilde{s}_m)_{m\geq -1}$ be the solutions to equations (2.6)-(2.8) respectively, such that

$$s_{-1}=0,\ s_0=1, \tag{2.9}$$
$$\widehat{s}_{-1}=0,\ \widehat{s}_0=1, \tag{2.10}$$
$$\widetilde{s}_{-1}=0,\ \widetilde{s}_0=1. \tag{2.11}$$

Then, from (1.4), the general solutions to equations (2.6)-(2.8) can be written in the following form

$$w_m=bw_{-1}s_{m-1}+w_0s_m,\ m\geq -1, \tag{2.12}$$
$$\widehat{w}_m=d\widehat{w}_{-1}\widehat{s}_{m-1}+\widehat{w}_0\widehat{s}_m,\ m\geq -1, \tag{2.13}$$
$$\widetilde{w}_m=f\widetilde{w}_{-1}\widetilde{s}_{m-1}+\widetilde{w}_0\widetilde{s}_m,\ m\geq -1, \tag{2.14}$$

s_{-2}, $\widehat{s}_{-2}$, $\widetilde{s}_{-2}$ are calculated by using the following relations $s_{m-1}=\frac{s_{m+1}-as_m}{b}$, $\widehat{s}_{m-1}=\frac{\widehat{s}_{m+1}-c\widehat{s}_m}{d}$, $\widetilde{s}_{m-1}=\frac{\widetilde{s}_{m+1}-e\widetilde{s}_m}{f}$, respectively, for $m=-1$.

From the equations in (2.5) and the equation (2.12)-(2.14), it follows that

$$q_m=\frac{bw_{-1}s_m+w_0s_{m+1}}{bw_{-1}s_{m-1}+w_0s_m}=\frac{bs_m+q_{-1}s_{m+1}}{bs_{m-1}+q_{-1}s_m},\quad m\geq -1, \tag{2.15}$$

$$\widehat{q}_m = \frac{d\widehat{w}_{-1}\widehat{s}_m + \widehat{w}_0\widehat{s}_{m+1}}{d\widehat{w}_{-1}\widehat{s}_{m-1} + \widehat{w}_0\widehat{s}_m} = \frac{d\widehat{s}_m + \widehat{q}_{-1}\widehat{s}_{m+1}}{d\widehat{s}_{m-1} + \widehat{q}_{-1}\widehat{s}_m}, \quad m \geq -1, \tag{2.16}$$

$$\widetilde{q}_m = \frac{f\widetilde{w}_{-1}\widetilde{s}_m + \widetilde{w}_0\widetilde{s}_{m+1}}{f\widetilde{w}_{-1}\widetilde{s}_{m-1} + \widetilde{w}_0\widetilde{s}_m} = \frac{f\widetilde{s}_m + \widetilde{q}_{-1}\widetilde{s}_{m+1}}{f\widetilde{s}_{m-1} + \widetilde{q}_{-1}\widetilde{s}_m}, \quad m \geq -1. \tag{2.17}$$

From (2.15)-(2.17), we obtain

$$u_m^{(k)} = \frac{bs_m + u_{-1}^{(k)}s_{m+1}}{bs_{m-1} + u_{-1}^{(k)}s_m},\ m \geq -1, \tag{2.18}$$

$$v_m^{(k)} = \frac{d\widehat{s}_m + v_{-1}^{(k)}\widehat{s}_{m+1}}{d\widehat{s}_{m-1} + v_{-1}^{(k)}\widehat{s}_m},\ m \geq -1, \tag{2.19}$$

$$t_m^{(k)} = \frac{f\widetilde{s}_m + t_{-1}^{(k)}\widetilde{s}_{m+1}}{f\widetilde{s}_{m-1} + t_{-1}^{(k)}\widetilde{s}_m},\ m \geq -1, \tag{2.20}$$

for $k \in \{1, 2\}$. From (2.18)-(2.20), we get

$$u_{2m+k} = \frac{bs_m + u_{k-2}s_{m+1}}{bs_{m-1} + u_{k-2}s_m},\ m \geq -1, \tag{2.21}$$

$$v_{2m+k} = \frac{d\widehat{s}_m + v_{k-2}\widehat{s}_{m+1}}{d\widehat{s}_{m-1} + v_{k-2}\widehat{s}_m},\ m \geq -1, \tag{2.22}$$

$$t_{2m+k} = \frac{f\widetilde{s}_m + t_{k-2}\widetilde{s}_{m+1}}{f\widetilde{s}_{m-1} + t_{k-2}\widetilde{s}_m},\ m \geq -1, \tag{2.23}$$

for $k \in \{1, 2\}$. From (2.2), we have that

$$\begin{aligned} x_{n+1} &= \frac{y_n}{u_{n+1}} = \frac{z_{n-1}}{u_{n+1}v_n} = \frac{x_{n-2}}{u_{n+1}v_nt_{n-1}} = \frac{y_{n-3}}{u_{n+1}v_nt_{n-1}u_{n-2}} \\ &= \frac{z_{n-4}}{u_{n+1}v_nt_{n-1}u_{n-2}v_{n-3}} = \frac{x_{n-5}}{u_{n+1}v_nt_{n-1}u_{n-2}v_{n-3}t_{n-4}},\ n \geq 3, \end{aligned} \tag{2.24}$$

$$\begin{aligned} y_{n+1} &= \frac{z_n}{v_{n+1}} = \frac{x_{n-1}}{v_{n+1}t_n} = \frac{y_{n-2}}{v_{n+1}t_nu_{n-1}} = \frac{z_{n-3}}{v_{n+1}t_nu_{n-1}v_{n-2}} \\ &= \frac{x_{n-4}}{v_{n+1}t_nu_{n-1}v_{n-2}t_{n-3}} = \frac{y_{n-5}}{v_{n+1}t_nu_{n-1}v_{n-2}t_{n-3}u_{n-4}},\ n \geq 3, \end{aligned} \tag{2.25}$$

$$\begin{aligned} z_{n+1} &= \frac{x_n}{t_{n+1}} = \frac{y_{n-1}}{t_{n+1}u_n} = \frac{z_{n-2}}{t_{n+1}u_nv_{n-1}} = \frac{x_{n-3}}{t_{n+1}u_nv_{n-1}t_{n-2}} \\ &= \frac{y_{n-4}}{t_{n+1}u_nv_{n-1}t_{n-2}u_{n-3}} = \frac{z_{n-5}}{t_{n+1}u_nv_{n-1}t_{n-2}u_{n-3}v_{n-4}},\ n \geq 3. \end{aligned} \tag{2.26}$$

From (2.24)-(2.26), we get

$$x_{6m+l} = \frac{x_{6(m-1)+l}}{u_{6m+l}v_{6m+l-1}t_{6m+l-2}u_{6m+l-3}v_{6m+l-4}t_{6m+l-5}}, \quad m \in \mathbb{N}_0, \tag{2.27}$$

$$y_{6m+l} = \frac{y_{6(m-1)+l}}{v_{6m+l}t_{6m+l-1}u_{6m+l-2}v_{6m+l-3}t_{6m+l-4}u_{6m+l-5}}, \quad m \in \mathbb{N}_0, \tag{2.28}$$

$$z_{6m+l} = \frac{z_{6(m-1)+l}}{t_{6m+l}u_{6m+l-1}v_{6m+l-2}t_{6m+l-3}u_{6m+l-4}v_{6m+l-5}}, \quad m \in \mathbb{N}_0, \tag{2.29}$$

for $l = \overline{4, 9}$. Multiplying the equalities which are obtained from (2.27)-(2.29), from 0 to m, it follows that

$$x_{6m+2i+j} = x_{2i+j-6}$$

$$\times \prod_{p=0}^{m} \frac{1}{u_{6p+2i+j}v_{6p+2i+j-1}t_{6p+2i+j-2}u_{6p+2i+j-3}v_{6p+2i+j-4}t_{6p+2i+j-5}}, \tag{2.30}$$

$$\begin{aligned} y_{6m+2i+j} =& y_{2i+j-6} \\ & \times \prod_{p=0}^{m} \frac{1}{v_{6p+2i+j}t_{6p+2i+j-1}u_{6p+2i+j-2}v_{6p+2i+j-3}t_{6p+2i+j-4}u_{6p+2i+j-5}}, \end{aligned} \tag{2.31}$$

$$\begin{aligned} z_{6m+2i+j} =& z_{2i+j-6} \\ & \times \prod_{p=0}^{m} \frac{1}{t_{6p+2i+j}u_{6p+2i+j-1}v_{6p+2i+j-2}t_{6p+2i+j-3}u_{6p+2i+j-4}v_{6p+2i+j-5}}, \end{aligned} \tag{2.32}$$

where $m \in \mathbb{N}_0$, $i = \overline{2,4}$ and $j \in \{0,1\}$. By substituting the formulas in (2.21)-(2.23) into (2.30)-(2.32) and by using equations in (2.2), we obtain

$$\begin{aligned} x_{6m+2i} = x_{2i-6} \prod_{p=0}^{m} & \frac{bx_0 s_{3p+i-2} + y_{-1} s_{3p+i-1}}{bx_0 s_{3p+i-1} + y_{-1} s_{3p+i}} \frac{dy_{-1}\widehat{s}_{3p+i-2} + z_{-2}\widehat{s}_{3p+i-1}}{dy_{-1}\widehat{s}_{3p+i-1} + z_{-2}\widehat{s}_{3p+i}} \\ & \times \frac{fz_0\widetilde{s}_{3p+i-3} + x_{-1}\widetilde{s}_{3p+i-2}}{fz_0\widetilde{s}_{3p+i-2} + x_{-1}\widetilde{s}_{3p+i-1}} \frac{bx_{-1} s_{3p+i-3} + y_{-2} s_{3p+i-2}}{bx_{-1} s_{3p+i-2} + y_{-2} s_{3p+i-1}} \\ & \times \frac{dy_0\widehat{s}_{3p+i-4} + z_{-1}\widehat{s}_{3p+i-3}}{dy_0\widehat{s}_{3p+i-3} + z_{-1}\widehat{s}_{3p+i-2}} \frac{fz_{-1}\widetilde{s}_{3p+i-4} + x_{-2}\widetilde{s}_{3p+i-3}}{fz_{-1}\widetilde{s}_{3p+i-3} + x_{-2}\widetilde{s}_{3p+i-2}}, \end{aligned} \tag{2.33}$$

$$\begin{aligned} x_{6m+2i+1} = x_{2i-5} \prod_{p=0}^{m} & \frac{bx_{-1} s_{3p+i-1} + y_{-2} s_{3p+i}}{bx_{-1} s_{3p+i} + y_{-2} s_{3p+i+1}} \frac{dy_0\widehat{s}_{3p+i-2} + z_{-1}\widehat{s}_{3p+i-1}}{dy_0\widehat{s}_{3p+i-1} + z_{-1}\widehat{s}_{3p+i}} \\ & \times \frac{fz_{-1}\widetilde{s}_{3p+i-2} + x_{-2}\widetilde{s}_{3p+i-1}}{fz_{-1}\widetilde{s}_{3p+i-1} + x_{-2}\widetilde{s}_{3p+i}} \frac{bx_0 s_{3p+i-3} + y_{-1} s_{3p+i-2}}{bx_0 s_{3p+i-2} + y_{-1} s_{3p+i-1}} \\ & \times \frac{dy_{-1}\widehat{s}_{3p+i-3} + z_{-2}\widehat{s}_{3p+i-2}}{dy_{-1}\widehat{s}_{3p+i-2} + z_{-2}\widehat{s}_{3p+i-1}} \frac{fz_0\widetilde{s}_{3p+i-4} + x_{-1}\widetilde{s}_{3p+i-3}}{fz_0\widetilde{s}_{3p+i-3} + x_{-1}\widetilde{s}_{3p+i-2}}, \end{aligned} \tag{2.34}$$

$$\begin{aligned} y_{6m+2i} = y_{2i-6} \prod_{p=0}^{m} & \frac{dy_0\widehat{s}_{3p+i-2} + z_{-1}\widehat{s}_{3p+i-1}}{dy_0\widehat{s}_{3p+i-1} + z_{-1}\widehat{s}_{3p+i}} \frac{fz_{-1}\widetilde{s}_{3p+i-2} + x_{-2}\widetilde{s}_{3p+i-1}}{fz_{-1}\widetilde{s}_{3p+i-1} + x_{-2}\widetilde{s}_{3p+i}} \\ & \times \frac{bx_0 s_{3p+i-3} + y_{-1} s_{3p+i-2}}{bx_0 s_{3p+i-2} + y_{-1} s_{3p+i-1}} \frac{dy_{-1}\widehat{s}_{3p+i-3} + z_{-2}\widehat{s}_{3p+i-2}}{dy_{-1}\widehat{s}_{3p+i-2} + z_{-2}\widehat{s}_{3p+i-1}} \\ & \times \frac{fz_0\widetilde{s}_{3p+i-4} + x_{-1}\widetilde{s}_{3p+i-3}}{fz_0\widetilde{s}_{3p+i-3} + x_{-1}\widetilde{s}_{3p+i-2}} \frac{bx_{-1} s_{3p+i-4} + y_{-2} s_{3p+i-3}}{bx_{-1} s_{3p+i-3} + y_{-2} s_{3p+i-2}}, \end{aligned} \tag{2.35}$$

$$\begin{aligned} y_{6m+2i+1} = y_{2i-5} \prod_{p=0}^{m} & \frac{dy_{-1}\widehat{s}_{3p+i-1} + z_{-2}\widehat{s}_{3p+i}}{dy_{-1}\widehat{s}_{3p+i} + z_{-2}\widehat{s}_{3p+i+1}} \frac{fz_0\widetilde{s}_{3p+i-2} + x_{-1}\widetilde{s}_{3p+i-1}}{fz_0\widetilde{s}_{3p+i-1} + x_{-1}\widetilde{s}_{3p+i}} \\ & \times \frac{bx_{-1} s_{3p+i-2} + y_{-2} s_{3p+i-1}}{bx_{-1} s_{3p+i-1} + y_{-2} s_{3p+i}} \frac{dy_0\widehat{s}_{3p+i-3} + z_{-1}\widehat{s}_{3p+i-2}}{dy_0\widehat{s}_{3p+i-2} + z_{-1}\widehat{s}_{3p+i-1}} \\ & \times \frac{fz_{-1}\widetilde{s}_{3p+i-3} + x_{-2}\widetilde{s}_{3p+i-2}}{fz_{-1}\widetilde{s}_{3p+i-2} + x_{-2}\widetilde{s}_{3p+i-1}} \frac{bx_0 s_{3p+i-4} + y_{-1} s_{3p+i-3}}{bx_0 s_{3p+i-3} + y_{-1} s_{3p+i-2}}, \end{aligned} \tag{2.36}$$

$$z_{6m+2i} = z_{2i-6} \prod_{p=0}^{m} \frac{fz_0\widetilde{s}_{3p+i-2} + x_{-1}\widetilde{s}_{3p+i-1}}{fz_0\widetilde{s}_{3p+i-1} + x_{-1}\widetilde{s}_{3p+i}} \frac{bx_{-1} s_{3p+i-2} + y_{-2} s_{3p+i-1}}{bx_{-1} s_{3p+i-1} + y_{-2} s_{3p+i}}$$

$$\times \frac{dy_0\widehat{s}_{3p+i-3}+z_{-1}\widehat{s}_{3p+i-2}}{dy_0\widehat{s}_{3p+i-2}+z_{-1}\widehat{s}_{3p+i-1}}\frac{fz_{-1}\widetilde{s}_{3p+i-3}+x_{-2}\widetilde{s}_{3p+i-2}}{fz_{-1}\widetilde{s}_{3p+i-2}+x_{-2}\widetilde{s}_{3p+i-1}} \tag{2.37}$$
$$\times \frac{bx_0s_{3p+i-4}+y_{-1}s_{3p+i-3}}{bx_0s_{3p+i-3}+y_{-1}s_{3p+i-2}}\frac{dy_{-1}\widehat{s}_{3p+i-4}+z_{-2}\widehat{s}_{3p+i-3}}{dy_{-1}\widehat{s}_{3p+i-3}+z_{-2}\widehat{s}_{3p+i-2}},$$

$$z_{6m+2i+1}=z_{2i-5}\prod_{p=0}^{m}\frac{fz_{-1}\widetilde{s}_{3p+i-1}+x_{-2}\widetilde{s}_{3p+i}}{fz_{-1}\widetilde{s}_{3p+i}+x_{-2}\widetilde{s}_{3p+i+1}}\frac{bx_0s_{3p+i-2}+y_{-1}s_{3p+i-1}}{bx_0s_{3p+i-1}+y_{-1}s_{3p+i}}$$
$$\times\frac{dy_{-1}\widehat{s}_{3p+i-2}+z_{-2}\widehat{s}_{3p+i-1}}{dy_{-1}\widehat{s}_{3p+i-1}+z_{-2}\widehat{s}_{3p+i}}\frac{fz_0\widetilde{s}_{3p+i-3}+x_{-1}\widetilde{s}_{3p+i-2}}{fz_0\widetilde{s}_{3p+i-2}+x_{-1}\widetilde{s}_{3p+i-1}} \tag{2.38}$$
$$\times\frac{bx_{-1}s_{3p+i-3}+y_{-2}s_{3p+i-2}}{bx_{-1}s_{3p+i-2}+y_{-2}s_{3p+i-1}}\frac{dy_0\widehat{s}_{3p+i-4}+z_{-1}\widehat{s}_{3p+i-3}}{dy_0\widehat{s}_{3p+i-3}+z_{-1}\widehat{s}_{3p+i-2}},$$

for $m\in\mathbb{N}_0$, $i=\overline{2,4}$.

Theorem 2.1. *The forbidden set of the initial values for system (1.8) is given by the set*

$$\mathbb{F}=\bigcup_{m\in\mathbb{N}_0}\bigcup_{i=0}^{1}\Big\{\frac{y_{i-2}}{x_{i-1}}=\widehat{f}^{-m-1}\left(-\frac{b}{a}\right),\quad \frac{z_{i-2}}{y_{i-1}}=g^{-m-1}\left(-\frac{d}{c}\right),$$
$$\frac{x_{i-2}}{z_{i-1}}=h^{-m-1}\left(-\frac{f}{e}\right)\Big\}\bigcup\bigcup_{j=0}^{2}\Big\{\left(\vec{x}_{-(2,0)},\vec{y}_{-(2,0)},\vec{z}_{-(2,0)}\right)\in\mathbb{R}^9: \tag{2.39}$$
$$x_{-j}=0\ \text{or}\ y_{-j}=0\ \text{or}\ z_{-j}=0\Big\},$$

where $\vec{x}_{-(2,0)}=(x_{-2},x_{-1},x_0)$, $\vec{y}_{-(2,0)}=(y_{-2},y_{-1},y_0)$, $\vec{z}_{-(2,0)}=(z_{-2},z_{-1},z_0)$,

Proof. At the beginning of Section 2, we have obtained that the set

$$\bigcup_{j=0}^{2}\Big\{\left(\vec{x}_{-(2,0)},\vec{y}_{-(2,0)},\vec{z}_{-(2,0)}\right)\in\mathbb{R}^9:\ x_{-j}=0\ \text{or}\ y_{-j}=0\ \text{or}\ z_{-j}=0\Big\},$$

where $\vec{x}_{-(2,0)}=(x_{-2},x_{-1},x_0)$, $\vec{y}_{-(2,0)}=(y_{-2},y_{-1},y_0)$, $\vec{z}_{-(2,0)}=(z_{-2},z_{-1},z_0)$, belongs to the forbidden set of the initial values for system (1.8). If $x_{-j}\neq 0$, $y_{-j}\neq 0$ and $z_{-j}\neq 0$, $j\in\{0,1,2\}$, then system (1.8) is undefined if and only if

$$bx_{n-1}+ay_{n-2}=0,\ dy_{n-1}+cz_{n-2}=0,\ fz_{n-1}+ex_{n-2}=0,\ n\in\mathbb{N}_0.$$

By taking into account the change of variables (2.2), we can write the corresponding conditions

$$u_{n-1}=-\frac{b}{a},\ v_{n-1}=-\frac{d}{c}\ \text{and}\ \ t_{n-1}=-\frac{f}{e},\ n\in\mathbb{N}_0. \tag{2.40}$$

Therefore, we can determine the forbidden set of the initial values for system (1.8) by using system (2.3). We know that the statements

$$u_{2m+i}=\widehat{f}^{m+1}\left(u_{i-2}\right), \tag{2.41}$$
$$v_{2m+i}=g^{m+1}\left(v_{i-2}\right), \tag{2.42}$$
$$t_{2m+i}=h^{m+1}\left(t_{i-2}\right), \tag{2.43}$$

where $m \in \mathbb{N}_0$, $i \in \{1,2\}$, $\widehat{f}(x) = \frac{ax+b}{x}$, $g(x) = \frac{cx+d}{x}$ and $h(x) = \frac{ex+f}{x}$, characterize the solutions of system (2.3). By using the conditions (2.40) and the statements (2.41)-(2.43), we have

$$u_{i-2} = \widehat{f}^{-m-1}\left(-\frac{b}{a}\right), \tag{2.44}$$

$$v_{i-2} = g^{-m-1}\left(-\frac{d}{c}\right), \tag{2.45}$$

$$t_{i-2} = h^{-m-1}\left(-\frac{f}{e}\right), \tag{2.46}$$

where $m \in \mathbb{N}_0$, $i \in \{1,2\}$ and $abcdef \neq 0$. This means that if one of the conditions in (2.44)-(2.46) holds, then $m-$th iteration or $(m+1)-$th iteration in system (1.8) can not be calculated. Consequently, desired result follows from (2.39). □

3. Some applications

In this section, we will give some applications for some special cases of the coefficients of the system (1.8).

Corollary 3.1. *Let $(x_n, y_n, z_n)_{n \geq -2}$ be a well-defined solution to the following system*

$$x_{n+1} = \frac{y_n y_{n-2}}{x_{n-1} + y_{n-2}},\ y_{n+1} = \frac{z_n z_{n-2}}{y_{n-1} + z_{n-2}},\ z_{n+1} = \frac{x_n x_{n-2}}{z_{n-1} + x_{n-2}},\ n \in \mathbb{N}_0. \tag{3.1}$$

Then

$$\begin{aligned} x_{6m+2i} = x_{2i-6} \prod_{p=0}^{m} & \frac{x_0 F_{3p+i-2} + y_{-1} F_{3p+i-1}}{x_0 F_{3p+i-1} + y_{-1} F_{3p+i}} \frac{y_{-1} F_{3p+i-2} + z_{-2} F_{3p+i-1}}{y_{-1} F_{3p+i-1} + z_{-2} F_{3p+i}} \\ & \times \frac{z_0 F_{3p+i-3} + x_{-1} F_{3p+i-2}}{z_0 F_{3p+i-2} + x_{-1} F_{3p+i-1}} \frac{x_{-1} F_{3p+i-3} + y_{-2} F_{3p+i-2}}{x_{-1} F_{3p+i-2} + y_{-2} F_{3p+i-1}} \\ & \times \frac{y_0 F_{3p+i-4} + z_{-1} F_{3p+i-3}}{y_0 F_{3p+i-3} + z_{-1} F_{3p+i-2}} \frac{z_{-1} F_{3p+i-4} + x_{-2} F_{3p+i-3}}{z_{-1} F_{3p+i-3} + x_{-2} F_{3p+i-2}}, \end{aligned} \tag{3.2}$$

$$\begin{aligned} x_{6m+2i+1} = x_{2i-5} \prod_{p=0}^{m} & \frac{x_{-1} F_{3p+i-1} + y_{-2} F_{3p+i}}{x_{-1} F_{3p+i} + y_{-2} F_{3p+i+1}} \frac{y_0 F_{3p+i-2} + z_{-1} F_{3p+i-1}}{y_0 F_{3p+i-1} + z_{-1} F_{3p+i}} \\ & \times \frac{z_{-1} F_{3p+i-2} + x_{-2} F_{3p+i-1}}{z_{-1} F_{3p+i-1} + x_{-2} F_{3p+i}} \frac{x_0 F_{3p+i-3} + y_{-1} F_{3p+i-2}}{x_0 F_{3p+i-2} + y_{-1} F_{3p+i-1}} \\ & \times \frac{y_{-1} F_{3p+i-3} + z_{-2} F_{3p+i-2}}{y_{-1} F_{3p+i-2} + z_{-2} F_{3p+i-1}} \frac{z_0 F_{3p+i-4} + x_{-1} F_{3p+i-3}}{z_0 F_{3p+i-3} + x_{-1} F_{3p+i-2}}, \end{aligned} \tag{3.3}$$

$$\begin{aligned} y_{6m+2i} = y_{2i-6} \prod_{p=0}^{m} & \frac{y_0 F_{3p+i-2} + z_{-1} F_{3p+i-1}}{y_0 F_{3p+i-1} + z_{-1} F_{3p+i}} \frac{z_{-1} F_{3p+i-2} + x_{-2} F_{3p+i-1}}{z_{-1} F_{3p+i-1} + x_{-2} F_{3p+i}} \\ & \times \frac{x_0 F_{3p+i-3} + y_{-1} F_{3p+i-2}}{x_0 F_{3p+i-2} + y_{-1} F_{3p+i-1}} \frac{y_{-1} F_{3p+i-3} + z_{-2} F_{3p+i-2}}{y_{-1} F_{3p+i-2} + z_{-2} F_{3p+i-1}} \\ & \times \frac{z_0 F_{3p+i-4} + x_{-1} F_{3p+i-3}}{z_0 F_{3p+i-3} + x_{-1} F_{3p+i-2}} \frac{x_{-1} F_{3p+i-4} + y_{-2} F_{3p+i-3}}{x_{-1} F_{3p+i-3} + y_{-2} F_{3p+i-2}}, \end{aligned} \tag{3.4}$$

$$\begin{aligned} y_{6m+2i+1} = y_{2i-5} \prod_{p=0}^{m} & \frac{y_{-1}F_{3p+i-1} + z_{-2}F_{3p+i}}{y_{-1}F_{3p+i} + z_{-2}F_{3p+i+1}} \frac{z_0F_{3p+i-2} + x_{-1}F_{3p+i-1}}{z_0F_{3p+i-1} + x_{-1}F_{3p+i}} \\ & \times \frac{x_{-1}F_{3p+i-2} + y_{-2}F_{3p+i-1}}{x_{-1}F_{3p+i-1} + y_{-2}F_{3p+i}} \frac{y_0F_{3p+i-3} + z_{-1}F_{3p+i-2}}{y_0F_{3p+i-2} + z_{-1}F_{3p+i-1}} \\ & \times \frac{z_{-1}F_{3p+i-3} + x_{-2}F_{3p+i-2}}{z_{-1}F_{3p+i-2} + x_{-2}F_{3p+i-1}} \frac{x_0F_{3p+i-4} + y_{-1}F_{3p+i-3}}{x_0F_{3p+i-3} + y_{-1}F_{3p+i-2}}, \end{aligned} \tag{3.5}$$

$$\begin{aligned} z_{6m+2i} = z_{2i-6} \prod_{p=0}^{m} & \frac{z_0F_{3p+i-2} + x_{-1}F_{3p+i-1}}{z_0F_{3p+i-1} + x_{-1}F_{3p+i}} \frac{x_{-1}F_{3p+i-2} + y_{-2}F_{3p+i-1}}{x_{-1}F_{3p+i-1} + y_{-2}F_{3p+i}} \\ & \times \frac{y_0F_{3p+i-3} + z_{-1}F_{3p+i-2}}{y_0F_{3p+i-2} + z_{-1}F_{3p+i-1}} \frac{z_{-1}F_{3p+i-3} + x_{-2}F_{3p+i-2}}{z_{-1}F_{3p+i-2} + x_{-2}F_{3p+i-1}} \\ & \times \frac{x_0F_{3p+i-4} + y_{-1}F_{3p+i-3}}{x_0F_{3p+i-3} + y_{-1}F_{3p+i-2}} \frac{y_{-1}F_{3p+i-4} + z_{-2}F_{3p+i-3}}{y_{-1}F_{3p+i-3} + z_{-2}F_{3p+i-2}}, \end{aligned} \tag{3.6}$$

$$\begin{aligned} z_{6m+2i+1} = z_{2i-5} \prod_{p=0}^{m} & \frac{z_{-1}F_{3p+i-1} + x_{-2}F_{3p+i}}{z_{-1}F_{3p+i} + x_{-2}F_{3p+i+1}} \frac{x_0F_{3p+i-2} + y_{-1}F_{3p+i-1}}{x_0F_{3p+i-1} + y_{-1}F_{3p+i}} \\ & \times \frac{y_{-1}F_{3p+i-2} + z_{-2}F_{3p+i-1}}{y_{-1}F_{3p+i-1} + z_{-2}F_{3p+i}} \frac{z_0F_{3p+i-3} + x_{-1}F_{3p+i-2}}{z_0F_{3p+i-2} + x_{-1}F_{3p+i-1}} \\ & \times \frac{x_{-1}F_{3p+i-3} + y_{-2}F_{3p+i-2}}{x_{-1}F_{3p+i-2} + y_{-2}F_{3p+i-1}} \frac{y_0F_{3p+i-4} + z_{-1}F_{3p+i-3}}{y_0F_{3p+i-3} + z_{-1}F_{3p+i-2}}, \end{aligned} \tag{3.7}$$

for $m \in \mathbb{N}_0$, $i = \overline{2,4}$, *where* $(F_m)_{m\geq -1}$ *is the solution to the following difference equation*

$$F_{m+1} = F_m + F_{m-1},\ m \in \mathbb{N}_0,$$

satisfying the initial conditions $F_{-1} = 0$, $F_0 = 1$. *The sequence* $(F_m)_{m\geq -1}$ *is called the well-known Fibonacci sequence in literature.*

Proof. System (3.1) is obtained from system (1.8) with $a = b = c = d = e = f = 1$. For these values of parameters a, b, c, d, e, f in equation (2.6), equation (2.7) and equation (2.8) are the same, that is

$$w_{m+1} = w_m + w_{m-1},\ m \in \mathbb{N}_0.$$

Hence, the sequences $(s_m)_{m\geq -1}$, $(\widehat{s}_m)_{m\geq -1}$ and $(\widetilde{s}_m)_{m\geq -1}$ satisfying conditions (2.9)-(2.11) are the same and so we have

$$s_m = \widehat{s}_m = \widetilde{s}_m = F_m,\ m \geq -1. \tag{3.8}$$

By using (3.8) in formulas (2.33)-(2.38), formulas (3.2)-(3.7) follow. □

Corollary 3.2. *Let* $(x_n, y_n, z_n)_{n\geq -2}$ *be a well-defined solution to the following system*

$$x_{n+1} = \frac{y_n y_{n-2}}{x_{n-1} - y_{n-2}},\ y_{n+1} = \frac{z_n z_{n-2}}{-y_{n-1} + z_{n-2}},\ z_{n+1} = \frac{x_n x_{n-2}}{-z_{n-1} - x_{n-2}},\ n \in \mathbb{N}_0. \tag{3.9}$$

Then

$$x_{6m+4} = x_{-2} \prod_{p=0}^{m} \frac{x_0F_{3p} - y_{-1}F_{3p+1}}{-x_0F_{3p+1} + y_{-1}F_{3p+2}} \frac{-y_{-1} + z_{-2}}{-y_{-1}} \frac{x_{-1}}{-z_0 - x_{-1}}$$

$$\times \frac{-x_{-1}F_{3p-1}+y_{-2}F_{3p}}{x_{-1}F_{3p}-y_{-2}F_{3p+1}} \frac{y_0}{x_{-2}}, \tag{3.10}$$

$$x_{6m+5} = x_{-1} \prod_{p=0}^{m} \frac{-x_{-1}F_{3p+1}+y_{-2}F_{3p+2}}{x_{-1}F_{3p+2}-y_{-2}F_{3p+3}} \frac{-y_0+z_{-1}}{-y_0} \frac{-z_{-1}-x_{-2}}{z_{-1}}$$
$$\times \frac{-x_0F_{3p-1}+y_{-1}F_{3p}}{x_0F_{3p}-y_{-1}F_{3p+1}} \frac{z_{-2}}{-y_{-1}+z_{-2}} \frac{z_0}{x_{-1}}, \tag{3.11}$$

$$x_{6m+6} = x_0 \prod_{p=0}^{m} \frac{-x_0F_{3p+1}+y_{-1}F_{3p+2}}{x_0F_{3p+2}-y_{-1}F_{3p+3}} \frac{y_{-1}}{z_{-2}} \frac{-z_0-x_{-1}}{z_0}$$
$$\times \frac{x_{-1}F_{3p}-y_{-2}F_{3p+1}}{-x_{-1}F_{3p+1}+y_{-2}F_{3p+2}} \frac{z_{-1}}{-y_0+z_{-1}} \frac{x_{-2}}{-z_{-1}-x_{-2}}, \tag{3.12}$$

$$x_{6m+7} = x_1 \prod_{p=0}^{m} \frac{x_{-1}F_{3p+2}-y_{-2}F_{3p+3}}{-x_{-1}F_{3p+3}+y_{-2}F_{3p+4}} \frac{y_0}{z_{-1}} \frac{z_{-1}}{x_{-2}}$$
$$\times \frac{x_0F_{3p}-y_{-1}F_{3p+1}}{-x_0F_{3p+1}+y_{-1}F_{3p+2}} \frac{-y_{-1}+z_{-2}}{-y_{-1}} \frac{x_{-1}}{-z_0-x_{-1}}, \tag{3.13}$$

$$x_{6m+8} = x_2 \prod_{p=0}^{m} \frac{x_0F_{3p+2}-y_{-1}F_{3p+3}}{-x_0F_{3p+3}+y_{-1}F_{3p+4}} \frac{z_{-2}}{-y_{-1}+z_{-2}} \frac{z_0}{x_{-1}}$$
$$\times \frac{-x_{-1}F_{3p+1}+y_{-2}F_{3p+2}}{x_{-1}F_{3p+2}-y_{-2}F_{3p+3}} \frac{-y_0+z_{-1}}{-y_0} \frac{-z_0-x_{-2}}{z_{-1}}, \tag{3.14}$$

$$x_{6m+9} = x_3 \prod_{p=0}^{m} \frac{-x_{-1}F_{3p+3}+y_{-2}F_{3p+4}}{x_{-1}F_{3p+4}-y_{-2}F_{3p+5}} \frac{z_{-1}}{-y_0+z_{-1}} \frac{x_{-2}}{-z_{-1}-x_{-2}}$$
$$\times \frac{-x_0F_{3p+1}+y_{-1}F_{3p+2}}{x_0F_{3p+2}-y_{-1}F_{3p+3}} \frac{y_{-1}}{z_{-2}} \frac{-z_0-x_{-1}}{z_0}, \tag{3.15}$$

$$y_{6m+4} = y_{-2} \prod_{p=0}^{m} \frac{-y_0+z_{-1}}{-y_0} \frac{-z_{-1}-x_{-2}}{z_{-1}} \frac{-x_0F_{3p-1}+y_{-1}F_{3p}}{x_0F_{3p}-y_{-1}F_{3p+1}}$$
$$\times \frac{z_{-2}}{-y_{-1}+z_{-2}} \frac{z_0}{x_{-1}} \frac{x_{-1}F_{3p-2}-y_{-2}F_{3p-1}}{-x_{-1}F_{3p-1}+y_{-2}F_{3p}}, \tag{3.16}$$

$$y_{6m+5} = y_{-1} \prod_{p=0}^{m} \frac{y_{-1}}{z_{-2}} \frac{-z_0-x_{-1}}{z_0} \frac{x_{-1}F_{3p}-y_{-2}F_{3p+1}}{-x_{-1}F_{3p+1}+y_{-2}F_{3p+2}}$$
$$\times \frac{z_{-1}}{-y_0+z_{-1}} \frac{x_{-2}}{-z_{-1}-x_{-2}} \frac{x_0F_{3p-2}-y_{-1}F_{3p-1}}{-x_0F_{3p-1}+y_{-1}F_{3p}}, \tag{3.17}$$

$$y_{6m+6} = y_0 \prod_{p=0}^{m} \frac{y_0}{z_{-1}} \frac{z_{-1}}{x_{-2}} \frac{x_0F_{3p}-y_{-1}F_{3p+1}}{-x_0F_{3p+1}+y_{-1}F_{3p+2}}$$
$$\times \frac{-y_{-1}+z_{-2}}{-y_{-1}} \frac{x_{-1}}{-z_0-x_{-1}} \frac{-x_{-1}F_{3p-1}+y_{-2}F_{3p}}{x_{-1}F_{3p}-y_{-2}F_{3p+1}}, \tag{3.18}$$

$$y_{6m+7} = y_1 \prod_{p=0}^{m} \frac{z_{-2}}{-y_{-1}+z_{-2}} \frac{z_0}{x_{-1}} \frac{-x_{-1}F_{3p+1}+y_{-2}F_{3p+2}}{x_{-1}F_{3p+2}-y_{-2}F_{3p+3}}$$
$$\times \frac{-y_0+z_{-1}}{-y_0} \frac{-z_{-1}-x_{-2}}{z_{-1}} \frac{-x_0F_{3p-1}+y_{-1}F_{3p}}{x_0F_{3p}-y_{-1}F_{3p+1}}, \tag{3.19}$$

$$y_{6m+8} = y_2 \prod_{p=0}^{m} \frac{z_{-1}}{-y_0 + z_{-1}} \frac{x_{-2}}{-z_{-1} - x_{-2}} \frac{-x_0 F_{3p+1} + y_{-1} F_{3p+2}}{x_0 F_{3p+2} - y_{-1} F_{3p+3}} \times \frac{y_{-1}}{z_{-2}} \frac{-z_0 - x_{-1}}{z_0} \frac{x_{-1} F_{3p} - y_{-2} F_{3p+1}}{-x_{-1} F_{3p+1} + y_{-2} F_{3p+2}}, \tag{3.20}$$

$$y_{6m+9} = y_3 \prod_{p=0}^{m} \frac{-y_{-1} + z_{-2}}{-y_{-1}} \frac{x_{-1}}{-z_0 - x_{-1}} \frac{x_{-1} F_{3p+2} - y_{-2} F_{3p+3}}{-x_{-1} F_{3p+3} + y_{-2} F_{3p+4}} \times \frac{y_0}{z_{-1}} \frac{z_{-1}}{x_{-2}} \frac{x_0 F_{3p} - y_{-1} F_{3p+1}}{-x_0 F_{3p+1} + y_{-1} F_{3p+2}}, \tag{3.21}$$

$$z_{6m+4} = z_{-2} \prod_{p=0}^{m} \frac{-z_0 - x_{-1}}{z_0} \frac{x_{-1} F_{3p} - y_{-2} F_{3p+1}}{-x_{-1} F_{3p+1} + y_{-2} F_{3p+2}} \frac{z_{-1}}{-y_0 + z_{-1}} \times \frac{x_{-2}}{-z_{-1} - x_{-2}} \frac{x_0 F_{3p-2} - y_{-1} F_{3p-1}}{-x_0 F_{3p-1} + y_{-1} F_{3p}} \frac{y_{-1}}{z_{-2}}, \tag{3.22}$$

$$z_{6m+5} = z_{-1} \prod_{p=0}^{m} \frac{z_{-1}}{x_{-2}} \frac{x_0 F_{3p} - y_{-1} F_{3p+1}}{-x_0 F_{3p+1} + y_{-1} F_{3p+2}} \frac{-y_{-1} + z_{-2}}{-y_{-1}} \times \frac{x_{-1}}{-z_0 - x_{-1}} \frac{-x_{-1} F_{3p-1} + y_{-2} F_{3p}}{x_{-1} F_{3p} - y_{-2} F_{3p+1}} \frac{y_0}{z_{-1}}, \tag{3.23}$$

$$z_{6m+6} = z_0 \prod_{p=0}^{m} \frac{z_0}{x_{-1}} \frac{-x_{-1} F_{3p+1} + y_{-2} F_{3p+2}}{x_{-1} F_{3p+2} - y_{-2} F_{3p+3}} \frac{-y_0 + z_{-1}}{-y_0} \times \frac{-z_{-1} - x_{-2}}{z_{-1}} \frac{-x_0 F_{3p-1} + y_{-1} F_{3p}}{x_0 F_{3p} - y_{-1} F_{3p+1}} \frac{z_{-2}}{-y_{-1} + z_{-2}}, \tag{3.24}$$

$$z_{6m+7} = z_1 \prod_{p=0}^{m} \frac{x_{-2}}{-z_{-1} - x_{-2}} \frac{-x_0 F_{3p+1} + y_{-1} F_{3p+2}}{x_0 F_{3p+2} - y_{-1} F_{3p+3}} \frac{y_{-1}}{z_{-2}} \times \frac{-z_0 - x_{-1}}{z_0} \frac{x_{-1} F_{3p} - y_{-2} F_{3p+1}}{-x_{-1} F_{3p+1} + y_{-2} F_{3p+2}} \frac{z_{-1}}{-y_0 + z_{-1}}, \tag{3.25}$$

$$z_{6m+8} = z_2 \prod_{p=0}^{m} \frac{x_{-1}}{-z_0 - x_{-1}} \frac{x_{-1} F_{3p+2} - y_{-2} F_{3p+3}}{-x_{-1} F_{3p+3} + y_{-2} F_{3p+4}} \times \frac{y_0}{x_{-2}} \frac{x_0 F_{3p} - y_{-1} F_{3p+1}}{-x_0 F_{3p+1} + y_{-1} F_{3p+2}} \frac{-y_{-1} + z_{-2}}{-y_{-1}}, \tag{3.26}$$

$$z_{6m+9} = z_3 \prod_{p=0}^{m} \frac{-z_{-1} - x_{-2}}{z_{-1}} \frac{x_0 F_{3p+2} - y_{-1} F_{3p+3}}{-x_0 F_{3p+3} + y_{-1} F_{3p+4}} \frac{z_{-2}}{-y_{-1} + z_{-2}} \times \frac{z_0}{x_{-1}} \frac{-x_{-1} F_{3p+1} + y_{-2} F_{3p+2}}{x_{-1} F_{3p+2} - y_{-2} F_{3p+3}} \frac{-y_0 + z_{-1}}{-y_0}, \tag{3.27}$$

for $m \in \mathbb{N}_0$, *where* $x_1 = \frac{y_0 y_{-2}}{x_{-1} - y_{-2}}$, $x_2 = \frac{z_0 z_{-2} y_{-1}}{(x_0 - y_{-1})(z_{-2} - y_{-1})}$, $x_3 = \frac{x_{-2} x_0 z_{-1}(x_{-1} - y_{-2})}{(x_{-2} + z_{-1})(z_{-1} - y_0)(x_{-1} - 2y_{-2})}$, $y_1 = \frac{z_0 z_{-2}}{z_{-2} - y_{-1}}$, $y_2 = \frac{x_0 x_{-2} z_{-1}}{(y_0 - z_{-1})(x_{-2} + z_{-1})}$, $y_3 = \frac{y_{-2} y_0 x_{-1}(z_{-2} - y_{-1})}{(x_{-1} - y_{-2})(x_{-1} + z_0) y_{-1}}$, $z_1 = \frac{x_0 x_{-2}}{-x_{-2} - z_{-1}}$, $z_2 = \frac{y_0 y_{-2} x_{-1}}{(z_0 + x_{-1})(y_{-2} - x_{-1})}$ *and* $z_3 = \frac{z_{-2} z_0 y_{-1}(x_{-2} + z_{-1})}{(y_{-1} - z_{-2})(x_0 - y_{-1}) z_{-1}}$.

Proof. System (3.9) is obtained from system (1.8) with $a = d = e = f = -1$, $b = c = 1$. For these values of parameters a, b, equation (2.6) becomes

$$w_{m+1} = -w_m + w_{m-1}, \; m \in \mathbb{N}_0. \tag{3.28}$$

Let

$$w_m = (-1)^m k_m, \ m \geq -1. \tag{3.29}$$

Employing (3.29) in (3.28), we get

$$k_{m+1} = k_m + k_{m-1}, \ m \in \mathbb{N}_0. \tag{3.30}$$

By considering (2.6) with conditions $a = -1$, $b = 1$ and (3.29) we obtain

$$s_{-1}^k = 0, \text{ and } s_0^k = 1. \tag{3.31}$$

From (3.31) and since s_m^k is a solution to equation (3.30), we have

$$s_m^k = F_m, \quad m \geq -1,$$

from which along with (3.29) it follows that

$$s_m = (-1)^m F_m, \quad m \geq -1. \tag{3.32}$$

For these values of parameters c, d, equation (2.7) becomes

$$\widehat{w}_{m+1} = \widehat{w}_m - \widehat{w}_{m-1}, \ m \in \mathbb{N}_0. \tag{3.33}$$

The solution $\widehat{s}_m$ to equation (3.33) satisfying the initial conditions in (2.10) is equal to

$$\widehat{s}_m = \frac{\widehat{\lambda}_1^{m+1} - \widehat{\lambda}_2^{m+1}}{\widehat{\lambda}_1 - \widehat{\lambda}_2}, \quad m \geq -1, \tag{3.34}$$

where

$$\widehat{\lambda}_{1,2} = \cos\frac{\pi}{3} \pm i \sin\frac{\pi}{3}$$

after some calculation in (3.34), it follows that

$$\widehat{s}_m = \frac{2}{\sqrt{3}} \sin\frac{(m+1)\pi}{3}, \ m \geq -1. \tag{3.35}$$

Formula (3.35) shows that the sequence $\widehat{s}_m$ is six periodic. Namely, we have

$$\widehat{s}_{6m-1} = \widehat{s}_{6m+2} = 0, \tag{3.36}$$
$$\widehat{s}_{6m} = \widehat{s}_{6m+1} = 1, \tag{3.37}$$
$$\widehat{s}_{6m+3} = \widehat{s}_{6m+4} = -1, \tag{3.38}$$

for $m \in \mathbb{N}_0$. Equalities (3.36)-(3.38) can be written as follows

$$\widehat{s}_{3m-1} = 0, \tag{3.39}$$
$$\widehat{s}_{3m} = (-1)^m, \tag{3.40}$$
$$\widehat{s}_{3m+1} = (-1)^m, \tag{3.41}$$

for $m \in \mathbb{N}_0$. For these values of parameters e, f, equation (2.8) becomes

$$\widetilde{w}_{m+1} = -\widetilde{w}_m - \widetilde{w}_{m-1}, \ m \in \mathbb{N}_0. \tag{3.42}$$

The solution $\widetilde{s}_m$ to equation (3.42) satisfying the initial conditions in (2.11) is equal to

$$\widetilde{s}_m = \frac{\widetilde{\lambda}_1^{m+1} - \widetilde{\lambda}_2^{m+1}}{\widetilde{\lambda}_1 - \widetilde{\lambda}_2}, \quad m \geq -1, \tag{3.43}$$

where

$$\widetilde{\lambda}_{1,2}=\cos\frac{2\pi}{3}\pm i\sin\frac{2\pi}{3},$$

after some calculation in (3.43), it follows that

$$\widetilde{s}_m=\frac{2}{\sqrt{3}}\sin\frac{2\,(m+1)\,\pi}{3},\ m\geq -1. \tag{3.44}$$

Formula (3.44) shows that the sequence $\widetilde{s}_m$ is three periodic. Namely, we have

$$\widetilde{s}_{3m}=1, \tag{3.45}$$
$$\widetilde{s}_{3m+1}=-1, \tag{3.46}$$
$$\widetilde{s}_{3m+2}=0, \tag{3.47}$$

for $m\in\mathbb{N}_0$. By using (3.32), (3.39)-(3.41), (3.45)-(3.47), in formulas (2.33)-(2.38), after some simple calculations are obtained formulas (3.10)-(3.27). □

Corollary 3.3. *Let $(x_n,y_n,z_n)_{n\geq -2}$ be a well-defined solution to the following system*

$$x_{n+1}=\frac{y_ny_{n-2}}{-x_{n-1}+3y_{n-2}},\ y_{n+1}=\frac{z_nz_{n-2}}{y_{n-1}+2z_{n-2}},\ z_{n+1}=\frac{x_nx_{n-2}}{z_{n-1}+4x_{n-2}},\ n\in\mathbb{N}_0. \tag{3.48}$$

Then

$$\begin{aligned}x_{6m+2i}=x_{2i-6}\prod_{p=0}^{m}&\frac{-x_0F_{6p+2i-3}+y_{-1}F_{6p+2i-1}}{-x_0F_{6p+2i-1}+y_{-1}F_{6p+2i+1}}\frac{y_{-1}P_{3p+i-2}+z_{-2}P_{3p+i-1}}{y_{-1}P_{3p+i-1}+z_{-2}P_{3p+i}}\\&\times\frac{z_0F_{9p+3i-7}+x_{-1}F_{9p+3i-4}}{z_0F_{9p+3i-4}+x_{-1}F_{9p+3i-1}}\frac{-x_{-1}F_{6p+2i-5}+y_{-2}F_{6p+2i-3}}{-x_{-1}F_{6p+2i-3}+y_{-2}F_{6p+2i-1}}\\&\times\frac{y_0P_{3p+i-4}+z_{-1}P_{3p+i-3}}{y_0P_{3p+i-3}+z_{-1}P_{3p+i-2}}\frac{z_{-1}F_{9p+3i-10}+x_{-2}F_{9p+3i-7}}{z_{-1}F_{9p+3i-7}+x_{-2}F_{9p+3i-4}},\end{aligned} \tag{3.49}$$

$$\begin{aligned}x_{6m+2i+1}=x_{2i-5}\prod_{p=0}^{m}&\frac{-x_{-1}F_{6p+2i-1}+y_{-2}F_{6p+2i+1}}{-x_{-1}F_{6p+2i+1}+y_{-2}F_{6p+2i+3}}\frac{y_0P_{3p+i-2}+z_{-1}P_{3p+i-1}}{y_0P_{3p+i-1}+z_{-1}P_{3p+i}}\\&\times\frac{z_{-1}F_{9p+3i-4}+x_{-2}F_{9p+3i-1}}{z_{-1}F_{9p+3i-1}+x_{-2}F_{9p+3i+2}}\frac{-x_0F_{6p+2i-5}+y_{-1}F_{6p+2i-3}}{-x_0F_{6p+2i-3}+y_{-1}F_{6p+2i-1}}\\&\times\frac{y_{-1}P_{3p+i-3}+z_{-2}P_{3p+i-2}}{y_{-1}P_{3p+i-2}+z_{-2}P_{3p+i-1}}\frac{z_0F_{9p+3i-10}+x_{-1}F_{9p+3i-7}}{z_0F_{9p+3i-7}+x_{-1}F_{9p+3i-4}},\end{aligned} \tag{3.50}$$

$$\begin{aligned}y_{6m+2i}=y_{2i-6}\prod_{p=0}^{m}&\frac{y_0P_{3p+i-2}+z_{-1}P_{3p+i-1}}{y_0P_{3p+i-1}+z_{-1}P_{3p+i}}\frac{z_{-1}F_{9p+3i-4}+x_{-2}F_{9p+3i-1}}{z_{-1}F_{9p+3i-1}+x_{-2}F_{9p+3i+2}}\\&\times\frac{-x_0F_{6p+2i-5}+y_{-1}F_{6p+2i-3}}{-x_0F_{6p+2i-3}+y_{-1}F_{6p+2i-1}}\frac{y_{-1}P_{3p+i-3}+z_{-2}P_{3p+i-2}}{y_{-1}P_{3p+i-2}+z_{-2}P_{3p+i-1}}\\&\times\frac{z_0F_{9p+3i-10}+x_{-1}F_{9p+3i-7}}{z_0F_{9p+3i-7}+x_{-1}F_{9p+3i-4}}\frac{-x_{-1}F_{6p+2i-7}+y_{-2}F_{6p+2i-5}}{-x_{-1}F_{6p+2i-5}+y_{-2}F_{6p+2i-3}},\end{aligned} \tag{3.51}$$

$$y_{6m+2i+1} = y_{2i-5} \prod_{p=0}^{m} \frac{y_{-1}P_{3p+i-1} + z_{-2}P_{3p+i}}{y_{-1}P_{3p+i} + z_{-2}P_{3p+i+1}} \frac{z_0F_{9p+3i-4} + x_{-1}F_{9p+3i-1}}{z_0F_{9p+3i-1} + x_{-1}F_{9p+3i+2}} \times \frac{-x_{-1}F_{6p+2i-3} + y_{-2}F_{6p+2i-1}}{-x_{-1}F_{6p+2i-1} + y_{-2}F_{6p+2i+1}} \frac{y_0P_{3p+i-3} + z_{-1}P_{3p+i-2}}{y_0P_{3p+i-2} + z_{-1}P_{3p+i-1}} \times \frac{z_{-1}F_{9p+3i-7} + x_{-2}F_{9p+3i-4}}{z_{-1}F_{9p+3i-4} + x_{-2}F_{9p+3i-1}} \frac{-x_0F_{6p+2i-7} + y_{-1}F_{6p+2i-5}}{-x_0F_{6p+2i-5} + y_{-1}F_{6p+2i-3}}, \tag{3.52}$$

$$z_{6m+2i} = z_{2i-6} \prod_{p=0}^{m} \frac{z_0F_{9p+3i-4} + x_{-1}F_{9p+3i-1}}{z_0F_{9p+3i-1} + x_{-1}F_{9p+3i+2}} \frac{-x_{-1}F_{6p+2i-3} + y_{-2}F_{6p+2i-1}}{-x_{-1}F_{6p+2i-1} + y_{-2}F_{6p+2i+1}} \times \frac{y_0P_{3p+i-3} + z_{-1}P_{3p+i-2}}{y_0P_{3p+i-2} + z_{-1}P_{3p+i-1}} \frac{z_{-1}F_{9p+3i-7} + x_{-2}F_{9p+3i-4}}{z_{-1}F_{9p+3i-4} + x_{-2}F_{9p+3i-1}} \times \frac{-x_0F_{6p+2i-7} + y_{-1}F_{6p+2i-5}}{-x_0F_{6p+2i-5} + y_{-1}F_{6p+2i-3}} \frac{y_{-1}P_{3p+i-4} + z_{-2}P_{3p+i-3}}{y_{-1}P_{3p+i-3} + z_{-2}P_{3p+i-2}}, \tag{3.53}$$

$$z_{6m+2i+1} = z_{2i-5} \prod_{p=0}^{m} \frac{z_{-1}F_{9p+3i-1} + x_{-2}F_{9p+3i+2}}{z_{-1}F_{9p+3i+2} + x_{-2}F_{9p+3i+5}} \frac{-x_0F_{6p+2i-3} + y_{-1}F_{6p+2i-1}}{-x_0F_{6p+2i-1} + y_{-1}F_{6p+2i+1}} \times \frac{y_{-1}P_{3p+i-2} + z_{-2}P_{3p+i-1}}{y_{-1}P_{3p+i-1} + z_{-2}P_{3p+i}} \frac{z_0F_{9p+3i-7} + x_{-1}F_{9p+3i-4}}{z_0F_{9p+3i-4} + x_{-1}F_{9p+3i-1}} \times \frac{-x_{-1}F_{6p+2i-5} + y_{-2}F_{6p+2i-3}}{-x_{-1}F_{6p+2i-3} + y_{-2}F_{6p+2i-1}} \frac{y_0P_{3p+i-4} + z_{-1}P_{3p+i-3}}{y_0P_{3p+i-3} + z_{-1}P_{3p+i-2}}, \tag{3.54}$$

for $m \in \mathbb{N}_0$, $i = \overline{2,4}$, *where* $(P_m)_{m\geq -1}$ *is the solution to the following difference equation*

$$P_{m+1} = 2P_m + P_{m-1},\ m \in \mathbb{N}_0,$$

satisfying the initial conditions $P_{-1} = 0$, $P_0 = 1$. *The sequence* $(P_m)_{m\geq -1}$ *is called the Pell sequence in literature.*

Proof. System (3.48) is obtained from system (1.8) with $a = 3$, $b = -1$, $c = 2$, $d = f = 1$, $e = 4$. For these values of parameters a, b, equation (2.6) becomes

$$w_{m+1} = 3w_m - w_{m-1},\ m \in \mathbb{N}_0. \tag{3.55}$$

The solution s_m to equation (3.55) satisfying the initial conditions in (2.9) is equal to

$$s_m = \frac{\lambda_1^{m+1} - \lambda_2^{m+1}}{\lambda_1 - \lambda_2},\quad m \geq -1, \tag{3.56}$$

where

$$\lambda_{1,2} = \frac{3 \pm \sqrt{5}}{2}.$$

Note that

$$\left(\frac{1 \pm \sqrt{5}}{2}\right)^2 = \frac{3 \pm \sqrt{5}}{2}. \tag{3.57}$$

Using (3.57) in (3.56), we obtain

$$s_m = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{2m+2} - \left(\frac{1-\sqrt{5}}{2}\right)^{2m+2}}{\left(\frac{1+\sqrt{5}}{2}\right)^{2} - \left(\frac{1-\sqrt{5}}{2}\right)^{2}} = F_{2m+1}, \quad m \geq -1. \tag{3.58}$$

For these values of parameters c, d, equation (2.7) becomes

$$\widehat{w}_{m+1} = 2\widehat{w}_m + \widehat{w}_{m-1},\ m \in \mathbb{N}_0. \tag{3.59}$$

Hence the sequence $\widehat{s}_m$ satisfying conditions (2.10) and we have

$$\widehat{s}_m = P_m,\ m \geq -1. \tag{3.60}$$

For these values of parameters e, f, equation (2.8) becomes

$$\widetilde{w}_{m+1} = 4\widetilde{w}_m + \widetilde{w}_{m-1},\ m \in \mathbb{N}_0. \tag{3.61}$$

The solution $\widetilde{s}_m$ to equation (3.61) satisfying the initial conditions in (2.11) is equal to

$$\widetilde{s}_m = \frac{\widetilde{\lambda}_1^{m+1} - \widetilde{\lambda}_2^{m+1}}{\widetilde{\lambda}_1 - \widetilde{\lambda}_2}, \quad m \geq -1, \tag{3.62}$$

where

$$\widetilde{\lambda}_{1,2} = 2 \pm \sqrt{5}.$$

Note that

$$\left(\frac{1 \pm \sqrt{5}}{2}\right)^3 = 2 \pm \sqrt{5}. \tag{3.63}$$

Using (3.63) in (3.62), we obtain

$$\widetilde{s}_m = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{3m+3} - \left(\frac{1-\sqrt{5}}{2}\right)^{3m+3}}{\left(\frac{1+\sqrt{5}}{2}\right)^{3} - \left(\frac{1-\sqrt{5}}{2}\right)^{3}} = \frac{F_{3m+2}}{2}, \quad m \geq -1. \tag{3.64}$$

By using (3.58), (3.60), (3.64), in formulas (2.33)-(2.38), after some simple calculations are obtained formulas (3.49)-(3.54).

□

4. Conclusion

In this paper, we have consider the following three-dimensional system of difference equations

$$x_{n+1} = \frac{y_n y_{n-2}}{bx_{n-1} + ay_{n-2}},\ y_{n+1} = \frac{z_n z_{n-2}}{dy_{n-1} + cz_{n-2}},\ z_{n+1} = \frac{x_n x_{n-2}}{fz_{n-1} + ex_{n-2}},\ n \in \mathbb{N}_0,$$

which is a generalization of both equations in (1.5) and systems in (1.6), (1.7), where the parameters a, b, c, d, e, f and the initial values x_{-i}, y_{-i}, z_{-i}, $i \in \{0, 1, 2\}$, are real numbers.

Firstly, we have obtained the explicit form of well defined solutions of the aforementioned system using suitable transformation reducing to the equations in Riccati type. Also, we describe the forbidden set of the initial values using the obtained formulas. In addition, the solutions of this system are related to both Fibanacci numbers and Pell numbers for some special cases of a, b, c, d, e, f.

Acknowledgements

The authors are thankful to the editor and reviewers for their constructive review.

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