Journal of Applied Analysis and Computation
Volume 12, Number 4, August 2022, 1259–1276
Website:http://www.jaac-online.com
DOI:10.11948/20200131

PARTIAL-APPROXIMATE CONTROLLABILITY OF HILFER FRACTIONAL BACKWARD EVOLUTION SYSTEMS*

Shouguo Zhu[1,†] and Gang Li[2]

Abstract This paper addresses a Hilfer fractional backward evolution model. We first develop the dual theory of resolvent. Then, we motivate the transformation technique and the resolvent method to formulate a suitable concept of mild solutions to this model. In addition, with the help of the dual properties of resolvent, we employ the variational technique to treat the partial-approximate controllability problem of the system. We end up analyzing a Hilfer fractional diffusion backward control system by using our theoretical results.

Keywords Hilfer fractional evolution equations, backward systems, resolvent, partial-approximate controllability.

MSC(2010) 34K37, 47A10, 93B05.

1. Introduction

Fractional differential abstract models have stirred a surge of research interest due to their broad applicability in formulating many physical practical problems. Many papers (see [2, 12, 14, 26–28, 32]) have been devoted to the systems.

Authors in [11] and [25] demonstrated that Riemann-Liouville type fractional equations are more appropriate to model some practical applications in materials with memory features than Caputo type. Moreover, Hilfer type systems, the extension of Riemann-Liouville type, play a central role in the theoretical simulation of dielectric relaxation in glass forming materials [13]. We thereby need focus on Hilfer type systems.

Backward abstract systems are of great importance in different fields like signals, control theory, optical tomography, etc (see [6]). They have been treated by some researchers relying on the semigroup method. For example, a backward stochastic evolution model was analyzed in [10]. A Caputo fractional backward abstract system was examined in [21]. However, the literature about Hilfer type fractional backward evolution equations is scarce, in spite of their practical importance. Thus, the paper aims at linking this gap.

†The corresponding author. Email address: sgzhu2015@163.com(S. Zhu)
1Wuxi Institute of Technology, Wuxi 214121, China
2School of Mathematical Sciences, Yangzhou University, Yangzhou 225002, China

*The authors were supported by the NSF of China (Nos. 11771378,11871064), the NSF of the JiangSu Higher Education Institutions (18KJB110019) and the Doctor Research Foundation (BT2019-03).

To treat fractional abstract models, the key step is to formulate the expression of solutions. [34,35] pointed out that the resolvent method introduced by [8,9] is an efficient and convenient approach in addressing these systems. Hence, we proceed to motivate the resolvent technique to investigate Hilfer type fractional backward evolution equations.

On the other hand, much attention has been paid to the approximate controllability problem of abstract systems due to its practical importance (refer to [3, 10, 16, 19]). Recently, by employing the variational approach, Mahmudov [19] analyzed the partial-approximate controllability problem of a Caputo fractional evolution equation. However, until now, there has little mention about the partial-approximate controllability result of fractional backward abstract systems of Hilfer type.

Motivated by the aforementioned above, we are interested in dealing with the partial-approximate controllability problem of the following Hilfer fractional backward system:

$$\begin{cases} D_b^{\beta,\gamma} y(s) = A^* y(s) + J_b^{\gamma(1-\beta)}(B^* u(s) + f(s, y(s))), \ s \in J' = [0, b), \\ \lim\limits_{s \to b^-} \Gamma(\beta + \gamma(1-\beta))(b-s)^{(1-\beta)(1-\gamma)} y(s) = y_1. \end{cases} \tag{1.1}$$

Here $0 < \beta < 1$, $0 \le \gamma \le 1$, $D_b^{\beta,\gamma}$ is the right-sided generalized Riemann-Liouville fractional derivative of β-order and γ-type, $J_b^{\gamma(1-\beta)}$ is the right-sided fractional integral of $\gamma(1-\beta)$ order and A^* is the dual operator of A, where A generates a β-order and γ-type fractional resolvent $\{R_{\beta,\gamma}(s)\}_{s>0}$. Moreover, $u \in L^2((0,b), U)$ and $B^* \in \mathscr{L}(U, H)$, U and H are two real Hilbert spaces.

The current work contains the following novelties:

(1)We develop the transformation technique and the resolvent method to explore the Hilfer fractional abstract backward system.

(2)We establish the relation between the dual theory of resolvent and the partial-approximation controllability problem. In addition, we address the problem by utilizing the variational technique and the resolvent method.

The remaining paper is built up in the following way. Section 2 provides some standard preliminary facts. Section 3 is intended to propose and explore the partial-approximate controllability problem of a Hilfer fractional backward evolution system. This work closes with a diffusion model.

2. Preliminaries

For later analysis, we provide here some needed notations and preliminary facts. We are given two separable Hilbert spaces H and U. Without any additional declaration, we always suppose that $0 < \beta < 1$, $0 \le \gamma \le 1$ and $\beta + \gamma(1-\beta) > \frac{1}{2}$. Set $\widetilde{y}(\cdot) = (b - \cdot)^{(1-\beta)(1-\gamma)} y(\cdot)$, $J = [0, b]$ and $J' = [0, b)$. We consider a Banach space

$$C^{\beta,\gamma}(J, H) = \{y \in C(J', H) | \widetilde{y}(b) = \lim_{\tau \to b^-} \widetilde{y}(\tau), \ \widetilde{y} \in C(J, H)\}$$

with the norm $\|y\|_{\beta,\gamma} = \sup\limits_{\tau \in J} \|\widetilde{y}(\tau)\|$. For an operator A, we employ the symbol A^* to denote its dual operator. In addition, we utilize the notation $\mathscr{L}(H, U)$ to mean the collection of all linear and continuous operators from H to U and the symbol $*$ to mean the convolution, i.e., $(f * g)(s) = \int_0^s f(s-\tau)g(\tau)\mathrm{d}\tau$.

Below, we begin with a brief introduction to some definitions in fractional calculus.

Definition 2.1 ([23])**.** The β-order left-sided and right-sided fractional integrals $J^\beta f(s)$ and $J_b^\beta f(s)$ are defined as

$$J^\beta f(s) = \frac{1}{\Gamma(\beta)} \int_0^s (s-\tau)^{\beta-1} f(\tau)\mathrm{d}\tau,\ s>0,\ \beta>0,$$

and

$$J_b^\beta f(s) = \frac{1}{\Gamma(\beta)} \int_s^b (\tau-s)^{\beta-1} f(\tau)\mathrm{d}\tau,\ s<b,\ \beta>0,$$

respectively.

Definition 2.2 ([15])**.** The β-order and γ-type left-sided and right-sided generalized Riemann-Liouville fractional derivatives (Hilfer fractional derivatives) $D^{\beta,\gamma} f(s)$ and $D_b^{\beta,\gamma} f(s)$ are given by

$$D^{\beta,\gamma} f(s) = J^{\gamma(1-\beta)} \frac{\mathrm{d}}{\mathrm{d}s} J^{(1-\beta)(1-\gamma)} f(s),\ s>0,$$

and

$$D_b^{\beta,\gamma} f(s) = -J_b^{\gamma(1-\beta)} \frac{\mathrm{d}}{\mathrm{d}s} J_b^{(1-\beta)(1-\gamma)} f(s),\ s<b,$$

respectively.

Remark 2.1. (i) When $\gamma=0$, the fractional derivatives $D^{\beta,\gamma} f(s)$ and $D_b^{\beta,\gamma} f(s)$ reduce to the left-sided and right-sided Riemann-Liouville fractional derivatives (see [23]):

$$D^{\beta,0} f(s) = \frac{\mathrm{d}}{\mathrm{d}s} J^{1-\beta} f(s) = D^\beta f(s),\ s>0,$$

and

$$D_b^{\beta,0} f(s) = -\frac{\mathrm{d}}{\mathrm{d}s} J_b^{1-\beta} f(s) = D_b^\beta f(s),\ s<b,$$

respectively.

(ii) When $\gamma=1$, the fractional derivatives $D^{\beta,\gamma} f(s)$ and $D_b^{\beta,\gamma} f(s)$ become the left-sided and right-sided Caputo fractional derivatives (see [23]):

$$D^{\beta,1} f(s) = J^{1-\beta} \frac{\mathrm{d}}{\mathrm{d}s} f(s) = {}^C D^\beta f(s),\ s>0,$$

and

$$D_b^{\beta,1} f(s) = -J_b^{1-\beta} \frac{\mathrm{d}}{\mathrm{d}s} f(s) = {}^C D_b^\beta f(s),\ s<b,$$

respectively.

We then review the concept of resolvent and exhibit some properties of the resolvent.

Definition 2.3 ([20]). A β-order and γ-type fractional resolvent on a Banach space V is a strongly continuous family $\{R_{\beta,\gamma}(s)\}_{s>0} \subseteq \mathscr{L}(V)$ satisfying that

(a) $\lim\limits_{s\to 0^+} \Gamma(\beta+\gamma(1-\beta))s^{(1-\beta)(1-\gamma)}R_{\beta,\gamma}(s)x = x,\ x \in V$;

(b) $R_{\beta,\gamma}(\tau)R_{\beta,\gamma}(s) = R_{\beta,\gamma}(s)R_{\beta,\gamma}(\tau),\ s,\ \tau > 0$;

(c) for $s,\ \tau > 0$, it holds

$$R_{\beta,\gamma}(\tau)J^{\beta}R_{\beta,\gamma}(s) - J^{\beta}R_{\beta,\gamma}(\tau)R_{\beta,\gamma}(s) = g_{\beta+\gamma(1-\beta)}(\tau)J^{\beta}R_{\beta,\gamma}(s) - g_{\beta+\gamma(1-\beta)}(s)J^{\beta}R_{\beta,\gamma}(\tau),$$

where $g_{\beta+\gamma(1-\beta)}(\cdot) = \frac{(\cdot)^{\beta+\gamma(1-\beta)-1}}{\Gamma(\beta+\gamma(1-\beta))}$ and $\mathscr{L}(V) = \mathscr{L}(V,V)$.

Moreover, by a generator of the resolvent $\{R_{\beta,\gamma}(s)\}_{s>0}$, we understand an operator $A : D(A) \subseteq V \to V$ satisfying that

$$Ax = \Gamma(2\beta+\gamma(1-\beta)) \lim_{s\to 0^+} \frac{s^{(1-\beta)(1-\gamma)}R_{\beta,\gamma}(s)x - \frac{x}{\Gamma(\beta+\gamma(1-\beta))}}{s^{\beta}},$$

where

$$D(A) = \left\{ x \in V : \lim_{s\to 0^+} \frac{s^{(1-\beta)(1-\gamma)}R_{\beta,\gamma}(s)x - \frac{x}{\Gamma(\beta+\gamma(1-\beta))}}{s^{\beta}} \text{ exists} \right\}.$$

Remark 2.2. (i) If $\gamma = 0$, the fractional resolvent $\{R_{\beta,\gamma}(s)\}_{s>0}$ reduces to the β-order resolvent [17] associated with Riemann-Liouville fractional fractional evolution systems.

(ii) If $\gamma = 1$, the fractional resolvent $\{R_{\beta,\gamma}(s)\}_{s>0}$ becomes the solution operator related to Caputo fractional evolution systems (see [24]).

Remark 2.3. If $\gamma \neq 1$, based on Definition 2.3, the resolvent $\{R_{\beta,\gamma}(s)\}_{s>0}$ is unbounded near the zero point. But we have $\sup_{s\in J} \|s^{(1-\beta)(1-\gamma)}R_{\beta,\gamma}(s)\| < \infty$, which is due to $s^{(1-\beta)(1-\gamma)}R_{\beta,\gamma}(s) \in C(J,V)$ and the uniform boundedness principle, where $\left(s^{(1-\beta)(1-\gamma)}R_{\beta,\gamma}(s)\right)|_{s=0} = \lim_{s\to 0^+}\left(s^{(1-\beta)(1-\gamma)}R_{\beta,\gamma}(s)\right)$. To facilitate our later analysis, we always put $M = \sup_{s\in J} \|s^{(1-\beta)(1-\gamma)}R_{\beta,\gamma}(s)\|$.

Lemma 2.1 ([20]). *For the resolvent $\{R_{\beta,\gamma}(s)\}_{s>0}$, we have*

(a) *$R_{\beta,\gamma}(s)D(A) \subseteq D(A)$ and $AR_{\beta,\gamma}(s)x = R_{\beta,\gamma}(s)Ax$ for $x \in D(A)$ and $s > 0$;*

(b) *$R_{\beta,\gamma}(s)x = g_{\beta+\gamma(1-\beta)}(s)x + J^{\beta}R_{\beta,\gamma}(s)Ax$ for $x \in D(A)$ and $s > 0$;*

(c) *$R_{\beta,\gamma}(s)x = g_{\beta+\gamma(1-\beta)}(s)x + AJ^{\beta}R_{\beta,\gamma}(s)x$ for $x \in V$ and $s > 0$;*

(d) *$\overline{D(A)} = V$.*

Lemma 2.2. *If $\{R_{\beta,\gamma}(s)\}_{s>0}$ is a resolvent, then for any $x \in V$, we have*

$$\Gamma(2\beta+\gamma(1-\beta))s^{1-2\beta-\gamma(1-\beta)}J^{\beta}R_{\beta,\gamma}(s)x = x.$$

Proof. For $x \in V$, we have

$$\begin{aligned} &\left\|\Gamma(2\beta+\gamma(1-\beta))h^{1-2\beta-\gamma(1-\beta)}J^{\beta}R_{\beta,\gamma}(h)x - x\right\| \\ \le &\left\|\frac{\Gamma(2\beta+\gamma(1-\beta))}{\Gamma(\beta)}\int_0^h h^{1-2\beta-\gamma(1-\beta)}(h-\tau)^{\beta-1}R_{\beta,\gamma}(\tau)x\mathrm{d}\tau - x\right\| \end{aligned}$$

$$
\begin{aligned}
\leq & \frac{\Gamma(2\beta+\gamma(1-\beta))}{\Gamma(\beta)\Gamma(\beta+\gamma(1-\beta))}\int_0^1 (1-\tau)^{\beta-1}\tau^{\beta+\gamma(1-\beta)-1}\mathrm{d}\tau \\
& \times \sup_{\tau\in[0,1]}\left\|\Gamma(\beta+\gamma(1-\beta))(h\tau)^{(1-\beta)(1-\gamma)}R_{\beta,\gamma}(h\tau)x-x\right\|.
\end{aligned}
$$

Thus, the conclusion is evident by Definition 2.3. □

As we all know, any C_0-semigroup is (M,ω) type. However, $\{R_{\beta,\gamma}(s)\}_{s>0}$ is not (M,ω) type based on its unboundedness on zero point, when $\gamma\neq 1$. Now, according to the summary of the practical applications, we propose the following definition to treat the singularity at zero:

Definition 2.4. For sufficiently small $s_0>0$, $\{R_{\beta,\gamma}(s)\}_{s>0}$ is $(\overline{M},\omega)$ type for $s\geq s_0$ if there exist two constants $\omega>0$ and $\overline{M}>0$ to ensure that

$$
\|R_{\beta,\gamma}(s)\|\leq \overline{M}e^{\omega s},\ \ s\geq s_0. \tag{2.1}
$$

For simplicity, we apply the symbol $A\in C_{s_0}^{\beta,\gamma}(\overline{M},\omega)$ to mean that $\{R_{\beta,\gamma}(s)\}_{s>0}$ is a resolvent satisfying *(2.1)*.

Lemma 2.3. *Let A generate a β-order and γ-type fractional resolvent $\{R_{\beta,\gamma}(s)\}_{s>0}$ on H and $A\in C_{s_0}^{\beta,\gamma}(\overline{M},\omega)$. Then $\{R^*_{\beta,\gamma}(s)\}_{s>0}$ is a β-order and γ-type fractional resolvent generated by A^*.*

Proof. We first check that $\lim\limits_{s\to 0^+}\Gamma(\beta+\gamma(1-\beta))s^{(1-\beta)(1-\gamma)}R^*_{\beta,\gamma}(s)x^*=x^*$ for any $x^*\in H$. Let $\|x\|=1$ and $s<s_0$. For any $x^*\in D(A^*)$ and $x\in H$, on account of Lemma 2.1, we find that

$$
\begin{aligned}
& |\langle\Gamma(\beta+\gamma(1-\beta))s^{(1-\beta)(1-\gamma)}R^*_{\beta,\gamma}(s)x^*-x^*,x\rangle| \\
=\ & \Gamma(\beta+\gamma(1-\beta))s^{(1-\beta)(1-\gamma)}|\langle x^*,R_{\beta,\gamma}(s)x-g_{\beta+\gamma(1-\beta)}(s)x\rangle| \\
=\ & \Gamma(\beta+\gamma(1-\beta))s^{(1-\beta)(1-\gamma)}|\langle A^*x^*,J^\beta R_{\beta,\gamma}(s)x\rangle| \\
\leq\ & \frac{M\Gamma(\beta+\gamma(1-\beta))}{\Gamma(\beta)}s^{(1-\beta)(1-\gamma)}\|A^*x^*\|\int_0^s (s-\tau)^{\beta-1}\tau^{\beta+\gamma(1-\beta)-1}\mathrm{d}\tau \\
\leq\ & \|A^*x^*\|\frac{\Gamma^2(\beta+\gamma(1-\beta))}{\Gamma(2\beta+\gamma(1-\beta))}Ms^\beta\to 0,\ \ s\to 0^+.
\end{aligned}
$$

Thus, for any $x^*\in H$, the limit $\lim\limits_{s\to 0^+}\Gamma(\beta+\gamma(1-\beta))s^{(1-\beta)(1-\gamma)}R^*_{\beta,\gamma}(s)x^*=x^*$ follows readily from, the well-known conclusion, $\overline{D(A^*)}=H$.

We then investigate the strong continuity of $\{R^*_{\beta,\gamma}(s)\}_{s>0}$. Let $\|x\|=1$, $s>0$ and $s+h>0$. For $x^*\in D(A^*)$, by virtue of Lemma 2.1, we get

$$
\begin{aligned}
& |\langle R^*_{\beta,\gamma}(s+h)x^*-R^*_{\beta,\gamma}(s)x^*,x\rangle| = |\langle x^*,R_{\beta,\gamma}(s+h)x-R_{\beta,\gamma}(s)x\rangle| \\
\leq & |\langle A^*x^*,J^\beta R_{\beta,\gamma}(s+h)x-J^\beta R_{\beta,\gamma}(s)x\rangle|+|\langle x^*,g_{\beta+\gamma(1-\beta)}(s+h)x-g_{\beta+\gamma(1-\beta)}(s)x\rangle| \\
\leq & \|A^*x^*\|\|J^\beta R_{\beta,\gamma}(s+h)x-J^\beta R_{\beta,\gamma}(s)x\|+\|x^*\|\|g_{\beta+\gamma(1-\beta)}(s+h)-g_{\beta+\gamma(1-\beta)}(s)\|.
\end{aligned}
$$

Below, we analyze the following two cases:

Case 1 For $0<s<s+h<s_0$, we have

$$
\begin{aligned}
& \|J^\beta R_{\beta,\gamma}(s+h)x-J^\beta R_{\beta,\gamma}(s)x\| \\
\leq\ & \frac{M}{\Gamma(\beta)}\int_0^s ((s-\tau)^{\beta-1}-(s+h-\tau)^{\beta-1})\tau^{\beta+\gamma(1-\beta)-1}\mathrm{d}\tau
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{M}{\Gamma(\beta)}\int_s^{s+h}(s+h-\tau)^{\beta-1}\tau^{\beta+\gamma(1-\beta)-1}\mathrm{d}\tau\\
\le\ &\frac{\Gamma(\beta+\gamma(1-\beta))M}{\Gamma(2\beta+\gamma(1-\beta))}(s^{2\beta+\gamma(1-\beta)-1}-(s+h)^{2\beta+\gamma(1-\beta)-1})+\frac{2Ms^{\beta+\gamma(1-\beta)-1}h^{\beta}}{\beta\Gamma(\beta)}\\
\to\ &0,\ h\to 0^+.
\end{aligned}
$$

Case 2 For $0<s_0\le s<s+h$, due to $A\in C_{s_0}^{\beta,\gamma}(\overline{M},\omega)$, we get

$$
\begin{aligned}
&\|J^{\beta}R_{\beta,\gamma}(s+h)x-J^{\beta}R_{\beta,\gamma}(s)x\|\\
\le\ &\frac{M}{\Gamma(\beta)}\int_0^{s_0}((s-\tau)^{\beta-1}-(s+h-\tau)^{\beta-1})\tau^{\beta+\gamma(1-\beta)-1}\mathrm{d}\tau\\
&+\frac{\overline{M}e^{\omega s}}{\beta\Gamma(\beta)}\left((s-s_0)^{\beta}-(s-s_0+h)^{\beta}+h^{\beta}\right)+\frac{\overline{M}e^{\omega(s+h)}h^{\beta}}{\beta\Gamma(\beta)}\\
\le\ &\frac{\Gamma(\beta+\gamma(1-\beta))M}{\Gamma(2\beta+\gamma(1-\beta))}(s^{2\beta+\gamma(1-\beta)-1}-(s+h)^{2\beta+\gamma(1-\beta)-1})+\frac{Ms^{\beta+\gamma(1-\beta)-1}h^{\beta}}{\beta\Gamma(\beta)}\\
&+\frac{\overline{M}e^{\omega s}}{\beta\Gamma(\beta)}\left((s-s_0)^{\beta}-(s-s_0+h)^{\beta}+h^{\beta}\right)+\frac{\overline{M}e^{\omega(s+h)}h^{\beta}}{\beta\Gamma(\beta)}\\
\to\ &0,\ h\to 0^+.
\end{aligned}
$$

Hence, we can deduce that $\lim\limits_{h\to0^+}\|R_{\beta,\gamma}^*(s+h)x^*-R_{\beta,\gamma}^*(s)x^*\|=0$. Additionally, adopting the similar technique yields $\lim\limits_{h\to0^-}\|R_{\beta,\gamma}^*(s+h)x^*-R_{\beta,\gamma}^*(s)x^*\|=0$. Thus, $\{R_{\beta,\gamma}^*(s)\}_{s>0}$ is strongly continuous on $D(A^*)$. As such, the strong continuity of $\{R_{\beta,\gamma}^*(s)\}_{s>0}$ can be obtained readily from $\overline{D(A^*)}=H$.

Therefore, according to the theory of dual operators in [7] and Definition 2.3, we can conclude that $\{R_{\beta,\gamma}^*(s)\}_{s>0}$ is a resolvent.

Finally, what is left is to check that A^* is the generator of $\{R_{\beta,\gamma}^*(s)\}_{s>0}$. Let $\mathcal{A}$ be the generator of $\{R_{\beta,\gamma}^*(s)\}_{s>0}$. We are reduced to showing that $\mathcal{A}=A^*$. For $x^*\in D(A^*)$ and $x\in H$, due to Lemma 2.2, we find that

$$
\begin{aligned}
&\lim_{s\to0^+}\left\langle\Gamma(2\beta+\gamma(1-\beta))\frac{s^{(1-\beta)(1-\gamma)}R_{\beta,\gamma}^*(s)x^*-\frac{x^*}{\Gamma(\beta+\gamma(1-\beta))}}{s^{\beta}},x\right\rangle\\
=&\lim_{s\to0^+}\left\langle x^*,\Gamma(2\beta+\gamma(1-\beta))\frac{s^{(1-\beta)(1-\gamma)}R_{\beta,\gamma}(s)x-\frac{x}{\Gamma(\beta+\gamma(1-\beta))}}{s^{\beta}}\right\rangle\\
=&\lim_{s\to0^+}\left\langle A^*x^*,\Gamma(2\beta+\gamma(1-\beta))s^{1-2\beta-\gamma(1-\beta)}J^{\beta}R_{\beta,\gamma}(s)x\right\rangle=\langle A^*x^*,x\rangle,
\end{aligned}
$$

which implies that $x^*\in D(\mathcal{A})$ and $\mathcal{A}x^*=A^*x^*$. As such, we obtain $A^*\subseteq\mathcal{A}$. Conversely, for $x^*\in D(\mathcal{A})$ and $x\in D(A)$, we have

$$
\begin{aligned}
\langle\mathcal{A}x^*,x\rangle&=\lim_{s\to0^+}\left\langle\Gamma(2\beta+\gamma(1-\beta))\frac{s^{(1-\beta)(1-\gamma)}R_{\beta,\gamma}^*(s)x^*-\frac{x^*}{\Gamma(\beta+\gamma(1-\beta))}}{s^{\beta}},x\right\rangle\\
&=\lim_{s\to0^+}\left\langle x^*,\Gamma(2\beta+\gamma(1-\beta))\frac{s^{(1-\beta)(1-\gamma)}R_{\beta,\gamma}(s)x-\frac{x}{\Gamma(\beta+\gamma(1-\beta))}}{s^{\beta}}\right\rangle\\
&=\lim_{s\to0^+}\left\langle x^*,\Gamma(2\beta+\gamma(1-\beta))s^{1-2\beta-\gamma(1-\beta)}J^{\beta}R_{\beta,\gamma}(s)Ax\right\rangle=\langle x^*,Ax\rangle.
\end{aligned}
$$

This gives $x^* \in D(A^*)$ and $\mathcal{A}x^* = A^*x^*$. Hence, we derive $\mathcal{A} \subseteq A^*$. Thus, we can deduce that $\mathcal{A} = A^*$. □

Remark 2.4. In Lemma 2.3, if we take $\gamma = 1$, then we can obtain the dual properties of the resolvents related to Caputo fractional Cauchy systems [31]. If $\gamma = 0$, then we get the dual results related to Riemann-Liouville systems [33]. Thus, we unify the dual theory of resolvents from Caputo systems and the theory from Riemann-Liouville systems.

To end this section, we propose the following properties of $\{R^*_{\beta,\gamma}(s)\}_{s>0}$:

Lemma 2.4. *Let $T_{\beta,\gamma}(s) = s^{(1-\beta)(1-\gamma)}R_{\beta,\gamma}(s)$ be compact and norm continuous for $s \in (0, b]$. Then for $s \in (0, b]$, we have*

(a) $\lim\limits_{\tau\to 0^+} \left\| T^*_{\beta,\gamma}(s+\tau) - \Big(\Gamma(\beta+\gamma(1-\beta))T^*_{\beta,\gamma}(\tau)\Big) T^*_{\beta,\gamma}(s) \right\| = 0;$

(b) $\lim\limits_{\tau\to 0^+} \left\| T^*_{\beta,\gamma}(s) - \Big(\Gamma(\beta+\gamma(1-\beta))T^*_{\beta,\gamma}(\tau)\Big) T^*_{\beta,\gamma}(s-\tau) \right\| = 0.$

Proof. Since $\{T_{\beta,\gamma}(s)\}_{s>0}$ is compact and norm continuous, we can obtain the compactness and equicontinuity of $\{T^*_{\beta,\gamma}(s)\}_{s>0}$, and thus we can derive the properties by the similar arguments employed in [12]. □

3. Partial-approximate controllability problems

This section is intended to focus attention on the partial-approximate controllability problem of (1.1). To achieve our aim, we need the following conditions:

(HA) $A \in C^{\beta,\gamma}_{s_0}(\overline{M}, \omega)$ and $\{s^{(1-\beta)(1-\gamma)}R_{\beta,\gamma}(s)\}_{s>0}$ is equicontinuous and compact.

(Hf) $f : J \times H \to H$ is continuous and there exists a constant $N > 0$ to guarantee that $\|f(s, x)\| \le N$ for all $(s, x) \in J \times H$.

(HB) $B^* \in \mathscr{L}(U, H)$ with $M_B = \|B^*\|$.

Under the condition (HA), it follows from Lemma 2.3 and Remark 2.3 that $\{R^*_{\beta,\gamma}(s)\}_{s>0}$ is a β-order and γ-type fractional resolvent generated by A^*, $M = \sup\limits_{s\in J} \|s^{(1-\beta)(1-\gamma)}R^*_{\beta,\gamma}(s)\|$, and $\{s^{(1-\beta)(1-\gamma)}R^*_{\beta,\gamma}(s)\}_{s>0}$ is equicontinuous and compact.

We begin with the following two important lemmas, which are helpful in formulating an appropriate notion of mild solutions to (1.1) and analyzing the partial-approximate controllability problem.

Lemma 3.1. *Let (HA) hold. Then $\int_\cdot^b R^*_{\beta,\gamma}(\tau - \cdot)h(\tau)\mathrm{d}\tau \in C(J, H)$, where $h \in L^2(J, H)$.*

Proof. Considering the unboundedness of $R^*_{\beta,\gamma}(s)$ on zero point, we first need to check the existence of $\int_s^b R^*_{\beta,\gamma}(\tau - s)h(\tau)\mathrm{d}\tau$, $s \in J$. Similar to the proof of Proposition 1.3.4 in [1], we can easily verify that $R^*_{\beta,\gamma}(\cdot - s)h(\cdot)$ is measurable on (s, b), $s \in [0, b)$. Additionally, we have

$$\left\| \int_s^b R^*_{\beta,\gamma}(\tau - s)h(\tau)\mathrm{d}\tau \right\| \le M \int_s^b (\tau - s)^{-(1-\beta)(1-\gamma)}\|h(\tau)\|\mathrm{d}\tau$$

$$\leq M\|h\|_{L^2}\sqrt{\frac{b^{2(\beta+\gamma(1-\beta))-1}}{2(\beta+\gamma(1-\beta))-1}}.$$

Hence, $\int_s^b R^*_{\beta,\gamma}(\tau-s)h(\tau)\mathrm{d}\tau$ exists almost everywhere.

We are now ready to investigate its continuity. Let $0\leq s_1<s_2\leq b$. Due to the strong continuity of $\{R^*_{\beta,\gamma}(s)\}_{s>0}$, the dominated convergence theorem and the absolute continuity of integration of $R^*_{\beta,\gamma}(\cdot-s)h(\cdot)$, we get

$$\begin{aligned}
&\left\|\int_{s_1}^b R^*_{\beta,\gamma}(\tau-s_1)h(\tau)\mathrm{d}\tau-\int_{s_2}^b R^*_{\beta,\gamma}(\tau-s_2)h(\tau)\mathrm{d}\tau\right\|\\
&\leq\left\|\int_{s_2}^b (R^*_{\beta,\gamma}(\tau-s_1)-R^*_{\beta,\gamma}(\tau-s_2))h(\tau)\mathrm{d}\tau\right\|+\left\|\int_{s_1}^{s_2} R^*_{\beta,\gamma}(\tau-s_1)h(\tau)\mathrm{d}\tau\right\|\\
&\to 0,\ s_1\to s_2.
\end{aligned}$$

Therefore, $\int_\cdot^b R^*_{\beta,\gamma}(\tau-\cdot)h(\tau)\mathrm{d}\tau\in C(J,H)$. □

Lemma 3.2. *Let (HA) hold. The map $\Lambda:L^2(J,H)\to C^{\beta,\gamma}(J,H)$, defined by*

$$(\Lambda h)(\cdot)=\int_\cdot^b R^*_{\beta,\gamma}(\tau-\cdot)h(\tau)\mathrm{d}\tau,$$

is compact.

Proof. Let $\|z_n\|_{L^2}\leq 1$. Fix $\widetilde{z_n}(\cdot)=(b-\cdot)^{(1-\beta)(1-\gamma)}(\Lambda z_n)(\cdot)$. Our problem reduces to investigating the compactness of $\{\widetilde{z_n}\}_{n\geq 1}$ in $C(J,H)$.

We first verify the equicontinuity of $\{\widetilde{z_n}\}_{n\geq 1}$. Let $t_1,\ t_2\in J$ with $t_1<t_2<b$. To simplify notation, put $T^*_{\beta,\gamma}(t)=t^{(1-\beta)(1-\gamma)}R^*_{\beta,\gamma}(t)$. For $\eta\in(0,b-t_2)$, we have

$$\begin{aligned}
&\|\widetilde{z_n}(t_1)-\widetilde{z_n}(t_2)\|\leq\left\|(b-t_1)^{(1-\beta)(1-\gamma)}(\Lambda z_n)(t_1)-(b-t_2)^{(1-\beta)(1-\gamma)}(\Lambda z_n)(t_2)\right\|\\
&\leq\left((b-t_1)^{(1-\beta)(1-\gamma)}-(b-t_2)^{(1-\beta)(1-\gamma)}\right)\|(\Lambda z_n)(t_2)\|\\
&\quad+b^{(1-\beta)(1-\gamma)}\|(\Lambda z_n)(t_1)-(\Lambda z_n)(t_2)\|\\
&\leq\left((b-t_1)^{(1-\beta)(1-\gamma)}-(b-t_2)^{(1-\beta)(1-\gamma)}\right)M\sqrt{\frac{b^{2(\beta+\gamma(1-\beta))-1}}{2(\beta+\gamma(1-\beta))-1}}\\
&\quad+b^{(1-\beta)(1-\gamma)}\left\|\int_{t_2+\eta}^b\left(T^*_{\beta,\gamma}(\tau-t_1)-T^*_{\beta,\gamma}(\tau-t_2)\right)(\tau-t_1)^{-(1-\beta)(1-\gamma)}z_n(\tau)\mathrm{d}\tau\right\|\\
&\quad+b^{(1-\beta)(1-\gamma)}\left\|\int_{t_2}^{t_2+\eta}\left(T^*_{\beta,\gamma}(\tau-t_1)-T^*_{\beta,\gamma}(\tau-t_2)\right)(\tau-t_1)^{-(1-\beta)(1-\gamma)}z_n(\tau)\mathrm{d}\tau\right\|\\
&\quad+b^{(1-\beta)(1-\gamma)}\left\|\int_{t_2}^b T^*_{\beta,\gamma}(\tau-t_2)\left((\tau-t_1)^{-(1-\beta)(1-\gamma)}-(\tau-t_2)^{-(1-\beta)(1-\gamma)}\right)z_n(\tau)\mathrm{d}\tau\right\|\\
&\quad+b^{(1-\beta)(1-\gamma)}\left\|\int_{t_1}^{t_2} R^*_{\beta,\gamma}(\tau-t_1)z_n(\tau)\mathrm{d}\tau\right\|\\
&\leq\left((b-t_1)^{(1-\beta)(1-\gamma)}-(b-t_2)^{(1-\beta)(1-\gamma)}\right)M\sqrt{\frac{b^{2(\beta+\gamma(1-\beta))-1}}{2(\beta+\gamma(1-\beta))-1}}
\end{aligned}$$

$$+\sup_{\tau\in[t_2+\eta,b]}\left\|T^*_{\beta,\gamma}(\tau-t_1)-T^*_{\beta,\gamma}(\tau-t_2)\right\|\sqrt{\frac{b}{2(\beta+\gamma(1-\beta))-1}}$$
$$+2Mb^{(1-\beta)(1-\gamma)}\left(\int_{t_2}^{t_2+\eta}(\tau-t_1)^{-2(1-\beta)(1-\gamma)}\mathrm{d}\tau\right)^{\frac{1}{2}}$$
$$+Mb^{(1-\beta)(1-\gamma)}\left(\int_{t_2}^{b}\left((\tau-t_1)^{-(1-\beta)(1-\gamma)}-(\tau-t_2)^{-(1-\beta)(1-\gamma)}\right)^2\mathrm{d}\tau\right)^{\frac{1}{2}}$$
$$+Mb^{(1-\beta)(1-\gamma)}\left(\frac{1}{2(\beta+\gamma(1-\beta))-1}\right)^{\frac{1}{2}}(t_2-t_1)^{\beta+\gamma(1-\beta)-\frac{1}{2}}.$$

Hence, by employing 2-mean continuity (see problem 23.9 on Page 445 in [30]) and the absolute continuity of integration of $(\cdot-t_1)^{-2(1-\beta)(1-\gamma)}$, we can infer that

$$\lim_{t_1\to t_2}\|\widetilde{z_n}(t_1)-\widetilde{z_n}(t_2)\|=0.$$

Moreover, for $t_2=b$, it is a simple matter to check that $\lim_{t_1\to t_2}\|\widetilde{z_n}(t_1)-\widetilde{z_n}(t_2)\|=0$. Therefore, $\{\widetilde{z_n}\}_{n\geq 1}$ is equicontinuous.

Now, for $t\in J$, we turn to deal with the relative compactness of $\{\widetilde{z_n}(t)\}_{n\geq 1}$ in H. Obviously, $\{\widetilde{z_n}(b)\}_{n\geq 1}=\{0\}$ is compact. We come to consider the case of $t\in[0,b)$. For $\varepsilon\in(0,b-t)$, $t\in[0,b)$, we conclude from the compactness of $T^*_{\beta,\gamma}(\varepsilon)$ that $\{\widetilde{z_n}^{\varepsilon}(t)\}_{n\geq 1}$ is relatively compact, where

$$\widetilde{z_n}^{\varepsilon}(t)=(b-t)^{(1-\beta)(1-\gamma)}(\Gamma(\beta+\gamma(1-\beta))T^*_{\beta,\gamma}(\varepsilon))\int_{t+\varepsilon}^{b}R^*_{\beta,\gamma}(\tau-t-\varepsilon)z_n(\tau)\mathrm{d}\tau.$$

For notational simplicity, we set

$$\begin{aligned}
&\Psi_t(\varepsilon,\tau)=T^*_{\beta,\gamma}(\tau-t-\varepsilon)(\Gamma(\beta+\gamma(1-\beta))T^*_{\beta,\gamma}(\varepsilon)),\\
&\Phi_t(\varepsilon,\tau)=T^*_{\beta,\gamma}(\tau-t)-\Psi_t(\varepsilon,\tau),\\
&\phi_t(\varepsilon,\tau)=(\tau-t-\varepsilon)^{-(1-\beta)(1-\gamma)}-(\tau-t)^{-(1-\beta)(1-\gamma)}.
\end{aligned}$$

Let $\delta\in(\varepsilon,b-t)$. We have

$$\begin{aligned}
&\|\widetilde{z_n}(t)-\widetilde{z_n}^{\varepsilon}(t)\|\\
&\leq b^{(1-\beta)(1-\gamma)}\int_{t+\delta}^{b}\|\Phi_t(\varepsilon,\tau)\|(\tau-t)^{-(1-\beta)(1-\gamma)}\|z_n(\tau)\|\mathrm{d}\tau\\
&\quad+b^{(1-\beta)(1-\gamma)}\int_{t+\varepsilon}^{t+\delta}\|\Phi_t(\varepsilon,\tau)\|(\tau-t)^{-(1-\beta)(1-\gamma)}\|z_n(\tau)\|\mathrm{d}\tau\\
&\quad+b^{(1-\beta)(1-\gamma)}\int_{t+\varepsilon}^{b}|\phi_t(\varepsilon,\tau)|\|\Psi_t(\varepsilon,\tau)\|\|z_n(\tau)\|\mathrm{d}\tau\\
&\quad+b^{(1-\beta)(1-\gamma)}\int_{t}^{t+\varepsilon}\|T^*_{\beta,\gamma}(\tau-t)\|(\tau-t)^{-(1-\beta)(1-\gamma)}\|z_n(\tau)\|\mathrm{d}\tau\\
&\leq b^{(1-\beta)(1-\gamma)}\int_{t+\delta}^{b}\|\Phi_t(\varepsilon,\tau)\|(\tau-t)^{-(1-\beta)(1-\gamma)}\|z_n(\tau)\|\mathrm{d}\tau\\
&\quad+(M+\Gamma(\beta+\gamma(1-\beta))M^2)b^{(1-\beta)(1-\gamma)}\left(\frac{\delta^{2(\beta+\gamma(1-\beta))-1}}{2(\beta+\gamma(1-\beta))-1}\right)^{\frac{1}{2}}
\end{aligned}$$

$$+\Gamma(\beta+\gamma(1-\beta))M^2 b^{(1-\beta)(1-\gamma)}\left(\int_{t+\varepsilon}^{b}|\phi_t(\varepsilon,\tau)|^2\mathrm{d}\tau\right)^{\frac{1}{2}}$$
$$+Mb^{(1-\beta)(1-\gamma)}\left(\frac{1}{2(\beta+\gamma(1-\beta))-1}\right)^{\frac{1}{2}}\varepsilon^{\beta+\gamma(1-\beta)-\frac{1}{2}}.$$

Based on Lemma 2.4, we get $\lim\limits_{\varepsilon\to 0}\|\Phi_t(\varepsilon,\tau)\|=0,\ \tau\in[t+\delta,b]$. Hence, employing the dominated convergence theorem, 2-mean continuity and the arbitrariness of δ yields $\lim\limits_{\varepsilon\to 0^+}\|\widetilde{z_n}(t)-\widetilde{z_n}^{\varepsilon}(t)\|=0$ As such, $\{\widetilde{z_n}(t)\}_{n\geq 1}$ is relatively compact.

Therefore, the compactness of Λ is evident by the Arzela-Ascoli theorem. □

We then formulate the concept of mild solutions to (1.1). For convenience, we analyze the following auxiliary system:

$$\begin{cases} D_b^{\beta,\gamma}y(s)=A^*y(s)+J_b^{\gamma(1-\beta)}h(s),\ s\in J'=[0,b),\\ \lim\limits_{s\to b^-}\Gamma(\beta+\gamma(1-\beta))(b-s)^{(1-\beta)(1-\gamma)}y(s)=y_1,\end{cases}\tag{3.1}$$

where $h\in L^2(J,H)$. Set $\mathcal{T}_b y(s)=y(b-s)$. By replacing the variable s by $b-s$, we can obtain that

$$\begin{cases} D^{\beta,\gamma}(\mathcal{T}_b y(s))=A^*\mathcal{T}_b y(s)+J^{\gamma(1-\beta)}\mathcal{T}_b h(s),\ s\in(0,b],\\ \lim\limits_{s\to 0^+}\Gamma(\beta+\gamma(1-\beta))s^{(1-\beta)(1-\gamma)}\mathcal{T}_b y(s)=y_1.\end{cases}\tag{3.2}$$

According to (3.2) and the dominated convergence theorem, we can easily obtain $J^{(1-\beta)(1-\gamma)}\mathcal{T}_b y(s)|_{s=0}=y_1$. Hence, utilizing the operator J^{β} on both sides of (3.2) and using $D^{\beta,\gamma}(\mathcal{T}_b y(s))=J^{\gamma(1-\beta)}D^{\beta+\gamma(1-\beta)}(\mathcal{T}_b y(s))$, we can derive that

$$\mathcal{T}_b y(s)=g_{\beta+\gamma(1-\beta)}(s)y_1+A^*J^{\beta}\mathcal{T}_b y(s)+J^{\beta+\gamma(1-\beta)}\mathcal{T}_b h(s).\tag{3.3}$$

Based on Lemmas 2.1 and 2.3, we have

$$\begin{aligned} g_{\beta+\gamma(1-\beta)}*\mathcal{T}_b y&=(R^*_{\beta,\gamma}-A^*(g_\beta*R^*_{\beta,\gamma}))*\mathcal{T}_b y\\ &=R^*_{\beta,\gamma}*(\mathcal{T}_b y-A^*(g_\beta*\mathcal{T}_b y))\\ &=R^*_{\beta,\gamma}*(y_1 g_{\beta+\gamma(1-\beta)}+g_{\beta+\gamma(1-\beta)}*\mathcal{T}_b h)\\ &=g_{\beta+\gamma(1-\beta)}*\left(y_1 R^*_{\beta,\gamma}+R^*_{\beta,\gamma}*\mathcal{T}_b h\right),\end{aligned}$$

which signifies that

$$\mathcal{T}_b y(s)=R^*_{\beta,\gamma}(s)y_1+\int_0^s R^*_{\beta,\gamma}(s-\tau)\mathcal{T}_b h(\tau)\mathrm{d}\tau, s\in(0,b],$$

and hence that

$$y(s)=R^*_{\beta,\gamma}(b-s)y_1+\int_s^b R^*_{\beta,\gamma}(\tau-s)h(\tau)\mathrm{d}\tau, s\in[0,b).$$

Therefore, we can define the mild solutions to system (1.1) as follows:

Definition 3.1. By a mild solution to system (1.1), we understand the function $y \in C^{\beta,\gamma}(J,H)$ which satisfies

$$y(s) = R^*_{\beta,\gamma}(b-s)y_1 + \int_s^b R^*_{\beta,\gamma}(\tau - s)(B^*u(\tau) + f(\tau, y(\tau)))\mathrm{d}\tau, s \in J'. \tag{3.4}$$

By virtue of Lemma 3.1, the concept of mild solution is reasonable. For simplicity, set

$$S(u) = \big\{y \in C^{\beta,\gamma}(J,H) : y \text{ satisfies (3.4)}\big\}.$$

Now, we are ready to propose and analyze the partial-approximate controllability problem of system (1.1).

Definition 3.2. The reachable set of system (1.1) is

$$K_0(f) = \{y(0,u,f) : y(\cdot,u,f) \in S(u)\}.$$

Moreover, If $\overline{K_0(f)} = H$, system (1.1) is approximately controllable on J.

Definition 3.3 ([5]). Given $y_0 \in E$ and $\varepsilon > 0$. System (1.1) is partial-approximately controllable if there exists a control $u \in L^2(J,U)$ to guarantee that $y \in S(u)$ satisfies $\|\Pi y(0) - y_0\| < \varepsilon$, where E is a closed subspace of H and the symbol Π means the projection from H onto E.

Remark 3.1. If $E = H$, the concept of partial-approximate controllability becomes the notion of approximate controllability. In addition, system (1.1) is approximately controllable if and only if $\varphi = 0$ whenever $BR_{\beta,\gamma}(t)\varphi = 0$ for any $t \in (0,b)$ (see [18]).

As we all know, the construction of control u is linked to the solutions of a sequence of optimal control problems and the construction of the cost functional depends on the expression of the mild solution. Following this idea, for $\varepsilon > 0$, $\zeta \in H$ and a closed subspace $E \subseteq H$, we put

$$J_\varepsilon(\varphi; y) = \frac{1}{2}\int_0^b \|BR_{\beta,\gamma}(\tau)\Pi^*\varphi\|^2 \mathrm{d}\tau + \varepsilon\|\varphi\| - \langle \varphi, h(y)\rangle, \varphi \in E, \tag{3.5}$$

where

$$h(y) = y_0 - \Pi R^*_{\beta,\gamma}(b)y_1 - \int_0^b \Pi R^*_{\beta,\gamma}(\tau) f(\tau, y(\tau))\mathrm{d}\tau.$$

Theorem 3.1. *Let (HA), (HB) and (Hf) hold. If, in addition, $\overline{K_0(0)} = H$, then system (1.1) is partial-approximately controllable on J.*

To deal with the partial-approximately controllable problem more transparent, we split our proof into the following lemmas.

Lemma 3.3. *For the functional J_ε, the following statements are valid:*

(*a*) *The map $\varphi \to J_\varepsilon(\varphi; y)$ is continuous and strictly convex;*

(*b*) *For any $r \geq 0$, we have*

$$\varliminf_{\|\varphi\|\to\infty} \inf_{z \in B(0,r)} \frac{J_\varepsilon(\varphi; z)}{\|\varphi\|} \geq \varepsilon,$$

where $B(0,r) = \{y \in C^{\beta,\gamma}(J,H) : \|y\|_{\beta,\gamma} \leq r\}$.

Proof. (a) Due to the definition of J_ε, it is apparent that the map is strictly convex and continuous.

(b) Assume that the statement is false. Then we can select sequences $\{z_n\} \subseteq B(0,r)$ and $\{\varphi_n\} \subseteq H$, with $\|\varphi_n\| \to \infty$, to guarantee that

$$\varliminf_{n\to\infty} \frac{J_\varepsilon(\varphi_n; z_n)}{\|\varphi_n\|} < \varepsilon. \tag{3.6}$$

Thanks to Lemma 3.2, $\{h(z_n)\}_{n\geq 1}$ is relatively compact. Thus, we can pick a subsequence from $\{h(z_n)\}_{n\geq 1}$, still denoted by it, such that

$$h(z_n) \to h \text{ in } H. \tag{3.7}$$

Putting $\widetilde{\varphi_n} = \frac{\varphi_n}{\|\varphi_n\|}$, we get $\|\widetilde{\varphi_n}\| = 1$. Hence, we can assume that

$$\widetilde{\varphi_n} \xrightarrow{w} \widetilde{\varphi} \text{ in } H.$$

As such, according to the compactness of $\{s^{(1-\beta)(1-\gamma)} R_{\beta,\gamma}(s)\}_{s>0}$, we can deduce that

$$(\cdot)^{(1-\beta)(1-\gamma)} BR_{\beta,\gamma}(\cdot)\Pi^* \widetilde{\varphi_n} \to (\cdot)^{(1-\beta)(1-\gamma)} BR_{\beta,\gamma}(\cdot)\Pi^* \widetilde{\varphi} \text{ in } C(J,H). \tag{3.8}$$

To facilitate our analysis, we rewrite (3.5) as

$$\frac{J_\varepsilon(\varphi_n; z_n)}{\|\varphi_n\|} = \frac{\|\varphi_n\|}{2} \int_0^b \|BR_{\beta,\gamma}(\tau)\Pi^* \widetilde{\varphi_n}\|^2 \mathrm{d}\tau + \varepsilon \|\widetilde{\varphi_n}\| - \langle \widetilde{\varphi_n}, h(z_n)\rangle. \tag{3.9}$$

By employing $\|\varphi_n\| \to \infty$, (3.6)-(3.9) and the Fatou lemma, we see at once that

$$\int_0^b \|BR_{\beta,\gamma}(\tau)\Pi^* \widetilde{\varphi}\|^2 \mathrm{d}\tau \leq \varliminf_{n\to\infty} \int_0^b \|BR_{\beta,\gamma}(\tau)\Pi^* \widetilde{\varphi_n}\|^2 \mathrm{d}\tau = 0.$$

Based on $\overline{K_0(0)} = H$, we get $\widetilde{\varphi} = 0$, which implies that $\widetilde{\varphi_n} \xrightarrow{w} 0$, and hence that

$$\varepsilon > \varliminf_{n\to\infty} \frac{J_\varepsilon(\varphi_n; z_n)}{\|\varphi_n\|} \geq \varliminf_{n\to\infty} \left(\varepsilon \|\widetilde{\varphi_n}\| - \langle \widetilde{\varphi_n}, h(z_n)\rangle\right) \geq \varepsilon,$$

which is impossible and proves (b). □

Lemma 3.4. *For any $z \in C^{\beta,\gamma}(J,H)$, the functional $J_\varepsilon(\cdot; z)$ possesses a unique minimum $\widehat{\varphi}_\varepsilon$ that can formulate a map $\Phi_\varepsilon : C^{\beta,\gamma}(J,H) \to H$, defined by $\Phi_\varepsilon(z) = \widehat{\varphi}_\varepsilon$. In addition, the following statements of Φ_ε are valid:*

- (a) *For any $z \in B(0,r)$, there exists $R_\varepsilon > 0$ to ensure that $\|\Phi_\varepsilon(z)\| < R_\varepsilon$.*
- (b) *For any $z_n, z \in B(0,r)$ satisfying $z_n \to z$ in $C^{\beta,\gamma}(J,H)$, we have*

$$\lim_{n\to\infty} \|\Phi_\varepsilon(z_n) - \Phi_\varepsilon(z)\| = 0.$$

Proof. Owing to Lemma 3.1, the map Φ_ε is reasonable. Next, we check the statements of Φ_ε.

For statement (a). Due to (b) of Lemma 3.3, we can select a constant $R_\varepsilon > 0$ to ensure that

$$\inf_{z\in B(0,r)} \frac{J_\varepsilon(\varphi; z)}{\|\varphi\|} \geq \frac{\varepsilon}{2}, \quad \text{for any } \|\varphi\| \geq R_\varepsilon. \tag{3.10}$$

Employing (3.5) and the definition of Φ_ε leads to

$$J_\varepsilon(\Phi_\varepsilon(z); z) \leq J_\varepsilon(0; z) = 0. \tag{3.11}$$

Hence, combining (3.10) with (3.11) gives $\|\Phi_\varepsilon(z)\| < R_\varepsilon$.

For statement (b). Put

$$\overline{\eta}_{\varepsilon,n} = \Phi_\varepsilon(z_n) \text{ and } \overline{\eta}_\varepsilon = \Phi_\varepsilon(\mathrm{z}). \tag{3.12}$$

By virtue of statement (a), $\{\overline{\eta}_{\varepsilon,n}\}_{n\geq 1}$ is bounded. We thereby can suppose that $\overline{\eta}_{\varepsilon,n} \stackrel{w}{\to} \widetilde{\eta}$. Moreover, due to (HA) and (Hf), we have

$$\begin{aligned}
&\|h(z_n) - h(z)\| \\
\leq\ & M \int_0^b \tau^{-(1-\beta)(1-\gamma)} \|\Pi\| \|f(\tau, z_n(\tau)) - f(\tau, z(\tau))\| \mathrm{d}\tau \\
\leq\ & M \sqrt{\frac{b^{2(\beta+\gamma(1-\beta))-1}}{2(\beta+\gamma(1-\beta))-1}} \|\Pi\| \left(\int_0^b \|f(\tau, z_n(\tau)) - f(\tau, z(\tau))\|^2 \mathrm{d}\tau \right)^{\frac{1}{2}} \\
\to\ & 0.
\end{aligned}$$

Thus, using (3.12) and the definition of J deduces that

$$\begin{aligned}
J_\varepsilon(\overline{\eta}_\varepsilon; z) \leq J_\varepsilon(\widetilde{\eta}; z) &\leq \varliminf_{n\to\infty} J_\varepsilon(\overline{\eta}_{\varepsilon,n}; z_n) \\
&\leq \varlimsup_{n\to\infty} J_\varepsilon(\overline{\eta}_{\varepsilon,n}; z_n) \leq \lim_{n\to\infty} J_\varepsilon(\overline{\eta}_\varepsilon; z_n) = J_\varepsilon(\overline{\eta}_\varepsilon; z),
\end{aligned}$$

which indicates that $J_\varepsilon(\overline{\eta}_\varepsilon; z) = J_\varepsilon(\widetilde{\eta}; z)$, and hence we conclude from the uniqueness of the minimum that $\overline{\eta}_\varepsilon = \widetilde{\eta}$. As such, we have

$$\begin{cases}
\lim\limits_{n\to\infty} J_\varepsilon(\overline{\eta}_{\varepsilon,n}; z_n) = J_\varepsilon(\overline{\eta}_\varepsilon; z), \\
\lim\limits_{n\to\infty} \int_0^b \|BR_{\beta,\gamma}(\tau)\Pi^* \overline{\eta}_{\varepsilon,n}\|^2 \mathrm{d}\tau = \int_0^b \|BR_{\beta,\gamma}(\tau)\Pi^*\overline{\eta}_\varepsilon\|^2 \mathrm{d}\tau, \\
\lim\limits_{n\to\infty} \langle \overline{\eta}_{\varepsilon,n}, h(z_n)\rangle = \langle \overline{\eta}_\varepsilon, h(z)\rangle, \ \|\overline{\eta}_\varepsilon\| \leq \lim\limits_{n\to\infty} \|\overline{\eta}_{\varepsilon,n}\|,
\end{cases}$$

which implies that

$$\lim_{n\to\infty} \|\overline{\eta}_{\varepsilon,n}\| = \|\overline{\eta}_\varepsilon\|.$$

Noting that H is a Hilbert space, we achieve $\overline{\eta}_{\varepsilon,n} \to \overline{\eta}_\varepsilon$, in light of $\overline{\eta}_{\varepsilon,n} \stackrel{w}{\to} \overline{\eta}$. □

Now, we return to Theorem 3.1 and analyze this problem.

Proof of Theorem 3.1. For any $y \in C^{\beta,\gamma}(J, H)$, set $J_\varepsilon(\overline{\eta}; y) = \min\limits_{\eta\in H} J_\varepsilon(\eta; y)$. Due to Lemma 3.4, we have

$$\overline{\eta} = \Phi_\varepsilon(y) \text{ and } J_\varepsilon(\overline{\eta}; y) \leq J_\varepsilon(\overline{\eta} + \lambda\eta; y)$$

for any $\eta \in H$ and $\lambda \in \mathbb{R}$. Applying the definition of J_ε deduces that

$$\begin{aligned}
\lambda\langle \eta, h(y)\rangle \leq &\frac{\lambda^2}{2} \int_0^b \langle BR_{\beta,\gamma}(\tau)\Pi^*\eta, BR_{\beta,\gamma}(\tau)\Pi^*\eta\rangle \mathrm{d}\tau \\
&+ \lambda \int_0^b \langle BR_{\beta,\gamma}(\tau)\Pi^*\overline{\eta}, BR_{\beta,\gamma}(\tau)\Pi^*\eta\rangle \mathrm{d}\tau
\end{aligned}$$

$$+\varepsilon(\|\overline{\eta}+\lambda\eta\|-\|\overline{\eta}\|). \tag{3.13}$$

Dividing (3.13) by $\lambda > 0$ and letting $\lambda \to 0^+$, we come to the result that

$$\begin{aligned}\langle \eta, h(y)\rangle \leq & \int_0^b \langle BR_{\beta,\gamma}(\tau)\Pi^*\overline{\eta}, BR_{\beta,\gamma}(\tau)\Pi^*\eta\rangle \mathrm{d}\tau + \varepsilon \varliminf_{\lambda\to 0^+} \frac{\|\overline{\eta}+\lambda\eta\|-\|\overline{\eta}\|}{\lambda} \\ \leq & \int_0^b \langle BR_{\beta,\gamma}(\tau)\Pi^*\overline{\eta}, BR_{\beta,\gamma}(\tau)\Pi^*\eta\rangle \mathrm{d}\tau + \varepsilon\|\eta\|.\end{aligned}$$

Repeating the same calculations with $\lambda \to 0^-$ gives rise to

$$\left|\int_0^b \langle BR_{\beta,\gamma}(\tau)\Pi^*\overline{\eta}, BR_{\beta,\gamma}(\tau)\Pi^*\eta\rangle \mathrm{d}\tau - \langle \eta, h(y)\rangle\right| \leq \varepsilon\|\eta\|. \tag{3.14}$$

Set

$$\begin{aligned}&\omega(s;y) = R^*_{\beta,\gamma}(b-s)y_1 + \int_s^b R^*_{\beta,\gamma}(\tau-s)f(\tau,y(\tau))\mathrm{d}\tau, s\in J',\\ &h(y) = y_0 - \Pi\omega(0;y) \text{ and } \overline{u}(\tau) = BR_{\beta,\gamma}(\tau)\Pi^*\Phi_\varepsilon(y),\\ &Q(y)(s) = \omega(s;y) + \int_s^b R^*_{\beta,\gamma}(\tau-s)B^*\overline{u}(\tau)\mathrm{d}\tau.\end{aligned}$$

Then, the proof will be completed by verifying that the map $Q : C^{\beta,\gamma}(J,H) \to C^{\beta,\gamma}(J,H)$ admits fixed points. Let us check this. Due to (HA) and Lemma 3.2, $\{h(y) : y \in B(0,r)\}$ is compact. From Lemma 3.4, one can acquire the boundedness and continuity of $\{\Phi_\varepsilon(y) : y \in B(0,r)\}$. Thus, we can conclude that Q is a continuous and compact map with the image being bounded. We thereby achieve that Q admits fixed points, by the Schauder fixed point theorem. Let y be a fixed point. Then, we have

$$\int_0^b \Pi R^*_{\beta,\gamma}(\tau)B^*BR_{\beta,\gamma}(\tau)\Pi^*\overline{\eta}\mathrm{d}\tau - h(y) = \Pi y(0) - y_0,$$

which together with (3.14) implies that for any $\eta \in H$,

$$|\langle \eta, \Pi y(0) - y_0\rangle| \leq \varepsilon\|\eta\|.$$

We thereby achieve that

$$\|\Pi y(0) - y_0\| \leq \varepsilon$$

and the proof is finished. □

Remark 3.2. By the variational approach and semigroup technique, [19] displayed the partial-approximate controllability result of the fractional evolution equation with nonlocal conditions. In our work, by adopting the resolvent technique and the variational approach, we have displayed the partial-approximate controllability result for backward system (1.1). Due to Remarks 2.1, 2.2 and 2.4, we emphasize that our results can unify the partial-approximate controllability problem of fractional evolution systems with Caputo type and the problem of fractional equations with Riemann-Liouville type.

Remark 3.3. If we take $\gamma = 1$ and $E = H$ in Theorem 3.1, then we obtain the approximate controllability result in [21]. Emphasis here is that our approach differs from the approach in [21]. The author in [21] constructed the control function related to the resolvent condition established in [4] to deal with the approximate controllability problem.

4. An application

We here provide a partial-approximate controllability example to illustrate the usefulness of our theoretical results.

Consider the following Hilfer fractional backward control system:

$$\begin{cases} D_1^{\beta,\gamma} y(s,x) = \frac{\partial^2}{\partial x^2} y(s,x) + J_1^{\gamma(1-\beta)}(f(s,y(s,x)) + u(s,x)), \ x \in (0,1), \ s \in [0,1), \\ y(s,0) = y(s,1) = 0, \\ \lim\limits_{s \to 1^-} \Gamma(\beta + \gamma(1-\beta))(1-s)^{(1-\beta)(1-\gamma)} y(s,x) = g(x) = \sum\limits_{k=1}^{\infty} c_k \sin k\pi x. \end{cases} \tag{4.1}$$

Fix $H = U = L^2(0,1)$, $e_k(x) = \sqrt{2}\sin(k\pi x)$, $k = 1, 2, \cdots$ and $A = \frac{\partial^2}{\partial x^2}$ with

$$D(A) = \{\xi \in H : \xi', \ \xi'' \in H \text{ and } \xi(0) = \xi(1) = 0\}.$$

Then A is a self-adjoint operator and A generates an analytic compact semigroup $\{T(s)\}_{s>0}$ (see [22]):

$$T(s)g(x) = \sum_{n=1}^{\infty} e^{-n^2\pi^2 s} \langle g, e_n \rangle e_n(x). \tag{4.2}$$

Moreover, A also generates a β-order and γ-type resolvent $\{R_{\beta,\gamma}(s)\}_{s>0}$ (see [20]):

$$R_{\beta,\gamma}(s)g(x) = \sum_{n=1}^{\infty} s^{\beta+\gamma(1-\beta)-1} E_{\beta,\beta+\gamma(1-\beta)}(-n^2\pi^2 s^\beta) \langle g, e_n \rangle e_n(x). \tag{4.3}$$

By employing (4.2), (4.3) and Laplace transformations, we can assert that

$$R_{\beta,\gamma}(s) = J^{\gamma(1-\beta)} \left(s^{\beta-1} \int_0^\infty \beta\tau \Psi_\beta(\tau) T(s^\beta \tau) \mathrm{d}\tau \right), \tag{4.4}$$

where

$$\Psi_\beta(\tau) = \sum_{k=0}^{\infty} \frac{(-\tau)^k}{k!\Gamma(-\beta k + 1 - \beta)} = \frac{1}{\beta} \tau^{-1-\frac{1}{\beta}} \varpi_\beta(\tau^{-\frac{1}{\beta}}),$$

$$\varpi_\beta(\tau) = \frac{1}{\pi} \sum_{m=0}^{\infty} (-1)^m \tau^{-(m+1)\beta-1} \frac{\Gamma((m+1)\beta+1)}{(m+1)!} \sin\left((m+1)\pi\beta\right), \ \tau \in \mathbb{R}_+.$$

On account of the compactness of $\{T(s)\}_{s>0}$ and $\|T(s)\| \le 1$, we deduce that $\{s^{(1-\beta)(1-\gamma)} R_{\beta,\gamma}(s)\}_{s>0}$ is equicontinuous and compact (see [29, 32]). In addition, due to (4.4), we can select $M > 0$ and $\omega > 0$ to guarantee that $\|R_{\beta,\gamma}(s)\| \le Me^{\omega s}$ for $s \ge s_0$, which indicates that (HA) holds.

Since A is self-adjoint, we can deduce that A^* generates an analytic compact semigroup $\{T^*(s)\}_{s>0} = \{T(s)\}_{s>0}$, and hence that A^* also generates a β-order and γ-type resolvent $\{R^*_{\beta,\gamma}(s)\}_{s>0} = \{R_{\beta,\gamma}(s)\}_{s>0}$.

Set $y(s)(x) = y(s,x)$ and $B^* = I$. Consider the following control map:

$$\Lambda_0^1 = \int_0^1 R^*_{\beta,\gamma}(s) R_{\beta,\gamma}(s)\mathrm{d}s.$$

Let $R_{\beta,\gamma}(s)y = 0, y \in H$. Then we have

$$\langle \Lambda_0^1 y, y\rangle = \int_0^1 \|R_{\beta,\gamma}(s)y\|^2 \mathrm{d}s = 0,$$

which implies that

$$\sum_{n=1}^{\infty} \int_0^1 s^{2(\beta+\gamma(1-\beta))-2} E^2_{\beta,\beta+\gamma(1-\beta)}(-n^2\pi^2 s^\beta)\mathrm{d}s \langle y, e_n\rangle^2 = 0.$$

According to

$$\begin{aligned}
&\int_0^1 s^{2(\beta+\gamma(1-\beta))-2} E^2_{\beta,\beta+\gamma(1-\beta)}(-n^2\pi^2 s^\beta)\mathrm{d}s \\
\geq &\left(\int_0^1 s^{\beta+\gamma(1-\beta)-1} E_{\beta,\beta+\gamma(1-\beta)}(-n^2\pi^2 s^\beta)\mathrm{d}s\right)^2 \\
\geq &E^2_{\beta,\beta+\gamma(1-\beta)+1}(-n^2\pi^2) > 0,
\end{aligned}$$

we have $y = 0$, which shows that $\overline{K_0(f)} = H$.

Define $f : [0,1] \times H \to H$ by $f(t,y)(x) = f(t, y(t,x))$. Let condition (Hf) hold. Then due to Theorem 3.1, system (4.1) is partial-approximately controllable.

References

[1] W. Arendt, C. Batty, M. Hieber and F. Neubrander, *Vector-valued Laplace Transforms and Cauchy Problems*, Birkhäuser Verlag, Basel, 2001.

[2] E. Bajlekova, *Fractional evolution equations in Banach spaces (Ph.D. thesis)*, University Press Facilities, Eindhoven University of Technology, 2001.

[3] P. Balasubramaniam and P. Tamilalagan, *Approximate controllability of a class of fractional neutral stochastic integro-differential inclusions with infinite delay by using Mainardi's function*, Appl. Math. Comput., 2015, 256, 232–246.

[4] A. E. Bashirov and N. I. Mahmudov, *On concepts of controllability for linear deterministic and stochastic systems*, SIAM J. Control Optim., 1999, 37, 1808–1821.

[5] A. E. Bashirov, N. I. Mahmudov, N. Semi and H. Etikan, *Partial controllability concepts*, Int. J. Control, 2007, 80, 1–7.

[6] J. Baumeister, *Stable Solution of Inverse Problems*, Braunschweig, Vieweg, 1987.

[7] J. B. Conway, *A Course in Functional Analysis (2nd edition)*, Springer-Verlag, New York, 1990.

[8] G. Da Prato and M. Iannelli, *Linear abstract integrodifferential equations of hyperbolic type in Hilbert spaces*, Rend. Sem. Mat. Padova, 1980, 62, 191–206.

[9] G. Da Prato and M. Iannelli, *Linear integrodifferential equations in Banach spaces*, Rend. Sem. Mat. Padova, 1980, 62, 207–219.

[10] J. P. Dauer, N. I. Mahmudov and M. M. Matar, *Approximate controllability of backward stochastic evolution equations in Hilbert spaces*, J. Math. Anal. Appl., 2006, 323, 42–56.

[11] M. Du and Z. Wang, *Initialized fractional differential equations with Riemann-Liouville fractional-order derivative*, Eur. Phys. J. Special Topics, 2011, 193, 49–60.

[12] Z. Fan, *Characterization of compactness for resolvents and its applications*, Appl. Math. Comput., 2014, 232, 60–67.

[13] H. Gu and J. J. Trujillo, *Existence of mild solution for evolution equation with Hilfer fractional derivative*, Appl. Math. Comput., 2015, 257, 344–354.

[14] Y. Jiang, Q. Zhang, A. Chen and Z. Wei, *Sensitivity analysis of optimal control problems governed by nonlinear Hilfer fractional evolution inclusions*, Appl. Math. Optim., 2021, 84, 3045–3082.

[15] R. Kamocki, *On generalized fractional integration by parts formulas and their applications to boundary value problems*, Georgian Math. J., 2019. D oi:10.1515/gmj-2019-2006.

[16] S. Kumar and N. Sukavanam, *Approximate controllability of fractional order semilinear systems with bounded delay*, J. Diff. Equ., 2012, 252(11), 6163–6174.

[17] K. Li and J. Peng, *Fractional resolvents and fractional evolution equations*, Appl. Math. Lett., 2012, 25, 808–812.

[18] X. Li and J. Yong, *Optimal Control Theory for Infinite Dimensional Systems*, Birkhäuser, Basel, 1994.

[19] N. I. Mahmudov, *Partial-approximate controllability of nonlocal fractional evolution equations via approximating method*, Appl. Math. Comput., 2018, 334, 227–238.

[20] Z. Mei, J. Peng and Y. Zhang, *On general fractional abstract Cauchy problem*, Commun. Pur. Appl. Anal., 2013, 12(6), 2753–2772.

[21] G. Mophou, *Controllability of a backward fractional semilinear differential quation*, Appl. Math. Comput., 2014, 242, 168–178.

[22] Z. Pazy, *Semigroups of linear operators and applications to partial differential equations*, Springer-Verlag, New York, 1983.

[23] I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, 1999.

[24] J. Prüss, *Evolutionary Integral Equations and Applications*, Birkhäuser, Basel, 1993.

[25] J. Sabatier, C. Farges and J. C. Trigeassou, *Fractional systems state space description: some wrong ideas and proposed solutions*, J. Vib. Control, 2014, 20(7), 1076–1084.

[26] J. Wang and Y. Zhou, *A class of fractional evolution equations and optimal controls*, Nonlinear Anal., 2011, 12, 262–272.

[27] J. Wang and Y. Zhou, *Existence and controllability results for fractional semilinear differential inclusions*, Nonlinear Anal., 2011, 12, 3642–3653.

[28] R. Wang, D. Chen and T. Xiao, *Abstract fractional Cauchy problems with almost sectorial operators*, J. Diff. Equ., 2012, 252, 202–235.

[29] M. Yang and Q. Wang, *Approximate controllability of Hilfer fractional differential inclusions with nonlocal conditions*, Math. Methods Appl. Sci., 2016. Doi:10.1002/mma.4040.

[30] E. Zeidler, *Nonlinear Functional Analysis and Its Application II/A*, Springer-Verlag, New York, 1990.

[31] J. Zhang and Y. Li, *Duality theory of regularized resolvents operator family*, J. Appl. Anal. Comput., 2011, 1, 279–290.

[32] Y. Zhou and F. Jiao, *Existence of mild solutions for fractional neutral evolution equations*, Comput. Math. Appl., 2010, 59(3), 1063–1077.

[33] S. Zhu and G. Li, *Duality theory of fractional resolvents and applications to backward fractional control systems*, Fract. Calc. Appl. Anal., 2021, 24(2), 542–558.

[34] S. Zhu, Z. Fan and G. Li, *Optimal controls for Riemann-Liouville fractional evolution systems without Lipschitz assumption*, J. Optim. Theory Appl., 2017, 174, 47–64.

[35] S. Zhu, Z. Fan and G. Li, *Topological characteristics of solution sets for fractional evolution equations and applications to control systems*, Topol. Methods Nonlinear Anal., 2019. Doi:10.12775/TMNA.2019.033.