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LAGRANGE INTERPOLATION ON TIME SCALES

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Abstract In this paper, we introduce the Lagrange interpolation polynomials on time scales. We define an alternative type of interpolation functions called σ-Lagrange interpolation polynomials. We discuss some properties of these polynomials and show that on some special time scales, including the set of real numbers, these two types of interpolation polynomials coincide. We apply our results on some particular examples.

Keywords Time scale, Lagrange interpolation, delta derivative.

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1. Introduction

Initiated by Hilger [16, 17], the studies on measure chains and time scales aim to unify the continuous and discrete calculus. Consequently, both one-variable and multi-vaviable calculus on time scales have been completely developed [4, 5]. The dynamic and integral equations on time scales have been studied in details both theoretically and in connection with their applications in various sciences [3, 11, 12, 14]. Recently, a book on fractional dynamic equations on time scales have been published [13].

On the other hand, there is still a gap in the literature on numerical analysis and methods on time scales. In some recent studies, numerical methods such as Euler's method, Taylor series method for dynamic equations on time scales have been introduced [6, 15]. However, there are not many studies related with numerical methods on time scales reported so far. In particular, there is a gap in the studies on Lagrange polynomials and Lagrange interpolation method on time scales.

The Lagrange interpolation is a very useful method to represent a discrete set of data by a polynomial. Therefore, it is widely used both alone [7, 26] and in combination with various numerical algorithms to solve problems of different kind modelled by differential, integral or fractional differential equations, optimization problems and other engineering problems [18, 19, 22]. Recently, Lagrange polynomials have been used for deriving a formula for representing a given set of numerical data on a pair of variables by a suitable polynomial(see [9] and references therein). Barycentric Lagrange interpolation is used for solving Volterra integral equations of the second kind(see [20] and references therein). Barycentric Lagrange interpolation is used for vibration analysis of the plate with the regular and irregular domain(see [23] and references therein), for investigation of a 2D higher-order

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time-fractional telegraph equation with nonlocal boundary condition(see [24] and references therein). Barycentric Lagrange interpolation collocation method is used for investigation of a hyperchaotic system(see [25] and references therein). The pseudospectral methods on the other hand, are also based on Lagrange polynomial interpolation (see [8, 10, 21]). These methods have been succesfully applied to Sturm- Liouville and Schrödinger type eigenvalue problems in both one and two dimensions (see [1, 2] for details).

In this study, we present the Lagrange interpolation polynomials on time scales. We propose two types of Lagrange interpolation polynomials which we define in Sections 3 and 4 and we give examples. Section 5 contains conclusion and directions for further study.

2. Preliminaries

The two main features of time scales are unification of the discrete and continuous problems and extension of the existing theory on continuous problems to discrete ones. Numerical methods on the other hand, are usually based on discretization of the continuous structures involved in the problem. In this sense, development of numerical methods on time scales is mainly related with the extension feature of time scales, so that, a numerical method constructed for an arbitrary time scale reduces to its known counterpart whenever the time scale under consideration is the set of real numbers.

In this study, we assume that the readers are familiar with the basic notions on time scales. However, for the sake of completeness, we briefly recall some of these concepts below.

Definition 2.1 ([4]).

1. Any nonempty closed subset of the set of real numbers $\mathbb{R}$ is called a time scale and is usually denoted by $\mathbb{T}$.
2. The function $\sigma : \mathbb{T} \longrightarrow \mathbb{T}$ defined as $\sigma(t) = \inf\{s \in \mathbb{T} : s > t\},\ t \in \mathbb{T}$, is called the forward jump operator.
3. The function $\rho : \mathbb{T} \longrightarrow \mathbb{T}$, defined as $\rho(t) = \sup\{s \in \mathbb{T} : s < t\},\ t \in \mathbb{T}$, is called the backward jump operator.
4. The function $\mu : \mathbb{T} \longrightarrow [0, \infty)$ defined as $\mu(t) = \sigma(t) - t,\ t \in \mathbb{T}$, is called the graininess function.
5. A point $t \in \mathbb{T}$ is
   (a) right scattered if $\sigma(t) > t$;
   (b) left scattered if $\rho(t) < t$;
   (c) isolated if it is both right and left scattered;
   (d) right dense if $\sigma(t) = t$ and $t < \sup \mathbb{T}$;
   (e) left dense if $\rho(t) = t$ and $t > \inf \mathbb{T}$;
   (f) dense if it is both right and left dense.

In this study, we assume that $\mathbb{T}$ is a time scale with forward jump operator σ, backward jump operator ρ, graininess function μ and delta differentiation operator Δ. For readers who are not familiar with time scale calculus we suggest the monographs [4, 5].

In order to give the definiton and the properties of the Lagrange interpolation polynomial we need the definition of generalized zeros of functions on time scales and the Rolle Theorem on time scales. Below we discuss the related theoretical background.

Definition 2.2 ([4]). Let $y : \mathbb{T} \to \mathbb{R}$ be $(k-1)$ times delta differentiable function for some $k \in \mathbb{N}$. We say that y has a generalized zero (GZ) of order greater than or equal to k at $t \in \mathbb{T}^{\kappa^{k-1}}$ provided

$$y^{\Delta^i}(t) = 0, \quad i \in \{0, \dots, k-1\}, \tag{2.1}$$

or

$$y^{\Delta^i}(t) = 0 \quad \text{for} \quad i \in \{0, \dots, k-2\} \quad \text{and} \quad y^{\Delta^{k-1}}(\rho(t)) y^{\Delta^{k-1}}(t) < 0 \tag{2.2}$$

holds.

Remark 2.1. Note that in the above definition, the case (2.1) is related with dense points and in case (2.2) t must be left-scattered. Otherwise, $\rho(t) = t$ and (2.2) yields

$$0 > y^{\Delta^{k-1}}(\rho(t)) y^{\Delta^{k-1}}(t) = \left(y^{\Delta^{k-1}}(t)\right)^2 \geq 0,$$

which is a contradiction.

Theorem 2.1 ([4]). *The condition* (2.2) *holds if and only if*

$$y^{\Delta^j}(t) = 0, \quad j \in \{0, \dots, k-2\}, \quad \text{and} \quad (-1)^{k-1} y(\rho(t)) y^{\Delta^{k-1}}(t) < 0. \tag{2.3}$$

Theorem 2.2 ([4]). *Let $j \in \mathbb{N}_0$ and $t \in \mathbb{T}^{\kappa^j}$. Then*

$$y^{\Delta^i}(t) = 0, \quad 0 \leq i \leq j, \tag{2.4}$$

if and only if

$$y^{\Delta^i}(\sigma^l(t)) = 0, \quad 0 \leq i \leq j - l, \quad 0 \leq l \leq j. \tag{2.5}$$

In this case,

$$y^{\Delta^{j+1-l}}(\sigma^l(t)) = \prod_{s=0}^{l-1} \mu(\sigma^s(t)) y^{\Delta^{j+1}}(t). \tag{2.6}$$

Definition 2.3 ([4]). If y has a GZ of order greater than or equal to k at t we will say that y has at least k GZs, counting multiplicities. By Theorem 2.2, it follows that if y has a GZ of order greater than or equal to k at t, then y has a GZ of order greater than or equal to $k-1$ at $\sigma(t)$. Therefore, if y has a GZ of order greater than or equal to k_1 at t_1 and a GZ of order greater than or equal to k_2 at t_2 and $\sigma^{k_1}(t_1) < t_2$, then we will say that y has at least $k_1 + k_2$ GZs, counting multiplicities.

Theorem 2.3 (Rolle's Theorem, [4]). *If y has at least $k \in \mathbb{N}$ GZs on $[a, b]$, counting multiplicities, then y^Δ has at least $k-1$ GZs on $[a, b]$, counting multiplicities.*

3. Lagrange Interpolation

In this section, we will define the Lagrange interpolation polynomial.

Let $\mathcal{P}_n$, $n \in \mathbb{N}_0$, denote the set of all polynomials of degree $\leq n$ defined over the set $\mathbb{R}$ of real numbers. Assume that $n \in \mathbb{N}$ and $x_i \in \mathbb{T}$, $i \in \{0, 1, \ldots, n\}$, are distinct and y_i, $i \in \{0, 1, \ldots, n\}$, are real numbers. We will find $p_n \in \mathcal{P}_n$ such that $p_n(x_i) = y_i$, $i \in \{0, 1, \ldots, n\}$.

Theorem 3.1. *Suppose that $n \in \mathbb{N}$. Then there exist polynomials $L_k \in \mathcal{P}_n$, $k \in \{0, 1, \ldots, n\}$, such that*

$$L_k(x_i) = \begin{cases} 1 & \text{if} \quad i = k \\ 0 & \text{if} \quad i \neq k, \end{cases}$$

$i, k \in \{0, 1, \ldots, n\}$. Moreover,

$$p_n(x) = \sum_{k=0}^{n} L_k(x) y_k$$

satisfies the condition $p_n(x_i) = y_i$, $i \in \{0, 1, \ldots, n\}$, $p_n \in \mathcal{P}_n$.

Proof. Define

$$L_k(x) = C_k \sum_{i=0, i \neq k}^{n} (x - x_i),$$

where $C_k \in \mathbb{R}$, $k \in \{0, 1, \ldots, n\}$, will be determined below. We have $L_k(x_i) = 0$, $i \in \{0, 1, \ldots, n\}$, $i \neq k$, and

$$\begin{aligned} L_k(x_k) &= C_k \prod_{i=0, i \neq k}^{n} (x_k - x_i) \\ &= 1, \quad k \in \{0, 1, \ldots, n\}. \end{aligned}$$

Thus,

$$C_k = \frac{1}{\prod_{i=0, i \neq k}^{n} (x_k - x_i)}, \quad k \in \{0, 1, \ldots, n\},$$

and hence

$$L_k(x) = \prod_{i=0, i \neq k}^{n} \frac{x - x_i}{x_k - x_i}, \quad k \in \{0, 1, \ldots, n\}. \tag{3.1}$$

We have that $L_k \in \mathcal{P}_n$, $k \in \{0, 1, \ldots, n\}$, and $p_n \in \mathcal{P}_n$. This completes the proof. □

Theorem 3.2. *Assume that $n \in \mathbb{N}_0$. Let $x_i \in \mathbb{T}$, $i \in \{0, 1, \ldots, n\}$, be distinct and $y_i \in \mathbb{R}$, $i \in \{0, 1, \ldots, n\}$. Then there exists a unique polynomial $p_n \in \mathcal{P}_n$ such that*

$$p_n(x_i) = y_i, \quad i \in \{0, 1, \ldots, n\}.$$

Proof. The existence of the polynomial p_n follows by Theorem 3.1. Suppose that there exist two polynomials $p_n, q_n \in \mathcal{P}_n$ such that

$$p_n(x_i) = q_n(x_i) = y_i, \quad i \in \{0, 1, \ldots, n\}.$$

Then the polynomial $h_n = p_n - q_n$ has $n + 1$ distinct roots. Therefore $h_n \equiv 0$ or $p_n \equiv q_n$. This completes the proof. □

Definition 3.1. Assume that $n \in \mathbb{N}_0$. Let $x_i \in \mathbb{T}$, $i \in \{0, 1, \ldots, n\}$, be distinct and $y_i \in \mathbb{R}$, $i \in \{0, 1, \ldots, n\}$. The polynomial

$$p_n(x) = \sum_{k=0}^{n} L_k(x) y_k,$$

where L_k, $k \in \{0, 1, \ldots, n\}$, are defined with (3.1), will be called the Lagrange interpolation polynomial of degree n with interpolation points (x_i, y_i), $i \in \{0, 1, \ldots, n\}$.

Definition 3.2. Assume that $n \in \mathbb{N}_0$. Let $x_i \in [a, b] \subset \mathbb{T}$, $i \in \{0, 1, \ldots, n\}$, be distinct and $f : [a, b] \to \mathbb{R}$ be a given function. The polynomial

$$p_n(x) = \sum_{k=0}^{n} L_k(x) f(x_k),$$

where L_k, $k \in \{0, 1, \ldots, n\}$, are defined in (3.1), will be called the Lagrange interpolation polynomial of degree n with interpolation points x_i, $i \in \{0, 1, \ldots, n\}$, for the function f.

Suppose that $n \in \mathbb{N}_0$ and $x_j \in \mathbb{T}$, $j \in \{0, 1, \ldots, n\}$, are distinct points. For $x \in \mathbb{T}$, we denote the polynomials

$$\pi_{n+1}(x) = \prod_{j=0}^{n} (x - x_j), \quad \Pi_{n+1}^{k}(x) = \pi_{n+1}^{\Delta^k}(x), \quad k \in \mathbb{N}_0,$$

which we will employ in the next theorem related with the error in the Lagrange interpolation.

Theorem 3.3. *Suppose that $n \in \mathbb{N}_0$, $a, b \in \mathbb{T}$, $a < b$, $x_j \in [a, b]$, $j \in \{0, 1, \ldots, n\}$, are distinct and $f : [a, b] \to \mathbb{R}$, $f^{\Delta^k}(x)$ exist for any $x \in [a, b]$ and for any $k \in \{1, \ldots, n+1\}$. Then for any $x \in [a, b]$ there exists $\xi = \xi(x) \in (a, b)$ such that*

$$f(x) - p_n(x) = \frac{f^{\Delta^{n+1}}(\xi)}{\Pi_{n+1}^{n+1}(\xi)} \pi_{n+1}(x)$$

or

$$F_{min,n+1}(\xi) \le \frac{f(x) - p_n(x)}{\pi_{n+1}(x)} \le F_{max,n+1}(\xi),$$

where

$$F_{max,n+1}(\xi) = \max \left\{ \frac{f^{\Delta^{n+1}}(\xi)}{\Pi_{n+1}^{n+1}(\xi)}, \frac{f^{\Delta^{n+1}}(\rho(\xi))}{\Pi_{n+1}^{n+1}(\rho(\xi))} \right\},$$

$$F_{min,n+1}(\xi) = \min \left\{ \frac{f^{\Delta^{n+1}}(\xi)}{\Pi_{n+1}^{n+1}(\xi)}, \frac{f^{\Delta^{n+1}}(\rho(\xi))}{\Pi_{n+1}^{n+1}(\rho(\xi))} \right\}.$$

Proof. Let p_n be the Lagrange interpolation polynomial for the function f with interpolation points x_j, $j \in \{0, 1, \ldots, n\}$. Define the function

$$\phi(t) = f(t) - p_n(t) - \frac{f(x) - p_n(x)}{\pi_{n+1}(x)} \pi_{n+1}(t), \quad t \in [a, b].$$

Then

$$\begin{aligned}\phi(x_j) &= f(x_j) - p_n(x_j) - \frac{f(x) - p_n(x)}{\pi_{n+1}(x)}\pi_{n+1}(x_j)\\ &= f(x_j) - f(x_j)\\ &= 0, \quad j \in \{0, 1, \dots, n\},\end{aligned}$$

and $\phi(x) = 0$. Thus, $\phi : [a, b] \to \mathbb{R}$ has at least $n + 2$ generalized zeros (GZs). Hence and the Rolle theorem 2.3, it follows that $\phi^{\Delta^{n+1}}$ has at least one GZ on (a, b). Therefore there exists $\xi = \xi(x) \in (a, b)$ such that

$$\phi^{\Delta^{n+1}}(\xi) = 0 \quad \text{or} \quad \phi^{\Delta^{n+1}}(\rho(\xi))\phi^{\Delta^{n+1}}(\xi) < 0.$$

Note that

$$\phi^{\Delta^{n+1}}(t) = f^{\Delta^{n+1}}(t) - \frac{f(x) - p_n(x)}{\pi_{n+1}(x)}\pi_{n+1}^{\Delta^{n+1}}(t), \quad t \in [a, b].$$

1. Let $\phi^{\Delta^{n+1}}(\xi) = 0$. Then

$$f^{\Delta^{n+1}}(\xi) = \frac{f(x) - p_n(x)}{\pi_{n+1}(x)}\pi_{n+1}^{\Delta^{n+1}}(\xi)$$

or

$$\begin{aligned}f(x) - p_n(x) &= \frac{f^{\Delta^{n+1}}(\xi)}{\pi_{n+1}^{\Delta^{n+1}}(\xi)}\pi_{n+1}(x)\\ &= \frac{f^{\Delta^{n+1}}(\xi)}{\Pi_{n+1}^{n+1}(\xi)}\pi_{n+1}(x).\end{aligned}$$

2. Let

$$\phi^{\Delta^{n+1}}(\rho(\xi))\phi^{\Delta^{n+1}}(\xi) < 0.$$

Then

$$\begin{aligned}\phi^{\Delta^{n+1}}(\rho(\xi)) &= f^{\Delta^{n+1}}(\rho(\xi)) - \frac{f(x) - p_n(x)}{\pi_{n+1}(x)}\pi_{n+1}^{\Delta^{n+1}}(\rho(\xi))\\ &= f^{\Delta^{n+1}}(\rho(\xi)) - \frac{f(x) - p_n(x)}{\pi_{n+1}(x)}\Pi_{n+1}^{n+1}(\rho(\xi)),\end{aligned}$$

and

$$\phi^{\Delta^{n+1}}(\xi) = f^{\Delta^{n+1}}(\xi) - \frac{f(x) - p_n(x)}{\pi_{n+1}(x)}\Pi_{n+1}^{n+1}(\xi).$$

Hence,

$$\begin{aligned}0 &> \phi^{\Delta^{n+1}}(\rho(\xi))\phi^{\Delta^{n+1}}(\xi)\\ &= \left(f^{\Delta^{n+1}}(\rho(\xi)) - \frac{f(x) - p_n(x)}{\pi_{n+1}(x)}\Pi_{n+1}^{n+1}(\rho(\xi))\right)\end{aligned}$$

$$\times\left(f^{\Delta^{n+1}}(\xi)-\frac{f(x)-p_n(x)}{\pi_{n+1}(x)}\Pi_{n+1}^{n+1}(\xi)\right)$$

$$=\left(\frac{f(x)-p_n(x)}{\pi_{n+1}(x)}\right)^2\Pi_{n+1}^{n+1}(\rho(\xi))\Pi_{n+1}^{n+1}(\xi)$$

$$-\frac{f(x)-p_n(x)}{\pi_{n+1}(x)}\left(\Pi_{n+1}^{n+1}(\rho(\xi))f^{\Delta^{n+1}}(\xi)+\Pi_{n+1}^{n+1}(\xi)f^{\Delta^{n+1}}(\rho(\xi))\right)$$

$$+f^{\Delta^{n+1}}(\rho(\xi))f^{\Delta^{n+1}}(\xi).$$

Hence,

$$F_{min,n+1}(\xi)\leq\frac{f(x)-p_n(x)}{\pi_{n+1}(x)}\leq F_{max,n+1}(\xi).$$

This completes the proof.

□

Remark 3.1. Suppose that all conditions of Theorem 3.3 hold. If

$$\lim_{n\to\infty}\max_{x\in[a,b]}\left(\frac{f^{\Delta^{n+1}}(\xi)}{\Pi_{n+1}^{n+1}(\xi)}\pi_{n+1}(x)\right)=0$$

and

$$\lim_{n\to\infty}\max_{x\in[a,b]}\left(\frac{f^{\Delta^{n+1}}(\rho(\xi))}{\Pi_{n+1}^{n+1}(\rho(\xi))}\pi_{n+1}(x)\right)=0,$$

then

$$\lim_{n\to\infty}\max_{x\in[a,b]}|f(x)-p_n(x)|=0.$$

4. σ-Lagrange Interpolation

In this section, we will show that a given function can be approximated with the so called σ-Lagrange polynomials. We will show that there are classes of time scales for which the Lagrange interpolation polynomials and the σ-Lagrange interpolation polynomials are different and classes of time scales for which these two types of polynomials coincide.

By $\mathcal{P}_n^\sigma$, $n\in\mathbb{N}_0$, we will denote the set of all functions in the form

$$a_n(\sigma(x))^n+a_{n-1}(\sigma(x))^{n-1}+\cdots+a_1(x)\sigma(x)+a_0, \tag{4.1}$$

where $a_j\in\mathbb{R}$, $j\in\{0,1,\ldots,n\}$.

Remark 4.1. Notice that depending on the time scale and on the forward jump operator σ, the expression above may not be a polynomial in the classical sense. For example, on $\mathbb{T}=\mathbb{N}_0^2=\{0,1,4,9,\ldots\}$ we have $\sigma(x)=(\sqrt{x}+1)^2$ and hence, on this time scale the expression (4.1) is not a polynomial.

Definition 4.1. We will call the functions of the form (4.1) σ-polynomials.

Let $a,b\in\mathbb{T}$, $a<b$.

Definition 4.2. Let $n \in \mathbb{N}_0$. The points $x_j \in [a, b)$, $j \in \{0, 1, \dots, n\}$, will be called σ-distinct if $\sigma(x_n) \le b$ and

$$\sigma(x_0) < \sigma(x_1) < \dots < \sigma(x_n).$$

Example 4.1. Let $\mathbb{T} = \{-1, 1\} \bigcup \left\{1 + \left(\frac{1}{2}\right)^n : n \in \mathbb{N}_0\right\} \bigcup \{3, 4, 5\}$ and $a = -1$, $b = 5$. Take the points

$$x_0 = -1, \quad x_1 = 1, \quad x_2 = 3.$$

Then

$$\sigma(x_0) = 1, \quad \sigma(x_1) = 1, \quad \sigma(x_2) = 4.$$

Thus, the points $\{x_0, x_1, x_2\}$ are not σ-distinct.

Example 4.2. Let $\mathbb{T} = 2^{\mathbb{N}_0}$, $a = 1$, $b = 16$. Take the points

$$x_0 = 1, \quad x_1 = 2, \quad x_2 = 4.$$

Then

$$\sigma(x_0) = 2, \quad \sigma(x_1) = 4, \quad \sigma(x_2) = 8.$$

Therefore, $\{x_0, x_1, x_2\}$ are σ-distinct points.

As in the previous section, one can prove the following result.

Theorem 4.1. *Suppose that $n \in \mathbb{N}$ and $x_j \in \mathbb{T}$, $j \in \{0, 1, \dots, n\}$, are σ-distinct. Then there exist unique σ-polynomials $L_{\sigma k} \in \mathcal{P}_n^\sigma$, $k \in \{0, 1, \dots, n\}$, such that*

$$L_{\sigma k}(x_i) = \begin{cases} 1 & \text{if} \quad i = k \\ 0 & \text{if} \quad i \ne k, \end{cases}$$

$i, k \in \{0, 1, \dots, n\}$. Moreover,

$$\begin{aligned} p_{\sigma n}(x) &= \sum_{k=0}^{n} L_{\sigma k}(x) y_k \\ &= \sum_{k=0}^{n} \left(\prod_{j=0, j \ne k}^{n} \frac{\sigma(x) - \sigma(x_j)}{\sigma(x_k) - \sigma(x_j)} \right) y_k \end{aligned}$$

satisfies the condition $p_{\sigma n}(x_i) = y_i$, $i \in \{0, 1, \dots, n\}$, $p_{\sigma n} \in \mathcal{P}_n^\sigma$.

Definition 4.3. Assume that $n \in \mathbb{N}_0$. Let $x_i \in \mathbb{T}$, $i \in \{0, 1, \dots, n\}$, be σ-distinct and $y_i \in \mathbb{R}$, $i \in \{0, 1, \dots, n\}$. The σ-polynomial

$$p_{\sigma n}(x) = \sum_{k=0}^{n} L_{\sigma k}(x) y_k,$$

where $L_{\sigma k}$, $k \in \{0, 1, \dots, n\}$, are defined in Theorem 4.1, will be called the σ-Lagrange interpolation polynomial of degree n with σ-interpolation points (x_i, y_i), $i \in \{0, 1, \dots, n\}$.

Definition 4.4. Assume that $n \in \mathbb{N}_0$. Let $x_i \in [a,b] \subset \mathbb{T}$, $i \in \{0,1,\ldots,n\}$, be σ-distinct and $f : [a,b] \to \mathbb{R}$ be a given function. The σ-polynomial

$$p_{\sigma n}(x) = \sum_{k=0}^{n} L_{\sigma k}(x) f(x_k),$$

where $L_{\sigma k}$, $k \in \{0,1,\ldots,n\}$, are defined in Theorem 4.1, will be called the σ-Lagrange interpolation polynomial of degree n with σ-interpolation points x_i, $i \in \{0,1,\ldots,n\}$, for the function f.

Example 4.3. Let $\mathbb{T} = \{-2,-1,0,3,7\}$,

$$a = -2, \quad b = 7, \quad x_0 = -2, \quad x_1 = 0,$$

and $f : \mathbb{T} \to \mathbb{R}$ be defined by

$$f(x) = x + 3, \quad x \in \mathbb{T}.$$

We will find the σ-Lagrange interpolation polynomial for the function f with σ-interpolation points x_0, x_1. We have

$$\begin{aligned}
\sigma(x_0) &= \sigma(-2) = -1,\\
\sigma(x_1) &= \sigma(0) = 3,\\
L_{\sigma 0}(x) &= \frac{\sigma(x) - \sigma(x_1)}{\sigma(x_0) - \sigma(x_1)} = \frac{\sigma(x) - 3}{-1-3} = -\frac{1}{4}(\sigma(x) - 3),\\
L_{\sigma 1}(x) &= \frac{\sigma(x) - \sigma(x_0)}{\sigma(x_1) - \sigma(x_0)} = \frac{\sigma(x) + 1}{3+1} = \frac{1}{4}(\sigma(x) + 1),\\
f(x_0) &= f(-2) = 1,\\
f(x_1) &= f(0) = 3.
\end{aligned}$$

Thus,

$$\begin{aligned}
p_{\sigma 1}(x) &= f(x_0) L_{\sigma 0}(x) + f(x_1) L_{\sigma 1}(x)\\
&= -\frac{1}{4}(\sigma(x) - 3) + \frac{3}{4}(\sigma(x) + 1)\\
&= \frac{1}{2}(\sigma(x) + 3), \quad x \in [-2, 7].
\end{aligned}$$

Next,

$$\begin{aligned}
p_{\sigma 1}(0) &= \frac{1}{2}(\sigma(0) + 3) = \frac{1}{2}(3+3) = 3,\\
p_{\sigma 1}(-1) &= \frac{1}{2}(\sigma(-1) + 3) = \frac{1}{2}(0+3) = \frac{3}{2}.
\end{aligned}$$

Now, we will find the Lagrange interpolation polynomial for the function f with interpolation points x_0, x_1. We have

$$\begin{aligned}
L_0(x) &= \frac{x - x_1}{x_0 - x_1} = -\frac{x}{2},\\
L_1(x) &= \frac{x - x_0}{x_1 - x_0} = \frac{x+2}{2}, \quad x \in [-2, 7].
\end{aligned}$$

Therefore,

$$\begin{aligned} p_1(x) &= f(x_0)L_0(x) + f_1(x)L_1(x) \\ &= -\frac{1}{2}x + 3\frac{x+2}{2} \\ &= \frac{3x+6-x}{2} \\ &= x+3, \quad x \in [-2,7]. \end{aligned}$$

Then

$$p_1(0) = 3, \quad p_1(-1) = 2.$$

We see that

$$p_1(0) = p_{\sigma 1}(0), \quad p_1(-1) \neq p_{\sigma 1}(-1),$$

which follows from the fact that 0 is one of the interpolation points while -1 is not.

Remark 4.2. By the above example we see that there are cases for which the Lagrange interpolation polynomial and the σ-Lagrange interpolation polynomial for a function f are different.

Now, we will describe the classes of time scales for which the σ-Lagrange and Lagrange interpolation polynomials coincide.

Theorem 4.2. *Let $\mathbb{T}$ be a time scale such that $\sigma(t) = ct + d$ for any $t \in \mathbb{T}$ and some constants c, d. Let also, $n \in \mathbb{N}$ and*

$$a = x_0 < x_1 < \ldots < x_n = b, \quad x_j \in \mathbb{T}, \quad j \in \{0, 1, \ldots, n\},$$

are σ-interpolation and interpolation points. Then

$$L_k(x) = L_{\sigma k}(x), \quad x \in [a, b], \quad k \in \{0, 1, \ldots, n\}.$$

Proof. Since $\sigma(t) = ct + d$ for any $t \in \mathbb{T}$, then

$$\begin{aligned} L_{\sigma k}(x) &= \prod_{j=0, j\neq k}^{n} \frac{\sigma(x) - \sigma(x_j)}{\sigma(x_k) - \sigma(x_j)} \\ &= \prod_{j=0, j\neq k}^{n} \frac{(cx+d) - (cx_j + d)}{(cx_k + d) - (cx_j + d)} \\ &= \prod_{j=0, j\neq k}^{n} \frac{x - x_j}{x_k - x_j} \\ &= L_k(x), \quad k \in \{0, 1, \ldots, n\}, \quad x \in [a, b]. \end{aligned}$$

This completes the proof. □

Remark 4.3. 1. Theorem 4.2 states that on time scales with a linear forward jump operator, the Lagrange and σ-Lagrange interpolation polynomials coincide. Examples of such time scales are $\mathbb{T} = 3\mathbb{Z}$, where $\sigma(t) = t + 3$, or $\mathbb{T} = q^{\mathbb{N}_0}$, $q > 1$, where $\sigma(t) = qt$. On time scales such as $\mathbb{T} = \mathbb{N}_0^2$, where $\sigma(t) = (\sqrt{t} + 1)^2$, or $\mathbb{T} = \left\{\frac{1}{n} \ : \ n \in \mathbb{N}\right\}$, where $\sigma(t) = \dfrac{t}{1-t}$, the Lagrange and σ-Lagrange polynomials are different which provides an alternative way to represent a discrete set of data.

2. From Theorem 4.2, it is clear that the uniqueness of interpolating polynomial of degree n is not violated. Indeed, if σ is not a linear function, the σ-Lagrange interpolation polynomial of a function f is not a polynomial of degree n or not a polynomial at all. Therefore, for any function f defined on a time scale $\mathbb{T}$ there is a unique interpolation polynomial of degree n.

Suppose that $n \in \mathbb{N}_0$ and $x_j \in \mathbb{T}$, $j \in \{0, 1, \ldots, n\}$, are σ-distinct points. For $x \in \mathbb{T}$, define the σ-polynomials

$$\pi_{\sigma n+1}(x) = \prod_{j=0}^{n}(\sigma(x) - \sigma(x_j)), \quad \Pi^k_{\sigma n+1}(x) = \pi^{\Delta^k}_{\sigma n+1}(x), \quad k \in \mathbb{N}_0,$$

to be employed in the error discussion of σ-Lagrange interpolation.

Theorem 4.3. *Suppose that $n \in \mathbb{N}_0$, $a, b \in \mathbb{T}$, $a < b$, $x_j \in [a, b]$, $j \in \{0, 1, \ldots, n\}$, are σ-distinct and $f : [a, b] \to \mathbb{R}$, $f^{\Delta^k}(x)$ exist for any $x \in [a, b]$ and for any $k \in \{1, \ldots, n+1\}$. Then for any $x \in [a, b]$ there exists $\xi = \xi(x) \in (a, b)$ such that*

$$f(x) - p_{\sigma n}(x) = \frac{f^{\Delta^{n+1}}(\xi)}{\Pi^{n+1}_{\sigma n+1}(\xi)}\pi_{\sigma n+1}(x)$$

or

$$F_{\sigma min,n+1}(\xi) \leq \frac{f(x) - p_{\sigma n}(x)}{\pi_{\sigma n+1}(x)} \leq F_{\sigma max,n+1}(\xi),$$

where

$$F_{\sigma max,n+1}(\xi) = \max\left\{\frac{f^{\Delta^{n+1}}(\xi)}{\Pi^{n+1}_{\sigma n+1}(\xi)}, \frac{f^{\Delta^{n+1}}(\rho(\xi))}{\Pi^{n+1}_{\sigma n+1}(\rho(\xi))}\right\},$$

$$F_{\sigma min,n+1}(\xi) = \min\left\{\frac{f^{\Delta^{n+1}}(\xi)}{\Pi^{n+1}_{\sigma n+1}(\xi)}, \frac{f^{\Delta^{n+1}}(\rho(\xi))}{\Pi^{n+1}_{\sigma n+1}(\rho(\xi))}\right\}.$$

Proof. Let $p_{\sigma n}$ be the σ-Lagrange interpolation polynomial for the function f with interpolation points x_j, $j \in \{0, 1, \ldots, n\}$. Define the function

$$\phi(t) = f(t) - p_{\sigma n}(t) - \frac{f(x) - p_{\sigma n}(x)}{\pi_{\sigma n+1}(x)}\pi_{\sigma n+1}(t), \quad t \in [a, b].$$

From here, the proof repeats the corresponding proof of Theorem 3.3 and we omit it. This completes the proof. □

Remark 4.4. Suppose that all conditions of Theorem 4.3 hold. If

$$\lim_{n\to\infty} \max_{x\in[a,b]} \left(\frac{f^{\Delta^{n+1}}(\xi)}{\Pi^{n+1}_{\sigma n+1}(\xi)}\pi_{\sigma n+1}(x)\right) = 0$$

and

$$\lim_{n\to\infty} \max_{x\in[a,b]} \left(\frac{f^{\Delta^{n+1}}(\rho(\xi))}{\Pi^{n+1}_{\sigma n+1}(\rho(\xi))}\pi_{\sigma n+1}(x)\right) = 0,$$

then

$$\lim_{n\to\infty} \max_{x\in[a,b]} |f(x) - p_{\sigma n}(x)| = 0.$$

Example 4.4. Let $\mathbb{T} = 2^{\mathbb{N}_0} = \{1, 2, 4 \cdots\}$ and $f(x) = e_\alpha(x, 1)$ be the exponential function where $\alpha \in \mathbb{R}$. Then

$$e_\alpha(x, 1) = \prod_{s \in [1, x)} (1 + s\alpha).$$

We will compute the third degree Lagrange and σ-Lagrange interpolation polynomials for $e_\alpha(x, 1)$ by taking the interpolation points as

$$x_0 = 1, x_1 = 2, x_2 = 4, x_3 = 8.$$

Notice that these points are σ-distinct. The values of $e_\alpha(x, 1)$ at these points are

$$\begin{aligned} &e_\alpha(1, 1) = 1, && e_\alpha(2, 1) = (1 + \alpha), \\ &e_\alpha(4, 1) = (1 + \alpha)(1 + 2\alpha), && e_\alpha(8, 1) = (1 + \alpha)(1 + 2\alpha)(1 + 4\alpha). \end{aligned}$$

Since $\sigma(x) = 2x$, we have

$$L_k(x) = \prod_{i=0, i \neq k}^{3} \frac{x - x_i}{x_k - x_i},$$

and,

$$L_{\sigma k}(x) = \prod_{i=0, i \neq k}^{3} \frac{\sigma(x) - \sigma(x_i)}{\sigma(x_k) - \sigma(x_i)} = \prod_{i=0, i \neq k}^{3} \frac{2(x - x_i)}{2(x_k - x_i)} = L_k(x).$$

Therefore,

$$\begin{aligned} L_0(x) = L_{\sigma 0}(x) &= -\frac{1}{21}(x - 2)(x - 4)(x - 8), \\ L_1(x) = L_{\sigma 1}(x) &= \frac{1}{12}(x - 1)(x - 4)(x - 8), \\ L_2(x) = L_{\sigma 2}(x) &= -\frac{1}{24}(x - 1)(x - 2)(x - 8), \\ L_3(x) = L_{\sigma 3}(x) &= \frac{1}{168}(x - 1)(x - 2)(x - 4). \end{aligned}$$

We conclude that both interpolatiom polynomials are identical and have the form

$$\begin{aligned} p_3(x) = p_{\sigma 3}(x) = &-\frac{1}{21}(x - 2)(x - 4)(x - 8) + \frac{(1 + \alpha)}{12}(x - 1)(x - 4)(x - 8) \\ &-\frac{(1 + \alpha)(1 + 2\alpha)}{24}(x - 1)(x - 2)(x - 8) \\ &+\frac{(1 + \alpha)(1 + 2\alpha)(1 + 4\alpha)}{168}(x - 1)(x - 2)(x - 4). \end{aligned}$$

5. Conclusion

In this study, for the first time the polynomial interpolation on time scales has been introduced. We expect that this work will focus attention on numerical methods on time scales and increase the research in this direction. The interpolation problem on time scales may, for example, be discussed by means of using backward jump operator ρ instead of the forward jump operator σ and thus, define ρ-Lagrange interpolation polynomial. Another perspective is using divided differences and σ-divided differences to introduce Newton interpolation polynomial and σ-Newton interpolation polynomial, respectively.

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