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OPTIMAL FEEDBACK CONTROL FOR SECOND-ORDER EVOLUTION EQUATIONS*

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Abstract The goal of this paper is to provide systematic approaches to study new results of optimal feedback control for second-order evolution equations. We firstly give some existence results of mild solutions for the equations by applying the Banach's fixed point theorem and the Leray-Schauder alternative fixed point theorem with Lipschitz conditions and different types of boundedness conditions. Next, by using the Filippove theorem and the Cesari property, a new set of sufficient assumptions are formulated to guarantee the existence results of feasible pairs for the feedback control systems. Finally, we apply our main results to the problems of controllability, Clarke's subdifferential inclusions and differential variational inequalities.

Keywords Optimal feedback control, second-order evolution equation, existence, feasible pairs, Cesari property.

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1. Introduction

Optimal control deals with the problem of finding a control law for a given system such that a certain optimality criterion is achieved. A control problem includes a cost functional that is a function of state and control variables. The famous work is the monograph by Lions [17]. This monograph discussed many statements on the solvability of control systems, such as elliptic partial differential equations, parabolic partial differential equations and so on. Also, the second order evolution equations control systems were obtained in it by using the calculus of variations.

In the last decades, optimal control theory has been greatly applied in many fields such as engineering, economies, computers and ecology, especially towards system with feedback control ([1–4, 9, 21, 27]). A feedback controlled system is a system that compares its output to a desired value, and then automatically takes "corrective action". Many modern conveniences, including automobile cruise control systems and thermostats, rely heavily on feedback. While its uses continue to

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grow, the utility of feedback control was first shown more than two thousand years ago. Such problems arise naturally from a wide variety of practical situations. Very recently, feedback control systems governed by several kinds of evolution equations have been investigated in many works, for instance, the optimal feedback of evolution systems were considered in [13, 16, 24, 25]. To the fractional control systems were discussed by [12, 18, 19]. Zhang and Jia focused the pedestrian counter flow in bidirectional corridors with multiple inflows [27], Zvyagin at. el. studied the optimal feedback control for (fractional) Viogt fluid models and thermoviscoelastic model of the motion of water polymer solutions [28–30]. For more details, one can see the references therein.

To the authors' best knowledge, no results are available for the optimal feedback control of second-order evolution equations. This fact is the main motivation of our work. In this paper we will concentrate on the case with feedback control, and establish sufficient conditions for the existence of feasible pairs to the optimal feedback control by relying on the Filippove theorem and the Cesari property.

Let X be a reflexive Banach space and V be a separable Banach space. In the sequel, we will study the problem with the following form:

$$\begin{cases} x''(t) = A(t)x(t) + f(t, x(t), u(t)), \qquad t \in [0, T], \\ u(t) \in U(t, x(t)), \qquad t \in [0, T], \\ x(0) = x_0, \ x'(0) = x_1. \end{cases} \tag{1.1}$$

where $\{A(t)\}_{t\in[0,+\infty)}$ is a family of closed densely defined linear operators in X. $f : [0, T] \times X \times V \to X$ is a given function to be specified later. The control function u takes values in the space V and $U : [0, T] \times X \to 2^V$ is a feedback multifunction.

In (1.1), if $u \equiv 0$, Kozak [15] had given the fundamental solutions for the second order evolution equation and discussed the existence results under the continuousness of the nonlinear function in 1995. But, we only need that the nonlinear function is Lipschitz continuous in our results. The results are based on the properties of evolution operators and fixed point theorems. Moreover, the existence result of admissible trajectories are proved involving the compactness of evolution operator $S(\cdot, \cdot)$ with the help of the Cesari property. Then, we present the existence of optimal feedback controls for the problem. We remark that our results obtained in this paper could be widely applied in many practical problems.

The paper is organized as follows. In Section 2 we recall useful definitions and preliminaries. In Section 3 we obtain some new existence results for second-order evolution equations. Theorems 3.2, 3.3, 3.4 are the main existence results of this section. In Section 4, we obtain some new existence results for the feedback control system. Theorems 4.1, 4.2, 4.3 are the main existence results of this part. In Section 5, we give a result for optimal feedback control problem. In the last section, we apply our main results to the problems of controllability, Clarke's subdifferential inclusions and differential variational inequalities.

2. Preliminaries

Let X be a Banach space and the norm of X be denoted by $\|\cdot\|_X$. For $T > 0$, let $C([0, T]; X)$ denote the Banach space of all continuous functions from $[0, T]$ into

X with the norm $\|x\|_C = \sup_{t\in[0,T]} \|x(t)\|_X$ and $L^2([0,T];X)$ denote the Banach space of all square integrable functions from $[0,T]$ into X with the norm $\|x\|_{L^2} = \left(\int_0^T \|x(t)\|_X^2 dt\right)^{\frac{1}{2}}$. We denote by " $\to$ " the strong convergence and " $\rightharpoonup$ " the weak convergence.

Definition 2.1 ([5, 14]). Let X and Y be two Banach space. A multifunction $F: X \to 2^Y$ with closed values is said to be

(i) measurable, if $F^{-1}(D) := \{x \in [0,T] | F(x) \cap D \neq \emptyset\} \in \mathcal{Q}$ for every closed set $D \subset X$, where $\mathcal{Q}$ denotes the σ-field of Lebesgue measurable sets on $[0,T]$.

(ii) upper semicontinuous (u.s.c. for short), if for every open subset $O \subset Y$ the set $F_+^{-1}(O) = \{x \in X : F(x) \subset O\}$ is open in X.

(iii) closed, if for any $(x_n, y_n) \in Gr(F) := \{(x,y) \in X \times Y : x \in X, y \in F(x)\}$ with $x_n \to \overline{x}$ in X, $y_n \to \overline{y}$ in Y we have $(\overline{x}, \overline{y}) \in Gr(F)$.

Definition 2.2 ([16]). Let X and Y be two metric spaces. A multifunction $F: X \to 2^Y$ is said to be pseudo-continuous at $x \in X$ if

$$\bigcap_{\delta>0} \overline{F(O_\delta(x))} = F(x),$$

where $O_\delta(x) = \{y \in X | \|y - x\| \leq \delta\}$. We say that F is pseudo-continuous on X if it is pseudo-continuous at each point $x \in X$.

Remark 2.1. (i) Let $F: X \to 2^Y$ be a multifunction with closed values. Then F is pseudo-continuous if and only if the graph

$$\mathcal{G} = \{(x,y) \in X \times Y | y \in F(x)\}$$

is closed in $X \times Y$.
(ii) If $F: X \to 2^Y$ is u.s.c. with closed values, then it is pseudo-continuous.

Definition 2.3 ([16]). Let X be a metric space, Y be a Banach space and $F: X \to 2^Y$ be a multifunction. We say F possesses the Cesari property at $x_0 \in X$, if

$$\bigcap_{\delta>0} \overline{co} F(O_\delta(x_0)) = F(x_0),$$

where $\overline{co} D$ is the closed convex hull of D. If F has the Cesari property at every point $x \in Z \subset X$, we simply say that F has the Cesari property on Z.

Lemma 2.1 ([16], Proposition 4.2). *Let X be a metric space and Y be a Banach space. If $F: X \to 2^Y$ is u.s.c. with closed convex values, then F has the Cesari property on X.*

To get the existence results, we are base on the following two well-known results.

Lemma 2.2 ([10], Theorem 6.5.4). *Let x ve a closed convex subset of a Banach space X such that $0 \in D$. Let $F: D \to D$ be a completely continuous map. Then the set $\{x \in D : x = \lambda F(x), 0 < \lambda < 1\}$ is unbounded or the map F has a fixed point in D.*

Lemma 2.3 ([7], Corollary 3.1). *Let $\mathcal{A}: \overline{B_r(0)} \to X$ and $\mathcal{B}: \overline{B_r(0)} \to \mathcal{P}_{kc}(X)$ be two operators such that*

(1) $\mathcal{A}$ is a is a single-valued contraction mapping with coefficient $\lambda < \dfrac{1}{2}$;

(2) $\mathcal{B}$ is compact and u.s.c.

Then one of the following conclusions holds

(i) there exist an element $\omega \in \overline{B_r(0)} \setminus B_r(0)$ such that $\rho\omega \in \mathcal{A}\omega + \mathcal{B}\omega$ for some $\rho > 1$;

(ii) the operator inclusion $u \in \mathcal{A}u + \mathcal{B}u$ has a solution in $\overline{B_r(0)}$.

3. Existence results

At first, we consider the following second-order evolution equation.

$$\begin{cases} x''(t) = A(t)x(t) + f(t, x(t), u(t)), \qquad t \in [0, T], \\ x(0) = x_0, \ x'(0) = x_1. \end{cases} \tag{3.1}$$

Let us recall that a two parameter family $\{S(t,s)\}_{(t,s)\in[0,T]\times[0,T]}$ (see [15]), where $S(t,s) : X \to X$ is a bounded linear operator, is an evolution system if the following conditions are satisfied:

(S1) For each $x \in X$ the map $(t,s) \mapsto S(t,s)x$ is continuously differential, and

(i) for each $t \in [0,T]$, $S(t,t) = 0$,

(ii) for all $t, s \in [0,T]$ and for each $x \in X$

$$\frac{\partial}{\partial t} S(t,s)|_{t=s} x = x, \ \frac{\partial}{\partial s} S(t,s)|_{t=s} x = -x.$$

(S2) for all $t, s \in [0,T]$, if $x \in D(A)$ then $S(t,s)x \in D(A)$, the map $(t,s) \mapsto S(t,s)x$ is twice continuously differential, and

(i) $\dfrac{\partial^2}{\partial t^2} S(t,s)x = A(t)S(t,s)x,$

(ii) $\dfrac{\partial^2}{\partial s^2} S(t,s)x = S(t,s)A(s)x,$

(iii) $\dfrac{\partial^2}{\partial s \partial t} S(t,s)|_{t=s} x = 0.$

(S3) for all $t, s \in [0,T]$, if $x \in D(A)$ then $\dfrac{\partial}{\partial s} S(t,s)x \in D(A)$, there exist $\dfrac{\partial^3}{\partial t^2 \partial s} S(t,s)x$, $\dfrac{\partial^3}{\partial s^2 \partial t} S(t,s)x$, and

(i) $\dfrac{\partial^3}{\partial t^2 \partial s} S(t,s)x = A(t) \dfrac{\partial}{\partial s} S(t,s)x,$

(ii) $\dfrac{\partial^3}{\partial s^2 \partial t} S(t,s)x = \dfrac{\partial}{\partial t} S(t,s)A(s)x,$

(iii) $\dfrac{\partial^2}{\partial s \partial t} S(t,s)|_{t=s} x = 0,$

(iv) the map $(t,s) \to A(t)\dfrac{\partial}{\partial s}S(t,s)x$ is continuous.

Throughout this work we assume that there exists an evolution operator $S(t,s)$ associated to the operator $A(t)$. To abbreviate the text, we introduce the operator $C(t,s) = -\dfrac{\partial}{\partial s}S(t,s)$. In addition, we set M_S and M_C for positive constants such that $\sup\limits_{s,t\in[0,T]} \|S(t,s)\| \le M_S$ and $\sup\limits_{s,t\in[0,T]} \|C(t,s)\| \le M_C$. Furthermore, we denote by L_S a positive constant such that

$$\|S(t+h,s) - S(t,s)\| \le L_S|h|, \qquad \forall s,t,t+h \in [0,T].$$

Definition 3.1. A function $x : [0,T] \to X$ is said to be a mild solution of problem (3.1) if $x \in C([0,T];X)$ and

$$x(t) = C(t,0)x_0 + S(t,0)x_1 + \int_0^t S(t,s)f(s,x(s),u(s))ds, \qquad t \in [0,T].$$

Assume that

(H_f) $f(\cdot,x,u) : [0,T] \to \mathbb{R}$ is measurable for every $x \in X, u \in V$.

(H_{f0}) There exists a constant $L_f > 0$ such that

$$\begin{aligned}&\|(f(t,x_1,u) - f(t,x_2,u)\|_X \le L_f\|x_1 - x_2\|_X,\\ &\|(f(t,0,u)\|_X \le L_f\end{aligned}$$

for all $x_1, x_2 \in X, u \in V$, a.e. $t \in [0,T]$.

Now we give the following existence result.

Theorem 3.1 ([15], Theorem 4.1)**.** *Assume that $(H_f),(H_{f0})$ are satisfied. Then for every $u \in L^2([0,T];V)$, problem (3.1) has a unique mild solution in $C([0,T];X)$.*

Consider the following boundedness condition and compactness condition.

(H_{f1}) There exist a function $\phi_1 \in L^2([0,T];\mathbb{R}_+)$ and constants $L_1, N_1 > 0$ such that

$$\|f(t,x,u)\| \le \phi_1(t) + L_1\|x\|_X + N_1\|u\|_V$$

for all $x \in X,\ u \in V$, a.e. $t \in [0,T]$.

(H_S) For $0 < s \le t \le T$, $S(t,s)$ is compact.

Similar to the proof of Lemma 3.3.2 in [16], we have the following result.

Lemma 3.1. *If condition (H_S) holds, then operator $G : L^p([0,T];X) \to C([0,T];X)$ for some $p > 1$, given by*

$$(Gf)(\cdot) = \int_0^{\cdot} S(\cdot,s)f(s)ds, \tag{3.2}$$

is compact for $f \in L^p([0,T];X)$.

Now, we are in the position to prove the following existence result of mild solution for problem (3.1) without Lipschitz condition in (H_{f0}).

Theorem 3.2. *Assume that the hypotheses $(H_f),(H_{f1}),(H_S)$ are satisfied. Then for each given control function $u \in L^2([0,T];V)$, the problem (3.1) has a mild solution on $C([0,T];X)$.*

Proof. Define the operator $\mathcal{F} : C([0,T];X) \to C([0,T];X)$ by

$$(\mathcal{F}x)(t) = C(t,0)x_0 + S(t,0)x_1 + \int_0^t S(t,s)f(s,x(s),u(s))ds, \quad t \in [0,T]. \tag{3.3}$$

It is clear to see that the fixed points of $\mathcal{F}$ are mild solutions of problem (3.1).

We will apply Leray-Schauder alternative fixed point theorem (see [10], Theorem 6.5.4) to obtain the existence of the fixed points of $\mathcal{F}$. Let

$$\Delta = \{x \in C([0,T];X) | \exists \lambda \in (0,1), x = \lambda \mathcal{F} x\}. \tag{3.4}$$

At first, we claim that the set Δ is bounded.

Let $x^\lambda \in \Delta$ and $t \in [0,T]$. From the hypothesis (H_{f1}) and the Hölder inequality, we obtain

$$\begin{aligned}
&\|x^\lambda(t)\|_X = \|\lambda(\mathcal{F}x^\lambda)(t)\|_X \\
&\le \|C(t,0)x_0\| + \|S(t,0)x_1\| + \int_0^t \|S(t,s)f(s,x(s),u(s))\|ds \\
&\le M_C\|x_0\| + M_S\|x_1\| + M_S \int_0^t [\|\phi_1(s)\| + L_1\|x^\lambda(s)\|_X + N_1\|u(s)\|_V]ds \\
&\le M_C\|x_0\| + M_S\|x_1\| + M_S\sqrt{T}\Big[\|\phi_1\|_{L^2(J,\mathbb{R}_+)} + N_1\|u\|_{L^2([0,T];V)}\Big] \\
&\quad + L_1 M_S \int_0^t \|x^\lambda(s)\|_X ds.
\end{aligned}$$

Thus, by applying the standard Gronwall inequality, we have

$$\|x^\lambda(t)\|_X \le M_1 e^{M_2} \qquad \text{for some } M_1, M_2 > 0,$$

which implies that the set Δ is bounded.

Moreover, by (H_S) and Lemma 3.1, it is clear that $\mathcal{F}$ is a compact operator. Therefore, by applying Leray-Schauder alternative fixed point theorem, we can deduce that the problem (3.1) has a mild solution on $[0,T]$. The proof is complete. □

Consider the second boundedness condition for f.

(H_{f2}) There exist a function $\phi_2 \in L^2([0,T];\mathbb{R}_+)$, a continuous non-decreasing function $\Omega : [0,\infty) \to [0,\infty)$ and a constant $N_2 > 0$ such that

$$\|f(t,x,u)\| \le \phi_2(t)\Omega(\|x\|_X) + N_2\|u\|_V$$

for all $x \in X,\ u \in V$, a.e. $t \in [0,T]$.

Theorem 3.3. *Assume that the hypotheses $(H_f), (H_S), (H_{f2})$ are satisfied. Then for each given control function $u \in L^2([0,T];V)$, the problem (3.1) has a mild solution on $C([0,T];X)$ provided*

$$\int_{M^*}^{\infty} \frac{1}{\Omega(s)} ds > M_S \int_0^T \phi_2(s)ds, \tag{3.5}$$

where

$$M^* = M_C\|x_0\| + M_S\|x_1\| + M_S N_2 \sqrt{T}\|u\|_{L^2([0,T];V)}.$$

Proof. Let $\digamma$ and Δ are defined by (3.3) and (3.4). From the proof of Theorem 3.2, we only prove that the set Δ is bounded.

Let $x^\lambda \in \Delta$ and $t \in [0,T]$. From the hypothesis (H_{f2}) and the Hölder inequality, we obtain

$$\begin{aligned}&\|x^\lambda(t)\|_X = \|\lambda(\digamma x^\lambda)(t)\|_X\\ &\leq \|C(t,0)x_0\| + \|S(t,0)x_1\| \int_0^t \|S(t,s)f(s,x(s),u(s))\|ds\\ &\leq M_C\|x_0\| + M_S\|x_1\| + M_S\int_0^t [\phi_2(t)\Omega(\|x(s)\|_X) + N_2\|u(s)\|_V]ds\\ &\leq M_C\|x_0\| + M_S\|x_1\| + M_S N_2\sqrt{T}\|u\|_{L^2([0,T];V)} + M_S\int_0^t \phi_2(s)\Omega(\|x^\lambda(s)\|_X)ds.\end{aligned}$$

Let $\alpha(t) = M^* + M_S\int_0^t \phi_2(s)\Omega(\|x^\lambda(s)\|_X)ds$. Then we see that

$$\alpha'(t) \leq M_S\phi_2(t)\Omega(\alpha(t)),$$

and subsequently, upon integrating over $[0,T]$, we obtain

$$\int_{M^*}^{\alpha(t)} \frac{1}{\Omega(s)}ds \leq M_S\int_0^t \phi_2(s)ds \leq M_S\int_0^T \phi_2(s)ds < \int_{M^*}^{\infty}\frac{1}{\Omega(s)}ds.$$

This estimate shows that $\{x^\lambda\}$ is bounded, and hence the set Δ is bounded.

Therefore, the problem (3.1) has a mild solution on $[0,T]$. The proof is complete. □

Furthermore, we consider following compactness condition.

(H_{S1}) For every $t \in [0,T], u \in V$ and every $r > 0$, the set $Q(t) = \{S(t,s)f(s,\psi,u) : s \in [0,T], \|\psi\| \leq r\}$ is relatively compact in X.

We define a multifunction $F : [0,T] \times X \to 2^X$ as

$$F(t,x) = f(t,x,U(t,x)),$$

where U is given in (1.1). Also, define the multifunction $\mathcal{G} : L^2([0,T];X) \to 2^{L^2([0,T];X)}$ by

$$\mathcal{G}(x) = \{v \in L^2([0,T];X) : v(t) \in F(t,x(t)) \quad \text{for a.e. } t \in [0,T]\},$$

for $x \in L^2([0,T];X)$. By the work of [22, Lemma 5.3], we know that $\mathcal{G}$ has nonempty, convex and weakly compact values. According to [23, Lemma 11], $\mathcal{G}$ has the following property: if $x_n \to x$ in $L^2([0,T],X), v_n \rightharpoonup v$ in $L^2([0,T],X)$ and $v_n \in \mathcal{G}(x_n)$, then $v \in \mathcal{G}(x)$

Theorem 3.4. *Assume that $(H_f), (H_{f1}), (H_{S1})$ are satisfied. Then for each given control function $u \in L^2([0,T];V)$, the problem (3.1) has a mild solution on $C([0,T];X)$.*

Proof. Let $\digamma$ and Δ be defined by (3.3) and (3.4). We will prove that the operator $\digamma$ satisfies all the condition of Lemma 2.3.

Let $\digamma = \digamma_0 + \digamma_1$, where

$$(\digamma_0 x)(t) = C(t,0)x_0 + S(t,0)x_1, \qquad t \in [0,T], \tag{3.6}$$

$$(\digamma_1 x)(t) = \int_0^t S(t,s)f(s,x(s),u(s))ds, \qquad t \in [0,T]. \tag{3.7}$$

Obviously, x is a fixed point of $\digamma$ if only and if it is a mild solution of system (3.1). According to Lemma 2.3, we need to prove that (i) of lemma 2.3 is not satisfied.

In fact, suppose that $\rho x \in \digamma_0 x + \digamma_1 x$ with $\rho > 1$ such that

$$\rho x(t) = C(t,0)x_0 + S(t,0)y_0 + \int_0^t S(t,s)f(s,x(s),u(s))\ ds.$$

Then by the similar proof of Theorem 3.2, we get that

$$\|x(t)\|_X \leq M_1 e^{M_2} \qquad \text{for some } M_1, M_2 > 0.$$

This implies

$$\|x\|_{C([0,T],X)} \leq M_1 e^{M_2} =: r.$$

Furthermore, we write

$$B_r(0) = \{x \in C([0,T],X) : \|x\|_{C([0,T],X)} < r+1\}.$$

Obviously, $B_r(0)$ is an open subset of $C([0,T],X)$.

Next, we will check that the operator $\digamma_0$ is a single-valued contraction mapping with coefficient $\lambda < \frac{1}{2}$ and the operator $\digamma_1$ is compact and u.s.c. For convenience, we divide the proof into two steps.

Step 1. We can easily see that $\digamma_0 : \overline{B_r(0)} \to X$ is a single-valued contraction mapping with coefficient $\lambda < \frac{1}{2}$.

Step 2. $\digamma_1$ is compact and u.s.c. We will divide the proof into four claims.

Claim 2.1. Operator $\digamma_1$ is bounded. In fact, for all $x \in B_r(0)$ and $\eta \in \digamma_1 x$, from condition (H_{f1}) and well-known Hölder inequality, we have

$$\begin{aligned}\|\eta(t)\|_X &\leq \int_0^t \|S(t,s)f(s,x(s),u(s))\|ds \\ &\leq M_S \int_0^t (\phi_1(s) + L_1\|x(s)\|_X + N_1\|u(s)\|_V)\, ds \\ &\leq M_S\sqrt{3}\left(\sqrt{T}\|\phi_1\|_{L^2([0,T];\mathbb{R}_+)} + L_1 r + N_1\|u\|_{L^2([0,T];V)}\right) := \ell,\end{aligned}$$

for all $t \in [0,T]$, which proves the claim.

Claim 2.2. The set $\{\digamma_1 x | x \in B_r(0)\}$ is equicontinuous for each $r > 0$.

To get this aim, let $0 < t_1 \leq t_2 \leq T$ and enough small $\delta > 0$. Then

$$\begin{aligned}&\|\eta(t_2) - \eta(t_1)\|_X \\ \leq &\int_0^{t_1} \|(S(t_2,s) - S(t_1,s))f(s,x(s),u(s))\|ds + \int_{t_1}^{t_2} \|S(t_2,s)f(s,x(s),u(s))\|ds \\ \leq &\int_0^{t_1} \|S(t_2,s) - S(t_1,s)\|\,(\phi_1(s) + L_1 r + N_1\|u(s)\|_V)\,ds \\ &+ M_S \int_{t_1}^{t_2} (\phi_1(s) + L_1\|x(s)\|_X + N_1\|u(s)\|_V)\,ds \\ \leq &\sup_{s\in[0,t_1]} \|S(t_2,s) - S(t_1,s)\|\sqrt{3}\left(\sqrt{T}\|\phi_1\|_{L^2([0,T];\mathbb{R}_+)} + L_1 r + N_1\|u\|_{L^2([0,T];V)}\right) \\ &+ M_S\sqrt{3}\left(\|\phi_1\|_{L^2([0,T];\mathbb{R}_+)}\sqrt{t_2 - t_1} + L_1 r(t_2 - t_1) + N_1\|u\|_{L^2([0,T];V)}\sqrt{t_2 - t_1}\right)\end{aligned}$$

According to the fact that the operator $S(t,s)$ is continuous in t in the uniform topology, we can easily get that the right hand side of the above inequality is independent of $x \in B_r(0)$ and tends to zero as $t_2 \to t_1$. Hence, we finish this claim.

Claim 2.3 We show that $\mathcal{F}_1$ is a compact map. We will prove that $\mathcal{F}_1$ maps bounded sets into relatively compact ones. Let $r > 0$ and

$$B(r) = \{x \in C([0,T];X) : \|x\|_C \leq r\}.$$

From the mean value theorem, we see that for $x \in B(r)$,

$$(\mathcal{F}_1 x)(t) \in t\overline{co}\{S(t,s)f(s,\psi,u(s)) : s \in [0,T], \|\psi\| \leq r\}, \qquad t \in [0,T],$$

which implies that the set $\{(\mathcal{F}_1 x)(t) : x \in B(r)\}$ is relatively compact for each $t \in [0,T]$. Moveover, from the proof of Lemma 4.2.1 in [14], we know that the set $\{(\mathcal{F}_1 x) : x \in B(r)\}$ is equicontinuous. Therefore, the relative compactness of the set $\{(\mathcal{F}_1 x) : x \in B(r)\}$ follows from the well known Arzela-Ascoli criterion, which shows that $\mathcal{F}_1$ is a compact map.

Claim 2.4. Operator $\mathcal{F}_1$ has a closed graph and it is u.s.c.

Set $x_n \in C([0,T];X), z_n \in \mathcal{F}_1 x_n$ be such that $x_n \to \widetilde{x}, z_n \to \widetilde{z}$. We shall show that $\widetilde{z} \in \mathcal{F}_1 \widetilde{x}$. Let $z_n \in \mathcal{F}_1 x_n$. There is $f_n \in \mathcal{G}(x_n)$ such that

$$z_n(t) = \int_0^t S(t,s)f_n(s)ds \quad \text{for all } t \in [0,T]. \tag{3.8}$$

By the hypothesis (H_{f1}), one can show that $\{f_n\}_{n\geq 1}$ is bounded in $L^2([0,T];X)$. Thus, passing to a subsequence if necessary, we may assume that

$$f_n \rightharpoonup \bar{f} \quad \text{in } t \in L^2([0,T];X). \tag{3.9}$$

According to (3.8), (3.9) and condition (H_{S1}), we get

$$z_n(t) \to \int_0^t S(t,s)\bar{f}(s)ds. \tag{3.10}$$

By the fact of $z_n \to \widetilde{z}$ and $z_n \in \mathcal{G}(x_n)$, applying the properties of $\mathcal{G}$ and (3.10), we get $\bar{f} \in \mathcal{G}(\widetilde{x})$. Hence, $\widetilde{z} \in \mathcal{F}_1 \widetilde{x}$, which implies the thesis. Then, thanks to [22, Property 3.3.12(2)] we obtain that operator $\mathcal{F}_1$ is u.s.c.

By all the above steps, we know that all the hypotheses of Lemma 2.3 hold. Hence, we know that there exists a fixed point x of $\mathcal{F}$. Therefore, the problem (3.1) has a mild solution on $[0,T]$. The proof is complete. □

Similarly, we have the following result.

Theorem 3.5. *Assume that* $(H_f), (H_{f2}), (H_{S1})$ *and (3.5) are satisfied. Then for each given control function* $u \in L^2([0,T];V)$*, the problem (3.1) has a mild solution on* $C([0,T];X)$.

4. Feedback control system

The following definition will be used in this part.

Definition 4.1. A pair (x,u) is said to be feasible if (x,u) satisfies (1.1) for $t \in [0,T]$.

To the readers' convenience, we denote
$V[0,T] = \{u : [0,T] \to V | u \text{ is measurable}\}$,
$X[0,T] = \{(x,u) \in C([0,T];X) \times V[0,T] |\ (x,u) \text{ is feasible for (1.1)}\}$.

(H_U) The feedback multifunction $U : [0,T] \times X \to 2^V$ is such that

(i) U is pseudo-continuous;

(ii) for a.e. $t \in [0,T],\ x \in X$, the following property holds:

$$\bigcap_{\delta>0} \overline{co} f(t, O_\delta(x), U(O_\delta(t,x))) = f(t,x,U(t,x)).$$

Now, we are in the position to present the existence result of feasible pairs for problem (1.1).

Theorem 4.1. *Assume that all the assumptions of one of Theorems 3.2, 3.3 and (H_U) are satisfied. Then the set $X[0,T]$ is nonempty.*

Proof. For any $k > 0$, let $t_j = \frac{j}{k}T,\ 0 \le j \le k-1$. We set

$$u_k(t) = \sum_{j=0}^{k-1} u^j \chi_{[t_j,t_{j+1})}(t), \qquad t \in [0,T],$$

where $\chi_{[t_j,t_{j+1})}$ is the character function of interval $[t_j, t_{j+1})$. The sequence $\{u^j\}$ is constructed as follows.

Firstly, we take $u^0 \in U(0,x_0)$. By Theorem 3.2, there exists x_k given by

$$x_k(t) = C(t,0)x_0 + S(t,0)x_1 + \int_0^t S(t,s)f(s,x_k(s),u^0)ds, \qquad t \in [0, \frac{T}{k}].$$

Then take $u^1 \in U(\frac{T}{k}, x_k(\frac{T}{k}))$. We can repeat this procedure to obtain x_k on $[\frac{T}{k}, \frac{2T}{k}]$, etc. By induction, we end up with the following:

$$\begin{cases} x_k(t) = C(t,0)x_0 + S(t,0)x_1 + \displaystyle\int_0^t S(t,s)f(s,x_k(s),u_k(s))ds, \qquad t \in [0,T], \\ u_k(t) \in U(\dfrac{jT}{k}, x_k(\dfrac{jT}{k})),\ t \in [\dfrac{jT}{k}, \dfrac{(j+1)T}{k}),\ 0 \le j \le k-1. \end{cases}$$

From the proof of Theorem 3.2, it is easy to prove that there exists $r_0 > 0$ such that

$$\|x_k\|_C \le r_0.$$

By (H_S), we deduce that the sequence $\{x_k\}$ is relatively compact in $C([0,T];X)$. Thus we may assume

$$x_k \to \overline{x} \qquad \text{in } C([0,T];X). \tag{4.1}$$

Moreover, it comes from (H_{f1}) that there exists $r_1 > 0$ such that

$$\|f(\cdot, x_k(\cdot), u_k(\cdot))\|_{L^2} \le r_2.$$

Therefore, there is subsequence of $\{f(\cdot, x_k(\cdot), u_k(\cdot))\}$, denoted by $\{f(\cdot, x_k(\cdot), u_k(\cdot))\}$ again, such that

$$f(\cdot, x_k(\cdot), u_k(\cdot)) \rightharpoonup \overline{f} \qquad \text{in } L^2([0,T];X). \tag{4.2}$$

By (H_S) and Lemma 3.1 we have that for any $t \in [0,T]$,

$$\int_0^t S(t,s)f(s,x_k(s),u_k(s))ds \to \int_0^t S(t,s)\overline{f}(s)ds,$$

and hence

$$\overline{x}(t) = C(t,0)x_0 + S(t,0)x_1 + \int_0^t S(t,s)\overline{f}(s)ds, \qquad t \in [0,T].$$

By (4.2), for $\delta > 0$ there exists a $k_0 > 0$ such that

$$x_k(t) \in O_\delta(\overline{x}(t)), \qquad t \in [0,T],\ k \geq k_0. \tag{4.3}$$

By the definition of u_k for k large enough, we have

$$u_k(t) \in U(t_j, x_k(t_j)) \subset U(O_\delta(t,\overline{x}(t))), \qquad \forall t \in [\frac{jT}{k}, \frac{(j+1)T}{k}),\ 0 \leq j \leq k-1. \tag{4.4}$$

Secondly, by (4.2) and Mazur Theorem ([16], Chapter 2, Corollary 2.8), let $a_{il} \geq 0$ and $\sum_{i\geq 1} a_{il} = 1$ such that

$$\phi_l = \sum_{i\geq 1} a_{il} f(\cdot, x_{i+l}(\cdot), u_{i+l}(\cdot)) \to \overline{f} \qquad \text{in } L^2([0,T];X).$$

Then, there is a subsequence of $\{\phi_l\}$, denoted by $\{\phi_l\}$ again, such that

$$\phi_l(t) \to \overline{f}(t) \qquad \text{in } X, \qquad \text{a.e. } t \in [0,T].$$

Hence, from (4.3) and (4.4), for l large enough,

$$\phi_l(t) \in \mathrm{co} f(t, O_\delta(\overline{x}(t)), U(O_\delta(t,\overline{x}(t)))), \qquad \text{a.e. } t \in [0,T].$$

Thus, for any $\delta > 0$,

$$\overline{f}(t) \in \overline{\mathrm{co}} f(t, \overline{x}(t), U(t,\overline{x}(t))), \qquad \text{a.e. } t \in [0,T].$$

In virtue of (H_U) and Corollary 2.18 of [16], we obtain that $U(\cdot,\overline{x}(\cdot))$ is Souslin measurable. By the Fillippove theorem (Chapter 2, Corollary 2.26, [16]), there exists a measurable function $\overline{u} \in V[0,T]$ such that

$$\begin{cases} \overline{u}(t) \in U(t,\overline{x}(t)), & \text{a.e. } t \in [0,T], \\ \overline{f}(t) = f(t,\overline{x}(t),\overline{u}(t)), & \text{a.e. } t \in [0,T]. \end{cases}$$

Therefore, $(\overline{x},\overline{u}) \in X[0,T]$. The proof is complete. □

Theorem 4.2. *Assume that all the assumptions of one of Theorems 3.4, 3.5 and (H_U) are satisfied. Then the set $X[0,T]$ is nonempty.*

Proof. Similar to the proof of Theorem 4.1, we have

$$x_n(t) = C(t,0)x_0 + S(t,0)x_1 + \int_0^t S(t,s)f(s,x_n(s),u_n(s))ds, \qquad t \in [0,T].$$

$$u_n(t) \in U(\tau_j, x_n(\tau_j)), \qquad t \in (\tau_j, \tau_{j+1}],\ 0 \le j \le n-1.$$

Moreover, there exist $r_0, r_1 > 0$ such that

$$\|x_n\|_C \le r_0 \qquad \text{and} \qquad \|f(\cdot, x_n(\cdot), u_n(\cdot))\|_{L^2} \le r_1.$$

Then, there is a subsequence $\{f(\cdot, x_n(\cdot), u_n(\cdot))\}$ such that

$$f(\cdot, x_n(\cdot), u_n(\cdot)) \rightharpoonup \overline{f}(\cdot) \qquad \text{in } L^2([0,T];X).$$

Since the operator $G : L^2([0,T];X) \to C([0,T];X)$ defined by (3.2) is linear and continuous, we obtain

$$\int_0^t S(t,s)f(s, x_n(s), u_n(s))ds \rightharpoonup \int_0^t S(t,s)\overline{f}(s)ds, \qquad t \in [0,T].$$

From the proof of Theorem 3.4 it follows that the set $\{x_n\}$ is relatively compact in $C([0,T];X)$. Then

$$x_n(\cdot) \to \overline{x}(\cdot) \qquad \text{in } C([0,T];X),$$

and hence

$$\overline{x}(t) = C(t,0)x_0 + S(t,0)x_1 + \int_0^t S(t,s)\overline{f}(s)ds, \qquad t \in [0,T].$$

The rest proof is similar to the proof of Theorem 4.1. □

Similarly, we also have the following result.

Theorem 4.3. *Assume that all the assumptions of Theorem 3.1 and* (H_S), (H_U) *are satisfied. Then the set* $X[0,T]$ *is nonempty.*

Consider the following compactness hypothesis on the feedback multifunction U.

(H_{U1}) $U : [0,T]\times X \to 2^K$ is pseudo-continuous, where K is a compact subset of V.

Theorem 4.4. *Assume that all the assumptions of one of Theorems 3.2, 3.3 and* (H_{U1}) *are satisfied. Then the set* $X[0,T]$ *is nonempty.*

Proof. Similar to the proof of Theorem 4.1, we have

$$\begin{cases} x_k(t) = C(t,0)x_0 + S(t,0)x_1 + \displaystyle\int_0^t S(t,s)f(s, x_k(s), u_k(s))ds, & t \in [0,T], \\ u_k(t) \in U(\dfrac{jT}{k}, x_k(\dfrac{jT}{k})), \qquad t \in [\dfrac{jT}{k}, \dfrac{(j+1)T}{k}),\ 0 \le j \le k-1, & \end{cases}$$

and

$$\|x_k\|_C \le r_0.$$

We claim that the sequence $\{x_k\}$ is relatively compact in $C([0,T];X)$.

Let $\varepsilon > 0$. From (H_{f1}) it follows that there exists $C > 0$ such that

$$\|f(t, x_k(t), u_k(t))\| \le C \qquad \text{for a.e. } t \in [0,T].$$

Let $B = \{x \in E_2 : \|x\| \le C + \|x_0\|\}$. Since $C(t,0)$ is strongly continuous and $\psi \in L^2([0,T];\mathbb{R}_+)$, it is possible to choose δ such that

$$\|(C(t,0) - I)x\| \le \frac{\varepsilon}{2M(1+T)}, \qquad \forall t \in [0,\delta], x \in B$$

and

$$\int_0^\delta \psi(s)ds < \frac{\varepsilon}{2M(1+r_0)}.$$

For $t_1, t_2 \in [0,T], 0 < t_2 - t_1 < \delta$ we have that

$$\begin{aligned}
&\|x_k(t_2) - x_k(t_1)\| \\
\leq &\|(C(t_2,0) - C(t_1,0))x_0\| \\
&+\|\int_0^{t_2} S(t_2,s)f(s,x_k(s),u_k(s))ds - \int_0^{t_1} S(t_1,s)f(s,x_k(s),u_k(s))ds\| \\
\leq &\|C(t_1,0)(C(t_2-t_1,0)-I)x_0\| + \|\int_{t_1}^{t_2} S(t_2,s)f(s,x_k(s),u_k(s))ds\| \\
&+\|\int_0^{t_1}(S(t_2,s)-S(t_1,s))f(s,x_k(s),u_k(s))ds\| \\
\leq &\frac{\varepsilon}{2M(1+T)}M + M_S\int_{t_1}^{t_2}\psi(s)(1+\|x_k(s)\|_X)ds \\
&+\|\int_0^{t_1} S(t_1,s)S(t_2-t_1,s)f(s,x_k(s),u_k(s))ds\| \\
\leq &\frac{\varepsilon}{2M(1+T)}M + M(1+r_0)\int_{t_1}^{t_2}\psi(s)ds + \frac{\varepsilon}{2M(1+T)}MT < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

Then $\{x_k\}$ is is equicontinuous. Moreover, since the set K is compact, we deduce that the sequence $\{x_k(t)\}$ is relatively compact in X for every $t \in [0,T]$. Therefore, the relative compactness of the sequence $\{x_k\}$ follows from the well known Arzela-Ascoli criterion. Thus we have

$$x_k(\cdot) \to \overline{x}(\cdot) \qquad \text{in } C([0,T];X)$$

for some $\overline{x} \in C([0,T];X)$. The rest of the proof follows from the proof of Theorem 4.2. □

5. Optimal control problem

In this section, we consider an optimal control problem stated as follows.
Problem (φ): find a pair $(x^0,u^0) \in X[0,T]$ such that

$$\varphi(x^0,u^0) \leq \varphi(x,u), \qquad \forall (x,u) \in X[0,T],$$

where $\varphi(x,u) = \int_0^T h(t,x(t),u(t))dt$.

We make the following assumptions on h.
(H_h) $h : [0,T] \times X \times V \to \mathbb{R} \cup \{\pm\infty\}$ is such that

(i) h is Borel measurable in (t,x,u);

(ii) $h(t,\cdot,\cdot)$ is lower semicontinuous on $X \times V$ for a.e. $t \in [0,T]$ and there exists a constant $M_1 > 0$ such that

$$h(t,x,u) \geq -M_1, \qquad (t,x,u) \in [0,T] \times X \times V.$$

For any $(t,x) \in [0,T] \times X$, we set

$$\varepsilon(t,x) = \{(z^0, z^1) \in \mathbb{R} \times X \times V | z^0 \geq h(t,x,z^1),\ z^1 = f(t,x,u),\ u \in U(t,x)\}.$$

In order to obtain the existence result of optimal state-control pairs for Problem (φ), we assume that:
(H_ε) : for a.e. $t \in [0,T]$, the map $\varepsilon(t,\cdot) : X \to 2^{\mathbb{R}\times V}$ is such that

$$\bigcap_{\delta>0} \overline{co}\varepsilon(t, O_\delta(x)) = \varepsilon(t,x), \qquad \forall x \in X.$$

Theorem 5.1. *If all the assumptions of one of Theorems 4.1, 4.2, 4.3, 4.4 and (H_h), (H_ε) are satisfied, then Problem (φ) admits at least one optimal state-control pair.*

Proof. Without considering the situation $\inf\{\varphi(x,u)|(x,u) \in X[0,T]\} = +\infty$, we assume that $\inf\{\varphi(x,u)|(x,u) \in X[0,T]\} = m < +\infty$. By (ii) of (H_h), we have $\varphi(x,u) \geq m \geq -M_1 T > -\infty$. Then there exists a sequence $\{(x^n, u^n)\}_{n\geq 1} \subset X[0,T]$ such that

$$\varphi(x^n, u^n) \to m.$$

From the proof of Theorem 4.1, we obtain that

$$x^n \to \overline{x} \qquad \text{in } C([0,T];X),$$

and

$$f(\cdot, x^n(\cdot), u^n(\cdot)) \rightharpoonup \overline{f}(\cdot) \qquad \text{in } L^2([0,T];X),$$

where

$$\overline{x}(t) = C(t,0)x_0 + S(t,0)x_1 + \int_0^t S(t,s)\overline{f}(s)ds, \qquad t \in [0,T].$$

By Mazur Theorem again, let $b_{kl} \geq 0$ and $\sum_{k\geq 1} b_{kl} = 1$ such that

$$\phi_l(\cdot) = \sum_{k\geq 1} b_{kl} f(\cdot, x^{k+l}(\cdot), u^{k+l}(\cdot)) \to \overline{f}(\cdot) \qquad \text{in } L^2([0,T];X).$$

Let

$$\psi_l(\cdot) = \sum_{k\geq 1} q_{kl} h(\cdot, x^{k+l}(\cdot), u^{k+l}(\cdot)), \qquad q_{kl} \geq 0,\ \sum_{k\geq 1} q_{kl} = 1,$$

and

$$\overline{h}(t) = \underline{\lim}_{l\to+\infty} \psi_l(t) \geq -M_1, \qquad \text{a.e. } t \in [0,T].$$

For any $\delta > 0$ and l large enough, we have

$$(\psi_l(t), \phi_l(t)) \in \mathrm{co}\varepsilon(t, O_\delta(\overline{x}(t))), \qquad \text{a.e. } t \in [0,T].$$

Then

$$(\overline{h}(t), \overline{f}(t)) \in \overline{co}\varepsilon(t, O_\delta(\overline{x}(t))), \qquad \text{a.e. } t \in [0,T].$$

It comes from (H_ε) that

$$(\overline{h}(t), \overline{f}(t)) \in \varepsilon(t, \overline{x}(t)), \qquad \text{a.e. } t \in [0,T].$$

Thus, we can find a measurable $\overline{u} : [0,T] \to V$ such that for $t \in [0,T]$,

$$\begin{cases} \overline{h}(t) \geq h(t, \overline{x}(t), \overline{u}(t)), \\ \overline{f}(t) = f(t, \overline{x}(t), \overline{u}(t)), \\ \overline{u}(t) \in U(t, \overline{x}(t)). \end{cases}$$

Therefore,

$$(\overline{x}, \overline{u}) \in X[0,T].$$

By Fatou's Lemma, we obtain

$$\begin{aligned} \int_0^b \overline{h}(t)dt &= \int_0^T \underline{\lim}_{l\to+\infty} \psi_l(t)dt \leq \underline{\lim}_{l\to+\infty} \int_0^b \psi_l(t)dt \\ &= \underline{\lim}_{l\to+\infty} \int_0^b \sum_{k\geq 1} q_{kl} h(t, x^{k+l}(t), u^{k+l}(t))dt \\ &= \underline{\lim}_{l\to+\infty} \sum_{k\geq 1} q_{kl} \int_0^b h(t, x^{k+l}(t), u^{k+l}(t))dt \\ &= \underline{\lim}_{l\to+\infty} \sum_{k\geq 1} q_{kl} \underline{\lim}_{l\to+\infty} \int_0^b h(t, x^{k+l}(t), u^{k+l}(t))dt \\ &= m. \end{aligned}$$

Therefore,

$$m \leq \varphi(\overline{x}, \overline{u}) = \int_0^b h(t, \overline{x}(t), \overline{u}(t))dt \leq m,$$

i.e.,

$$\int_0^b h(t, \overline{x}(t), \overline{u}(t))dt = m = \inf_{(x,u)\in X[0,T]} \varphi(x,u).$$

Thus, $(\overline{x}, \overline{u})$ is an optimal state-control pair. The proof is complete. □

6. Applications

In this section, we applied our previous results to a controllability result for semilinear evolution equations, existence results for Clarke's subdifferential inclusions and a class of differential variational inequalities.

6.1. Controllability result

Consider the controllability of the following semilinear evolution equation.

$$\begin{cases} x''(t) = A(t)x(t) + f'(t, x(t)) + Bu(t), \qquad t \in [0,T], \\ x(0) = x_0, \end{cases} \tag{6.1}$$

where $\{A(t)\}_{t\in[0,+\infty)}$ is a family of closed densely defined linear operators in X, B is a linear operator from V into $L^2([0,T]; X)$.

Now we give the definition of controllability as follows.

Definition 6.1. System (6.1) is said to be controllable on $[0,T]$, if for every $x_0, x_T \in X$, there exists a control $u \in L^2([0,T];V)$ such that a mild solution x of system (6.1) satisfies $x(T) = x_T$.

We need to make the following assumptions.
$(H_{f'})$: $f' : [0,T] \times X \to X$ is Borel measurable on $[0,T] \times X$ and continuous on X, there exist a function $\phi_4 \in L^2([0,T];\mathbb{R}_+)$ and constants $L_4 > 0$ such that

$$\|f'(t,x)\| \leq \phi_4(t) + L_4\|x\|_X$$

for all $x \in X$, a.e. $t \in [0,T]$.
(H_W) The linear operator $W : L^2([0,T];V) \to X$ defined by

$$Wu = \int_0^T S(T,s)Bu(s)ds$$

has an invertible operator W^{-1} which takes values in $L^2([0,T];V)/\ker W$, where $\ker W = \{x \in L^2([0,T];V) : Wx = 0\}$ and there exists a positive constant M_W such that $\|W^{-1}\| \leq M_W$.

The following theorem is the main result of this section.

Theorem 6.1. *Assume that (H_S), $(H_{f'})$ and (H_W) are satisfied. Then the system (6.1) is controllable on $[0,T]$.*

Proof. For any $x \in C([0,T];X)$, $x_T \in X$, we can define the feedback control $U : [0,T] \times X \to 2^V$ by

$$\overline{U}(t,x) = W^{-1}\bigg(x_T - C(T,0)x_0 - S(T,0)x_1 - \int_0^T S(T,s)f'(s,x)ds\bigg)(t), \quad t \in [0,T].$$

We show that, using this control, the operator $\Phi : C([0,T];X) \to C([0,T];X)$, defined by

$$\Phi(x) = C(t,0)x_0 + S(t,0)x_1 + \int_0^t S(t,s)[f'(s,x(s)) + B\overline{U}(t,x(s))]ds, \quad t \in [0,T]$$

has a fixed point x, which is a mild solution of system (6.1).

It is east to verify that $\overline{U}$ is continuous on $[0,T] \times X$, and hence (H_U) hold. Then, by applying Theorem 4.1, we obtain that the operator Φ has a fixed point. Therefore, system (6.1) is controllable on $[0,T]$. $\square$

6.2. Clarke's subdifferential inclusion

Let us recall the definition of the Clarke's subdifferential for a locally Lipschitz function $j : K \subset X \to \mathbb{R}$, where K is a nonempty subset of a Banach space X (one can see [6, 8, 22]). We denote by $j^0(x;y)$ the Clarke's generalized directional derivative of j at the point $x \in K$ in the direction $y \in X$, that is

$$j^0(x;y) := \limsup_{\lambda \to 0^+,\ \zeta \to x} \frac{j(\zeta + \lambda y) - j(\zeta)}{\lambda}.$$

Recall also that the Clarke's subdifferential or generalized gradient of j at $x \in K$, denoted by $\partial j(x)$, is a subset of X^* given by

$$\partial j(x) := \{x^* \in X^* : j^0(x;y) \geq \langle x^*, y\rangle,\ \forall y \in X\}.$$

Lemma 6.1 ([22], Proposition 3.23)**.** *If $j : K \to \mathbb{R}$ is locally Lipschitz function, then*
(i) the function $(x, y) \mapsto j^0(x; y)$ is u.s.c. from $K \times X$ into $\mathbb{R}$;
(ii) for every $x \in K$ the gradient $\partial j(x)$ is a nonempty, convex and weakly* compact subset of X^* which is bounded by the Lipschitz constant $L_x > 0$ of j near x;
(iii) the graph of ∂j is closed in $X \times X^*_{w^*}$;
(iv) the multifunction ∂j is u.s.c. from K into $X^*_{w^*}$.

Consider the following Clarke's subdifferential inclusion:

$$\begin{cases} x''(t) = A(t)x(t) + g(t, x(t)) + \gamma^* u(t), \qquad t \in (0, b], \\ u(t) \in \partial j(t, \gamma x(t)), \qquad \text{a.e. } t \in [0, T], \\ x(0) = x_0 \in X, \end{cases} \tag{6.2}$$

where $\{A(t)\}_{t\in[0,+\infty)}$ is a family of closed densely defined linear operators in X, $j : [0, T] \times Y \to \mathbb{R}$ is a locally Lipschitz function with respect to the second variable with Y being a separable reflexive Banach space, $\partial j(t, \cdot)$ denotes the Clarke's subdifferential of $j(t, \cdot)$ for $t \in [0, T]$ and $\gamma : X \to Y$ is a linear, continuous and compact operator.

We need to make the following assumptions.
(H_g) $g(\cdot, x) : [0, T] \to \mathbb{R}$ is measurable for every $x \in X$, and there exists a constant $L_g > 0$ such that

$$\|(g(t, x_1) - g(t, x_2)\|_X \le L_g \|x_1 - x_2\|_X,$$
$$\|(g(t, 0)\|_X \le L_g$$

for all $x_1, x_2 \in X, u \in V$, a.e. $t \in [0, T]$.
(H_j) $j : [0, T] \times Y \to \mathbb{R}$ is continuous on $[0, T]$ and locally Lipschitz continuous on Y, and there exist a function $\phi_5 \in L^2([0, T]; \mathbb{R}_+)$ and constants $L_5 > 0$ such that

$$\|\partial j(t, y)\| \le \phi_5(t) + L_5\|y\|_Y$$

for all $y \in Y$, a.e. $t \in [0, T]$.

We have the following result.

Theorem 6.2. *If $(H_S), (H_g), (H_j)$ hold, then the system (6.2) has a solution.*

Proof. From the properties of ∂j in Lemma 6.1 and the compactness of γ, it follows that the multifunction $U : [0, T] \times X \to 2^{Y^*}$, defined by $U(t, x) = \partial j(t, \gamma x)$ for $t \in [0, T], x \in X$, satisfies the condition (H_U). The result of this theorem is a consequence of Theorem 4.1. □

6.3. Differential variational inequalities

Consider the following second-order differential variational inequality:

$$\begin{cases} x''(t) = A(t)x(t) + f(t, x(t), u(t)), \qquad t \in (0, T], \\ u(t) \in SOL(K, g(t, x(t), \cdot), \phi), \qquad \text{a.e. } t \in (0, T], \\ x(0) = x_0, \ x'(0) = x_1, \end{cases} \tag{6.3}$$

where $\{A(t)\}_{t\in[0,+\infty)}$ is a family of closed densely defined linear operators in X, $SOL(K, g(t, x(t), \cdot), \phi)$ denotes the solution set of the following mixed variational inequality in V: find $u : [0, T] \to K \subset V$ such that

$$\langle g(t, x(t), u(t)), v - u(t)\rangle_V + \phi(v) - \phi(u(t)) \geq 0, \qquad \forall v \in K,\ t \in [0, T].$$

Let $F : [0, T] \times X \to 2^V$ given by $F(t, x) = f(t, x, U(t, x))$. Then we have the following result.

Lemma 6.2 ([16], Proposition 4.5)**.** *Let $U : [0, T] \times X \to 2^V$ be u.s.c. taking closed set values and $f(t, x, u)$ be uniformly continuous in (x, u) for any $t \in [0, T]$. Then the following are equivalent:*

(i) $(H_U)(ii)$ holds;

(ii) $F(t, \cdot)$ has the Cesari property for a.e. $t \in [0, T]$;

(iii) $F(t, x)$ is closed and convex for all $x \in X$, a.e. $t \in [0, T]$.

Lemma 6.3 ([20], Theorem 3.4)**.** *Assume that X, V are real separable reflexive Banach spaces and K is a nonempty, compact and convex subset of V. Assume that $g : [0, T] \times X \times K \to V^*$ is such that $g(\cdot, \cdot, u)$ is continuous from $[0, T] \times X$ to V^* endowed with the weak* topogy whenever $u \in K$. In addition, we assume that for every $(t, x) \in [0, T] \times X$ the mappings $Q := g(t, x, \cdot)$ and $\phi : V \to \mathbb{R} \cup \{+\infty\}$ satisfy the following hypotheses:*
(i) $Q : K \to V^*$ is monotone on K and satisfies

$$\liminf_{\lambda\to 0^+} \langle Q(\lambda u + (1 - \lambda)v), v - u\rangle \leq \langle Qv, v - u\rangle, \qquad u, v \in K;$$

(ii) ϕ is convex, lower semicontinuous, and $\not\equiv +\infty$.
Then the multifunction $U : [0, T] \times X \to K$ defined by

$$U(t, x) := \{u \in K : \langle g(t, x, u), v - u\rangle + \phi(v) - \phi(u) \geq 0, \qquad \forall v \in K\}. \tag{6.4}$$

(i) U is u.s.c.;

(ii) U is superpositionally measurable.

Lemma 6.4 ([20], Lemma 4.2)**.** *Assume that X, V are real separable reflexive Banach spaces and K is a nonempty, compact and convex subset of V. Assume that the hypotheses of Lemma 6.3 and $H(1)$ and $H(5)$ are satisfied. Suppose that, in addition, $f(t, x, D)$ is convex for every convex $D \subset E_2$, all $x \in X$, a.e. $t \in [0, T]$. Then there hold:*

(i) $F(t, x)$ is a closed convex subset of X for all $(t, x) \in [0, T] \times X$;

(ii) $F(\cdot, x)$ has a strongly measurable selection for a.e. $t \in [0, T]$;

(iii) $F(t, \cdot)$ is u.s.c for a.e. $t \in [0, T]$.

Let $X'[0, T] = \{(x, u) \in C([0, T]; X) \times V[0, T]|\ (x, u) \text{ is feasible for } (6.3)\}$. The following result is a consequence of Theorem 4.1, Lemma 6.2 and Lemma 6.4.

Theorem 6.3. *If all the assumptions of Lemma 6.4 are satisfied and (H_S) hold, then $X'[0, T]$ is nonempty.*

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