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NEW OSCILLATION CRITERIA FOR FIRST ORDER LINEAR DIFFERENTIAL EQUATIONS WITH NON-MONOTONE DELAYS

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Abstract This paper is concerned with the oscillation of the first order linear delay differential equation $x'(t) + q(t)x(\tau(t)) = 0$, $t \geq t_0$, where $q, \tau \in C([t_0, \infty), [0, \infty))$, $\tau(t) \leq t$, such that $\lim_{t\to\infty} \tau(t) = \infty$. Several new oscillation criteria of iterative and non-iterative types are obtained. Two examples are presented to show the strength and applicability of these criteria over known ones.

Keywords Oscillation criteria, delay differential equations, non-monotone delay.

MSC(2010) 34C10, 34K11, 34K05.

1. Introduction

The study of the oscillation theory of delay differential equations has attracted a large number of researchers since the pioneering work of Myshkis [26]. The reader is referred to the monographs [1, 16, 17] as well as the papers [2–15, 18–31] for a considerable account of results. One of the principal problems in oscillation theory is to obtain sufficient criteria for the oscillation of certain equation. In this paper, we investigate the oscillatory character of the delay differential equation

$$x'(t) + q(t)x(\tau(t)) = 0, \qquad t \geq t_0, \tag{1.1}$$

where $q, \tau \in C([t_0, \infty), [0, \infty))$, $\tau(t) \leq t$, such that $\lim_{t\to\infty} \tau(t) = \infty$.

Equation (1.1) is called oscillatory, if each of its solutions possesses arbitrary large zeros. Due to the linearity of Eq.(1.1), it will be non-oscillatory if it has an eventually positive solution. Although Eq.(1.1) looks simple, its oscillatory behavior has not yet been fully characterized except for the constant coefficients case

$$x'(t) + \bar{p}x(t - \bar{\tau}) = 0,$$

which is known to be oscillatory if and only if $\bar{p}\bar{\tau} > \frac{1}{e}$, see [17]. Many oscillation criteria were established for the non-autonomous case with nondecreasing delay arguments, see for example [6, 9, 10, 14, 19, 20, 24, 25, 27, 28] and the references cited therein.

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The oscillatory behavior of equations with non-monotone delays is very challenging and not an easy extension of those results for equations with monotone delays. A remarkable result in this direction is due to Braverman and Karpuz [5]. It states, in its simplest interpretation, that the classic condition for the oscillation of Eq.(1.1) in monotone delay case

$$\limsup_{t\to\infty} \int_{\tau(t)}^{t} q(u)du > 1, \tag{1.2}$$

is not applicable for the non-monotone case in general. This highlights the importance of obtaining necessary and/or sufficient conditions for the oscillation of the general form (1.1). In [13–15, 27] sharp oscillatory condition that improves (1.2) is obtained for a particular class of Eq.(1.1) with coefficients enjoying the slowly varying property. In this work, we obtain new sufficient criteria for the oscillation of Eq.(1.1) when τ is not assumed to be monotone and q need not to be of slowly varying type.

We assume the existence of a non-decreasing continuous function $h(t)$ such that $\tau(t) \le h(t) \le t$ for all $t \ge t_1$, and some $t_1 \ge t_0$. A particular case of the function h is the following one which has been employed in many known results

$$\delta(t) = \sup_{u\le t} \tau(u), \quad t \ge t_0. \tag{1.3}$$

We also, assume that $\lambda(\zeta)$ is the smaller real root of the equation $\lambda = \mathrm{e}^{\lambda\zeta}$,

$$\begin{aligned}
D(u) &= \frac{1-u-\sqrt{1-2u-u^2}}{2}, \quad 0 \le u \le \frac{1}{\mathrm{e}},\\
k &= \liminf_{t\to\infty} \int_{\tau(t)}^{t} q(u)\, du \le \frac{1}{\mathrm{e}},\\
k^* &= \liminf_{t\to\infty} \int_{h(t)}^{t} q(u)\, du,\\
\rho &= \begin{cases} 1, & k^* = 0,\\ \lambda(k^*) - \epsilon, & k^* > 0, \qquad \epsilon \in (0,\, \lambda(k^*)), \end{cases}
\end{aligned} \tag{1.4}$$

and

$$L^* = \limsup_{t\to\infty} \int_{h(t)}^{t} q(u)\, du.$$

For an easy reference we recall some known oscillation criteria of Eq.(1.1) with non-monotone delay. The first one is due to Koplatadze and Kvinikadze [22];

$$\limsup_{t\to\infty} \int_{\delta(t)}^{t} q(u)\mathrm{e}^{\int_{\delta(u)}^{\delta(t)} q(u_1)V_n(u_1)du_1} du > 1 - D(k), \tag{1.5}$$

where

$$V_1(t) = 0, \quad V_n(t) = \mathrm{e}^{\int_{\tau(t)}^{t} q(u)V_{n-1}(u)du},\ n = 2, 3, \ldots.$$

Braverman and Karpuz [5] obtained the condition

$$\limsup_{t\to\infty} \int_{\delta(t)}^{t} q(u)\ \mathrm{e}^{\int_{\tau(u)}^{\delta(t)} q(u_1)\ du_1} du > 1. \tag{1.6}$$

Stavroulakis [29] improved the preceding condition and (1.5) with $n = 2$ by the criterion

$$\limsup_{t\to\infty} \int_{\delta(t)}^{t} q(u)\, e^{\int_{\tau(u)}^{\delta(t)} q(u_1)\, du_1} du > 1 - D(k). \tag{1.7}$$

Infante etc [18] improved (1.6) by each one of the conditions

$$\limsup_{t\to\infty} \int_{h(t)}^{t} q(u)\, e^{\int_{\tau(u)}^{h(t)} q(u_1)\, e^{\int_{\tau(u_1)}^{u_1} q(u_2)du_2} du_1} du > 1, \tag{1.8}$$

and

$$\limsup_{\epsilon\to 0^+} \left(\limsup_{t\to\infty} \int_{h(t)}^{t} q(u) e^{(\lambda(k)-\epsilon)\int_{\tau(u)}^{h(t)} q(u_1)du_1} du \right) > 1. \tag{1.9}$$

El-Morshedy and Attia [11] derived the condition

$$\limsup_{t\to\infty} \left(\int_{h(t)}^{t} B_n(u)du + D(k^*) e^{\int_{h(t)}^{t} \sum_{i=0}^{n-1} B_i(u)du} \right) > 1, \tag{1.10}$$

where

$$B_0(t) = q(t),\ B_1(t) = B_0(t) \int_{\tau(t)}^{t} B_0(u) e^{\int_{\tau(u)}^{t} B_0(u_1)du_1} du,$$

and

$$B_n(t) = B_{n-1}(t) \int_{h(t)}^{t} B_{n-1}(u) e^{\int_{h(u)}^{t} B_{n-1}(u_1)du_1} du, \quad n = 2, 3, \ldots.$$

Chatzarakis [7] improved (1.7) by the condition

$$\limsup_{t\to\infty} \int_{\delta(t)}^{t} q(u) e^{\int_{\tau(u)}^{\delta(t)} q(u_1)Q_n(u_1)du_1} du > 1 - D(k), \tag{1.11}$$

where, $n \in \mathbb{N}$,

$$\begin{aligned} Q_0(t) &= q(t), \\ Q_n(t) &= q(t) \left[1 + \int_{\tau(t)}^{t} q(u) e^{\int_{\tau(u)}^{\delta(t)} Q_{n-1}(u_1)du_1} du \right]. \end{aligned}$$

Bereketoglu etc [4] derived the following improvement of (1.11),

$$\limsup_{t\to\infty} \int_{h(t)}^{t} q(u) e^{\int_{\tau(u)}^{h(t)} P_n(u_1)du_1} du > \begin{cases} 1 & \text{or} \\ 1 - D(k^*), \end{cases} \tag{1.12}$$

for some $n \in \mathbb{N}$, where

$$\begin{aligned} P_0(t) &= q(t), \\ P_n(t) &= q(t) \left[1 + \int_{h(t)}^{t} q(u) e^{\int_{\tau(u)}^{t} P_{n-1}(u_1)du_1} du \right]. \end{aligned}$$

Attia etc [3] established the following general criterion that improves conditions (1.8), (1.11) and (1.12),

$$\limsup_{t\to\infty}\int_{h(t)}^{t} q(u)\mathrm{e}^{\int_{\tau(u)}^{h(t)} q(u_1)\Psi i,j(u_1)du_1}du > 1-D(k^*), \quad \text{for some } i,j\in\mathbb{N}, \tag{1.13}$$

where the double sequence $\{\Psi_{i,j}(t)\}$ is defined by

$$\Psi_{i,j}(t) = 1+\int_{\tau(t)}^{t} q(u)\mathrm{e}^{\int_{\tau(u)}^{t} q(u_1)\Phi_{i-1,j}(u_1)du_1}du,$$

with

$$\Phi_{k,l}(t) = \mathrm{e}^{\int_{\tau(t)}^{t} q(u)\Phi_{k,l-1}(u)du}, \quad k=1,\,2,\ldots,i-1,\; l=1,2,\ldots,j,$$

and

$$\begin{aligned}
\Phi_{0,0}(t) &= (\lambda(k^*)-\epsilon)\left(1+(\lambda(k^*)-\epsilon)\int_{\tau(t)}^{h(t)} q(u)du\right), \qquad \epsilon\in(0,\,\lambda(k^*)),\\
\Phi_{0,l}(t) &= \mathrm{e}^{\int_{\tau(t)}^{t} q(u)\Phi_{0,l-1}(u)du}, \quad l=1,\,2,\ldots,j,\\
\Phi_{k,0}(t) &= \Psi_{k,j}(t), \quad k=1,2,\ldots,i-1.
\end{aligned}$$

Very Recently, Attia [2] established the condition

$$\limsup_{t\to\infty}\int_{h(t)}^{t} q(u)\mathrm{e}^{\int_{\tau(u)}^{h(t)} q(u_1)\Omega_n(u_1)du_1}du > 1-D(k^*), \tag{1.14}$$

where $\{\Omega_n(t)\}_{n=0}^{\infty}$ and $\{\Upsilon_n(t)\}_{n=1}^{\infty}$ are defined as follows:

$$\begin{aligned}
&\Omega_0(t) = \rho,\\
&\Upsilon_1(t) = \int_{h(t)}^{t} q(u_1)\mathrm{e}^{\int_{\tau(u_1)}^{h(t)} q(u_2)\Omega_0(u_2)du_2}\, du_1, \quad \Omega_1(t) = \frac{1}{1-\Upsilon_1(t)},\\
&\Upsilon_2(t) = \int_{h(t)}^{t} q(u_1)\, du_1 + \Omega_1(h(t))\int_{h(t)}^{t} q(u_1)\int_{\tau(u_1)}^{h(t)} q(u_2)\mathrm{e}^{\int_{\tau(u_2)}^{h^2(t)} q(u_3)\Omega_1(u_3)du_3}du_2du_1,\\
&\Omega_2(t) = \frac{1}{1-\Upsilon_2(t)},
\end{aligned}$$

and

$$\begin{aligned}
&\Upsilon_n(t)\\
&= \int_{h(t)}^{t} q(u_1)\, du_1 + \Omega_{n-1}(h(t))\int_{h(t)}^{t} q(u_1)\int_{\tau(u_1)}^{h(t)} q(u_2)du_2\, du_1 + \ldots\\
&\quad + \prod_{i=2}^{n-1}\left(\Omega_{n-1}(h^{i-1}(t))\right)\int_{h(t)}^{t} q(u_1)\int_{\tau(u_1)}^{h(t)} q(u_2)\cdots\int_{\tau(u_{n-2})}^{h^{n-2}(t)} q(u_{n-1})du_{n-1}\ldots\, du_1\\
&\quad + \prod_{i=2}^{n}\left(\Omega_{n-1}(h^{i-1}(t))\right)\\
&\quad \times\int_{h(t)}^{t} q(u_1)\int_{\tau(u_1)}^{h(t)} q(u_2)\cdots\int_{\tau(u_{n-1})}^{h^{n-1}(t)} q(u_n)\mathrm{e}^{\int_{\tau(u_n)}^{h^n(t)} q(u_{n+1})\Omega_{n-1}(u_{n+1})du_{n+1}}du_n\ldots\, du_1,
\end{aligned}$$

$$\Omega_n(t) = \frac{1}{1-\Upsilon_n(t)}, \quad n = 3, 4, \ldots .$$

The above-mentioned criteria are of iterative integral types. These kind of criteria are very powerful but look fairly applicable for large iterates. Therefore, in this work, we derive new oscillation criteria of both iterative and non-iterative types. Moreover, the strength of each type of our conditions over the above-mentioned ones is supported by an illustrative example.

2. Preliminary Results

The following four lemmas are very crucial for proving the results of the next section. The first one can be proved by using the non-increasing nature of $x(t)$ and Lemma 2.1.2 in [12].

Lemma 2.1. *Let $x(t)$ be an eventually positive solution of Eq.(1.1). Then*

$$\frac{x(h(t))}{x(t)} \geq \rho, \qquad \text{for all sufficiently large } t, \tag{2.1}$$

where ρ is defined by (1.4).

Lemma 2.2 ([30]). *Assume that $x(t)$ is an eventually positive solution of Eq.(1.1). Then*

$$\liminf_{t\to\infty} \frac{x(t)}{x(h(t))} \geq D(k^*).$$

Lemma 2.3. *Assume that $x(t)$ is a solution of Eq.(1.1) and n is a positive integer. Then*

$$x(h(t)) = x(t) + \sum_{i=1}^{n-1} x(h^i(t))Q_i^n(t) + x(h^n(t))\bar{Q}_n^n(t), \tag{2.2}$$

where

$$Q_i^n(t) = \int_{h(t)}^{t} q(u_1)\int_{\tau(u_1)}^{h(t)} q(u_2)\int_{\tau(u_2)}^{h^2(t)} \cdots \int_{\tau(u_{i-1})}^{h^{i-1}(t)} q(u_i)du_i\, du_{i-1}\ldots\, du_1,\ i=1,\ldots,n-1,$$

and

$$\bar{Q}_n^n = \int_{h(t)}^{t} q(u_1)\int_{\tau(u_1)}^{h(t)} q(u_2)\int_{\tau(u_2)}^{h^2(t)} \cdots \int_{\tau(u_{n-1})}^{h^{n-1}(t)} q(u_n)e^{\int_{\tau(u_n)}^{h^n(t)} q(u_{n+1})\frac{x(\tau(u_{n+1}))}{x(u_{n+1})}du_{n+1}} du_n du_{n-1}\ldots du_1.$$

Proof. Integrating (1.1) from $h(t)$ to t, we have

$$x(t) - x(h(t)) + \int_{h(t)}^{t} q(u_1)x(\tau(u_1))du_1 = 0. \tag{2.3}$$

Again integrating (1.1) from $\tau(v)$ to $h(t)$, $v \leq t$, it follows that

$$x(\tau(v)) = x(h(t)) + \int_{\tau(v)}^{h(t)} q(u_2)x(\tau(u_2))du_2.$$

Substituting into (2.3),

$$x(t)-x(h(t))+x(h(t))\int_{h(t)}^{t} q(u_1)du_1+\int_{h(t)}^{t} q(u_1)\int_{\tau(u_1)}^{h(t)} q(u_2)x(\tau(u_2))du_2\ du_1=0. \tag{2.4}$$

Notice that, $\tau(v) \leq h^2(t)$, for all $v \leq h(t)$. Hence, integrating (1.1) from $\tau(v)$ to $h^2(t)$, we obtain

$$x(\tau(v)) = x(h^2(t)) + \int_{\tau(v)}^{h^2(t)} q(u_3)x(\tau(u_3))du_3.$$

This, together with (2.4), leads to

$$x(t) - x(h(t)) + x(h(t)) \int_{h(t)}^{t} q(u_1)du_1 + x(h^2(t)) \int_{h(t)}^{t} q(u_1) \int_{\tau(u_1)}^{h(t)} q(u_2)du_2\ du_1$$
$$+ \int_{h(t)}^{t} q(u_1) \int_{\tau(u_1)}^{h(t)} q(u_2) \int_{\tau(u_2)}^{h^2(t)} q(u_3)x(\tau(u_3))du_3\ du_2\ du_1 = 0.$$

Similarly, we obtain

$$\begin{aligned} &x(h(t)) \\ &=x(t)+\sum_{i=1}^{n-1} x(h^i(t))\int_{h(t)}^{t}q(u_1)\int_{\tau(u_1)}^{h(t)}q(u_2)\int_{\tau(u_2)}^{h^2(t)}\cdots\int_{\tau(u_{i-1})}^{h^{i-1}(t)}q(u_i)du_i\ du_{i-1}\ldots du_1 \\ &\quad+\int_{h(t)}^{t} q(u_1) \int_{\tau(u_1)}^{h(t)} q(u_2) \int_{\tau(u_2)}^{h^2(t)} \cdots \int_{\tau(u_{n-1})}^{h^{n-1}(t)} q(u_n)x(\tau(u_n))du_n\ du_{n-1}\ldots\ du_1. \end{aligned} \tag{2.5}$$

Dividing both sides of (1.1) by $x(t)$, where t is sufficiently large, and then integrating from $\tau(v)$ to $h^n(t)$, we get

$$x(\tau(v)) = x(h^n(t))e^{\int_{\tau(v)}^{h^n(t)} q(u_{n+1})\frac{x(\tau(u_{n+1}))}{x(u_{n+1})}du_{n+1}},\ \text{where } v \leq h^{n-1}(t).$$

Substituting into (2.5), we finally obtain (2.2). □

Next, we define the finite sequence $\{Q_i^n(t)\}_{i=1}^n$, for $i = 1, 2, \ldots, n-1$ as in Lemma 2.3, while $Q_n^n(t)$ is defined as follows:

$$\begin{aligned} &Q_n^n(t) \\ &=\int_{h(t)}^{t}q(u_1)\int_{\tau(u_1)}^{h(t)}q(u_2)\int_{\tau(u_2)}^{h^2(t)}\cdots\int_{\tau(u_{n-1})}^{h^{n-1}(t)}q(u_n)e^{\rho\int_{\tau(u_n)}^{h^n(t)}q(u_{n+1})du_{n+1}}du_n\ du_{n-1}\ldots du_1. \end{aligned}$$

Lemma 2.4. *Assume that $x(t)$ is an eventually positive solution of Eq.(1.1), and $M := \liminf_{t\to\infty} \frac{x(t)}{x(h(t))}$. Then*

$$\limsup_{t\to\infty} \left(\sum_{i=1}^{n} \frac{x(h^i(t))}{x(h(t))}Q_i^n(t)\right) \leq 1 - M.$$

Proof. Using (2.2) and Lemma 2.1, we obtain

$$x(t) - x(h(t)) + \sum_{i=1}^{n} x(h^i(t))Q_i^n(t) \le 0.$$

Dividing by $x(h(t))$,

$$\sum_{i=1}^{n} \frac{x(h^i(t))}{x(h(t))} Q_i^n(t) \le 1 - \frac{x(t)}{x(h(t))}.$$

Consequently

$$\limsup_{t\to\infty} \left(\sum_{i=1}^{n} \frac{x(h^i(t))}{x(h(t))} Q_i^n(t) \right) \le 1 - M.$$

□

3. Main Results

In this section, we derive sufficient criteria for the oscillation of Eq.(1.1) as well as two applications of our results.

Theorem 3.1. *Assume that $n \in \mathbb{N}$ and*

$$\limsup_{t\to\infty} \left(\sum_{i=1}^{n} \left(\prod_{j=2}^{i} W\left(h^{j-1}(t)\right) \right) Q_i^n(t) \right) > 1 - D(k^*), \tag{3.1}$$

where $W(t) = \frac{1}{1 - \int_{h(t)}^{t} q(u_1) \exp\left(\int_{\tau(u_1)}^{h(t)} \frac{q(u_2)}{1 - Q_1^1(u_2)} du_2 \right) du_1}$, *by convention we set* $\prod_{j=2}^{1} W(h^j(t)) = 1$. *Then Eq.*(1.1) *is oscillatory.*

Proof. Let $x(t)$ be a non-oscillatory solution of Eq.(1.1). Due to the linearity of Eq.(1.1), we can chose $x(t)$ to be eventually positive. The representation (2.2) with $n = 1$, leads to

$$x(t) = x(h(t)) - x(h(t)) \int_{h(t)}^{t} q(u_1) e^{\int_{\tau(u_1)}^{h(t)} q(u_2) \frac{x(\tau(u_2))}{x(u_2)} du_2} du_1.$$

Dividing by $x(t)$ and rearranging, we get

$$\frac{x(h(t))}{x(t)} = \frac{1}{1 - \int_{h(t)}^{t} q(u_1) e^{\int_{\tau(u_1)}^{h(t)} q(u_2) \frac{x(\tau(u_2))}{x(u_2)} du_2} du_1}. \tag{3.2}$$

Using Lemma 2.1, we arrive at

$$\frac{x(h(t))}{x(t)} \ge \frac{1}{1 - \int_{h(t)}^{t} q(u_1) e^{\rho \int_{\tau(u_1)}^{h(t)} q(u_2) du_2} du_1} = \frac{1}{1 - Q_1^1(t)}.$$

Substituting into (3.2),

$$\frac{x(h(t))}{x(t)} \ge \frac{1}{1 - \int_{h(t)}^{t} q(u_1) e^{\int_{\tau(u_1)}^{h(t)} \frac{q(u_2)}{1 - Q_1^1(u_2)} du_2} du_1} = W(t). \tag{3.3}$$

But

$$\frac{x(h^i(t))}{x(h(t))} = \prod_{j=2}^{i} \frac{x(h^j(t))}{x(h^{j-1}(t))}, \quad \text{for } i = 2, 3, \ldots, n. \tag{3.4}$$

Then (3.3) yields

$$\frac{x(h^i(t))}{x(h(t))} \geq \prod_{j=2}^{i} W(h^{j-1}(t)).$$

Combining this inequality with Lemmas 2.4 and using Lemma 2.2, we obtain

$$\limsup_{t\to\infty} \left(\sum_{i=1}^{n} \left(\prod_{j=2}^{i} W\left(h^{j-1}(t)\right) \right) Q_i^n(t) \right) \leq 1 - D(k^*).$$

The proof is complete. □

Notice that (3.4) and Lemma 2.1 yield $\frac{x(h^i(t))}{x(h(t))} \geq \rho^{i-1}$. Therefore, using Lemmas 2.2 and 2.4, we obtain the following result.

Theorem 3.2. *Assume that* $n \in \mathbb{N}$ *and*

$$\limsup_{t\to\infty} \left(\sum_{i=1}^{n} \rho^{i-1} Q_i^n(t) \right) > 1 - D(k^*).$$

Then Eq.(1.1) *is oscillatory.*

The following example shows the strength of condition (3.1), when $k = 0$.

Example 3.1. Consider the delay differential equation

$$x'(t) + q(t)x(\tau(t)) = 0, \qquad t \geq 2, \tag{3.5}$$

where

$$\tau(t) = t - 1 - \alpha \sin^2(\nu\pi t),$$

and

$$q(t) := \begin{cases} 0, \quad t \in [a_r, b_r], \\ \beta\,(t - b_r)\sin^2\left(\frac{\pi}{2}\sqrt{(t-b_r)(b_r+1-t)+1}\right), \quad t \in [b_r, b_r+1], \\ \beta, \quad t \in [b_r+1, b_r+6], \\ \beta\left(1 - \frac{1}{a_{r+1}-b_r-6}(t - b_r - 6)\right)\cos^2\left(\pi\sqrt{(t-b_r-6)(a_{r+1}-t)+1}\right), \\ t \in [b_r+6, a_{r+1}], \end{cases}$$

$r \in \mathbb{N}$, $0 \leq a_r < b_r - 1 - \alpha$, $b_r + 6 < a_{r+1}$ such that $\lim_{r\to\infty} a_r = \infty$, $\alpha = 0.0001$, $\beta = 0.449$ and $\nu = 20000$. We choose $h(t) = \delta(t)$ (that is defined by (1.3)), then

$$t - 1 - \alpha \leq \tau(t) \leq h(t) = \delta(t) \leq t - 1.$$

Therefore

$$0 \leq \int_{\tau(b_r)}^{b_r} q(u)du \leq \int_{b_r-1-\alpha}^{b_r} q(u)du = 0.$$

Then, $k = k^* = \liminf_{t\to\infty} \int_{\tau(t)}^{t} q(u)du = 0$, and hence, $\rho = 1$ (from (1.4)). Clearly

$$
\begin{aligned}
Q_1^2(b_r+6) &= \int_{h(b_r+6)}^{b_r+6} q(u_1)du_1 \\
&\geq \int_{b_r+5}^{b_r+6} q(u_1)du_1 = \int_{b_r+5}^{b_r+6} \beta du_1 = 0.449, \\
Q_2^2(b_r+6) &= \int_{h(b_r+6)}^{b_r+6} q(u_1) \int_{\tau(u_1)}^{h(b_r+6)} q(u_2)\mathrm{e}^{\rho\int_{\tau(u_2)}^{h^2(b_r+6)} q(u_3)du_3} du_2 du_1 \\
&\geq \int_{b_r+5}^{b_r+6} q(u_1) \int_{u_1-1}^{b_r+5-\alpha} q(u_2)\mathrm{e}^{\int_{u_2-1}^{b_r+4-2\alpha} q(u_3)du_3} du_2 du_1 > 0.117713.
\end{aligned}
$$

Also, for $b_r + 3 \leq v \leq b_r + 4 - \alpha$

$$
\begin{aligned}
Q_1^1(v) &= \int_{h(v)}^{v} q(u_1)\mathrm{e}^{\rho\int_{\tau(u_1)}^{h(v)} q(u_2)du_2} du_1 \geq \int_{v-1}^{v} q(u_1)\mathrm{e}^{\int_{u_1-1}^{v-1-\alpha} q(u_2)du_2} du_1 \\
&= \int_{v-1}^{v} \beta\mathrm{e}^{\int_{u_1-1}^{v-1-\alpha} \beta du_2} du_1 > 0.56671.
\end{aligned}
$$

Consequently

$$
\begin{aligned}
W(h(b_r+6)) &= \frac{1}{1 - \int_{h^2(b_r+6)}^{h(b_r+6)} q(u_1)\mathrm{e}^{\int_{h(u_1)}^{h^2(b_r+6)} \frac{q(u_2)}{1-Q_1^1(u_2)}du_2} du_1} \\
&\geq \frac{1}{1 - \int_{b_r+4}^{b_r+5-\alpha} q(u_1)\mathrm{e}^{\int_{u_1-1}^{b_r+4-2\alpha} \frac{q(u_2)}{1-Q_1^1(u_2)}du_2} du_1} > 4.71362.
\end{aligned}
$$

Then

$$
\begin{aligned}
&\sum_{i=1}^{2} \left(\prod_{j=2}^{i} W\left(h^{j-1}(b_r+6)\right) \right) Q_i^2(b_r+6) \\
=&Q_1^2(b_r+6) + W(h(b_r+6))Q_2^2(b_r+6) > 1.001.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&\limsup_{t\to\infty} \sum_{i=1}^{2} \left(\prod_{j=2}^{i} W\left(h^{j-1}(t)\right) \right) Q_i^2(t) \\
\geq &\lim_{r\to\infty} \sum_{i=1}^{2} \left(\prod_{j=2}^{i} W\left(h^{j-1}(b_r+5)\right) \right) Q_i^2(b_r+5) > 1.
\end{aligned}
$$

It follows from (3.1) that Eq.(3.5) is oscillatory. However, since

$$
\int_{\tau(t)}^{t} q(u_1)du_1 \leq \int_{t-1-\alpha}^{t} \beta\, du_1 = (1+\alpha)\,\beta.
$$

Then

$$
V_4(t) = \mathrm{e}^{\int_{\tau(t)}^{t} q(u_1)\mathrm{e}^{\int_{\tau(u_1)}^{u_1} q(u_2)du_2} du_1} \leq \mathrm{e}^{\beta(1+\alpha)\mathrm{e}^{\beta(1+\alpha)}} < 2.021.
$$

From this and $D(k)=0$, we obtain

$$\limsup_{t\to\infty}\int_{\delta(t)}^{t} q(u)\ \mathrm{e}^{\int_{\tau(u)}^{\delta(t)} q(u_1)V_4(u_1)\,du_1} du < 0.731463 < 1-D(k). \tag{3.6}$$

On the other hand

$$\limsup_{t\to\infty}\int_{\delta(t)}^{t} q(u)\ \mathrm{e}^{\int_{\delta(u)}^{\delta(t)} q(u_1)V_4(u_1)\,du_1} du \le \limsup_{t\to\infty}\int_{\delta(t)}^{t} q(u)\ \mathrm{e}^{\int_{\tau(u)}^{\delta(t)} q(u_1)V_4(u_1)\,du_1} du,$$

and

$$\begin{aligned}&\limsup_{t\to\infty}\int_{\delta(t)}^{t} q(u)\,\mathrm{e}^{\int_{\tau(u)}^{\delta(t)} q(u_1)\ \mathrm{e}^{\int_{\tau(u_1)}^{u_1} q(u_2)du_2} du_1} du\\ \le&\limsup_{t\to\infty}\int_{\delta(t)}^{t} q(u)\ \mathrm{e}^{\int_{\tau(u)}^{\delta(t)} q(u_1)V_4(u_1)\,du_1} du.\end{aligned}$$

This, together with (3.6), implies respectively that (1.5) with $n=4$ and (1.8) are not satisfied. As a result, conditions (1.6), (1.7) and (1.9) can not be applied. Since

$$\begin{aligned}\int_{h(t)}^{t} B_1(u)du &\le \int_{t-1-\alpha}^{t} q(u)\int_{u-1-\alpha}^{u} q(u_1)\mathrm{e}^{\int_{u_1-1-\alpha}^{u} q(u_2)du_2} du_1 du\\ &\le \int_{t-1-\alpha}^{t}\beta\int_{u-1-\alpha}^{u}\beta\mathrm{e}^{\int_{u_1-1-\alpha}^{u}\beta du_2} du_1 du < 0.3988.\end{aligned}$$

Then

$$\limsup_{t\to\infty}\left(\int_{h(t)}^{t} B_1(u)du + D(k^*)\mathrm{e}^{\int_{h(t)}^{t} B_0(u)du}\right) < 0.3988 < 1.$$

Therefore, condition (1.10) is not satisfied.

Finally

$$\begin{aligned}P_1(t) &= q(t)\left[1+\int_{h(t)}^{t} q(u_1)\mathrm{e}^{\int_{\tau(u_1)}^{t} P_0(u_2)du_2} du_1\right]\\ &\le \beta\left[1+\int_{t-1-\alpha}^{t}\beta\mathrm{e}^{\int_{u_1-1-\alpha}^{t}\beta du_2} du_1\right] < 0.84776,\\ \Psi_{1,1}(t) &\le 1+\int_{\tau(t)}^{t} q(u_1)\mathrm{e}^{\int_{\tau(u_1)}^{t} q(u_2)\mathrm{e}^{\int_{\tau(u_2)}^{u_2} q(u_3)(1+\alpha)du_3} du_2} du_1\\ &\le 1+\int_{t-1-\alpha}^{t}\beta\mathrm{e}^{\int_{u_1-1-\alpha}^{t}\beta\mathrm{e}^{\beta(1+\alpha)^2} du_2} du_1 < 2.31694,\end{aligned}$$

and

$$\begin{aligned}\Omega_1(t) &= \frac{1}{1-\int_{h(t)}^{t} q(u_1)\mathrm{e}^{\rho\int_{\tau(u_1)}^{h(t)} q(u_2)du_2} du_1} \le \frac{1}{1-\int_{t-1-\alpha}^{t} q(u_1)\mathrm{e}^{\int_{u_1-1-\alpha}^{t-1} q(u_2)du_2} du_1}\\ &= \frac{1}{1-\int_{t-1-\alpha}^{t}\beta\mathrm{e}^{\int_{u_1-1-\alpha}^{t-1}\beta du_2} du_1} < 2.31.\end{aligned}$$

Then

$$\limsup_{t\to\infty} \int_{h(t)}^{t} q(u)\mathrm{e}^{\int_{\tau(u)}^{h(t)} P_1(u_1)du_1} du < 0.707 < 1 - D(k),$$

$$\limsup_{t\to\infty} \int_{h(t)}^{t} q(u)\mathrm{e}^{\int_{\tau(u)}^{h(t)} \Psi_{1,1}(u_1)q(u_1)du_1} du < 0.8 < 1 - D(k),$$

and

$$\limsup_{t\to\infty} \int_{h(t)}^{t} q(u)\mathrm{e}^{\int_{\tau(u)}^{h(t)} \Omega_1(u_1)q(u_1)du_1} du < 0.78865 < 1 - D(k).$$

Hence the conditions (1.11), (1.12) with $n = 1$, (1.13) with $i = j = 1$, and (1.14) with $n = 1$ can not be applied.

Theorem 3.3. *Assume that $n \in \mathbb{N}$, $k^* > 0$ and*

$$\int_{h(u_1)}^{h(t)} q(u)du \geq \int_{u_1}^{t} q(u)du, \quad \text{for all} \quad u_1 \in [h(t), t]. \tag{3.7}$$

If

$$\limsup_{t\to\infty} \left(\sum_{i=1}^{n} \rho^{i-1} Q_i^n(t) \right) > k^* + \frac{1}{\lambda(k^*)}, \tag{3.8}$$

then Eq.(1.1) *is oscillatory..*

Proof. Assume the existence of an eventually positive solution $x(t)$ of Eq.(1.1). For sufficiently small $\epsilon_1 > 0$ and sufficiently large t, Lemma 2.4 implies that

$$\sum_{i=1}^{n} \frac{x(h^i(t))}{x(h(t))} Q_i^n(t) \leq 1 - M + \epsilon_1.$$

Using (3.4) and Lemma 2.1, it follows that

$$\sum_{i=1}^{n} \rho^{i-1} Q_i^n(t) \leq 1 - M + \epsilon_1.$$

On the other hand under assumption (3.7), in [29, Remark 2.3] it is shown that

$$\liminf_{t\to\infty} \frac{x(t)}{x(h(t))} \geq 1 - k^* - \frac{1}{\lambda(k^*)}.$$

Recall that this is denoted by M. So, $1 - M + \epsilon_1 \leq k^* + \frac{1}{\lambda(k^*)} + \epsilon_1$, and

$$\sum_{i=1}^{n} \rho^{i-1} Q_i^n(t) \leq k^* + \frac{1}{\lambda(k^*)} + \epsilon_1.$$

Then

$$\limsup_{t\to\infty} \left(\sum_{i=1}^{n} \rho^{i-1} Q_i^n(t) \right) \leq k^* + \frac{1}{\lambda(k^*)} + \epsilon_1.$$

As ϵ_1 goes to zero, we have

$$\limsup_{t\to\infty} \left(\sum_{i=1}^{n} \rho^{i-1} Q_i^n(t)\right) \leq k^* + \frac{1}{\lambda(k^*)}.$$

This contradiction completes the proof. □

The following particular case of Theorem 3.3 provides a non-iterative criterion.

Corollary 3.1. *Assume that* (3.7) *holds and*

$$L^* > \frac{\ln\left(2 + k^*\lambda(k^*)\right)}{\lambda(k^*)}, \tag{3.9}$$

where $k^* > 0$. *Then Eq.*(1.1) *is oscillatory.*

Proof. If Eq.(1.1) has a non-oscillatory solution, then condition (3.8) does not hold. That is, for $n = 1$, we have

$$\limsup_{t\to\infty} \int_{h(t)}^{t} q(u_1) \mathrm{e}^{(\lambda(k^*)-\epsilon)\int_{h(u_1)}^{h(t)} q(u)du} du_1 \leq k^* + \frac{1}{\lambda(k^*)},$$

hence for any $\epsilon > 0$ and sufficiently large t, we obtain

$$\int_{h(t)}^{t} q(u_1) \mathrm{e}^{(\lambda(k^*)-\epsilon)\int_{h(u_1)}^{h(t)} q(u)du} du_1 < k^* + \frac{1}{\lambda(k^*)} + \epsilon.$$

This, in view of (3.7), implies that

$$\int_{h(t)}^{t} q(u_1) \mathrm{e}^{(\lambda(k^*)-\epsilon)\int_{u_1}^{t} q(u)du} du_1 < k^* + \frac{1}{\lambda(k^*)} + \epsilon.$$

By evaluating the integral, we obtain

$$\frac{1}{\lambda(k^*) - \epsilon}\left(\mathrm{e}^{(\lambda(k^*)-\epsilon)\int_{h(t)}^{t} q(u)du} - 1\right) < k^* + \frac{1}{\lambda(k^*)} + \epsilon,$$

that is

$$\mathrm{e}^{(\lambda(k^*)-\epsilon)\int_{h(t)}^{t} q(u)du} < 1 + (\lambda(k^*) - \epsilon)\left(k^* + \frac{1}{\lambda(k^*)} + \epsilon\right).$$

Consequently

$$\int_{h(t)}^{t} q(u)du < \frac{\ln\left(1 + (\lambda(k^*) - \epsilon)\left(k^* + \frac{1}{\lambda(k^*)} + \epsilon\right)\right)}{\lambda(k^*) - \epsilon}.$$

Therefore

$$\limsup_{t\to\infty} \int_{h(t)}^{t} q(u)du \leq \frac{\ln\left(1 + (\lambda(k^*) - \epsilon)\left(k^* + \frac{1}{\lambda(k^*)} + \epsilon\right)\right)}{\lambda(k^*) - \epsilon}.$$

As ϵ goes to zero, we obtain

$$L^* \leq \frac{\ln\left(2 + k^*\lambda(k^*)\right)}{\lambda(k^*)}.$$

This contradicts (3.9). □

Remark 3.1.

(i) Condition (3.9) improves the condition

$$L^* > \min\left\{ \frac{-1+\sqrt{3+2k^*\lambda(k^*)}}{\lambda(k^*)},\ 1+k^*+\frac{1}{\lambda(k^*)}-\sqrt{1+\left(k^*+\frac{1}{\lambda(k^*)}\right)^2} \right\}, \tag{3.10}$$

due to Attia etc [3].

(ii) The conclusion of Theorem 3.3 is still valid, if $q(t) > 0$ and condition (3.7) is replaced by (see Remark 1 in [20])

$$\liminf_{t\to\infty} \frac{q(h(t))h'(t)}{q(t)} = 1. \tag{3.11}$$

Next, we show that condition (3.9) can be easily applied to study the oscillation of certain equation, while the conditions (1.5)-(1.12) and (3.10) fail to do so.

Example 3.2. Consider the equation

$$x'(t) + px(\tau(t)) = 0, \quad t \geq b + \frac{1}{p\mathrm{e}} + \frac{\alpha}{p}, \tag{3.12}$$

where

$$\tau(t) := t - b\sin^2\left(\pi\sqrt{pt}\right) - \frac{1}{p\mathrm{e}} - \frac{\alpha}{p}\sin^2\left(\eta\pi pt\right),$$

and $b, p > 0$ and $pb = 0.40416 - \frac{1}{\mathrm{e}}$, $\alpha = 0.0001$, and $\eta = 20000$. Let

$$h(t) = t - b\sin^2\left(\pi\sqrt{tp}\right) - \frac{1}{p\mathrm{e}}.$$

Then condition (3.11) is satisfied. Clearly

$$\begin{aligned} k = k^* &= \liminf_{t\to\infty} \int_{h(t)}^{t} q(u)du \\ &= \liminf_{t\to\infty} p\left(b\sin^2\left(\pi\sqrt{tp}\right) + \frac{1}{p\mathrm{e}}\right) = \frac{1}{\mathrm{e}}, \end{aligned}$$

it follows that $\lambda(k^*) = \mathrm{e}$, so

$$\frac{\ln\left(2 + k^*\lambda(k^*)\right)}{\lambda(k^*)} = \frac{\ln(3)}{\mathrm{e}} < 0.404157.$$

Also, we have

$$\begin{aligned} L^* &= \limsup_{t\to\infty} \int_{h(t)}^{t} q(u)du \\ &= \limsup_{t\to\infty} p\left(b\sin^2\left(\pi\sqrt{tp}\right) + \frac{1}{p\mathrm{e}}\right) = 0.40416. \end{aligned}$$

Consequently, Corollary 3.1 implies the oscillation of Eq.(3.12). Choose $h(t) = \delta(t)$, then

$$t - b - \frac{1}{p\mathrm{e}} - \frac{\alpha}{p} \leq \tau(t) \leq \delta(t) \leq t - \frac{1}{p\mathrm{e}}.$$

Clearly

$$\begin{aligned} L^* &\le \limsup_{t\to\infty} \int_{\tau(t)}^{t} q(u)du \\ &= \limsup_{t\to\infty} \left(bp\sin^2\left(\pi\sqrt{tp}\right) + \frac{1}{\mathrm{e}} + \alpha\sin^2\left(\eta\pi pt\right) \right) \\ &\le bp + \frac{1}{\mathrm{e}} + \alpha = 0.40426, \\ & \frac{-1+\sqrt{3+2k^*\lambda(k^*)}}{\lambda(k^*)} > 0.45472, \end{aligned}$$

and

$$1 + k^* + \frac{1}{\lambda(k^*)} - \sqrt{1 + \left(k^* + \frac{1}{\lambda(k^*)}\right)^2} > 0.49425.$$

Then condition (3.10) is not satisfied. Let

$$A(t) = \int_{\tau(t)}^{t} q(u)du = p\left(b\sin^2\left(\pi\sqrt{tp}\right) + \frac{1}{p\mathrm{e}} + \frac{\alpha}{p}\sin^2\left(\eta\pi pt\right)\right).$$

Then $A(t)$ is not slowly varying at infinity which means that Theorem 3 in [13] can not be applied. Also, it is clear that

$$\int_{h(t)}^{t} q(u)\mathrm{e}^{\lambda(k)\int_{\tau(u)}^{h(t)} q(u_1)\ du_1} du \le \int_{t-b-\frac{1}{p\mathrm{e}}-\frac{\alpha}{p}}^{t} p\mathrm{e}^{\mathrm{e}\int_{u-b-\frac{1}{p\mathrm{e}}-\frac{\alpha}{p}}^{t-\frac{1}{p\mathrm{e}}} p\,du_1} du < 0.812581.$$

Then

$$\limsup_{t\to\infty} \int_{h(t)}^{t} q(u)\mathrm{e}^{\lambda(k)\int_{\tau(u)}^{h(t)} q(u_1)\,du_1} du < 0.812582 < 1 - D(k) < 0.86346. \tag{3.13}$$

Therefore, conditions (1.6), (1.7) and (1.9) can not be applied.

Since

$$\mathrm{e}^{\int_{\tau(t)}^{t} q(u)du} \le \mathrm{e}^{\int_{t-b-\frac{1}{p\mathrm{e}}-\frac{\alpha}{p}}^{t} p\,du} < 1.498194 < \lambda(k) = \mathrm{e},$$

and

$$\begin{aligned} 1 + \int_{h(t)}^{t} q(u)\mathrm{e}^{\int_{\tau(u)}^{t} q(u_1)du_1} du &\le 1 + \int_{t-b-\frac{1}{q\mathrm{e}}-\frac{\alpha}{p}}^{t} p\mathrm{e}^{\int_{u-b-\frac{1}{p\mathrm{e}}-\frac{\alpha}{p}}^{t} p\,du_1} du \\ &< 1.746391 < \lambda(k) = \mathrm{e}. \end{aligned}$$

It follows from these inequalities and (3.13) that none of the conditions (1.5) with $n = 3$, (1.8) and (1.11), (1.12) with $n = 1$ is satisfied.

Finally, let

$$I(t) = \int_{\tau(t)}^{t} q(u) \int_{\tau(u)}^{u} q(u_1)\mathrm{e}^{\int_{\tau(u_1)}^{u} q(u_2)du_2} du_1 du.$$

Then

$$I(t) \le \int_{t-b-\frac{1}{p\mathrm{e}}-\frac{\alpha}{p}}^{t} p \int_{u-b-\frac{1}{p\mathrm{e}}-\frac{\alpha}{p}}^{u} p\mathrm{e}^{\int_{u_1-b-\frac{1}{p\mathrm{e}}-\frac{\alpha}{p}}^{u} p\,du_2} du_1 du < 0.301736.$$

Since

$$\limsup_{t\to\infty}\left(\int_{h(t)}^{t} B_1(u)du + D(k^*)\mathrm{e}^{\int_{h(t)}^{t} q(u)\,du}\right) \le \limsup_{t\to\infty}\left(I(t) + D(k)\mathrm{e}^{\int_{\tau(t)}^{t} p\,du}\right) < 0.5064 < 1.$$

Then condition (1.10) fails to apply when $n = 1$.

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