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A NOTE ON TOPOLOGICALLY TRANSITIVE TREE MAPS*

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Abstract In this note, W is a tree. $F : W \to W$ is a continuous map. $\mathcal{K}(W) = \{C \subset W : C \neq \emptyset$ and C is compact$\}$ is endowed with a Hausdorff metric. The paper gives a sufficient and necessary condition under which F is topologically transitive. Furthermore, it is shown that both a topologically transitive tree map $F : W \to W$ and the continuous map $\overline{F}$ on $\mathcal{K}(W)$ which is induced by F are cofinitely sensitive, where $\overline{F}(C) = \{F(x) : x \in C\}$ for any $C \in \mathcal{K}(W)$.

Keywords Sensitivity, topologically mixing, topologically weak mixing, topological transitive, cofinitely sensitive.

MSC(2010) 37B40, 37D45, 54H20.

1. Introduction

In the following, for simplicity, "dynamical system" is denoted by "D.S.", "topologically weak mixing" is denoted by "TWM", "topologically mixing" is denoted by "TM", and "topologically transitive" is denoted by "TT".

As one knows, sensitivity characterizes the unpredictability of chaotic phenomenon. This kind of property is one of the essential conditions of various definitions of a chaotic system. Hence, when is a system sensitive? From [1, 2, 8, 9, 13, 16, 18, 22, 24, 32, 34, 37–40, 44, 47, 50, 53, 60, 64] one can see the above question has attracted some attention. Throughout the paper, a D.S. is a continuous map $F : W \to W$ on a compact metric space (W, d) which has at least two points. In [1], Abraham et al. established that if a measure-preserving map F on a metric probability space $(W, d, \mathcal{B}(W), \mu)$ with supp $\mu = W$ is either topologically mixing

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or weak-mixing, and satisfies that for every nonempty open set $U \subset W$, there exists a sequence $\{M_j\}$ with positive upper density such that

$$U \cap (\bigcap_{j\geq 0} T^{-m_j} U) \neq \emptyset,$$

then F is sensitive. Motivated by [1], He et al. relaxed the conditions of the above result and obtained that a measure-preserving map F (resp. a measure-preserving semi-flow φ) on $(W, d, \mathcal{B}(W), \mu)$ with supp $\mu = W$ is weak-mixing, then it is sensitive [22]. Gu [16] showed that if W is a nontrivial compact metric space, a pair (F, μ) satisfies the large deviations principle, and F is a continuous and topologically strongly ergodic map of W, then F is sensitive. In addition, if W is a nontrivial metric space and a map F on W is topologically mixing, then F is sensitive [22]. Motivated by [16, 22], we presented several sufficient conditions of sensitivity (see [37]), which relax and extend the conditions of the above results in [1, 16, 18, 22, 37, 40]. Inspired by [37, 44], we discussed some stronger forms of sensitivity for measure-preserving maps and semi-flows on probability spaces and defined a new form of sensitivity, which is called to be ergodic sensitivity (see [18]). Moreover, we obtained that, on a metric probability space with a fully supported measure, if a measure-preserving map is weak mixing, then it is ergodically sensitive and multi-sensitive. And if it is strong mixing, then it is cofinitely sensitive, where it is not required that the map is continuous and the space is compact. Similar results for measure-preserving semi-flows are obtained, where it is required in a result about ergodic sensitivity that the space is compact in some sense and the semi-flow is continuous. Additionally, relationships between some sensitive properties of a map and its iterations are discussed, including syndetic sensitivity, cofinite sensitivity, ergodic sensitivity as well as usual sensitivity, n-sensitivity, and multi-sensitivity. Also, it was verified that multi-sensitivity, cofinite sensitivity, and ergodic sensitivity can be lifted up by a semiopen factor map. In [9], Degirmenci and Kocak explored how chaos conditions on maps carry over to their products. First they presented a counterexample showing that the product of two chaotic maps (in the sense of Devaney) need not be chaotic. Then they remarked that if two maps (or even one of them) is sensitive, so does their product. Likewise, if two maps possess dense periodic points, so does their product. On the other side, the product of two TT maps need not be TT. They also gave sufficient conditions under which the product of two chaotic maps is Devaney chaotic. Inspired by [9], we mainly considered how chaos conditions on semi-flows carry over to their products (see [39]) and extended the results of [9] to semi-flows. In particular, we gave the notion of ergodic sensitivity and proved that for any two (not-necessarily continuous) maps (resp. semi-flows) on the metric spaces W_1 and W_2, their product is ergodically sensitive if and only if one of them is ergodically sensitive.

According to [44], one can see that a D.S. (W, F) is sensitive if for every region S of W, there are $u, v \in S$ with $u \neq v$ and some $k \in \mathbb{Z}^+$ satisfying that the kth iterates of them under F are significantly separated. The largeness of the set of all $k \in \mathbb{Z}^+$ which satisfy that sensitivity happens can be considered as a measure of how sensitive the system is. Particularly, when this set is very thin and has arbitrarily large gaps between consecutive entries, we have some excuse for considering such a system as practically non-sensitive [44]. For continuous maps on metric spaces which are compact, Moothathu [44] gave a preliminary research of some stronger forms of sensitivity formulated by using large subsets of $\mathbb{Z}^+$ and mainly explored syndetic sensitivity and cofinite sensitivity. Moreover, [44] gave a transitive and sensitive map, where the map is not syndetically sensitive and established the following result: (1) Every non-minimal map which is syndetically transitive is syndetically sensitive. (2) Every sensitive map of the unit interval is cofinitely sensitive. (3) Every sensitive subshift of finite type is cofinitely sensitive. (4) Every infinite subshift which is syndetically transitive is syndetically sensitive. (5) All Sturmian subshifts are not cofinitely sensitive.

In [28], the authors gave examples showing that sensitivity of a continuous surjection $F : W \to W$ does not imply sensitivity of the set-valued mapping $\overline{F} : \mathcal{K}(W) \to \mathcal{K}(W)$. Also, they proved that if F is continuous onto interval map, then sensitivity of $\overline{F}$ is equivalent to that of F. In [60] Xu et al. obtained that a mixing transformation $F : W \to W$ on a manifold W is sensitive and topologically transitive, and that a Devaney chaotic transformation F with some assumption is an expanding map, which means a few statistical properties in this transformation map. Motivated by [60], [32] extended the results to semi-flows.

Bauer and Sigmund [6] explored the interplay of chaos in dynamical systems (individual chaos) with the corresponding set-valued versions (collective chaos). For results on set-valued discrete systems we refer the reader to [4, 12, 14, 15, 20, 23, 26–28, 30, 48, 51, 58, 59] and references therein. For example, Banks [4] and Peris [48] independently obtained that F is TWM if and only if it is transitive, which is equivalent to the topologically weakly mixing property of $\overline{F}$. Kwietniak and Oprocha [23] established that under some nonrecurrence assumption, $\overline{F}$ always has positive topological entropy. Then, Lampart and Raitha [30] showed that the topological entropy of the induced set-valued map of a homeomorphism defined on an interval or on a circle is zero or infinity. Guirao et al. [14] got that $\overline{F}$ has the same chaos as F (distributional chaos, Li-Yorke chaos, ω-chaos, topological chaos (where chaoticity include positive topological entropy), specification property, exact Devaney chaos, total Devaney chaos). Hou et al. [20] proved that if F is a non-minimal M-system, then $\overline{F}$ is sensitive. In [17], Gu and Guo studied the relationships between the mixing property of $\overline{F}$ and the mixing property of F and discussed specification of $\overline{F}$. In [25], Liao et al. proved that F is TWM (resp. TM) if and only if so is $\overline{F}$, and that for a interval map F, $\overline{F}$ is chaotic (in the sense of Devaney) if and only if F is TWM. In [15], Gu showed that if $\overline{F}$ is sensitive then so is F, and that if $\overline{F}$ accessible (resp. Kato chaotic), then so is F. He also proved that F is sensitive (resp. Kato chaotic) if and only if so is $\overline{F}$ in w^e-topology. Then, Liu et al. [28] gave examples to show that the converse may not hold, i.e., the sensitivity of F does not necessarily imply the sensitivity of $\overline{F}$ and they proved that if F is a surjective continuous interval map, then the sensitivities of $\overline{F}$ and F are equivalent. In [33], we showed that $\overline{F}$ is syndetically sensitive (resp. multi-sensitive) if and only if F is syndetically sensitive (resp. multi-sensitive), and that if $\overline{F}$ is ergodically sensitive, then so is F. Inspired by Furstenberg families, the interplay of chaos in dynamical systems (individual chaos) with the corresponding set-valued versions (collective chaos), see [39, 44]. In [55], Wu et al. obtained a few sufficient and necessary conditions to ensure a dynamical system be $\mathfrak{F}$-sensitive or multi-sensitive and some other important results on sensitivity of hyperspatial dynamical systems. They obtained that $f \times g$ is multi-sensitive if and only if f or g is multi-sensitive, which gives a positive answer to a question posed in [39]. In [56], to answering tree open problems, Wu et al. proved that there is two non-syndetically sensitive cascades defined on complete metric spaces whose product is cofinitely sensitive, and that there is a syndetically sensitive semiflow (G, W) over a complete metric space W such that $(G(1), W)$ is not sensitive for some syndetic closed submonoid $G(1)$ of G. In [57], to answering two open problems, Wu and Zhang proved that there exists a monoid, on which neither the syndetic property nor the dual syndetic property holds. And there is a strongly mixing semi-flow with this monoid action which does not have thick sensitivity, syndetic sensitivity, thickly syndetic sensitivity, or thickly periodical sensitivity. Also, they obtained that there is a thickly sensitive cascade which is not multi-sensitive. In [35], we gave the definitions of collective accessibility and collectively Kato chaotic for a dynamical system and explored the relations between topologically weakly mixing and collective accessibility, or strong accessibility, or strongly Kato chaos. Moreover, we presented some same properties of F and $\overline{F}$ and proved that F is collectively accessible (or strongly accessible) if and only if so is $\overline{F}$ in w^e-topology. More recently, in [55], Salman etal. defined the concepts of sensitivity, multi-sensitivity, cofinite sensitivity, and syndetic sensitivity for nonautonomous dynamical systems on uniform spaces and established some sufficient conditions under which topological transitivity and dense periodic points imply sensitivity for nonautonomous systems on Hausdorff uniform spaces. They also considered sensitivity and other stronger versions of sensitivity for the systems induced on hyperspaces and for the product of nonautonomous dynamical systems on uniform spaces.

In [21], for a continuous map F of a tree W, Hosaka and Kato proved the following two results: (1) if $\Omega(F)$ is finite then it is the set of periodic points of F, where $\Omega(F)$ denotes the set of nonwandering points for F, (2) $\Omega(F)$ is contained in the closure of the set of eventually periodic points of F. Also, they gave some examples which imply that these results are not true for the case that W is a dendrite or a graph. In [29], for a strictly piecewise monotone continuous map F on a finite graph W, the authors studied the topological structure of the inverse limit space (W, F) by using F as a sole bonding map and obtained seven necessary and sufficient conditions under that the topological entropy of F is zero. In [62, 63] Ye studied properties of non-wandering points of tree maps and graph maps and showed that the depths of tree maps and graph maps are at most 3. Inspired by [62, 63], Mai and Sun [42] showed that the depth of a graph map is at most 2. In [7], for a continuous map F such that any vertex of W is a fixed point of F, Canovas and Hric proved that F is distributionally chaotic if and only if its topological entropy is positive. In [45], Naghmouchi proved that if F is a tree map having zero topological entropy and μ is an F-invariant Borel measure, then any scrambled set S has zero outer μ-measure (hence μ-measurable). Particularly, if S is measurable, it has zero μ-measure. In [43], for a tree map $F : W \to W$, Matviichuk investigated the dynamics of subcontinua of W under action of the map F. Particularly, he showed that a subcontinuum of W is asymptotically periodic or asymptotically degenerate. As an application of this result, he proved that zero topological entropy of the system (W, F) implies zero topological entropy of its functional envelope (endowed with the Hausdorff metric).

In this paper, it is shown that a continuous tree map $F : W \to W$ is TT if and only if the following condition (1) or condition (2) is satisfied.

(1) F is TM.

(2) There are an integer $m > 1$, a fixed point p of F with $V(p) \geq m$ and closed subtrees $W_1, W_2, \cdots, W_m \subset W$ which are non-degenerate such that $W = \bigcup_{s=1}^{m} W_s$,

$W_s \cap W_t = \{p\}$ for all $s \neq t$, $F(W_s) = W_{s+1}$ for all $s = 1, 2, \cdots, m-1$, $F(W_m) = W_1$ and $F^m|_{W_s}$ is TM for each $s = 1, 2, \cdots, m$.

Also, we get that a TT tree map $F : W \to W$ and the continuous map $\overline{F}$ on $\mathcal{K}(W)$ which is induced by F are cofinitely sensitive.

This note is organized as follows. In Section 2, some concepts and notations are recalled. Main results are obtained and proved in Section 3.

2. Preliminaries

A subset $A \subset \mathbb{Z}^+$ is said to be thick if for each $m \in \mathbb{Z}^+$, there exists a $b_m \in \mathbb{Z}^+$ satisfying that $\{b_m, \cdots, b_m + m\} \subset A$. A subset $A \subset \mathbb{Z}^+$ is called to be syndetic if $\mathbb{Z}^+ \backslash A$ is not thick. A subset $A \subset \mathbb{Z}^+$ is called to be cofinite if there exists a $D \in \mathbb{Z}^+$ satisfying that $\{D, D+1, \cdots\} \subset A$.

A tree is a graph contains no cycles (see [46]). Let W be a tree. The valence of a point $z \in W$ is the number of connected components of $W - \{z\}$. For $z \in W$, $V(z)$ denote the valence of this point and

$$O(W) = \{z \in W : V(z) \geq 3\}.$$

If a point has valence one it is an end. $E(W)$ denotes the set of endpoints of W. For $u, v \in W$, $[u, v]$ denotes the smallest closed connected subset which contains the two points u, v. For a subset $C \subset W$, C^0 and $\overline{C}$ be the interior and the closure of the set C, respectively.

A subset C of a tree W is said to be an interval if there is a homeomorphism $f : H \to C$, where H is $[0, 1]$, $(0, 1]$, $[0, 1)$ or $(0, 1)$. The set $f((0, 1))$ is said to be the interior of C. If $H = [0, 1]$, the interval C is said to be closed. If $H = (0, 1)$, the interval C is said to be open. We know that an open interval may not be an open set in W, and that the interior of this interval may not be equal to its interior in the topology of W. Clearly, any open interval C with $C \cap O(W) = \emptyset$ is open in W.

Let $F : W \to W$ be continuous and (W, d) be a metric space which is compact. It is well known that the set-valued mapping $\overline{F}$ which is induced by F on $\mathcal{K}(W) = \{C \subset W : C \text{ is compact and } C \notin \{\emptyset\}\}$ is defined by

$$\overline{F}(C) = F(C) = \{F(z) : z \in C\}$$

for any $C \in \mathcal{K}(W)$. Then $(\mathcal{K}(W), \overline{F})$ is a D.S., where the space $\mathcal{K}(W)$ which is endowed with the Hausdorff metric

$$H(C_1, C_2) = \max\{\sup\{d(z_1, C_2) : z_1 \in C_1\}, \sup\{d(z_2, C_1) : z_2 \in C_2\}\},$$

for any $C_1, C_2 \in \mathcal{K}(W)$ (see [20,28]).

For a D.S. (W, F) and any $S, T \subset W$, write

$$N_F(S, T) = \{n \in \mathbb{Z}^+ : S \cap F^{-k}(T) \neq \emptyset\} = \{k \in \mathbb{Z}^+ : F^k(S) \cap T \neq \emptyset\}.$$

A map $F : W \to W$ is TT if $N_F(S, T) \neq \emptyset$ for any $S, T \subset W$ with $S, T \notin \{\emptyset\}$ which are open.

F is TWM if $N_{F \times F}(S, T) \neq \emptyset$ for any $S, T \subset W \times W$ with $S, T \notin \{\emptyset\}$ which are open.

F is TM if $N_F(S, T)$ is cofinite for any $S, T \subset W$ with $S, T \notin \{\emptyset\}$ which are open.

For the above system, from the classical definition one can see that F is sensitive if there exists a $\tau > 0$, for any $z \in W$ and any open neighborhood T_z of z, there exists an integer $m \in \mathbb{Z}^+$ such that $\sup\{d(F^m(z), F^m(w)) : w \in T_z\} > \tau$. We may denote this in the following way. For any $T \subset W$ and any $\tau > 0$, put

$N_F(T, \tau) = \{m \in \mathbb{Z}^+ : \text{there exist } e, f \in T \text{ satisfying } d(F^m(e), F^m(f)) > \tau\}$.

Then,

(1) F is sensitive if there is $\tau > 0$ which satisfies that for any $T \subset W$ with $T \notin \{\emptyset\}$ which is open, $N_F(T, \delta) \notin \{\emptyset\}$.

(2) F is syndetically sensitive if there is $\delta > 0$ which satisfies that for any $T \subset W$ with $T \notin \{\emptyset\}$ which is open, $N_F(T, \delta)$ is syndetic.

(3) F is said to be cofinitely sensitive if there is $\delta > 0$ which satisfies that for any $T \subset W$ with $T \notin \{\emptyset\}$ which is open, $N_F(T, \delta)$ is cofinite.

By [44], one can see that cofinitely sensitive is stronger than syndetically sensitive, and syndetically sensitive is stronger than sensitive.

3. Main results

The following Lemmas are given for proving the main results.

Lemma 3.1 (see [61]). *Assume that (W, F) is a dynamical system, and that F is a continuous surjection. If $\overline{P(F)} = W$ (where $P(F)$ is the set of all period points of F), then F is TWM if and only if F^n is TT for any integer $n > 0$.*

Lemma 3.2 (see [3]). *Assume that $F : W \to W$ is a transitive tree map. Then (1) or (2) is true:*

(1) F^s is TT for any integer $s \geq 1$.

(2) There are an integer $m \geq 2$, a fixed point p of F with $V(p) \geq m$ and closed subtrees $W_1, W_2, \cdots, W_m \subset W$ which are non-degenerate such that $W = \bigcup_{s=1}^{m} W_s$, $W_s \cap W_t = \{p\}$ for any $s \neq t$, $F(W_s) = W_{s+1}$ for all $s = 1, 2, \cdots, m-1$, and $F(W_m) = W_1$.

Lemma 3.3 (see [62]). *Let F be a tree, and $F : W \to W$ be a continuous map. If F is TT, then $\overline{P(F)} = W$.*

Lemma 3.4 (see [5, 54]). *Let W be a compact metric space which has at least two points, and let $F : W \to W$ be continuous. Then the following are equivalent:*

(1) F is TT.

(2) There is $z \in W$ satisfying

$$\omega(z, F) = W.$$

(3) For any open subset $S \subset W$ with $S \notin \{\emptyset\}$,

$$\overline{\bigcup_{n=0}^{\infty} F^n(S)} = W.$$

The following Lemma is from [31]. For completeness, we give its proof here.

Lemma 3.5. *Let W be a tree, $F : W \to W$ is continuous. Then the TWM of F is equivalent to the TM of F.*

Proof. Obviously, if F is TM then F is topologically total ergodicity. It is enough to verify that if F is TWM then F is TM. Using $E(W)$ and $O(W)$ to denote the set of all ends of W and the set of all branching points of W, respectively. Let

$$W - O(W) = \bigcup_{j=1}^{i=t} I_j,$$

where I_j denotes a connected component of $W - O(W)$ for any $j \in \{1, 2, \cdots, t\}$. Let $S \subset W - O(W)$ and $T \subset W - O(W)$ be connected and open. Without loss of generality, one can suppose that $S \subset I_k^o$ for some $k \in \{1, 2, \cdots, t\}$. It is clear that F is TT. By Lemma 3.3,

$$\overline{P(F)} = W.$$

For any distinct $p, z \in S \cap P(F)$, suppose that p is a periodic point of period m_1 and z is a periodic point of period m_2. Then we have

$$O_F(p) \cup O_F(z) \subset W - E(W).$$

Suppose that $u \in T$ is a periodic point of F with period m_3 and m is a common multiple of m_1, m_2 and m_3. So, one has

$$O_F(v) \cup O_F(z) \cup \{u\} \subset \{y \in W : F^m(y) = y\}.$$

Let $g = F^m$ and

$$E = \bigcup_{n=0}^{\infty} g^n(T).$$

Clearly, E is connected. Since g is TT, by Lemma 3.4, $\overline{E} = W$ and $E \supset I_k$. Therefore, for any $y \in O_F(p) \cup O_F(z)$, there exists an $s_y > 0$ satisfying

$$y \in g^{s_y}(T).$$

Let

$$s = \max\{s_y : y \in O_F(p) \cup O_F(z)\}.$$

Since

$$O_F(p) \cup O_F(z) \subset \{x \in W : g(x) = x\},$$

then

$$O_F(p) \cup O_F(z) \subset g^s(T) = F^{sm}(T).$$

One can easily see that

$$F^n(T) \supset O_F(p) \cup O_F(z)$$

for any integer $n > sm - 1$. Thus, $F^n(T) \supset [p, z]$ for any integer $n > sm - 1$. This deduces that

$$F^n(T) \cap S \neq \emptyset$$

for any integer $n > sm - 1$. $\square$

For any transitive continuous map over a tree, the following results are obtained.

Theorem 3.1. *Let (W, F) be a D.S. and W be a tree. Then F is TT if and only if (1) or (2) is hold.*

(1) F is TM.

(2) There is an integer $m \geq 2$, a fixed point p of F with $V(p) \geq m$ and some closed subtrees $W_1, W_2, \cdots, W_m \subset W$ which are non-degenerate satisfy that $W = \bigcup_{s=1}^{m} W_s$, $W_s \cap W_t = \{p\}$ for any integer $s \neq t$, $F(W_s) = W_{s+1}$ for all $s = 1, 2, \cdots, m-1$, $F(W_m) = W_1$ and $F^m|_{W_s}$ is TM for any $s \in \{1, 2, \cdots, m\}$.

Proof. Assume that F is TT. Then, by Lemma 3.2 one knows that exactly one of the following two cases is true:

Case 1. F^k is TT for any integer $k \geq 1$.

By Lemmas 3.1, 3.3 and 3.5, F is TM.

Case 2. There exists an interger $m \geq 2$, a fixed point z of F with $V(z) \geq m$ and some closed subtrees which are non-degenerate $W_1, W_2, \cdots, W_m \subset W$ satisfy that $W = \bigcup_{s=1}^{m} W_s$, $W_s \cap W_t = \{z\}$ for all $s \neq t$, $F(W_s) = W_{s+1}$ for any $s \in \{1, 2, \cdots, m-1\}$ and $F(W_m) = W_1$.

Since F is TT, $F^m|_{W_s}$ is TT for any $s \in \{1, 2, \cdots, m\}$. And because z is a fixed point of $F^m|_{W_s}$ and $W_s - \{z\}$ is connected for every $s = 1, 2, \cdots, m$, by Corollary 2.5 in [26], $F^m|_{W_s}$ is TM for every $s = 1, 2, \cdots, m$.

Hypothesis that the converse is true. Since topological mixing implies topological transitivity, it suffices to show that case 2 implies that F is TT. Let $S, T \subset W$ with $S, T \notin \{\emptyset\}$ be open such that S and T are connected sets. Without loss of generality, we may assume that $S, T \subset W_i$ for some $i \in \{1, 2, \cdots, k\}$ or $S \subset W_i$ and $T \subset W_j$ for some $i \in \{1, 2, \cdots, k\}$ and some $j \in \{1, 2, \cdots, k\}$, where $i < j$. If $S, T \subset W_i$ for some $i \in \{1, 2, \cdots, k\}$, then, by the topological mixing of $F^k|_{W_i}$, there exists an integer $n > 0$ such that $F^{mk}(S) \cap T \neq \emptyset$ for any integer $m \geq n$. If $S \subset W_i$ and $T \subset W_j$ for some $i \in \{1, 2, \cdots, k\}$ and some $j \in \{1, 2, \cdots, k\}$ with $i < j$, then, by

$$F^{j-i}(W_i) = W_j$$

one can get that $F^{j-i}(S) \subset W_j$. Since S is connected and F is sensitive, $F^{j-i}(S)$ is a non-degenerate and connected set. So, there exists a nonempty open set S' satisfying that S' is connected and $S' \subset F^{j-i}(S)$. By the topological mixing of $F^k|_{W_j}$, there exists an integer $l > 0$ such that $F^{km}(S') \cap T \neq \emptyset$ for any integer $m \geq l$. Therefore, $F^{km+j-i}(S) \cap T \supset F^{km}(S') \cap T \neq \emptyset$. Consequently, by the above argument one has that case 2 implies F is TT. $\square$

By Proposition 2 and Theorem 1 in [44], a TM (resp. syndetically transitive but not minimal) map is cofinitely sensitive (resp. syndetically sensitive). Furthermore, it is well known that every TT tree map is sensitive. The following will show that a TT tree map $F : W \to W$ is cofinitely sensitive (Theorem 3.2). First we recall a conclusion in [36] (the following Lemma 3.6). For completeness, the proof is given here.

Lemma 3.6. *Let (W, F) be a D.S.. Then F^n is cofinitely sensitive for some $n \geq 2$ if and only if F is cofinitely sensitive.*

Proof. Clearly, cofinite sensitivity of F implies cofinite sensitivity of F^n, where $n = 1, 2, \cdots$. So, it suffices to prove that if F^n is cofinitely sensitive with δ as a constant of sensitivity for some $n \geq 2$, then F is cofinitely sensitive. Since F^j

is uniformly continuous for any $j \in \{0, 1, \cdots, n\}$, there is a $\epsilon \in (0, \delta)$, for any $j \in \{0, 1, \cdots, n\}$,

$$d(F^j(x), F^j(y)) \leq \varepsilon$$

implies

$$d(F^n(x), F^n(y)) \leq \delta.$$

Let $S \subset W$ be any open set with $S \notin \{\emptyset\}$. Then there is some integer $M > 0$ such that

$$\{M, M+1, \cdots\} \subset N_{F^n}(S, \delta).$$

Therefore, for any $m \in \{M, M+1, \cdots\}$, there exist two point $x, y \in U$ satisfying

$$d(F^{nm}(x), F^{nm}(y)) > \delta.$$

This implies that

$$d(F^{nm+i}(x), F^{nm+i}(y)) > \varepsilon$$

for all $i = 1, 2, \cdots, n$. Consequently, one can get that

$$\{nM, nM+1, \cdots\} \subset N_F(S, \delta).$$

Hence, by the definition, if F^n is cofinitely sensitive for some $n \geq 2$, then F is cofinitely sensitive. $\square$

Theorem 3.2. *Let W be a tree, $F : W \to W$ is continuous. If F is TT, then F is cofinitely sensitive.*

Proof. Since F is TT, by Theorem 3.1, one can deduced that exactly one of Case 1 or Case 2 is satisfied:

Case 1. F is TM.

By Proposition 2 in [44], F is cofinitely sensitive.

Case 2. There exists an interger $m > 1$, a fixed point z of F with $V(z) \geq m$ and some closed subtrees $W_1, W_2, \cdots, W_m \subset W$ which are non-degenerate and satisfy that $W = \bigcup_{s=1}^{m} W_s$, $W_s \cap W_t = \{z\}$ for all $s \neq t$, $F(W_s) = W_{s+1}$ for all $s = 1, 2, \cdots, m-1$, $F(W_m) = W_1$ and $F^m|_{W_s}$ is TM for each $s = 1, 2, \cdots, m$.

By Proposition 2 in [44], $F^m|_{W_s}$ is cofinitely sensitive for any $s \in \{1, 2, \cdots, m\}$. Let $S \subset W$ be any open set with $S \notin \{\emptyset\}$. Then there is $i \in \{1, 2, \cdots, k\}$ such that $S \cap W_i \neq \emptyset$. So, $S \cap W_i$ is a nonempty and open subset of W_i. Clearly,

$$N_{F^k|_{W_i}}(S \cap W_i, \delta) \subset N_{F^k}(S, \delta),$$

where δ is a constant of sensitivity of $F^k|_{W_i}$ for all $1 \leq i \leq k$. Since $F^m|_{W_i}$ is cofinitely sensitive, then

$$N_{F^m|_{W_i}}(S \cap W_i, \delta)$$

is cofinite. Therefore, $N_{F^m}(S, \delta)$ is cofinite. By the definition, one gets that F^m is cofinitely sensitive. By Lemma 3.6, F is cofinitely sensitive. $\square$

Theorem 3.3. *Let W be a tree, $F : W \to W$ is continuous. If F is TT, then the continuous map $\overline{F} : \mathcal{K}(W) \to \mathcal{K}(W)$ is cofinitely sensitive.*

Proof. Since F is TT, by Theorem 3.1 one can deduced that Case 1 or Case 2 is satisfied:

Case 1. F is TM.

By [26], $\overline{F}$ is TM if and only if F is TT. So, $\overline{F}$ is TT. Consequently, by Proposition 2 in [11], one knows that $\overline{F}$ is cofinitely sensitive.

Case 2. There exists a $k > 1$, a fixed point y of F with $V(y) \geq k$ and some closed subtrees $W_1, W_2, \cdots, W_k \subset W$ which are non-degenerate and satisfy that $W = \bigcup_{i=1}^{k} W_i$, $W_i \cap W_j = \{y\}$ for all $i \neq j$, $F(W_i) = W_{i+1}$ for all $i = 1, 2, \cdots, k-1$, $F(W_k) = W_1$ and $F^k|_{W_i}$ is TM for each $i = 1, 2, \cdots, k$.

Clearly,

$$\overline{F^k} = \overline{F}^{\,k}.$$

It follows from [28] that $\overline{F^k}|_{\overline{W_i}}$ is TM for each $i = 1, 2, \cdots, k$. By Proposition 2 in [44], $\overline{F^k}|_{\overline{W_i}}$ is cofinitely sensitive for each $i = 1, 2, \cdots, k$. Clearly,

$$\mathcal{K}(W) \supset \bigcup_{i=1}^{k} \mathcal{K}(W_i).$$

For any positive integer n, let G_i be nonempty and open subset of W, $i = 1, 2, \cdots, n$. Write

$$B_W(G_1, G_2, \cdots, G_n) = \{S \in \mathcal{K}(W) : S \subset \bigcup_{i=1}^{n} G_i, S \cap G_i \neq \emptyset, 1 \leq i \leq n\}$$

and

$$G_{ij} = G_j \cap W_i,$$

for every $1 \leq i \leq k$ and every $1 \leq j \leq n$. Then one has

$$B_{W_i}(G_{i1}, G_{i2}, \cdots, G_{in}) = \{S \in \mathcal{K}(W_i) : S \subset \bigcup_{j=1}^{n} G_{ij}, S \cap G_{ij} \neq \emptyset, 1 \leq j \leq n\},$$

for each $1 \leq i \leq k$. Since

$$\mathcal{K}(W) \supset \bigcup_{i=1}^{k} \mathcal{K}(W_i),$$

$$B_W(G_1, G_2, \cdots, G_n) \supset \bigcup_{i=1}^{k} B_{W_i}(G_{i1}, G_{i2}, \cdots, G_{in}).$$

So, for any $\delta > 0$,

$$N_{\overline{F^k}}(B_W(G_1, G_2, \cdots, G_n), \delta) \supset \bigcup_{i=1}^{k} N_{\overline{F^k|_{W_i}}}(B_{W_i}(G_{i1}, G_{i2}, \cdots, G_{in}), \delta). \tag{3.1}$$

Note that F is TM if and only if $\overline{F}$ is TM too. Therefore, by Proposition 2 in [44], one can conclude that if F is TM, then $\overline{F}$ is cofinitely sensitive, which implies

$$\overline{F^k|_{W_i}} = \overline{F|_{W_i}}^{\,k}$$

is cofinitely sensitive. By (3.1), it is easy to get that

$$\overline{F^k} = \overline{F}^{\,k}$$

is cofinitely sensitive. It is known from Lemma 3.6 that $\overline{F}$ is cofinitely sensitive. □

Theorem 3.4. *Let (W, F) be a D.S. and W be a tree. Then F is TT if and only if (a) or (b) is true:*

(a) The induced continuous map $\overline{F} : \mathcal{K}(W) \to \mathcal{K}(W)$ is TM.

(b) There exists a $k > 1$, a fixed point y of F with $V(y) \geq k$ and some closed subtrees $W_1, W_2, \cdots, W_k \subset W$ which are non-degenerate and satisfy that $W = \bigcup_{i=1}^{k} W_i$, $W_i \cap W_j = \{y\}$ for any integer $i \neq j$, $F(W_i) = W_{i+1}$ for all $i = 1, 2, \cdots, k-1$, $F(W_k) = W_1$ and each induced map $\overline{F}^k|_{A_j}$ is TM for each $j = 1, 2, \cdots, k$, where $A_j = W_j$.

Proof. By the above Theorem 3.1 and Theorem 3.6 in [17] (or Theorem 3.5 in [25]), Theorem 3.4 is true. □

The w^e-topology on $\mathcal{K}(W)$ generated by the sets $e(B)$, where B is an open subset in W and $e(B) = \{S \in \mathcal{K}(W) : S \subset W\}$ (see [15, 49]). In [15], R. Gu obtained that a continuous map F over a compact metric space is Kato chaotic if and only if so is $\overline{F}$ in w^e-topology. Inspired by this result we have the following result in w^e-topology.

Theorem 3.5. *Let (W, F) be a D.S. and W be a tree. Then F is TT if and only if exactly one of the following is hold.*

(a) The induced continuous map $\overline{F} : \mathcal{K}(W) \to \mathcal{K}(W)$ is TM in the w^e-topology.

(b) There exists a $k > 1$, a fixed point y of F with $V(y) \geq k$ and some closed subtrees

$$W_1, W_2, \cdots, W_k \subset W$$

which are non-degenerate such that

$$W = \bigcup_{i=1}^{k} W_i,\ \ W_i \cap W_j = \{y\}$$

for any integer $i \neq j$,

$$F(W_i) = W_{i+1}$$

for all $i = 1, 2, \cdots, k-1$,

$$F(W_k) = W_1$$

and each induced map $\overline{F}^k|_{A_j}$ is TM in the w^e-topology for each $j = 1, 2, \cdots, k$, where $A_j = W_j$.

Proof. By the definitions, one can easily show that a continuous F on a compact metric space W is TM if and only if so is $\overline{F}$ in the w^e-topology. By Theorem 3.1, Theorem 3.5 holds. □

As examples of our main theorems, we give the following Example 3.1 and Example 3.2.

Example 3.1. Suppose that W is the m-star having branching point 0 and endpoints $x_1, x_2, \cdots, x_m$, and that each branch of the star has length 1. Let y_j be the midpoint of $[0, x_j]$. We define a continuous function $F : W \to W$ by $F(y_j) = y_{j+1}$ for every $j \in \{1, 2, \cdots, m-1\}$, $F(y_m) = y_1$, $F(0) = 0$ and $F(tx_j) = (1-t)x_{j+1}$ for any $t \in [0,1]$ and every $j \in \{1, 2, \cdots, m-1\}$, where $[0, x_j] = \{tx_j | t \in [0,1]\}$ for every $j \in \{1, 2, \cdots, m\}$. Then the map W is TT and cofinitely sensitive, and $\overline{W}$ is cofinitely sensitive.

Proof. Let $W_j = [0, x_j]$. Clearly, $F^m|_{W_j}$ is topologically conjugate to the tent map. So, $F^m|_{W_j}$ is TM. By Theorem 3.1, F^m is TT, which implies that F is TT. By Theorems 3.2 and 3.3, W is cofinitely sensitive, and $\overline{W}$ is cofinitely sensitive. □

Example 3.2. Suppose that F is the tent map. Then the map W is TT and cofinitely sensitive, and $\overline{W}$ is cofinitely sensitive.

Proof. Since the tent map is TT. By Theorems 3.1, 3.2 and 3.3, Example 3.2 is true. □

Statements and Declarations

All authors agree to publish this paper. There is no conflict of interest.

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