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BÄCKLUND TRANSFORMATIONS AND ROUGE WAVES IN THE FRAME OF A FRACTIONAL ORDER MODEL IN MAGNETIZED DUSTY PLASMA

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Abstract Dusty plasma has become a hot topic in physics in recent years because of its wide application in the space environment, industrial processing and fusion reaction. The (3+1)-dimensional modified Zakharov-Kuznetsov (mZK) equation describing waves propagation in dusty plasma is derived via the reduced perturbation method, based on the governing equation. Furthermore, integer-order equation is derived as the fractional modified Zakharov-Kuznetsov (TF-mZK) equation. The exact solution and Bäcklund transformation are obtained by the fractional transformation and Bell polynomials. Finally, the rouge wave phenomenon in magnetized dusty plasma is described, and the effects of fractional order, phase velocity and dust-cyclotron frequency on the propagation characteristics of dust acoustic rogue waves are analyzed.

Keywords Time-fractional modified Zakharov-Kuznetsov equation, Bell polynomials, Bäcklund transformation, rouge waves, dusty plasma.

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1. Introduction

With the progress of science and technology, researchers have discovered a complex plasma, that is, dusty plasma. [23, 27, 34] Plasma is a form of matter mainly composed of free electrons and charged ions, which exists widely in the universe. Dusty plasma is special in that it contains charged dust particles compared with ordinary ionized bodies. These charged dust particles vary greatly in size, and their movements are influenced by electromagnetic forces and gravity. Due to these characteristics of dusty plasma, the dusty plasma system presents many complex physical phenomena.

Dusty plasmas widely exist in nature, laboratory and astronomical environment. In the early 1980s, spokes were observed in Saturn's rings from the photos returned by Voyager 2 spacecraft, these spokes are composed of fine particles. Hill et al. [17] proposed for the first time that interplanetary dust particles enter Jupiter's plasma

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layer to be charged. For the first time, in 1989 Selwyn et al. [30] reported the dust pollution in the process of plasma etching conductor chip. Before that, people mistakenly thought that the chip was polluted in the air. In 1994, Yi et al. [8] and Morfill et al. [31] obtained the crystal structure of dusty plasma for the first time in their experiments almost simultaneously.

Dusty plasma can produce not only abundant wave modes but also nonlinear coherent structures. When the balance between the nonlinear effect and dispersion effect is reached, dust acoustic waves [1, 19] will be formed. Dust acoustic wave is an important nonlinear coherent structure in dusty plasma, and it is the focus of research on wave motion in dusty plasmas. In 1990, Rao et al. [26] predicted the wave motion pattern in dusty plasma in theory, until 1995, Barkan et al. [5] confirmed this wave motion pattern for the first time in experiments. Since then, many scholars have devoted themselves to the research of dust acoustic waves. Jharna et al. [32] obtained the solitary wave solution and dynamic transition of dust ion acoustic waves based on the damped Korteweg de Vries (DKdV) equation, Mahmoud et al. [20] studied the nonlinear dust acoustic waves produced by the interaction of flowing protons and electrons with dusty plasma, El-Taibany et al. [12] derived a Korteweg-de Vries (KdV) equation

$$u_t + r_1 u u_x + r_2 u_{xxx} = 0,$$

and researched the nonlinear propagation of dust acoustic waves in a variable size grains dusty plasma. In 1974, Zakharov and Kuznetsov derived the ZK equation from magnetized plasma containing cold ions and hot isothermal electrons. Later, the stability of the periodic wave solution of the ZK equation in plane and solitary traveling wave solution has also been studied by many physicists. Munro and Parkes derived the mZK equation when the ions or electrons in the plasma did not satisfy the Boltzmann distribution. Recently, Popel et al. [24] took dusty plasma research to a new level by studying the effect of the Earth's magnetotail magnetic field on dusty plasma on the surface of the Moon illuminated by sunlight. At present, most of the researches on dusty plasma are based on low-dimensional physical models. We know that the propagation space of waves in nature should be multidimensional in general. Therefore£¬for more general theoretical and practical problems, we still need to consider multidimensional models.

Recently, El-Shiekh [13] established a two-dimensional Kadomtsev-Petviashvili Burgers(KPB) equation

$$(u_t + r_1 u u_x + r_2 u_{xxx})_x + r_3 u_{yy} + r_4 u_{xx} = 0,$$

to study bright and dark solitons, periodic soliton wave and shock wave in dusty plasma and quantum plasma.

Fractional order calculus [2, 3] may still be unfamiliar to most people, but it was put forward as early as 300 years ago. In 1695, in a famous letter to Leibniz, L'Hospital wrote, "For a simple linear function f(x)=x, what happens if the derivative degree of the function is a fraction instead of an integer? ". It is acknowledged that the fractional differential is mentioned for the first time. For a long time, there are many researches on fractional calculus in the field of mathematical pure theory.

Nowadays, the methods of solving fractional differential equations are becoming more and more perfect, with the wide application of fractional calculus in biological system, thermal systems and mechanical systems. Mahmoud [21] derived the time fractional Gardner equation

$$D_\tau^\omega u + r_1 u u_x r_2 u^2 u_x + r_3 u_{xxx} = 0,$$

and discussed the time fractional effect of dust sound bilayer waves. Veeresha et al. [33] obtained the solutions for fractional potential KdV and Benjamin equations via q-homotopy analysis transform method, Devendra et al. [10] found nonlinear fractional differential equations of variable order by using Bernoulli wavelet method. In the past, most scholars used integer order models to study wave propagation in dusty plasma. In some practical cases, fractional-order equations can describe some complex plasma processes and phenomena more accurately than integer-order equations.

Until now the linear wave theory in plasma has been mature and systematic, and the exploration of various nonlinear wave modes has attracted more and more scholars' interest. In 1965, Draper [11] first proposed the concept of strange waves. Rogue waves [6, 9, 35] have very high peaks and short duration. There is no sign before rogue waves appear, they will pop up in one area at one time, and then disappear quickly. Generally, a single wave with a wave height greater than twice the effective wave height can be called a rogue wave. Because these characteristics are different from other nonlinear waves, the study of rogue waves has become a hot topic. For the first time in 2011, the rogue waves in plasmas was observed in experiments by Bailung et al. [7]. After that, Almutalk et al. [4] obtained the numerical solution of dusty acoustic super rogue waves in a strongly coupled dusty plasma, Sun and Tian [28] investigated the Dust ion-acoustic rogue waves in an ultracold quantum dusty plasma, Mouhammadoul et al. [22] found the influence of the different plasma parameters of the highly energetic rogue wave.

The letter is organized as follows. In Section 2, we derive a (3+1)-dimensional mZK equation by the reduced perturbation method [15]. In Section 3, by means of semi-inverse method, Euler-Lagrange equation and fractional variational principle [14], the TF-mZK equation is obtained from integer order mZK equation. In Section 4, exact solution and Bäcklund transformation are given via the definition and properties of the bell polynomials [29]. The phenomenon of rouge waves in magnetized dusty plasma, effects of fractional order, phase velocity and dust-cyclotron frequency on the propagation characteristics of dust acoustic rogue waves are analyzed in Section 5.

2. Mathematical model and derivation of the mZK equation

The governing equations are based on a magnetized dusty plasma system consisting of dust particles, superthermal electrons, and two populations of ions with two distinct temperatures.

In order to derive the differential equation describing the propagation of dust acoustic wave in magnetized dusty plasma, it is first assumed that there is no dust particle collision effect in the system. Second, the second explanation is the acoustic wave propagates along the direction of x-axis, but the high-order transverse disturbance in the y-axis and z-axis directions is weak.

Since dusty plasma is electrically neutral at equilibrium

$$n_{e0} + n_{d0} Z_{d0} = n_{ih0} + n_{ic0},$$

where n_{e0}, n_{d0}, n_{ih0}, and n_{ic0} denote the unperturbed number densities of electrons, dust particles, cold ions and thermionic ions, respectively, and Z_{d0} denotes the unperturbed number of charges on dust particles.

On the basis of the above explanations, the nonlinear dynamics of dust acoustic waves is described by the normalized fluid equations

$$\begin{cases} \frac{\partial n_d}{\partial t} + \frac{\partial (n_d u_d)}{\partial x} + \frac{\partial (n_d v_d)}{\partial y} + \frac{\partial (n_d w_d)}{\partial z} = 0, \\ \frac{\partial u_d}{\partial t} + u_d \frac{\partial u_d}{\partial x} + v_d \frac{\partial u_d}{\partial y} + w_d \frac{\partial u_d}{\partial z} = \frac{\partial \phi}{\partial x}, \\ \frac{\partial v_d}{\partial t} + u_d \frac{\partial v_d}{\partial x} + v_d \frac{\partial v_d}{\partial y} + w_d \frac{\partial v_d}{\partial z} = \frac{\partial \phi}{\partial y} + \Omega w_d, \\ \frac{\partial w_d}{\partial t} + u_d \frac{\partial w_d}{\partial x} + v_d \frac{\partial w_d}{\partial y} + w_d \frac{\partial w_d}{\partial z} = \frac{\partial \phi}{\partial z} - \Omega v_d, \\ \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = n_d + (\gamma_{ic} + \gamma_{ih} - 1) n_e - \gamma_{ic} n_{ic} - \gamma_{ih} n_{ih}, \end{cases} \tag{2.1}$$

where the number density of the dust particles n_d is normalized by n_{d0}. u_d, v_d and w_d are the velocities of the dust-fluid along x, y and z directions, respectively, and normalized by dust acoustic speed $C_d = (\frac{Z_{d0} T_{ih}}{m_d})^{\frac{1}{2}}$, where m_d denotes dust particle mass, T_{ih} denotes effective temperature. The space variable (x, y, z) are normalized by the dusty plasma Debye radius $\lambda_d = (\frac{T_{\mathrm{ih}}}{4\pi n_{d0} Z_{d0} e^2})^{\frac{1}{2}}$, and the time variable t is normalized by the inverse of the dust-plasma frequency $\omega_{pd}^{-1} = (\frac{m_d}{4\pi n_{d0} Z_{d0}^2 e^2})^{\frac{1}{2}}$. The electrostatic potential ϕ is normalized by $\frac{T_{\mathrm{ih}}}{e}$, where e denotes the magnitude of the electric charge. Ω is the normalized dust-cyclotron frequency. n_e, n_{ic} and n_{ih} denote the number densities of the electrons, cold ions, and thermionic ions respectively, and normalized by $n_{d0} Z_{d0}$. γ_{ic} and γ_{ih} are the undisturbed number density ratios of cold ions to dust and thermionic ions to dust.

The densities of electrons and ions are as follows

$$\begin{aligned} & n_e = (1 - \frac{\phi}{\kappa - \frac{3}{2}})^{-\kappa + \frac{1}{2}}, \ n_{ic} = [1 - (q-1)\phi]^{\frac{(3q-1)}{2(q-1)}}, \\ & n_{ih} = [1 - \sigma(q-1)\phi]^{\frac{(3q-1)}{2(q-1)}}. \end{aligned} \tag{2.2}$$

According to Eq. (2.2) and Poisson's equation, we have

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = n_d - 1 + C_1 \phi + C_2 \phi^2 + C_3 \phi^3, \tag{2.3}$$

where

$$\begin{aligned} & C_1 = \gamma_e \frac{(\kappa - \frac{1}{2})}{(\kappa - \frac{3}{2})} + (\gamma_{ic} + \gamma_{ih}\sigma) \frac{(3q-1)}{2}, \\ & C_2 = \gamma_e \frac{(\kappa^2 - \frac{1}{4})}{2(\kappa - \frac{3}{2})^2} - (\gamma_{ic} + \gamma_{ih}\sigma^2) \frac{(3q-1)(q+1)}{8}, \\ & C_2 = \gamma_e \frac{(\kappa^2 - \frac{1}{4})(\kappa + \frac{3}{2})}{2(\kappa - \frac{3}{2})^3} + (\gamma_{ic} + \gamma_{ih}\sigma^3) \frac{(3q-1)(q+1)(3-q)}{48}. \end{aligned} \tag{2.4}$$

Next, the reduced perturbation method is used to process Eq. (2.1), and the mZK equation is obtained for studying the evolutionary properties. First, making the following scale analysis

$$\xi = \varepsilon(x - \lambda t),\ \eta = \varepsilon y,\ \zeta = \varepsilon z,\ \tau = \varepsilon^3 t. \tag{2.5}$$

According to Eq. (2.5), we obtain as

$$\frac{\partial}{\partial x} = \varepsilon\frac{\partial}{\partial \xi},\ \frac{\partial}{\partial y} = \varepsilon\frac{\partial}{\partial \eta},\ \frac{\partial}{\partial z} = \varepsilon\frac{\partial}{\partial \zeta},\ \frac{\partial}{\partial t} = \varepsilon^3\frac{\partial}{\partial \tau} - \varepsilon\lambda\frac{\partial}{\partial \xi}. \tag{2.6}$$

The dependent variables n_d, u_d, v_d, w_d and ϕ are expanded in the powers of ε as follows

$$\begin{cases} n_d = 1 + \varepsilon n_{d_1} + \varepsilon^2 n_{d_2} + \varepsilon^3 n_{d_3} + \cdots, \\ u_d = \varepsilon u_{d_1} + \varepsilon^2 u_{d_2} + \varepsilon^3 u_{d_3} + \cdots, \\ v_d = \varepsilon^2 v_{d_1} + \varepsilon^3 v_{d_2} + \cdots, \\ w_d = \varepsilon^2 w_{d_1} + \varepsilon^3 w_{d_2} + \cdots, \\ \phi = \varepsilon\phi_1 + \varepsilon^2\phi_2 + \varepsilon^3\phi_2 + \cdots. \end{cases} \tag{2.7}$$

Substituting Eqs. (2.6) and (2.7) into Eq. (2.1) yields

$$\begin{cases} \varepsilon^3\frac{\partial n_d}{\partial \tau} - \varepsilon\lambda\frac{\partial n_d}{\partial \xi} + \varepsilon\frac{\partial(n_d u_d)}{\partial \xi} + \varepsilon\frac{\partial(n_d v_d)}{\partial \eta} + \varepsilon\frac{\partial(n_d w_d)}{\partial \zeta} = 0, \\ \varepsilon^3\frac{\partial u_d}{\partial \tau} - \varepsilon\lambda\frac{\partial u_d}{\partial \xi} + \varepsilon u_d\frac{\partial u_d}{\partial \xi} + \varepsilon v_d\frac{\partial u_d}{\partial \eta} + \varepsilon w_d\frac{\partial u_d}{\partial \zeta} = \varepsilon\frac{\partial \phi}{\partial \xi}, \\ \varepsilon^3\frac{\partial v_d}{\partial \tau} - \varepsilon\lambda\frac{\partial v_d}{\partial \xi} + \varepsilon u_d\frac{\partial v_d}{\partial \xi} + \varepsilon v_d\frac{\partial v_d}{\partial \eta} + \varepsilon w_d\frac{\partial v_d}{\partial \zeta} = \varepsilon\frac{\partial \phi}{\partial \eta} + \Omega w_d, \\ \varepsilon^3\frac{\partial w_d}{\partial \tau} - \varepsilon\lambda\frac{\partial w_d}{\partial \xi} + \varepsilon u_d\frac{\partial w_d}{\partial \xi} + \varepsilon v_d\frac{\partial w_d}{\partial \eta} + \varepsilon w_d\frac{\partial w_d}{\partial \zeta} = \varepsilon\frac{\partial \phi}{\partial \zeta} - \Omega v_d, \\ \varepsilon^2\frac{\partial^2 \phi}{\partial \xi^2} + \varepsilon^2\frac{\partial^2 \phi}{\partial \eta^2} + \varepsilon^2\frac{\partial^2 \phi}{\partial \zeta^2} = n_d - 1 + C_1\phi + C_2\phi^2 + C_3\phi^3. \end{cases} \tag{2.8}$$

According to Eq. (2.8) and equating coefficients of like powers of the ε, at the lowest order

$$\varepsilon : n_{d_1} + C_1\phi_1 = 0. \tag{2.9}$$

$$\varepsilon^2 : \begin{cases} -\lambda\frac{\partial n_{d_1}}{\partial \xi} + \frac{\partial u_{d_1}}{\partial \xi} = 0,\ -\lambda\frac{\partial u_{d_1}}{\partial \xi} = \frac{\partial \phi_1}{\partial \xi}, \\ \frac{\partial \phi_1}{\partial \eta} + \Omega w_{d_1} = 0,\ \frac{\partial \phi_1}{\partial \zeta} - \Omega v_{d_1} = 0, \\ n_{d_2} + C_1\phi_2 + C_2\phi_1^2 = 0, \end{cases} \tag{2.10}$$

$$\varepsilon^3:\begin{cases}-\lambda\dfrac{\partial n_{d_2}}{\partial\xi}+\dfrac{\partial u_{d_2}}{\partial\xi}+\dfrac{\partial(n_{d_1}u_{d_1})}{\partial\xi}+\dfrac{\partial(v_{d_1})}{\partial\eta}+\dfrac{\partial(w_{d_1})}{\partial\zeta}=0,\\ -\lambda\dfrac{\partial u_{d_2}}{\partial\xi}+u_{d_1}\dfrac{\partial u_{d_1}}{\partial\xi}=\dfrac{\partial\phi_2}{\partial\xi},\\ -\lambda\dfrac{\partial v_{d_1}}{\partial\xi}=\dfrac{\partial\phi_2}{\partial\eta}+\Omega w_{d_2},\\ -\lambda\dfrac{\partial w_{d_1}}{\partial\xi}=\dfrac{\partial\phi_2}{\partial\zeta}-\Omega v_{d_2},\\ \dfrac{\partial^2\phi_1}{\partial\xi^2}+\dfrac{\partial^2\phi_1}{\partial\eta^2}+\dfrac{\partial^2\phi_1}{\partial\zeta^2}=n_{d_3}+C_1\phi_3+2C_2\phi_1\phi_2+C_3\phi_1^3.\end{cases}\tag{2.11}$$

According to Eq. (2.10) and Eq. (2.11), one has

$$\begin{aligned}&n_{d_1}=\frac{u_{d_1}}{\lambda},\ u_{d_1}=-\frac{\phi_1}{\lambda},\ C_1=\frac{1}{\lambda^2},\\ &n_{d_2}=-C_2\phi_1^2-C_1\phi_2,\ u_{d_2}=\frac{1}{2\lambda^3}\phi_1^2-\frac{1}{\lambda}\phi_2,\\ &v_{d_2}=-\frac{\lambda}{\Omega^2}\frac{\partial^2\phi_1}{\partial\xi\eta}+\frac{1}{\Omega}\frac{\partial\phi_2}{\partial\eta},\ w_{d_2}=-\frac{\lambda}{\Omega^2}\frac{\partial^2\phi_1}{\partial\xi\zeta}-\frac{1}{\Omega}\frac{\partial\phi_2}{\partial\eta},\ C_2=-\frac{3}{2\lambda^4}.\end{aligned}\tag{2.12}$$

At the highest order of the ε, we get

$$\varepsilon^4:\begin{cases}\dfrac{\partial n_{d_1}}{\partial\tau}-\lambda\dfrac{\partial n_{d_3}}{\partial\xi}+\dfrac{\partial u_{d_3}}{\partial\xi}+\dfrac{\partial(n_{d_1}u_{d_2})}{\partial\xi}+\dfrac{\partial(n_{d_2}u_{d_1})}{\partial\xi}+\dfrac{\partial n_{d_1}v_{d_1}}{\partial\eta}+\dfrac{\partial v_{d_2}}{\partial\eta}\\ +\dfrac{\partial n_{d_1}w_{d_1}}{\partial\zeta}+\dfrac{\partial w_{d_2}}{\partial\zeta}=0,\\ \dfrac{\partial u_{d_1}}{\partial\tau}-\lambda\dfrac{\partial u_{d_3}}{\partial\xi}+u_{d_1}\dfrac{\partial u_{d_2}}{\partial\xi}+u_{d_2}\dfrac{\partial u_{d_1}}{\partial\xi}+v_{d_1}\dfrac{\partial u_{d_1}}{\partial\eta}+w_{d_1}\dfrac{\partial u_{d_1}}{\partial\zeta}=\dfrac{\partial\phi_3}{\partial\xi}.\end{cases}\tag{2.13}$$

Substituting Eq. (2.12) into Eq. (2.13), the (3+1)-dimensional mZK equation is obtained as

$$\frac{\partial\phi_1}{\partial\tau}+a_1\phi_1^2\frac{\partial\phi_1}{\partial\xi}+a_2\frac{\partial^3\phi_1}{\partial^3\xi}+a_3(\frac{\partial^3\phi_1}{\partial\xi\eta\eta}+\frac{\partial^3\phi_1}{\partial\xi\zeta\zeta})=0,\tag{2.14}$$

where

$$a_1=-\frac{\lambda^2}{2}(\frac{6\lambda^2-6}{2\lambda^5}-3\lambda C_3),\ a_2=\frac{\lambda^3}{2},\ a_3=\frac{\lambda^3}{2}(1+\frac{1}{\Omega^2}).\tag{2.15}$$

Remark 2.1. When the parameter $a_3=0$, the KdV equation can be obtained from Eq. (2.14). The (3+1)-dimensional mZK equation is a generalized form of the traditional KdV equation. Further discussion of the new equation is of great significance to the study of dusty plasma.

3. Derivation of the TF-mZK equation

A new (3+1)-dimensional integer order mZK equation is obtained in the above section. In this section, the semi-inverse method and fractional variational principle are used to establish the (3+1)-dimensional generalized TF-mZK equation. Introducing the following definitions of fractional

Definition 3.1. The Riemann-Liouville fractional derivative [14]

$$D_\tau^\omega f(\tau)=\begin{cases}\dfrac{1}{\Gamma(n-\omega)}\dfrac{\mathrm{d^n}}{\mathrm{d}\tau^{\mathrm{n}}}\displaystyle\int_0^\tau(\tau-T)^{n-\omega-1}f(T)\mathrm{dT}, & \mathrm{n}-1<\alpha<\mathrm{n},\\ \dfrac{\partial^n f(t)}{\partial\tau^n}, & \omega=n.\end{cases}\tag{3.1}$$

Definition 3.2. The fractional integration by parts is given as [14]

$$\int_a^b(\mathrm{d}\tau)^\omega\mathrm{f}(\tau)\mathrm{D}_\tau^\omega\mathrm{g}(\tau)=\Gamma(1+\omega)[\mathrm{g}(\tau)\mathrm{f}(\tau)|_\mathrm{a}^\mathrm{b}-\int_\mathrm{a}^\mathrm{b}(\mathrm{d}\tau)^\omega\mathrm{g}(\tau)\mathrm{D}_\tau^\omega\mathrm{f}(\tau)],\quad f(\tau),g(\tau)\in[a,b].\tag{3.2}$$

Eq. (2.14) can be expressed by

$$\psi_\tau+a_1\psi^2\psi_\xi+a_2\psi_{\xi\xi\xi}+a_3(\psi_{\xi\eta\eta}+\psi_{\xi\zeta\zeta})=0.\tag{3.3}$$

Letting $\psi(\xi,\eta,\zeta,\tau)=\varphi_\xi(\xi,\eta,\zeta,\tau)$, where $\varphi(\xi,\eta,\zeta,\tau)$ denotes a potential function, the potential equation of Eq. (3.3) is

$$\varphi_{\xi\tau}+a_1\varphi_\xi^2\varphi_{\xi\xi}+a_2\varphi_{\xi\xi\xi\xi}+a_3(\varphi_{\xi\xi\eta\eta}+\varphi_{\xi\xi\zeta\zeta})=0.\tag{3.4}$$

Then, the semi-inverse method [20] is used to derive the Lagrangian form of Eq. (3.2). The functional form of Eq. (3.3) can be expressed as

$$\begin{aligned}J(\varphi)=\int_X\mathrm{d}\xi\int_\mathrm{Y}\mathrm{d}\eta\int_\mathrm{Z}\mathrm{d}\zeta\int_\mathrm{T}\mathrm{d}\tau[\varphi(\mathrm{b_1}\varphi_{\xi\tau}+\mathrm{b_2a_1}\varphi_\xi^2\varphi_{\xi\xi}\\+b_3a_2\varphi_{\xi\xi\xi\xi}+b_4a_3\varphi_{\xi\xi\eta\eta}+b_5a_3\varphi_{\xi\xi\zeta\zeta})],\end{aligned}\tag{3.5}$$

where $b_i(i=1,2,\ldots,5)$ denote the Lagrangian multipliers.

Applying integration by parts to Eq. (3.5), and letting $\varphi_\xi|_T=\varphi_\xi|_X=\varphi_{\xi\xi\xi}|_X=\varphi_{\xi\eta\eta}|_X=\varphi_{\xi\zeta\zeta}|_X=\varphi_\xi|_Y=\varphi_\xi|_Z=0$, we get

$$\begin{aligned}J(\varphi)=\int_X\mathrm{d}\xi\int_\mathrm{Y}\mathrm{d}\eta\int_\mathrm{Z}\mathrm{d}\zeta\int_\mathrm{T}\mathrm{d}\tau[-\mathrm{b_1}\varphi_\xi\varphi_\tau-\frac{1}{3}\mathrm{b_2a_1}\varphi_\xi^4+\mathrm{b_3a_2}\varphi_{\xi\xi}^2\\-b_4a_3\varphi_{\xi\eta}^2-b_5a_3\varphi_{\xi\zeta}^2].\end{aligned}\tag{3.6}$$

Lagrangian multipliers $b_i(i=1,2,\ldots,5)$ can be determined by the variation of Eq. (3.6) to make them optimal. Via the variation of Eq. (3.6) and the optimal conditions, integrate each term by parts, and get

$$\begin{aligned}&L(\xi,\eta,\zeta,\tau,\varphi_\tau,\varphi_\xi,\varphi_{\xi\eta},\varphi_{\xi\zeta},\varphi_{\xi\xi})\\&=\frac{\partial F}{\partial\varphi}-\frac{\partial}{\partial\tau}(\frac{\partial F}{\partial\varphi_\tau})-\frac{\partial}{\partial\xi}(\frac{\partial F}{\partial\varphi_\xi})+\frac{\partial^2}{\partial\xi^2}(\frac{\partial F}{\partial\varphi_{\xi\xi}})-\frac{\partial^2}{\partial\xi\eta}(\frac{\partial F}{\partial\varphi_{\xi\eta}})-\frac{\partial^2}{\partial\xi\zeta}(\frac{\partial F}{\partial\varphi_{\xi\zeta}})\\&=2b_1\varphi_{\xi\tau}+4b_2a_1\varphi_\xi^2\varphi_{\xi\xi}+2b_3a_2\varphi_{\xi\xi\xi\xi}+2b_4a_3\varphi_{\xi\xi\eta\eta}+2b_5a_3\varphi_{\xi\xi\zeta\zeta}=0.\end{aligned}\tag{3.7}$$

Eq. (3.7) is equivalent to Eq. (3.4), so Lagrangian multipliers $b_i(i=1,2,3,4,5)$ are

$$b_1=b_3=b_4=b_5=\frac{1}{2},\ b_2=\frac{1}{4}.\tag{3.8}$$

According to Eq. (3.6) and Eq. (3.8), the Lagrangian form of mZK equation can be obtained as

$$L(\varphi_\tau,\varphi_\xi,\varphi_{\xi\xi},\varphi_{\xi\eta},\varphi_{\xi\zeta}) = -\frac{1}{2}\varphi_\xi\varphi_\tau - \frac{1}{12}a_1\varphi_\xi^4 + \frac{1}{2}a_2\varphi_{\xi\xi}^2 - \frac{1}{2}a_3\varphi_{\xi\eta}^2 - \frac{1}{2}a_3\varphi_{\xi\zeta}^2. \tag{3.9}$$

Further the Lagrangian form of the TF-mZK equation is given by

$$\begin{aligned}&\mathcal{L}(D_\tau^\omega\varphi,\varphi_\xi,\varphi_{\xi\xi},\varphi_{\xi\eta},\varphi_{\xi\zeta})\\ =&-\frac{1}{2}\varphi_\xi D_\tau^\omega\varphi - \frac{1}{12}a_1\varphi_\xi^4 + \frac{1}{2}a_2\varphi_{\xi\xi}^2 - \frac{1}{2}a_3\varphi_{\xi\eta}^2 - \frac{1}{2}a_3\varphi_{\xi\zeta}^2,\end{aligned} \tag{3.10}$$

and, the functional is

$$J_\mathcal{L}(\varphi) = \int_X \mathrm{d}\xi \int_\mathrm{Y} \mathrm{d}\eta \int_\mathrm{Z} \mathrm{d}\zeta \int_\mathrm{T} (\mathrm{d}\tau)^\omega \mathcal{L}(\mathrm{D}_\tau^\omega\varphi,\varphi_\xi,\varphi_{\xi\xi},\varphi_{\xi\eta},\varphi_{\xi\zeta}). \tag{3.11}$$

Using the Agrawal's method [18], the variation of Eq. (3.11) with respect to $\psi(\xi,\eta,\zeta,\tau)$ is

$$\begin{aligned}\delta J_\mathcal{L}(\varphi) =& \int_X \mathrm{d}\xi \int_\mathrm{Y} \mathrm{d}\eta \int_\mathrm{Z} \mathrm{d}\zeta \int_\mathrm{T} (\mathrm{d}\tau)^\omega [(\frac{\partial\mathcal{L}}{\partial \mathrm{D}_\tau^\omega\varphi})\delta \mathrm{D}_\tau^\omega\varphi\\ &+(\frac{\partial\mathcal{L}}{\partial\varphi_\xi})\delta\varphi_\xi + (\frac{\partial\mathcal{L}}{\partial\varphi_{\xi\xi}})\delta\varphi_{\xi\xi} + (\frac{\partial\mathcal{L}}{\partial\varphi_{\xi\eta}})\delta\varphi_{\xi\eta} + (\frac{\partial\mathcal{L}}{\partial\varphi_{\xi\zeta}})\delta\varphi_{\xi\zeta}].\end{aligned} \tag{3.12}$$

According to Definition 3.2 and Eq.(3.12), we obtain

$$\begin{aligned}\delta J_\mathcal{L}(\varphi) =& \int_X d\xi \int_Y d\eta \int_Z d\zeta \int_T (d\tau)^\omega [-D_\tau^\omega(\frac{\partial\mathcal{L}}{\partial D_\tau^\omega\varphi}) - \frac{\partial}{\partial\xi}(\frac{\partial\mathcal{L}}{\partial\varphi_\xi})\\ &+\frac{\partial^2}{\partial\xi^2}(\frac{\partial\mathcal{L}}{\partial\varphi_{\xi\xi}}) - \frac{\partial^2}{\partial\xi\eta}(\frac{\partial\mathcal{L}}{\partial\varphi_{\xi\eta}}) - \frac{\partial^2}{\partial\xi\zeta}(\frac{\partial\mathcal{L}}{\partial\varphi_{\xi\zeta}})]\delta\varphi.\end{aligned} \tag{3.13}$$

Optimizing the variation of the functional $J_\mathcal{L}(\varphi)$, the Euler-Lagrange equation is given by

$$-D_\tau^\omega(\frac{\partial\mathcal{L}}{\partial D_\tau^\omega\varphi}) - \frac{\partial}{\partial\xi}(\frac{\partial\mathcal{L}}{\partial\varphi_\xi}) + \frac{\partial^2}{\partial\xi^2}(\frac{\partial\mathcal{L}}{\partial\varphi_{\xi\xi}}) - \frac{\partial^2}{\partial\xi\eta}(\frac{\partial\mathcal{L}}{\partial\varphi_{\xi\eta}}) - \frac{\partial^2}{\partial\xi\zeta}(\frac{\partial\mathcal{L}}{\partial\varphi_{\xi\zeta}}) = 0. \tag{3.14}$$

Substituting of Eq. (3.9) into Eq. (3.14) yields

$$D_\tau^\omega\varphi_\xi + a_1(\varphi_\xi)^2\varphi_{\xi\xi} + a_2\varphi_{\xi\xi\xi\xi} + a_3\varphi_{\xi\xi\eta\eta} + a_3\varphi_{\xi\xi\zeta\zeta} = 0. \tag{3.15}$$

And finally, substituting $\varphi_\xi(\xi,\eta,\zeta,\tau) = \psi(\xi,\eta,\zeta,\tau)$ into Eq. (3.15), (3+1)-dimensional TF-mZK equation is given by

$$D_\tau^\omega\psi + a_1\psi^2\psi_\xi + a_2\psi_{\xi\xi\xi} + a_3(\psi_{\xi\eta\eta} + \psi_{\xi\zeta\zeta}) = 0. \tag{3.16}$$

Remark 3.1. According to Definition 3.1, when the time fractional order $\omega = 1$, Eq. (3.16) is an integer order mZK equation. Therefore, compared with the integer-order model, Eq. (3.16) is more general. It is meaningful to use fractional equation to study complex physical phenomena about dusty plasma.

4. Solutions of the TF-mZK equation

Researchers usually use bilinear method to obtain the rogue wave solution of the equation. However, the application of bilinear methods requires great skill in selecting variable transformations. Therefore, in this section, we obtain the bilinear form of the equation through the relationship between Bell polynomials and bilinear derivatives, and then get the rogue wave solution.

Definition 4.1. Taking $f = f(x_1, x_2, \cdots, x_n)$ be a C^∞ function with n variables, the multi-dimensional Bell polynomials(generalized Bell polynomial or Y-polynomials) [29] are given by

$$Y_{n_1x_1,\cdots,n_lx_l}(f) \equiv Y_{n_1,\cdots,n_l}(f_{r_1x_1,\cdots,r_lx_l}) = \mathrm{e}^{-f}\partial_{x_1}^{n_1}\cdots\partial_{x_l}^{n_l}\mathrm{e}^{f}, \tag{4.1}$$

here

$$f_{r_1x_1,\cdots,r_lx_l} = \partial_{x_1}^{n_1}\cdots\partial_{x_l}^{n_l}f \quad (r_1 = 0,\cdots,n_1;\cdots;r_l = 0,\cdots,n_l). \tag{4.2}$$

Definition 4.2. Bell polynomials containing functions ϑ and ϖ are called multi-dimensional binary Bell polynomials [18]

$$\begin{aligned}\mathcal{Y}_{n_1x_1,\cdots,n_lx_l}(\vartheta,\varpi) &\equiv Y_{n_1,\cdots,n_l}(f) \mid f_{r_1x_1,\cdots,r_lx_l} \\ &= \begin{cases}\vartheta_{r_1x_1,\cdots,r_lx_l}, & r_1+r_2+\cdots+r_l \text{ is odd,}\\ \varpi_{r_1x_1,\cdots,r_lx_l}, & r_1+r_2+\cdots+r_l \text{ is even.}\end{cases}\end{aligned} \tag{4.3}$$

Theorem 4.1. *The $\mathcal{Y}$-polynomials and Hirota D-operator satisfy [18]*

$$\mathcal{Y}_{n_1x_1,\cdots,n_lx_l}(\vartheta = \ln\frac{F}{G}, \varpi = \ln FG) = (FG)^{-1}D_{x_1}^{n_1}\cdots D_{x_l}^{n_l}F\cdot G, \tag{4.4}$$

here $n_1+n_2+\cdots+n_l \geq 1$.

Next, introducing fractional-order transformations are

$$t = \frac{j_1\tau^\omega}{\Gamma(1+\omega)}, \tag{4.5}$$

where j_1 is an arbitrary constant.

With the Eq. (4.5), $\frac{\partial^\omega\psi}{\partial\tau^\omega} = j_1\frac{\partial\psi}{\partial t}$ is obtained. Therefore, Eq. (3.3) can be rewritten as

$$\psi_t + a_1\psi^2\psi_\xi + a_2\psi_{\xi\xi\xi} + a_3(\psi_{\xi\eta\eta} + \psi_{\xi\zeta\zeta}) = 0. \tag{4.6}$$

Let $\psi = \vartheta_\xi$, Eq. (4.6) can be expressed as

$$\vartheta_{\xi t} + a_1\vartheta_\xi^2\vartheta_{\xi\xi} + a_2\vartheta_{\xi\xi\xi\xi} + a_3(\vartheta_{\xi\xi\eta\eta} + \vartheta_{\xi\xi\zeta\zeta}) = 0. \tag{4.7}$$

Introducing constraints $\vartheta_\eta = p\vartheta_\xi$, $\vartheta_\zeta = q\vartheta_\xi$, where p and q are arbitrary constants. The following equation is obtained as

$$\vartheta_{\xi t} + a_1\vartheta_\xi^2\vartheta_{\xi\xi} + a_2\vartheta_{\xi\xi\xi\xi} + a_3(p\vartheta_{\xi\xi\xi\eta} + q\vartheta_{\xi\xi\xi\zeta}) = 0. \tag{4.8}$$

According to Eq. (4.7) and constraints $\vartheta_\eta = p\vartheta_\xi$, $\vartheta_\zeta = q\vartheta_\xi$, we obtain

$$\vartheta_t + \frac{a_1}{3}\vartheta_\xi^3 + a_2\vartheta_{\xi\xi\xi} + a_3(p^2\vartheta_{\xi\xi\xi} + q^2\vartheta_{\xi\xi\xi}) = 0. \tag{4.9}$$

When $a_1 = -6[a_2 + a_3(p^2 + q^2)]$, Eq. (4.9) can be expressed as

$$\vartheta_t+[a_2+a_3(p^2+q^2)](\vartheta_{\xi\xi\xi}+3\varpi_{\xi\xi}\vartheta_\xi+\vartheta_\xi^3)-3[a_2+a_3(p^2+q^2)](\vartheta_\xi^3+\varpi_{\xi\xi}\vartheta_\xi) = 0. \quad (4.10)$$

According to constraints and Eq. (4.10), we get

$$\begin{cases} \vartheta_\eta = p\vartheta_\xi, \\ \vartheta_\zeta = q\vartheta_\xi, \\ \vartheta_t + [a_2 + a_3(p^2 + q^2)](\vartheta_{\xi\xi\xi} + 3\varpi_{\xi\xi}\vartheta_\xi + \vartheta_\xi^3) = 0, \\ -3[a_2 + a_3(p^2 + q^2)](\vartheta_\xi^2 + \varpi_{\xi\xi}) = 0. \end{cases} \quad (4.11)$$

Then, the bell polynomial form of Eq. (4.6) is

$$\begin{cases} \mathcal{Y}_\eta(\vartheta, \varpi) = p\mathcal{Y}_\xi(\vartheta, \varpi), \\ \mathcal{Y}_\zeta((\vartheta, \varpi) = q\mathcal{Y}_\xi(\vartheta, \varpi), \\ -3[a_2 + a_3(p^2 + q^2)]\mathcal{Y}_{\xi\xi}(\vartheta, \varpi) = 0, \\ \mathcal{Y}_t(\vartheta, \varpi) + [a_2 + a_3(p^2 + q^2)]\mathcal{Y}_{\xi\xi\xi}(\vartheta, \varpi) = 0. \end{cases} \quad (4.12)$$

According to Theorem 4.1 and $\vartheta = \ln(g/f)$, $\varpi = \ln(gf)$, the bilinear form of Eq. (4.6) is given by

$$\begin{cases} D_\eta(g \cdot f) = pD_\xi(g \cdot f), \\ D_\zeta(g \cdot f) = qD_\xi(g \cdot f), \\ -3[a_2 + a_3(p^2 + q^2)]D_\xi^2(g \cdot f) = 0, \\ D_t(g \cdot f)) + [a_2 + a_3(p^2 + q^2)]D_\xi^3(g \cdot f) = 0. \end{cases} \quad (4.13)$$

Assuming f and g can be expanded as follows

$$\begin{cases} f = 1 + \epsilon f^{(1)} + \epsilon^2 f^{(2)} + \cdots, \\ g = 1 + \epsilon g^{(1)} + \epsilon^2 g^{(2)} + \cdots. \end{cases} \quad (4.14)$$

Substituting Eq. (4.14) into Eq. (4.13), we obtain

$$\epsilon : \begin{cases} g_\eta^{(1)} - f_\eta^{(1)} = p(g_\xi^{(1)} - f_\xi^{(1)}), \\ g_\zeta^{(1)} - f_\zeta^{(1)} = q(g_\xi^{(1)} - f_\xi^{(1)}), \\ -3[a_2 + a_3(p^2 + q^2)](g_{\xi\xi}^{(1)} - f_{\xi\xi}^{(1)}) = 0, \\ g_t^{(1)} - f_t^{(1)} + [a_2 + a_3(p^2 + q^2)](g_{\xi\xi\xi}^{(1)} - f_{\xi\xi\xi}^{(1)}) = 0. \end{cases} \quad (4.15)$$

Assume $f^{(1)} = d_1 \exp(\theta_1)$, $g^{(1)} = d_2 \exp(\theta_1)$, where d_1, d_2, d_3 are arbitrary constants, $\theta_1 = k_1\xi + l_1\eta + m_1\zeta + n_1 t + d_3$. Solving Eq. (4.15), we get

$$\begin{cases} l_1 = pk_1, \ m_1 = qk_1, \ d_1 = -d_2, \\ n_1 = -[a_2 + a_3(p^2 + q^2)]k_1^3. \end{cases} \quad (4.16)$$

Therefore, the solution is obtained as

$$\psi = [\ln(\frac{1 + g^{(1)}}{1 + f^{(1)}})]_x = \frac{2k_1 d_2 \exp(\theta_1)}{[1 + d_1 \exp(\theta_1)][1 + d_2 \exp(\theta_1)]}, \quad (4.17)$$

where $\theta_1 = k_1\xi + l_1\eta + m_1\zeta + n_1 \frac{\tau^\omega}{\Gamma(1+\omega)} + d_3$.

5. Bilinear Bӑcklund transformations

We further study the Bӑcklund transformation [16,25] of the equation with parameters based on the bell polynomial form of the equation deduced previously. The Bӑcklund transformation not only describes the integrability of the equation, but is also a transformation that relates one solution to another.

According to Eq. (4.11), we set

$$\begin{cases} P_1(\vartheta,\varpi) = \vartheta_\eta - p\vartheta_\xi, \\ P_2(\vartheta,\varpi) = \vartheta_\zeta - q\vartheta_\xi, \\ P_3(\vartheta,\varpi) = \vartheta_t + [a_2 + a_3(p^2+q^2)](\vartheta_{\xi\xi\xi} + 3\varpi_{\xi\xi}\vartheta_\xi + \vartheta_\xi^3) = 0, \\ P_4(\vartheta,\varpi) = -3[a_2 + a_3(p^2+q^2)](\vartheta_\xi^2 + \varpi_{\xi\xi}) = 0. \end{cases} \tag{5.1}$$

Assume that Eq. (5.1) has the following two different sets of solutions

$$\vartheta = \ln(g/f),\ \varpi = \ln(gf);\ \tilde{\vartheta} = \ln(\tilde{g}/\tilde{f}),\ \tilde{\varpi} = \ln(\tilde{g}\tilde{f}). \tag{5.2}$$

Introducing

$$\begin{aligned} &\vartheta_1 = \ln(\tilde{f}/f),\ \varpi_1 = \ln(\tilde{f}f),\ \vartheta_2 = \ln(\tilde{g}/g),\ \varpi_2 = \ln(\tilde{g}g), \\ &\vartheta_3 = \ln(\tilde{g}/f),\ \varpi_3 = \ln(\tilde{g}f),\ \vartheta' = \ln(\tilde{f}/g),\ \varpi' = \ln(f\tilde{g}), \end{aligned} \tag{5.3}$$

and considering the four conditions

$$\begin{cases} P_1(\vartheta,\varpi) - P_1(\tilde{\vartheta},\tilde{\varpi}) = 0, \\ P_2(\vartheta,\varpi) - P_2(\tilde{\vartheta},\tilde{\varpi}) = 0, \\ P_3(\vartheta,\varpi) - P_3(\tilde{\vartheta},\tilde{\varpi}) = 0, \\ P_4(\vartheta,\varpi) - P_4(\tilde{\vartheta},\tilde{\varpi}) = 0. \end{cases} \tag{5.4}$$

From Eq. (5.2) and Eq. (5.3), some relationships are obtained as

$$\begin{aligned} &\vartheta_1 - \vartheta_2 = \tilde{\vartheta} - \vartheta,\ 2\vartheta_3 - \vartheta_1 - \vartheta_2 = \tilde{\vartheta} + \vartheta, \\ &\varpi_1 - \varpi_2 + 2\vartheta_3 = \tilde{\varpi} - \varpi,\ \varpi_1 + \varpi_2 = \tilde{\varpi} - \varpi. \end{aligned} \tag{5.5}$$

Based on Eq. (5.4), one has

$$\begin{aligned} P_1(\vartheta,\varpi) - P_1(\tilde{\vartheta},\tilde{\varpi}) &= (\tilde{\vartheta} - \vartheta)_\eta - p(\tilde{\vartheta} - \vartheta)_\xi \\ &= (\vartheta_2 - \vartheta_1)_\eta - p(\vartheta_2 - \vartheta_1)_\xi. \end{aligned} \tag{5.6}$$

The following two Bell polynomials can be obtained from Eq. (5.5)

$$\mathcal{Y}_\eta(\vartheta_1,\varpi_1) - p\mathcal{Y}_\xi(\vartheta_1,\varpi_1) = 0,\ \mathcal{Y}_\eta(\vartheta_2,\varpi_2) - p\mathcal{Y}_\xi(\vartheta_2,\varpi_2) = 0. \tag{5.7}$$

Similarly, we have

$$P_2(\vartheta,\varpi) - P_2(\tilde{\vartheta},\tilde{\varpi}) = (\vartheta_2 - \vartheta_1)_\zeta - p(\vartheta_2 - \vartheta_1)_\xi, \tag{5.8}$$

and

$$\mathcal{Y}_\zeta(\vartheta_1,\varpi_1) - p\mathcal{Y}_\xi(\vartheta_1,\varpi_1) = 0,\ \mathcal{Y}_\zeta(\vartheta_2,\varpi_2) - p\mathcal{Y}_\xi(\vartheta_2,\varpi_2) = 0. \tag{5.9}$$

Then, the following constraints were introduced as

$$(\vartheta_3)_\xi = \lambda(t)\exp(\vartheta_1-\vartheta_2),\ (\vartheta_4)_\xi = U(t)\exp(\vartheta_2-\vartheta_1). \tag{5.10}$$

According to Eq. (5.9), we obtain

$$\begin{aligned}
&P_4(\vartheta,\varpi)-P_4(\tilde{\vartheta},\tilde{\varpi})\\
=&-3[a_2+a_3(p^2+q^2)][(\tilde{\varpi}_{\xi\xi}-\varpi_{\xi\xi})+(\tilde{\vartheta}_\xi^2-\vartheta_\xi^2)]\\
=&-3[a_2+a_3(p^2+q^2)][(\varpi_1-\varpi_2+2\vartheta_3)_{\xi\xi}+(2\vartheta_3-\vartheta_1-\vartheta_2)_\xi(\vartheta_2-\vartheta_1)_\xi]\\
=&3[a_2+a_3(p^2+q^2)][((\varpi_2)_{\xi\xi}+(\vartheta_2^2)_{\xi\xi})-((\varpi_1)_{\xi\xi}+(\vartheta_1^2)_{\xi\xi})].\\
&P_3(\vartheta,\varpi)-P_3(\tilde{\vartheta},\tilde{\varpi})\\
=&(\vartheta'-\vartheta)_t+[a_2+a_3(p^2+q^2)][((\vartheta')_{\xi\xi\xi}+3(\varpi')_{\xi\xi}(\vartheta')_\xi+(\vartheta')_\xi^3)-(\vartheta_{\xi\xi\xi}+3\varpi_{\xi\xi}\vartheta_\xi+\vartheta_\xi^3)]\\
=&(\vartheta_2-\vartheta_1)_t+[a_2+a_3(p^2+q^2)][((\vartheta_2)_{\xi\xi\xi}+3(\varpi_2)_{\xi\xi}(\vartheta_2)_\xi+(\vartheta_2)_\xi^3)+3\lambda(t)U(t)(\vartheta_2)_\xi]\\
&-[a_2+a_3(p^2+q^2)][((\vartheta_1)_{\xi\xi\xi}+3(\varpi_1)_{\xi\xi}(\vartheta_1)_\xi+(\vartheta_1)_\xi^3)+3\lambda(t)U(t)(\vartheta_1)_\xi].
\end{aligned} \tag{5.11}$$

From Eq. (5.11), six Bell polynomials can be obtained as follows

$$\begin{aligned}
&3[a_2+a_3(p^2+q^2)]\mathcal{Y}_{\xi\xi}(\vartheta_1,\varpi_1)=0,\\
&3[a_2+a_3(p^2+q^2)]\mathcal{Y}_{\xi\xi}(\vartheta_2,\varpi_3)=0,\\
&\mathcal{Y}(\vartheta_3)=\lambda(t)\exp(\vartheta_1-\vartheta_2),\\
&\mathcal{Y}_t(\vartheta_1)+[a_2+a_3(p^2+q^2)][\mathcal{Y}_{\xi\xi\xi}(\vartheta_1,\varpi_1)+3\lambda(t)U(t)\mathcal{Y}_\xi(\vartheta_1)]=0,\\
&\mathcal{Y}_t(\vartheta_2)+[a_2+a_3(p^2+q^2)][\mathcal{Y}_{\xi\xi\xi}(\vartheta_1,\varpi_1)+3\lambda(t)U(t)\mathcal{Y}_\xi(\vartheta_1)]=0,\\
&\mathcal{Y}(\vartheta_4)=U(t)\exp(\vartheta_2-\vartheta_1).
\end{aligned} \tag{5.12}$$

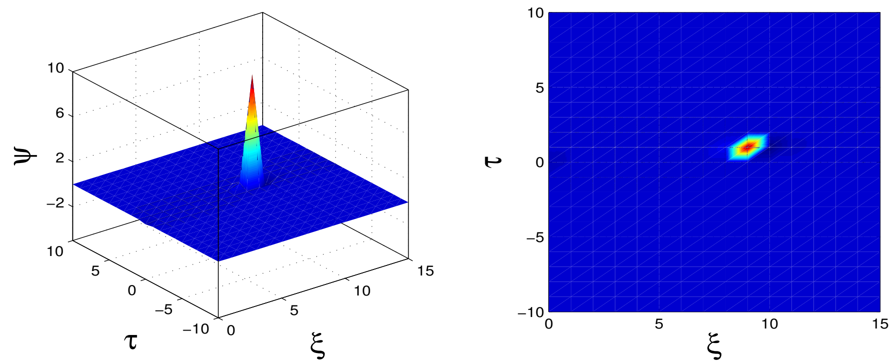
According to the Bäcklund transformation of the Bell polynomial form, the bilinear Bäcklund transformation can be written as

$$\begin{cases}
(D_\eta-pD_\xi)(\tilde{f}\cdot f)=0,(D_\eta-pD_\xi)(\tilde{g}\cdot g)=0,\\
(D_\zeta-pD_\xi)(\tilde{f}\cdot f)=0,(D_\zeta-pD_\xi)(\tilde{g}\cdot g)=0,\\
-3[a_2+a_3(p^2+q^2)]D_\xi^2(\tilde{f}\cdot f)=0,\\
-3[a_2+a_3(p^2+q^2)]D_\xi^2(\tilde{g}\cdot g)=0,\\
D_\xi(\tilde{g}\cdot f)=\lambda(t)(\tilde{f}\cdot g),D_\xi(\tilde{f}\cdot g)=U(t)(\tilde{g}\cdot f).\\
(D_t+[a_2+a_3(p^2+q^2)]D_\xi^3+3\lambda(t)U(t)D_\xi)(\tilde{f}\cdot f)=0,\\
(D_t+[a_2+a_3(p^2+q^2)]D_\xi^3+3\lambda(t)U(t)D_\xi)(\tilde{g}\cdot g)=0.
\end{cases} \tag{5.13}$$

6. Dust acoustic rogue waves

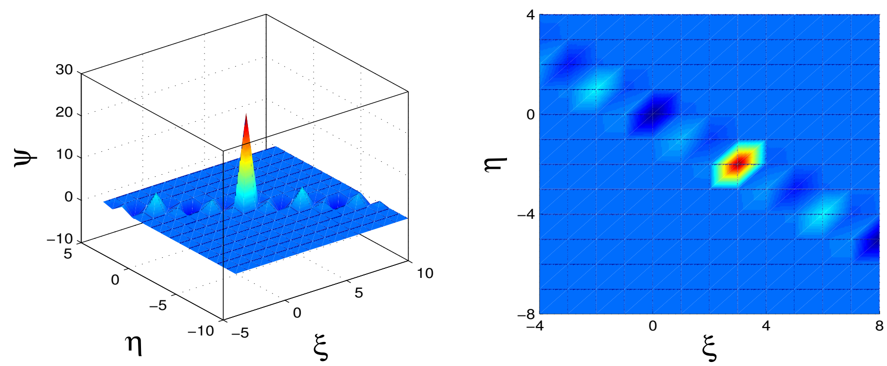
The existence and propagation characteristics of dust acoustic anomalies in fractional model of magnetized dusty plasma are studied. In particular, we analyzed the fractional effect, phase velocity and dust-cyclotron frequency on the propagation characteristics of dust acoustic rogue waves.

We discover the existence of dust acoustic rogue waves in Figure.1 (a), it is obvious that the peak of the wave is very sharp and the wave height is very high.



(a) (b)

Figure 1. The profile of Eq. (4.17) with $p = 1.6, k_1 = 1.99, q = 0.1, \lambda = 0.8, \Omega = 0.6, d_1 = 0.5, d_3 = 1, \omega = 1, \zeta = 0, \eta = 1$.



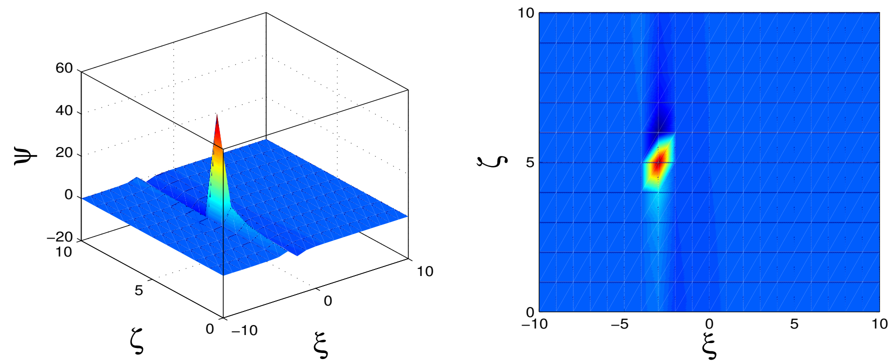
(a) (b)

Figure 2. The profile of Eq. (4.17) with $k_1 = 1.99, p = 1.6, d_1 = 0.5, d_3 = 1, \omega = 1, \tau = \zeta = 0$.

To show this whole progress more delicately, the vertical view is exhibited in Figure.1 (b), and we found the duration of wave is very short. (between $t = 0$ and $t = 2$).

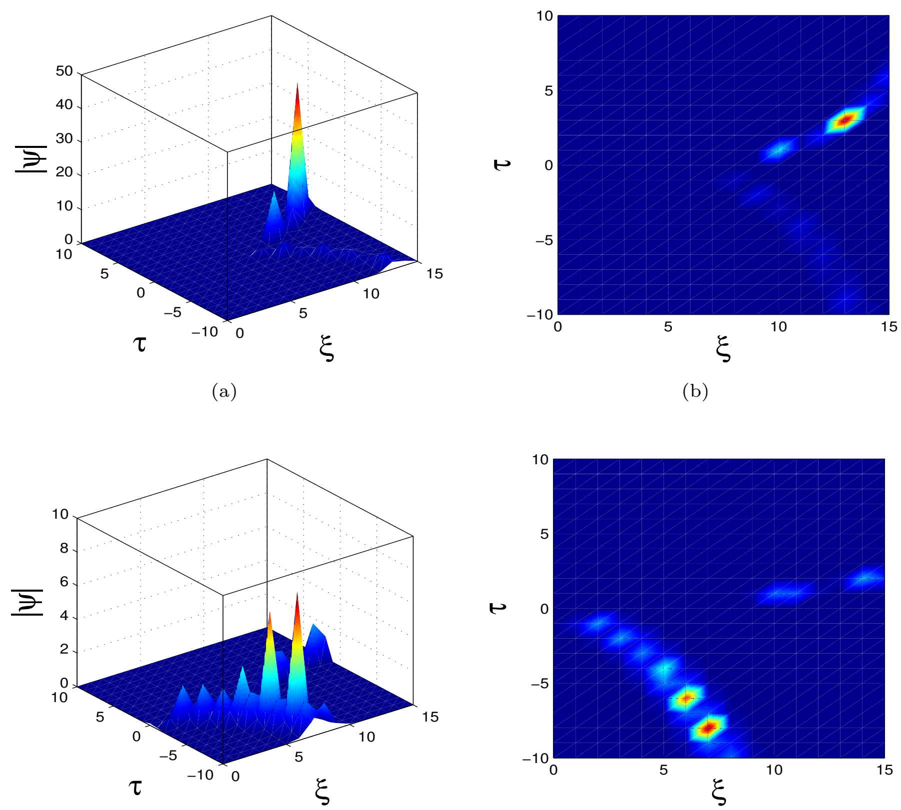
Dust acoustic rogue waves in space (x, y) at $t = 0$ is shown in Figure.2 (a), and the vertical view is exhibited in Figure.2 (b). Dust acoustic rogue waves in space (x, z) at $t = 0$ is shown in Figure.3 (a), and Figure.3 (b) is the vertical view. Dust acoustic rouge waves suddenly appear in space-time, and its amplitude is several times that of the surrounding waves. In this case, the energy is confined to a very small space and time range before reaching the formation of dust acoustic rogue waves. These characteristics more powerfully explain the existence of dust acoustic rogue waves.

Note that Figures 1, 2 and 3 are all obtained when the time fractional order $\omega = 1$, that is, we only analyzed the phenomenon of dust acoustic rogue waves in the integer order equation. To intuitively understand the propagation of acoustic rogue waves, the pictures in Figures 1, 2 and 3 show the three-dimensional pattern of rogue waves in the (ξ, τ, ψ) coordinate system. It can be seen from the three sets of figures that the amplitude of acoustic rogue waves decreases with the increase of



(a) (b)

Figure 3. The profile of Eq. (4.17) with $k_1 = 1.2, p = 1.65, q = 0.1, d_1 = 1.5, d_3 = 2.57, \omega = 1, \tau = \eta = 0$.



(a) (b) (c) (d)

Figure 4. The panels show the evolution of the dust acoustic rogue waves.The parameters are $k_1 = 1.99, p = 1.6, q = 0.1, \lambda = 0.8, \Omega = 0.6, d_1 = 0.5, d_3 = 1, \eta = 1$ and $\zeta = 0$.

propagation time. The reason for this phenomenon is that $a_1 < 0$ in the dissipation term $a_1\phi_1^2\frac{\partial\phi_1}{\partial\xi}$ means that the plasma system continuously releases energy to the outside world, so the amplitude of acoustic rogue waves decreases. The existence of dust acoustic rogue waves in the fractional model is further discussed, and the influence of different time fractional orders on dust acoustic rogue waves in the dusty plasma is discussed.

Figure.4 (a) and Figure.4 (c) are the figures of dust acoustic rogue waves when $\omega = 0.2$ and $\omega = 0.4$, respectively. Figure.4 (b) and Figure.4 (c) are vertical views. Compared with Figure.1 ($w = 1$), we find that ω can significantly change the amplitude, time point and spatial position of rogue waves. Further study on the effect of ω on dust acoustic rogue waves in Figure.5 (a)-(d), we continue to analyze the figures of dust acoustic rogue waves at $\omega = 0.6$ and $\omega = 0.8$.

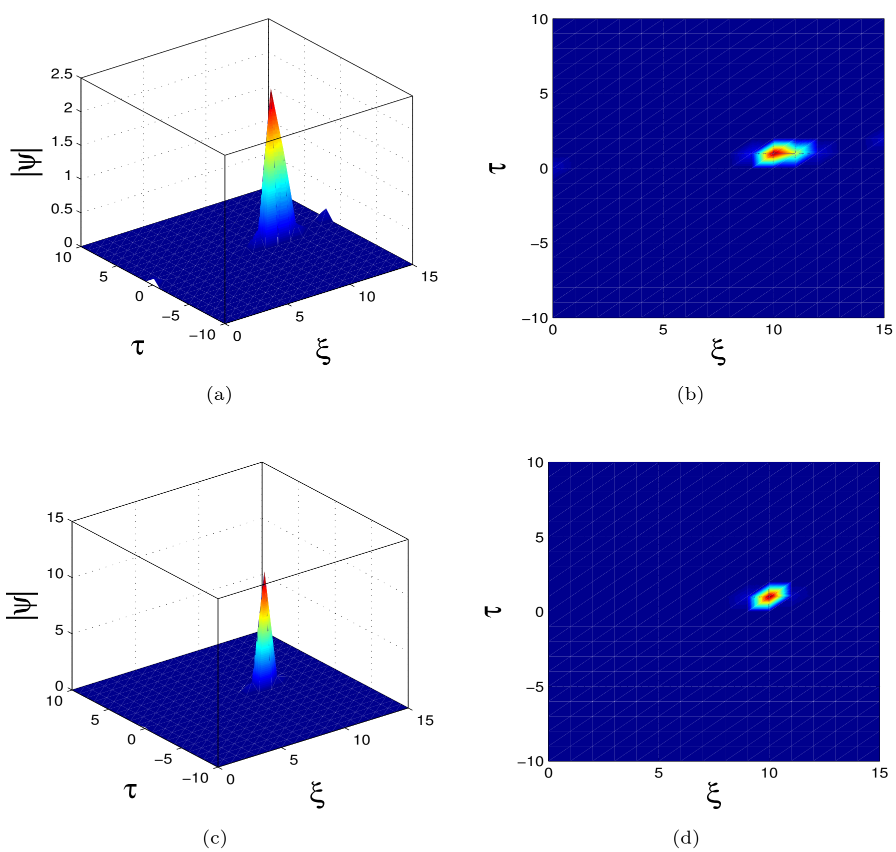
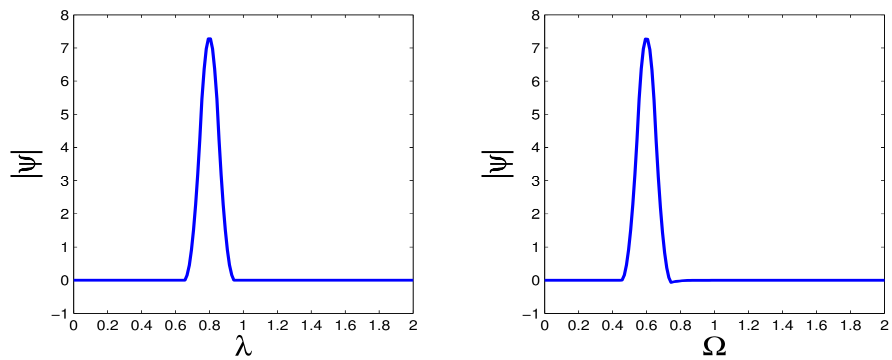


Figure 5. The panels show the evolution of the dust acoustic rogue waves. The parameters are $k_1 = 1.99, p = 1.6, q = 0.1, \lambda = 0.8, \Omega = 0.6, d_1 = 0.5, d_3 = 1, \eta = 1$ and $\zeta = 0$.

We observe that when $\omega = 0.6$ and $\omega = 0.8$., the amplitude, time point and spatial position of rogue waves are very close to those when $\omega = 1$. In particular, $\omega = 0.8$, the propagation characteristics of dust acoustic rogue waves are basically consistent with those shown in Figure. 1.

This result is consistent with previous inferences [25]. Therefore, we obtain for the first time the result that the propagation characteristics of dust acoustic rogue waves in the time fractional mZK equation will appear in the fractional model similar to the integer model when the fractional time order is close to 1, that is, when it is close to the integer order.

It is obvious from the Figure.6 that the phase velocity λ and the dust-cyclotron frequency Ω have an effect on the propagation of dust acoustic rogue waves. According to Figure.6 (a), in a certain range of values, the amplitude of dust acoustic rogue waves increases first and then decreases with the increase of λ value, and reaches the maximum value when $\lambda = 0.8$. Similarly, it can be seen from Figure.6 (b) that with the increase of Ω, the amplitude of dust acoustic rogue first increases and then decreases, and reaches the maximum when $\Omega = 0.6$.



(a) $\omega = 0.1$ (b) $\omega = 0.9$

Figure 6. The profile of Eq. (4.17) by choosing $k_1 = 1.9, p = 1.5, q = 0.5, d_1 = 1, d_3 = 1, \zeta = 0.\omega = 1, \xi = 9, \tau = \eta = 1,$

7. Conclusion

In conclusion, we obtain the (3+1)-dimensional integer order mZK equation, and further derive the (3+1)-dimensional TF-mZK equation with the semi-inverse method and fractional variational principle. Compared with the low dimensional model, the three-dimensional model can better describe the wave propagation in space. Compared with the integer-order model, the fractional-order model is more general, so the derived TF-mZK equation can better describe the wave propagation in dusty plasma.

Furthermore, with the help of fractional transformations and Bell polynomials, the Bäcklund transformation and exact solution of (3+1)-dimensional TF-mZK equation are obtained. The rouge waves in dust acoustic waves is found, and the influence of fractional order, phase velocity and dust-cyclotron frequency on the propagation characteristics of dust acoustic strange waves is analyzed. We found that the fractional order, phase velocity and dust-cyclotron frequency can all change the amplitude of dust acoustic strange waves.

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