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SUFFICIENT AND NECESSARY CONDITIONS FOR PERSISTENCE AND EXTINCTION OF A STOCHASTIC TWO-PREY ONE-PREDATOR SYSTEM

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Abstract This paper applies a new approach for stochastic Kolmogorov systems generalized by Hening and Nguyen to describe the dynamics of a stochastic two independent prey one predator system perturbed by white noise. Through calculating Lyapunov exponents, we thoroughly address the stability of the ergodic invariant probability measures. Sufficient and necessary conditions under which the species persist as well as conditions under which some species go extinct are established for this three dimensional models. One of the key points is that the critical cases for Lyapunov exponents being zero are considered. Finally, some numerical simulations illustrate the analytical results.

Keywords Two-prey one-predator system, invariant probability measure, Lyapunov exponent, persistence and extinction.

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1. Introduction

In ecosystem, it is impossible for any biological population to exist independently of other biological populations. The interaction between species is very important for the survival and development of the whole biological community. It not only affects the existence of each species, but also links each species into a complex life network, which determines the stability of the ecosystem. Predator-prey interaction is one of the basic relationships. The first mathematical model describing predator-prey interaction is Lotka-Volterra model, which is for one prey and one predator. From then on, based on Lotka-Volterra model, many scholars proposed different predator-prey models and studied their dynamical behaviors. For example, Lliber and Dong [13] studied a one-prey two-predator model and obtained the sufficient and necessary conditions for the principle competitive exclusion to hold and gave the global dynamics of three species in the first octant; Dubey and Upadhyay [1] investigated the persistence and extinction of another one-prey and two-predator system; Djomegni, Govinder and Goufo [2] proposed a two-prey one-predator model and studied the stability of steady states; Gard and Hallam [4], So [18], and Harrison

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[5] gave criteria for persistence and extinction and studied the global stability of equilibria for Lotka-Volterra food chain models.

As a matter of fact, the dynamics of species is inherently stochastic since population systems are always affected by the environmental noise. Hence, stochastic population models are more appropriate for describing the dynamical behaviors of species. One approach to derive the stochastic population systems is by adding noise to deterministic model. More and more attention has been paid to stochastic population models with parameter perturbations. For example, Mao et al. [16] studied a stochastic population system and showed that even a small amount of noise can suppress potential population explosions. Liu and Mandal [14] proposed a stochastic one-prey two-predator model and derived sufficient conditions for extinction of each species. They also showed that larger white noise may make deterministic system that coexist become extinct. Liu and Wang [15] considered the qualitative properties of a stochastic two competing prey and one predator system. Ji et al. [11] considered a stochastic one predator and two independent preys system, and established sufficient criteria for extinction and persistence in time average.

In a given population system, the exploitation of the stability of equilibria has been of great importance. However, most stochastic models have no traditional equilibrium state. Therefore, invariant distribution (stochastic equilibrium state) of stochastic biomathematical models has been received great attention. One traditional method to study invariant distribution is the Lyapunov function method [6]. For example, Zhang et al. [19] studied a stochastic Holling-II predator-prey system with hyperbolic mortality and obtained sufficient conditions for the existence of the ergodic invariant distribution; Zhao et al. [20] studied the invariant distribution of a stochastic competitive model in a polluted environment. In general, Lyapunov function method seems to have some difficulty in constructing suitable Lyapunov function, especially for high dimensional population systems. On the other hand, in most cases, Lyapunov function method only gets sufficient conditions for the existence of the ergodic invariant distribution. In the meanwhile, in a given ecosystem, an important and interesting problem is to determine which species go extinct and which are persistent. In literature [11, 14, 15], the authors gave sufficient conditions for the persistence and extinction of species by complex calculation. As we know, the existence of ergodic invariant distribution also implies the persistence of the species. While the Lyapunov function methods may become unsuitable for some stochastic population systems, a new approach based on studying the properties of the invariant measures of the process that are supported on the boundary of the domain developed by A. Hening and D. Nguyen [7–9] has been succeed in giving sharp sufficient conditions for both persistence and extinction of stochastic Kolmogorov systems. From then on, Zou et al. [21] studied the dynamics of a stochastic Holling II predator-prey system through analyzing the stability of ergodic invariant measures. Liu and Bai [12] obtained threshold of stochastic persistence and collapse of a stochastic mutualism model.

Borrowing the ideas of [7, 12, 21], we establish the following stochastic two-prey one-predator model

$$\begin{cases} dx_1(t) = \Big(r_1x_1(t)(1-\frac{x_1(t)}{K_1}) - \alpha x_1(t)x_3(t)\Big)\,dt + \sigma_1x_1(t)dB_1(t), \\ dx_2(t) = \Big(r_2x_2(t)(1-\frac{x_2(t)}{K_2}) - \beta x_2(t)x_3(t)\Big)\,dt + \sigma_2x_2(t)dB_2(t), \\ dx_3(t) = x_3(t)\left(-\sigma + C_1\alpha x_1(t) + C_2\beta x_2(t) - \gamma x_3(t)\right)dt + \sigma_3x_3(t)dB_3(t). \end{cases} \tag{1.1}$$

Here $x_1(t)$, $x_2(t)$ and $x_3(t)$ stand for population size at time t of the two preys and the predator, respectively. $r_i > 0$ $(i = 1, 2)$ are the intrinsic growth rate, and K_i $(i = 1, 2)$ are the carrying capacity of two preys, respectively. α and β are capture rates of the two preys. σ is the reduction rate of x_3 and C_1 (C_2) is the conversation rate of x_1 (x_2) into x_3, γ is intraspecific competition rate of species $x_3(t)$. All parameters in (1.1) remains positive. $B_i(t)$ are independent standard Brownian motions defined on the complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$ with filtration $\{\mathcal{F}_t\}_{t\geq 0}$ and σ_i^2 represent the intensities of white noise $\dot{B}_i(t)$, $i = 1, 2, 3$.

With this new method generalized by [7], we will thoroughly analyze the stability of ergodic invariant measures on invariant sets and establish sufficient and necessary criteria under which the species go to extinction as well as criteria under which species coexist, which is the distinctive contribution of this paper. The outline of the paper is summarized as follows: In Section 2, we will show the existence and uniqueness of the global positive solution of system (1.1), and give some preliminaries which are needed in this paper. Section 3 completely address the stability of ergodic invariant measures and obtain the sharp conditions for the persistence and extinction of each species. In Section 4, some numerical simulations are introduced to demonstrate the theoretical results. Finally, some conclusions are presented to end this paper.

2. The existence and uniqueness of global positive solution to (1.1) and some preliminaries

We rewrite (1.1) as

$$dx_i(t) = x_i(t) f_i(X(t))dt + \sigma_i x_i(t) dB_i(t), \quad i = 1, 2, 3,$$

where $X(t) = (x_1(t), x_2(t), x_3(t))$ is a stochastic process that takes values in $\mathbb{R}^3_+$, $f_1(X) = r_1 - \frac{r_1}{K_1}x_1 - \alpha x_3$, $f_2(X) = r_2 - \frac{r_2}{K_2}x_2 - \beta x_3$ and $f_3(X) = -\sigma + C_1\alpha x_1 + C_2\beta x_2 - \gamma x_3$. Throughout the paper, we set

$$\begin{aligned}
&\mathbb{R}^3_+ = \{(x_1, x_2, x_3) : x_i \geq 0, i = 1, 2, 3\}, \quad \mathbb{R}^{3,0}_+ = \{(x_1, x_2, x_3) : x_i > 0, i = 1, 2, 3\},\\
&\partial\mathbb{R}^3_+ = \mathbb{R}^3_+ \setminus \mathbb{R}^{3,0}_+, \quad \mathbb{R}^0_{i+} = \{(x_1, x_2, x_3) : x_i > 0, x_j = 0, j \neq i, i, j = 1, 2, 3\},\\
&\mathbb{R}^0_{12+} = \{(x_1, x_2, x_3) : x_1 > 0, x_2 > 0, x_3 = 0\},\\
&\mathbb{R}^0_{13+} = \{(x_1, x_2, x_3) : x_1 > 0, x_3 > 0, x_2 = 0\},\\
&\mathbb{R}^0_{23+} = \{(x_1, x_2, x_3) : x_2 > 0, x_3 > 0, x_1 = 0\}.\\
&A = \begin{pmatrix} \frac{r_1}{K_1} & 0 & \alpha \\ 0 & \frac{r_2}{K_2} & \beta \\ -C_1\alpha & -C_2\beta & \gamma \end{pmatrix}, \quad A^{(1)} = \begin{pmatrix} r_1 - \frac{\sigma_1^2}{2} & 0 & \alpha \\ r_2 - \frac{\sigma_2^2}{2} & \frac{r_2}{K_2} & \beta \\ -(\sigma + \frac{\sigma_3^2}{2}) & -C_2\beta & \gamma \end{pmatrix},\\
&A^{(2)} = \begin{pmatrix} \frac{r_1}{K_1} & r_1 - \frac{\sigma_1^2}{2} & \alpha \\ 0 & r_2 - \frac{\sigma_2^2}{2} & \beta \\ -C_1\alpha & -(\sigma + \frac{\sigma_3^2}{2}) & \gamma \end{pmatrix}, \quad A^{(3)} = \begin{pmatrix} \frac{r_1}{K_1} & 0 & r_1 - \frac{\sigma_1^2}{2} \\ 0 & \frac{r_2}{K_2} & r_2 - \frac{\sigma_2^2}{2} \\ -C_1\alpha & -C_2\beta & -(\sigma + \frac{\sigma_3^2}{2}) \end{pmatrix}.
\end{aligned}$$

Obviously, $\det(A) > 0$. Let A_{ij} be the algebraic cofactor of the element of row i and column j in determinant $det(A)$.

The random normalized occupation measures are defined as

$$\tilde{\Pi}_t(\cdot) := \frac{1}{t}\int_0^t \mathbf{I}_{\{X(s)\in\cdot\}} ds, \quad t > 0.$$

For a probability measure μ on $\partial\mathbb{R}^3_+$, we define the ith Lyapunov exponent of μ as

$$\lambda_i(\mu) = \int_{\partial\mathbb{R}^3_+} (f_i(X) - \frac{\sigma_i^2}{2})\mu(dX).$$

Before investigating the coexistence and extinction of system (1.1), we show that the solution is positive and globally exists.

Theorem 2.1. *For any initial value $(x_1(0), x_2(0), x_3(0)) \in \mathbb{R}^{3,0}_+$, there exists a unique solution $X(t) = (x_1(t), x_2(t), x_3(t)) \in \mathbb{R}^{3,0}_+$ of system (1.1) on $t \geq 0$ with probability 1.*

Proof. The method of this proof is standard, which is similar to Theorem 1 in [22]. For simplicity, we only give the Lyapunov function

$$V(x_1, x_2, x_3) = C_1(x_1 - 1 - \ln x_1) + C_2(x_2 - 1 - \ln x_2) + (x_3 - 1 - \ln x_3).$$

Applying Itô's formula, we have

$$dV = \mathcal{L}V dt + C_1\sigma_1(x_1 - 1)dB_1(t) + C_2\sigma_2(x_2 - 1)dB_2(t) + \sigma_3(x_3 - 1)dB_1(t),$$

where

$$\begin{aligned}\mathcal{L}V =& C_1(x_1 - 1)\left(r_1 - \frac{r_1}{K_1}x_1 - \alpha x_3\right) + \frac{C_1}{2}\sigma_1^2 + C_2(x_2 - 1)\left(r_2 - \frac{r_2}{K_2}x_2 - \beta x_3\right) \\ &+ \frac{C_2}{2}\sigma_2^2 + (x_3 - 1)(-\sigma + C_1\alpha x_1 + C_2\beta x_2 - \gamma x_3) + \frac{1}{2}\sigma_3^2 \\ \leq& -\frac{C_1 r_1}{K_1}x_1^2 + (C_1 r_1 + \frac{C_1 r_1}{K_1} - C_1\alpha)x_1 - \frac{C_2 r_2}{K_2}x_2^2 + (C_2 r_2 + \frac{C_2 r_2}{K_2} - C_2\beta)x_2 \\ &- \gamma x_3^2 + (C_1\alpha + C_2\beta + \gamma - \sigma)x_3 + \frac{C_1}{2}\sigma_1^2 + \frac{C_2}{2}\sigma_2^2 + \frac{1}{2}\sigma_3^2 \\ \leq& K,\end{aligned}$$

where K is a positive constant. □

In this paper, we will make use of the general results of Hening and Nguyen [7] to classify which species go extinct and which species are persistent by calculating the Lyapunov exponent. We firstly verify that Assumption 1.1 and Assumption 1.4 in [7] hold. It is clear that conditions (1) and (2) in Assumption 1.1 are satisfied for system (1.1), and we also have proved that system (1.1) has a unique global positive solution in Theorem 1. For $\mathbf{c} = (C_1, C_2, 1)$, there are $\tilde{b}_1 > 0$ and $\tilde{b}_2 > 0$ such that

$$\begin{aligned}&\frac{C_1 x_1 f_1(X) + C_2 x_2 f_2(X) + x_3 f_3(X)}{1 + C_1 x_1 + C_2 x_2 + x_3} - \frac{C_1^2\sigma_1^2 x_1^2 + C_2^2\sigma_2^2 x_2^2 + \sigma_3^2 x_3^2}{2(1 + C_1 x_1 + C_2 x_2 + x_3)^2} \\ &\leq -\tilde{b}_1(1 + C_1 x_1 + C_2 x_2 + x_3) - \tilde{b}_2,\end{aligned}$$

for sufficiently large $\|X\| = \sum_{i=1}^{3} |x_i|$. And we can also find a $\tilde{b}_3 > 0$

$$\begin{aligned}1 + \sum_{i=1}^{3}(|f_i(X)| + 1) =& 4 + |r_1 - \frac{r_1}{K_1}x_1 - \alpha x_3| + |r_2 - \frac{r_2}{K_2}x_2 - \beta x_3| \\ &+ | - \sigma + C_1\alpha x_1 + C_2\beta x_2 - \gamma x_3| \\ \le& \tilde{b}_3(1 + C_1x_1 + C_2x_2 + x_3).\end{aligned}$$

Therefore, we can deduce that there is a sufficiently small $\gamma_b > 0$ such that

$$\begin{aligned}&\frac{C_1x_1f_1(X) + C_2x_2f_2(X) + x_3f_3(X)}{1 + C_1x_1 + C_2x_2 + x_3} - \frac{C_1^2\sigma_1^2x_1^2 + C_2^2\sigma_2^2x_2^2 + \sigma_3^2x_3^2}{2(1 + C_1x_1 + C_2x_2 + x_3)^2} \\ &+ \gamma_b(4 + \sum_{i=1}^{3}|f_i(X)|) < 0\end{aligned}$$

for sufficiently large $\|X\|$. That is to say, Assumption 1.1 is satisfied.

On the other hand, it is easy to show that there is a sufficiently small $\delta > 0$ such that

$$\lim_{\|X\|\to\infty} \frac{3(|x_1| + |x_2| + |x_3|)^\delta}{4+|r_1-\frac{r_1}{K_1}x_1-\alpha x_3|+|r_2-\frac{r_2}{K_2}x_2-\beta x_3|+|-\sigma+C_1\alpha x_1+C_2\beta x_2-\gamma x_3|} = 0.$$

Therefore, Assumption 1.4 in [7] is also satisfied.

3. Coexistence and extinction

This section mainly shows the dynamics of system (1.1), give sharp conditions under which the populations converge to their ergodic stationary distribution and conditions under which some species go extinct. For simplicity, denote $\langle f(t)\rangle = \frac{1}{t}\int_0^t f(s)ds$. And

δ^* is the Dirac measure concentrated at $\mathbf{0}$.

$\mu_i(\cdot)$ is the probability measure on $\mathbb{R}^0_{i+}$, $i = 1, 2, 3$.

$\mu_{ij}(\cdot)$ is the probability measure on $\mathbb{R}^0_{ij+}$, $i, j = 1, 2, 3$, $i \neq j$.

We now give the main results.

Theorem 3.1. *Let $X(t) = (x_1(t), x_2(t), x_3(t))$ be the solution of system (1.1) with the initial value $X(0) = (x_1(0), x_2(0), x_3(0)) \in \mathbb{R}^{3,0}_+$. Then*

***I**. Assume $\lambda_1(\delta^*) = r_1 - \frac{\sigma_1^2}{2} < 0$ and $\lambda_2(\delta^*) = r_2 - \frac{\sigma_2^2}{2} < 0$. Then all the species go to extinction, that is to say,*

$$\lim_{t\to\infty} x_i(t) = 0, \quad a.s., \quad i = 1, 2, 3.$$

***II**. Assume $\lambda_1(\delta^*) = r_1 - \frac{\sigma_1^2}{2} < 0$ and $\lambda_2(\delta^*) = r_2 - \frac{\sigma_2^2}{2} > 0$. Then there are following cases:*

***Case 1**. If $\lambda_3(\mu_2) \le 0$, that is*

$$r_1 - \frac{\sigma_1^2}{2} < 0, \quad r_2 - \frac{\sigma_2^2}{2} > 0, \quad C_2\beta K_2(r_2 - \frac{\sigma_2^2}{2}) \le r_2(\sigma + \frac{\sigma_3^2}{2}), \tag{3.1}$$

then species $x_1(t)$ and $x_3(t)$ go extinct exponentially, $x_2(t)$ will persist in a long time, and

$$\lim_{t\to\infty} x_1(t) = 0, \quad \lim_{t\to\infty} \langle x_2(t)\rangle = \frac{K_2(r_2 - \sigma_2^2/2)}{r_2}, \quad \lim_{t\to\infty} x_3(t) = 0, \quad a.s.$$

Case 2. *If $\lambda_3(\mu_2) > 0$, that is*

$$r_1 - \frac{\sigma_1^2}{2} < 0, \quad r_2 - \frac{\sigma_2^2}{2} > 0, \quad C_2\beta K_2(r_2 - \frac{\sigma_2^2}{2}) > r_2(\sigma + \frac{\sigma_3^2}{2}), \tag{3.2}$$

then species $x_1(t)$ goes extinct exponentially, $x_2(t)$ and $x_3(t)$ will persist in a long time, and

$$\lim_{t\to\infty} x_1(t) = 0, \quad \lim_{t\to\infty} \langle x_2(t)\rangle = \frac{A_{11}^{(2)}}{A_{11}}, \lim_{t\to\infty} \langle x_3(t)\rangle = \frac{A_{11}^{(3)}}{A_{11}} \quad a.s.$$

III. *Assume $\lambda_1(\delta^*) = r_1 - \frac{\sigma_1^2}{2} > 0$ and $\lambda_2(\delta^*) = r_2 - \frac{\sigma_2^2}{2} < 0$. Then there are following cases:*

Case 3. *If $\lambda_3(\mu_1) \le 0$, that is*

$$r_1 - \frac{\sigma_1^2}{2} > 0, \quad r_2 - \frac{\sigma_2^2}{2} < 0, \quad C_1\alpha K_1(r_1 - \frac{\sigma_1^2}{2}) \le r_1(\sigma + \frac{\sigma_3^2}{2}), \tag{3.3}$$

then species $x_2(t)$ and $x_3(t)$ go extinct exponentially, $x_1(t)$ will persist in a long time, and

$$\lim_{t\to\infty} \langle x_1(t)\rangle = \frac{K_1(r_1 - \sigma_1^2/2)}{r_1}, \quad \lim_{t\to\infty} x_2(t) = 0, \quad \lim_{t\to\infty} x_3(t) = 0, \quad a.s.$$

Case 4. *If $\lambda_3(\mu_1) > 0$, that is*

$$r_1 - \frac{\sigma_1^2}{2} > 0, \quad r_2 - \frac{\sigma_2^2}{2} < 0, \quad C_1\alpha K_1(r_1 - \frac{\sigma_1^2}{2}) > r_1(\sigma + \frac{\sigma_3^2}{2}), \tag{3.4}$$

then species $x_2(t)$ goes extinct exponentially, $x_1(t)$ and $x_3(t)$ will persist in a long time, and

$$\lim_{t\to\infty} \langle x_1(t)\rangle = \frac{A_{22}^{(1)}}{A_{22}}, \quad \lim_{t\to\infty} x_2(t) = 0, \lim_{t\to\infty} \langle x_3(t)\rangle = \frac{A_{22}^{(3)}}{A_{22}} \quad a.s.$$

IV. *Assume $\lambda_1(\delta^*) = r_1 - \frac{\sigma_1^2}{2} > 0$ and $\lambda_2(\delta^*) = r_2 - \frac{\sigma_2^2}{2} > 0$. Then there are following cases:*

Case 5. *If $\lambda_3(\mu_1) \le 0$, $\lambda_3(\mu_2) \le 0$ and $\lambda_3(\mu_{12}) \le 0$, i.e.*

$$\begin{aligned}
&r_1 - \frac{\sigma_1^2}{2} > 0, \quad r_2 - \frac{\sigma_2^2}{2} > 0, \\
&C_1\alpha K_1(r_1 - \frac{\sigma_1^2}{2}) \le r_1(\sigma + \frac{\sigma_3^2}{2}), \quad C_2\beta K_2(r_2 - \frac{\sigma_2^2}{2}) \le r_2(\sigma + \frac{\sigma_3^2}{2}), \\
&-\sigma - \frac{\sigma_3^2}{2} + C_1\alpha \frac{K_1(r_1 - \frac{\sigma_1^2}{2})}{r_1} + C_2\beta \frac{K_2(r_2 - \frac{\sigma_2^2}{2})}{r_2} \le 0,
\end{aligned} \tag{3.5}$$

then species x_3 will be extinct, $x_1(t)$ and $x_2(t)$ will be persistent a.s., and

$$\lim_{t\to\infty}\langle x_1(t)\rangle=\frac{K_1(r_1-\frac{\sigma_1^2}{2})}{r_1},\quad \lim_{t\to\infty}\langle x_2(t)\rangle=\frac{K_2(r_2-\frac{\sigma_2^2}{2})}{r_2},\quad \lim_{t\to\infty}x_3(t)=0, a.s.$$

Case 6. *If $\lambda_3(\mu_1)>0$, $\lambda_3(\mu_2)\le 0$ and $\lambda_2(\mu_{13})\le 0$, i.e.,*

$$r_1-\frac{\sigma_1^2}{2}>0,\quad r_2-\frac{\sigma_2^2}{2}>0,\quad C_1\alpha K_1(r_1-\frac{\sigma_1^2}{2})>r_1(\sigma+\frac{\sigma_3^2}{2}), \tag{3.6}$$

$$C_2\beta K_2(r_2-\frac{\sigma_2^2}{2})\le r_2(\sigma+\frac{\sigma_3^2}{2}),\ r_2-\frac{\sigma_2^2}{2}-\beta\frac{C_1\alpha K_1(r_1-\sigma_1^2/2)-r_1(\sigma+\frac{\sigma_3^2}{2})}{\gamma r_1+C_1K_1\alpha^2}\le 0, \tag{3.7}$$

then species x_2 goes extinct, $x_1(t)$ and $x_3(t)$ will be persistent, and

$$\lim_{t\to\infty}\langle x_1(t)\rangle=\frac{A_{22}^{(1)}}{A_{22}},\quad \lim_{t\to\infty}x_2(t)=0, \lim_{t\to\infty}\langle x_3(t)\rangle=\frac{A_{22}^{(3)}}{A_{22}}\quad a.s.$$

Case 7. *If $\lambda_3(\mu_1)\le 0$, $\lambda_3(\mu_2)>0$ and $\lambda_1(\mu_{23})\le 0$, i.e.,*

$$\begin{aligned}&r_1-\frac{\sigma_1^2}{2}>0,\quad r_2-\frac{\sigma_2^2}{2}>0,\quad C_1\alpha K_1(r_1-\frac{\sigma_1^2}{2})\le r_1(\sigma+\frac{\sigma_3^2}{2}),\\ &C_2\beta K_2(r_2-\frac{\sigma_2^2}{2})>r_2(\sigma+\frac{\sigma_3^2}{2}),\ r_1-\frac{\sigma_1^2}{2}-\alpha\frac{C_2\beta K_2(r_2-\sigma_2^2/2)-r_2(\sigma+\frac{\sigma_3^2}{2})}{\gamma r_2+C_2K_2\beta^2}\le 0,\end{aligned} \tag{3.8}$$

then species x_1 goes extinct, $x_2(t)$ and $x_3(t)$ will be persistent, and

$$\lim_{t\to\infty}x_1(t)=0,\quad \lim_{t\to\infty}\langle x_2(t)\rangle=\frac{A_{11}^{(2)}}{A_{11}}, \lim_{t\to\infty}\langle x_3(t)\rangle=\frac{A_{11}^{(3)}}{A_{11}}\quad a.s.$$

Case 8. *If $\lambda_3(\mu_1)>0$, $\lambda_3(\mu_2)>0$, $\lambda_2(\mu_{13})\le 0$ $\lambda_1(\mu_{23})>0$, then*

$$\lim_{t\to\infty}\langle x_1(t)\rangle=\frac{A_{22}^{(1)}}{A_{22}},\quad \lim_{t\to\infty}x_2(t)=0, \lim_{t\to\infty}\langle x_3(t)\rangle=\frac{A_{22}^{(3)}}{A_{22}}\quad a.s.$$

Case 9. *If $\lambda_3(\mu_1)>0$, $\lambda_3(\mu_2)>0$, $\lambda_2(\mu_{13})>0$ $\lambda_1(\mu_{23})\le 0$, then*

$$\lim_{t\to\infty}x_1(t)=0,\quad \lim_{t\to\infty}\langle x_2(t)\rangle=\frac{A_{11}^{(2)}}{A_{11}}, \lim_{t\to\infty}\langle x_3(t)\rangle=\frac{A_{11}^{(3)}}{A_{11}}\quad a.s.$$

Case 10. *If one of the following is satisfied:*

- $\lambda_1(\delta^*)>0$, $\lambda_2(\delta^*)>0$, $\lambda_3(\mu_1)\le 0$, $\lambda_3(\mu_2)\le 0$ *and* $\lambda_3(\mu_{12})>0$.
- $\lambda_1(\delta^*)>0$, $\lambda_2(\delta^*)>0$, $\lambda_3(\mu_1)>0$, $\lambda_3(\mu_2)\le 0$ *and* $\lambda_2(\mu_{13})>0$.
- $\lambda_1(\delta^*)>0$, $\lambda_2(\delta^*)>0$, $\lambda_3(\mu_1)\le 0$, $\lambda_3(\mu_2)>0$ *and* $\lambda_1(\mu_{23})>0$.
- $\lambda_1(\delta^*)>0$, $\lambda_2(\delta^*)>0$, $\lambda_3(\mu_1)>0$, $\lambda_3(\mu_2)>0$, $\lambda_2(\mu_{13})>0$ *and* $\lambda_1(\mu_{23})>0$.

then

$$\lim_{t\to\infty}\langle x_1(t)\rangle=\frac{det(A^{(1)})}{det(A)},\quad \lim_{t\to\infty}\langle x_2(t)\rangle=\frac{det(A^{(2)})}{det(A)},\lim_{t\to\infty}\langle x_3(t)\rangle=\frac{det(A^{(3)})}{det(A)}\quad a.s.$$

Case 11. *If* $\lambda_3(\mu_1)>0$*,* $\lambda_3(\mu_2)>0$*,* $\lambda_2(\mu_{13})<0$ *and* $\lambda_1(\mu_{23})<0$*, then* $p_i^X>0$*,* $i=1,2$ *and* $p_1^X+p_2^X=1$*, where*

$$p_i^X=\mathbb{P}\left\{\mu(\omega)=\{\mu_{i3}\}\quad and \quad \lim_{t\to\infty}\frac{\ln X_i(t)}{t}=\lambda_j(\mu_{i3}),\ j\in\{1,2\}\backslash\{i\}\right\}.$$

Proof. Now we use Theorems 1.1-1.3 in literature [7] and [17, 21] to prove this theorem.

I. If $\lambda_1(\delta^*)=r_1-\frac{\sigma_1^2}{2}<0$ and $\lambda_2(\delta^*)=r_2-\frac{\sigma_2^2}{2}<0$, then δ^* is the unique invariant probability measure on $\mathbb{R}_+^3$. Hence all the species go to extinction a.s.

II. If $\lambda_1(\delta^*)=r_1-\frac{\sigma_1^2}{2}<0$ and $\lambda_2(\delta^*)=r_2-\frac{\sigma_2^2}{2}>0$, then there is a unique ergodic invariant probability measure μ_2 on $\mathbb{R}_{2+}^0$. By Lemma 2.1 in [7], it follows that

$$\lambda_2(\mu_2)=\int_{\mathbb{R}_{2+}^0}\left(r_2-\frac{\sigma_2^2}{2}-\frac{r_2}{K_2}x_2\right)\mu_2(dX)=0,$$

which leads to

$$\int_{\mathbb{R}_{2+}^0}x_2\mu_2(dX)=\frac{K_2(r_2-\sigma_2^2/2)}{r_2}.$$

Thus $\lambda_1(\mu_2)=r_1-\frac{\sigma_1^2}{2}<0$, by Theorem 1.2 in [7], there is no invariant probability measure on $\mathbb{R}_{12+}^0$ which means that $x_1(t)$ converges to 0 exponentially.

$$\lambda_3(\mu_2)=\int_{\mathbb{R}_{2+}^0}\left(-\sigma-\frac{\sigma_3^2}{2}+C_2\beta x_2\right)\mu_2(dX)=C_2\beta\frac{K_2(r_2-\sigma_2^2/2)}{r_2}-\left(\sigma+\frac{\sigma_3^2}{2}\right).$$

If $\lambda_3(\mu_2)<0$, there is no invariant probability measure on $\mathbb{R}_{23+}^0$. Using Theorems 1.1 and 1.3 in [7], we obtain that $x_3(t)$ also converges to 0 exponentially and the randomized occupation measure $\tilde{\Pi}_t(\cdot)$ converges weakly to μ_2 a.s.. By the ergodicity of μ_2 it follows that

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t x_2(s)ds=\int_{\mathbb{R}_{2+}^0}x_2\mu_2(dX)=\frac{K_2(r_2-\sigma_2^2/2)}{r_2}\quad a.s..$$

If $\lambda_3(\mu_2)=0$, using the similar method of paper [17]. Let $\hat{x}_i(t)$ be the solution to

$$d\hat{x}_i(t)=r_i\hat{x}_i(t)\left(1-\frac{\hat{x}_i(t)}{t}\right)dt+\sigma\hat{x}_i(t)dB_i(t),\quad \hat{x}_i(0)=x_i(0),\quad i=1,2. \tag{3.9}$$

By comparison theorem, $x_i(t)\le\hat{x}_i(t)$ a.s. System (3.9) has been well studied in literature [3], in which if $r_i-\frac{\sigma_i^2}{2}>0$, then

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t\hat{x}_i(s)ds=\frac{K_i(r_i-\frac{\sigma_i^2}{2})}{r_i},\quad a.s.,\ i=1,2. \tag{3.10}$$

Suppose there is an invariant probability measure μ_{23} on $\mathbb{R}^0_{23+}$. From the ergodicity it follows that

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t x_3(s)ds=\int_{\mathbb{R}^0_{23+}} x_3\mu_{23}(dX)=:\bar{x}_3>0.$$

From Itô's formula it follows that

$$\begin{aligned}\frac{\ln x_3(t)-\ln x_3(0)}{t}&=-\left(\sigma+\frac{\sigma_3^2}{2}\right)+\frac{\sigma_3 B_3(t)}{t}+\frac{C_1\alpha}{t}\int_0^t x_1(s)ds\\&\quad+\frac{C_2\beta}{t}\int_0^t x_2(s)ds-\frac{\gamma}{t}\int_0^t x_3(s)ds\\&=-\left(\sigma+\frac{\sigma_3^2}{2}\right)+\frac{\sigma_3 B_3(t)}{t}+\frac{C_1\alpha}{t}\int_0^t x_1(s)ds\\&\quad+\frac{C_2\beta}{t}\int_0^t \hat{x}_2(s)ds-\frac{C_2\beta}{t}\int_0^t(\hat{x}_2(s)-x_2(s))ds-\frac{\gamma}{t}\int_0^t x_3(s)ds.\end{aligned}$$

Based on results of system (3.9) and the property of Brownian motion, combining $\lim_{t\to\infty}x_1(t)=0$, we deduce that

$$\begin{aligned}\limsup_{t\to\infty}\frac{\ln x_3(t)}{t}&\le-\left(\sigma+\frac{\sigma_3^2}{2}\right)+C_2\beta\frac{K_2(r_2-\sigma_2^2/2)}{r_2}-\lim_{t\to\infty}\frac{\gamma}{t}\int_0^t x_3(s)ds\\&=\lambda_3(\mu_2)-\bar{x}_3=-\bar{x}_3<0,\quad a.s.\end{aligned}$$

This implies $\lim_{t\to\infty}x_3(t)=0$ a.s.. As a result, there is no invariant probability measure on $\mathbb{R}^0_{23+}$. Therefore, there exist only two invariant probability measure δ^* and μ_2 on $\mathbb{R}^3_+$. Hence case 1 is derived.

Case 2. If $\lambda_3(\mu_2)>0$, by Theorem 1.1 in [7], there is an ergodic invariant probability measure μ_{23} on $\mathbb{R}^0_{23+}$. In light of Lemma 2.1 in [7], we have

$$\begin{aligned}\lambda_2(\mu_{23})&=r_2-\frac{\sigma_2^2}{2}-\frac{r_2}{K_2}\int_{\mathbb{R}^0_{23+}}x_2\mu_{23}(dX)-\beta\int_{\mathbb{R}^0_{23+}}x_3\mu_{23}(dX)=0,\\\lambda_3(\mu_{23})&=-\left(\sigma+\frac{\sigma_3^2}{2}\right)+C_2\beta\int_{\mathbb{R}^0_{23+}}x_2\mu_{23}(dX)-\gamma\int_{\mathbb{R}^0_{23+}}x_3\mu_{23}(dX)=0.\end{aligned}$$

By calculation, we obtain

$$\begin{aligned}&\lim_{t\to\infty}\frac{1}{t}\int_0^t x_2(s)ds=\int_{\mathbb{R}^0_{23+}}x_2\mu_{23}(dX)=\frac{\gamma K_2(r_2-\frac{\sigma_2^2}{2})+K_2\beta(\sigma+\frac{\sigma_3^2}{2})}{\gamma r_2+C_2K_2\beta^2}=\frac{A_{11}^{(2)}}{A_{11}},\\&\lim_{t\to\infty}\frac{1}{t}\int_0^t x_3(s)ds=\int_{\mathbb{R}^0_{23+}}x_3\mu_{23}(dX)=\frac{C_2\beta K_2(r_2-\frac{\sigma_2^2}{2})-r_2(\sigma+\frac{\sigma_3^2}{2})}{\gamma r_2+C_2K_2\beta^2}=\frac{A_{11}^{(3)}}{A_{11}}\ a.s.\end{aligned}$$

In this case,

$$\lambda_1(\mu_{23})=r_1-\frac{\sigma_1^2}{2}-\alpha\frac{C_2\beta K_2(r_2-\frac{\sigma_2^2}{2})-r_2(\sigma+\frac{\sigma_3^2}{2})}{\gamma r_2+C_2K_2\beta^2}<0,$$

which implies that there is no invariant probability measure on $\mathbb{R}_+^{3,0}$. Therefore, $\tilde{\Pi}_t(\cdot)$ converges weakly to μ_{23} and

$$\lim_{t\to\infty} x_1(t) = 0, \quad \lim_{t\to\infty}\langle x_2(t)\rangle = \frac{A_{11}^{(2)}}{A_{11}}, \quad \lim_{t\to\infty}\langle x_3(t)\rangle = \frac{A_{11}^{(3)}}{A_{11}} \quad a.s..$$

This completes the proof of Case 2.

III. The proof of Case 3 and Case 4 is similar to those in Case 1 and Case 2, hence we omit it.

IV. If $\lambda_1(\delta^*) = r_1 - \frac{\sigma_1^2}{2} > 0$ and $\lambda_2(\delta^*) = r_2 - \frac{\sigma_2^2}{2} > 0$, there is a unique ergodic invariant probability measure μ_i on $\mathbb{R}_{i+}^0$, $i = 1, 2$. Hence $\lambda_1(\mu_2) = r_1 - \frac{\sigma_1^2}{2} > 0$ and $\lambda_2(\mu_1) = r_2 - \frac{\sigma_2^2}{2} > 0$, and there is a unique ergodic invariant probability measure μ_{12} on $\mathbb{R}_{12+}^0$. From $\lambda_1(\mu_{12}) = 0$ and $\lambda_2(\mu_{12}) = 0$, we deduce that

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t x_1(s)ds = \int_{\mathbb{R}_{12+}^0} x_1\mu_{12}(dX) = \frac{K_1(r_1 - \frac{\sigma_1^2}{2})}{r_1},$$

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t x_2(s)ds = \int_{\mathbb{R}_{12+}^0} x_2\mu_{12}(dX) = \frac{K_2(r_2 - \frac{\sigma_2^2}{2})}{r_2} \quad a.s.$$

Thus

$$\lambda_3(\mu_1) = -\sigma - \frac{\sigma_3^2}{2} + C_1\alpha\int_{\mathbb{R}_{12+}^0} x_1\mu_{12}(dX) = -\sigma - \frac{\sigma_3^2}{2} + C_1\alpha\frac{K_1(r_1 - \frac{\sigma_1^2}{2})}{r_1},$$

$$\lambda_3(\mu_2) = -\sigma - \frac{\sigma_3^2}{2} + C_2\beta\int_{\mathbb{R}_{12+}^0} x_2\mu_{12}(dX) = -\sigma - \frac{\sigma_3^2}{2} + C_2\beta\frac{K_2(r_2 - \frac{\sigma_2^2}{2})}{r_2},$$

$$\lambda_3(\mu_{12}) = -\sigma - \frac{\sigma_3^2}{2} + C_1\alpha\frac{K_1(r_1 - \frac{\sigma_1^2}{2})}{r_1} + C_2\beta\frac{K_2(r_2 - \frac{\sigma_2^2}{2})}{r_2}.$$

Case 5. If $\lambda_3(\mu_1) \le 0$, $\lambda_3(\mu_2) \le 0$ and $\lambda_3(\mu_{12}) < 0$, then $x_3(t)$ converges to 0 exponentially and $\tilde{\Pi}_t(\cdot)$ converges weakly to μ_{12} almost surely for any initial value $X(0) \in \mathbb{R}_+^{3,0}$. Moreover, we have

$$\lim_{t\to\infty}\langle x_1(t)\rangle = \frac{K_1(r_1 - \frac{\sigma_1^2}{2})}{r_1}, \quad \lim_{t\to\infty}\langle x_2(t)\rangle = \frac{K_2(r_2 - \frac{\sigma_2^2}{2})}{r_2}, \quad \lim_{t\to\infty} x_3(t) = 0 \quad a.s.$$

If $\lambda_3(\mu_{12}) = 0$. We also argue by contradiction. Suppose there is an invariant probability measure π on $\mathbb{R}_+^{3,0}$. From the ergodicity it follows that

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t x_3(s)ds = \int_{\mathbb{R}_+^{3,0}} x_3\pi(dX) =: x_3^* > 0.$$

From Itô's formula and 3.10, it follows that

$$\begin{aligned}\limsup_{t\to\infty}\frac{\ln x_3(t)}{t} &= -\left(\sigma + \frac{\sigma_3^2}{2}\right) + \lim_{t\to\infty}\frac{C_1\alpha}{t}\int_0^t \hat{x}_1(s)ds + \lim_{t\to\infty}\frac{C_2\beta}{t}\int_0^t \hat{x}_2(s)ds \\ &\quad - \lim_{t\to\infty}\frac{C_1\alpha}{t}\int_0^t(\hat{x}_1(s) - x_1(s))ds - \lim_{t\to\infty}\frac{C_2\beta}{t}\int_0^t(\hat{x}_2(s) - x_2(s))ds\end{aligned}$$

$$
\begin{aligned}
&-\lim_{t\to\infty}\frac{\gamma}{t}\int_0^t x_3(s)ds\\
\le&-\left(\sigma+\frac{\sigma_3^2}{2}\right)+C_1\alpha\frac{K_1(r_1-\frac{\sigma_1^2}{2})}{r_1}+C_2\beta\frac{K_2(r_2-\frac{\sigma_2^2}{2})}{r_2}\\
&-\lim_{t\to\infty}\frac{\gamma}{t}\int_0^t x_3(s)ds\\
=&\lambda_3(\mu_{12})-x_3^*=-x_3^*<0,\quad a.s.
\end{aligned}
$$

This implies $\lim_{t\to\infty} x_3(t)=0$ a.s.. As a result, there is no invariant probability measure on $\mathbb{R}_+^{3,0}$.

Thus Case 5 is derived.

Case 6. If $\lambda_3(\mu_1)>0$, $\lambda_3(\mu_2)\le 0$, then there is a unique ergodic invariant probability measure μ_{13} on $\mathbb{R}^0_{13+}$. In light of $\lambda_1(\mu_{13})=0$ and $\lambda_3(\mu_{13})=0$, that is

$$
\begin{aligned}
&r_1-\frac{\sigma_1^2}{2}-\frac{r_1}{K_1}\int_{\mathbb{R}^0_{13+}}x_1\mu_{13}(dX)-\alpha\int_{\mathbb{R}^0_{13+}}x_3\mu_{13}(dX)=0,\\
&-\left(\sigma+\frac{\sigma_3^2}{2}\right)+C_1\alpha\int_{\mathbb{R}^0_{13+}}x_1\mu_{13}(dX)-\gamma\int_{\mathbb{R}^0_{13+}}x_3\mu_{13}(dX)=0.
\end{aligned}
$$

This leads to

$$
\begin{aligned}
&\int_{\mathbb{R}^0_{13+}}x_1\mu_{13}(dX)=\frac{\gamma K_1(r_1-\frac{\sigma_1^2}{2})+\alpha K_1(\sigma+\frac{\sigma_3^2}{2})}{\gamma r_1+C_1K_1\alpha^2}=\frac{A_{22}^{(1)}}{A_{22}},\\
&\int_{\mathbb{R}^0_{13+}}x_3\mu_{13}(dX)=\frac{C_1K_1\alpha(r_1-\frac{\sigma_1^2}{2})-r_1(\sigma+\frac{\sigma_3^2}{2})}{\gamma r_1+C_1K_1\alpha^2}=\frac{A_{22}^{(3)}}{A_{22}}.
\end{aligned}
$$

In this case

$$
\begin{aligned}
&\lambda_3(\mu_{12})>\lambda_3(\mu_1)>0,\\
&\lambda_2(\mu_{13})=r_2-\frac{\sigma_2^2}{2}-\beta\frac{C_1K_1\alpha(r_1-\frac{\sigma_1^2}{2})-r_1(\sigma+\frac{\sigma_3^2}{2})}{\gamma r_1+C_1K_1\alpha^2}.
\end{aligned}
$$

If $\lambda_2(\mu_{13})<0$, then $x_2(t)$ converges to 0 and $\tilde{\Pi}_t(\cdot)$ converges weakly to μ_{13} almost surely for any initial value $X(0)\in\mathbb{R}_+^{3,0}$. Therefore, by the ergodicity of μ_{13}, we have

$$
\lim_{t\to\infty}\langle x_1(t)\rangle=\frac{A_{22}^{(1)}}{A_{22}},\quad \lim_{t\to\infty}x_2(t)=0,\quad \lim_{t\to\infty}\langle x_3(t)\rangle=\frac{A_{22}^{(3)}}{A_{22}}\quad a.s.
$$

If $\lambda_2(\mu_{13})=0$, we also argue by contradiction. Let $(\bar{x}_1(t),\bar{x}_3(t))$ be the solution of the following system

$$
\begin{aligned}
&d\bar{x}_1(t)=\left(r_1\bar{x}_1(t)(1-\frac{\bar{x}_1}{K_1})-\alpha\bar{x}_1(t)\bar{x}_3(t)\right)dt+\sigma_1\bar{x}_1(t)dB_1(t),\\
&d\bar{x}_3(t)=\bar{x}_3(t)(-\sigma+C_1\alpha\bar{x}_1(t)-\gamma\bar{x}_3(t))dt+\sigma_3\bar{x}_3(t)dB_3(t),
\end{aligned}
\tag{3.11}
$$

with initial value $(\bar{x}_1(0),\bar{x}_3(0))=(x_1(0),x_3(0))$. From comparison theorem, we have $\bar{x}_3(t)\le x_3(t)$ a.s. In the meanwhile, similar analysis results that if $\lambda_3(\mu_1)>$

0, system (3.11) has an invariant probability measure $\bar{\mu}_{13}$ on $\mathbb{R}^0_{13+}$. From the ergodicity, we deduce that

$$\lim_{t\to\infty}\langle \bar{x}_1(t)\rangle = \frac{A^{(1)}_{22}}{A_{22}}, \quad \lim_{t\to\infty}\langle \bar{x}_3(t)\rangle = \frac{A^{(3)}_{22}}{A_{22}} \quad a.s.$$

Now we discuss this case for $\lambda_2(\mu_{13}) = 0$. If there is an invariant probability measure π on $\mathbb{R}^{3,0}_+$, then

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t x_2(s)ds = \int_{\mathbb{R}^{3,0}_+} x_2\pi(dX) =: x_2^* > 0.$$

Using Itô's formula again and the results of (3.11), we have

$$\begin{aligned}\frac{\ln x_2(t) - \ln x_2(0)}{t} &= \left(r_2 - \frac{\sigma_2^2}{2}\right) - \frac{r_2}{K_2 t}\int_0^t x_2(s)ds - \frac{\beta}{t}\int_0^t x_3(s)ds + \frac{\sigma_2 B_2(t)}{t}\\ &\le \left(r_2 - \frac{\sigma_2^2}{2}\right) - \frac{r_2}{K_2 t}\int_0^t x_2(s)ds - \frac{\beta}{t}\int_0^t \bar{x}_3(s)ds + \frac{\sigma_2 B_2(t)}{t}.\end{aligned}$$

This implies that

$$\limsup_{t\to\infty}\frac{\ln x_2(t)}{t} \le \lambda_2(\mu_{13}) - x_2^* = -x_2^* < 0, \quad a.s.,$$

and $\lim_{t\to\infty} x_2(t) = 0$ a.s.. This leads to a contradiction. As a result, there is no invariant probability measure on $\mathbb{R}^{3,0}_+$ if $\lambda_2(\mu_{13}) = 0$. Hence Case 6 is completed.

The proof of Case 7 is similar to that of Case 6, here we omit it.

If $\lambda_3(\mu_1) > 0$ and $\lambda_3(\mu_2) > 0$, then there are unique ergodic invariant probability measures μ_{13} and μ_{23} on $\mathbb{R}^0_{13+}$ and $\mathbb{R}^0_{23+}$, respectively. And from above analysis, we obtain

$$\begin{aligned}\lambda_2(\mu_{13}) &= r_2 - \frac{\sigma_2^2}{2} - \beta\frac{C_1K_1\alpha(r_1 - \frac{\sigma_1^2}{2}) - r_1(\sigma + \frac{\sigma_3^2}{2})}{\gamma r_1 + C_1K_1\alpha^2},\\ \lambda_1(\mu_{23}) &= r_1 - \frac{\sigma_1^2}{2} - \alpha\frac{C_2K_2\beta(r_2 - \frac{\sigma_2^2}{2}) - r_2(\sigma + \frac{\sigma_3^2}{2})}{\gamma r_2 + C_2K_2\beta^2}.\end{aligned}$$

Furthermore, we have

- Case 8. If $\lambda_2(\mu_{13}) \le 0$ and $\lambda_1(\mu_{23}) > 0$, then $x_2(t)$ goes to extinction and $\tilde{\Pi}_t(\cdot)$ converges weakly to μ_{13}. Therefore, we deduce the result of Case 8.
- Case 9. If $\lambda_2(\mu_{13}) > 0$ and $\lambda_1(\mu_{23}) \le 0$, then $x_1(t)$ goes to extinction and $\tilde{\Pi}_t(\cdot)$ converges weakly to μ_{23}. Therefore, we deduce the result of Case 9.

Case 10. By the ergodic decomposition theorem, any invariant probability measure μ on $\partial\mathbb{R}^3_+$ is a convex combination of δ^*, μ_1, μ_2, μ_{12}, μ_{23} and μ_{13} (if they exist), that is to say, μ has the form $\mu = p_0\delta^* + p_1\mu_1 + p_2\mu_2 + p_{12}\mu_{12} + p_{13}\mu_{13} + p_{23}\mu_{23}$ with $0 \le p_0, p_1, p_2, p_{12}, p_{13}, p_{23}$ and $p_0 + p_1 + p_2 + p_{12} + p_{13} + p_{23} = 1$. An application of Theorem 1.1 in [7] to the convex combination μ, we deduce that the transition probability $\mathbb{P}_X(t, X, \cdot)$, $X \in \mathbb{R}^{3,0}_+$ of $(X(t))_{t\ge 0}$ converges to its unique invariant probability measure π^* on $\mathbb{R}^{3,0}_+$ if one of the following conditions is satisfied:

- $\lambda_1(\delta^*) > 0, \quad \lambda_2(\delta^*) > 0, \quad \lambda_3(\mu_1) \le 0, \quad \lambda_3(\mu_2) \le 0$ and $\lambda_3(\mu_{12}) > 0$.
- $\lambda_1(\delta^*) > 0, \quad \lambda_2(\delta^*) > 0, \quad \lambda_3(\mu_1) > 0, \quad \lambda_3(\mu_2) \le 0$ and $\lambda_2(\mu_{13}) > 0$.
- $\lambda_1(\delta^*) > 0, \quad \lambda_2(\delta^*) > 0, \quad \lambda_3(\mu_1) \le 0, \quad \lambda_3(\mu_2) > 0$ and $\lambda_1(\mu_{23}) > 0$.
- $\lambda_1(\delta^*) > 0, \quad \lambda_2(\delta^*) > 0, \quad \lambda_3(\mu_1) > 0, \quad \lambda_3(\mu_2) > 0, \quad \lambda_2(\mu_{13}) > 0$ and $\lambda_1(\mu_{23}) > 0$.

Furthermore, by calculation, we obtain that any above condition implies that

$$\det(A^{(1)}) > 0, \quad \det(A^{(2)}) > 0 \quad \text{and} \quad \det(A^{(3)}) > 0.$$

Since π^* is the unique ergodic invariant probability measure on $\mathbb{R}_+^{3,0}$, $x_i(t)$, $i = 1,2,3$ can go to neither 0 nor ∞ as $t \to \infty$. Thus if $X(0) = (x_1(0), x_2(0), x_3(0)) \in \mathbb{R}_+^{3,0}$, we have

$$\lim_{t\to\infty} \frac{\ln x_i(t)}{t} = 0, \quad a.s. \quad i = 1,2,3.$$

These implies that

$$\begin{aligned}
&r_1 - \frac{\sigma_1^2}{2} - \frac{r_1}{K_1} \int_{\mathbb{R}_+^{3,0}} x_1 \pi^*(dX) - \alpha \int_{\mathbb{R}_+^{3,0}} x_3 \pi^*(dX) = 0,\\
&r_2 - \frac{\sigma_2^2}{2} - \frac{r_2}{K_2} \int_{\mathbb{R}_+^{3,0}} x_2 \pi^*(dX) - \beta \int_{\mathbb{R}_+^{3,0}} x_3 \pi^*(dX) = 0,\\
&-(\sigma + \frac{\sigma_3^2}{2}) + C_1\alpha \int_{\mathbb{R}_+^{3,0}} x_1 \pi^*(dX) + C_2\beta \int_{\mathbb{R}_+^{3,0}} x_2 \pi^*(dX) - \gamma \int_{\mathbb{R}_+^{3,0}} x_3 \pi^*(dX) = 0.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\lim_{t\to\infty} \langle x_1(t) \rangle &= \int_{\mathbb{R}_+^{3,0}} x_1 \pi^*(dX) = \frac{\det(A^{(1)})}{\det(A)},\\
\lim_{t\to\infty} \langle x_2(t) \rangle &= \int_{\mathbb{R}_+^{3,0}} x_2 \pi^*(dX) = \frac{\det(A^{(2)})}{\det(A)},\\
\lim_{t\to\infty} \langle x_3(t) \rangle &= \int_{\mathbb{R}_+^{3,0}} x_3 \pi^*(dX) = \frac{\det(A^{(3)})}{\det(A)} \quad a.s.
\end{aligned}$$

This completes the proof of Case 10.

The proof of Case 11 can be directly obtained from Theorem 1.3 in [7].

The proof is completed. □

Remark 3.1. Theorem 2 gives sufficient and necessary conditions for persistence and extinction of each species. This is one of the main contributions of this paper, which generalize the results in paper [11]. A summary of these cases can be found in Table 1, where E and P mean extinction and persistence in the time of average, respectively.

4. Examples and numerical simulations

In this section,we will introduce some examples and numerical simulations to demonstrate the above theoretical results. Using the Milstein higher-order method devel-

Table 1. Extinction and persistence(Red parts are the required conditions)

$\lambda_1(\delta^*)$	$\lambda_2(\delta^*)$	$\lambda_3(\mu_1)$	$\lambda_3(\mu_2)$	$\lambda_3(\mu_{12})$	$\lambda_2(\mu_{13})$	$\lambda_1(\mu_{23})$	$\tilde{\Pi}_t(\cdot)$	x_1	x_2	x_3
<0	<0	Null	Null	Null	Null	Null	$\tilde{\Pi}_t(\cdot)\to\delta^*(\cdot)$	E	E	E
>0	<0	≤ 0	Null	Null	Null	Null	$\tilde{\Pi}_t(\cdot)\to\mu_1(\cdot)$	P	E	E
>0	<0	>0	Null	Null	<0	Null	$\tilde{\Pi}_t(\cdot)\to\mu_{13}(\cdot)$	P	E	P
<0	>0	Null	≤ 0	Null	Null	Null	$\tilde{\Pi}_t(\cdot)\to\mu_2(\cdot)$	E	P	E
<0	>0	Null	>0	Null	Null	<0	$\tilde{\Pi}_t(\cdot)\to\mu_{23}(\cdot)$	E	P	P
>0	>0	≤ 0	≤ 0	≤ 0	Null	Null	$\tilde{\Pi}_t(\cdot)\to\mu_{12}(\cdot)$	P	P	E
>0	>0	>0	≤ 0	>0	≤ 0	Null	$\tilde{\Pi}_t(\cdot)\to\mu_{13}(\cdot)$	P	E	P
>0	>0	≤ 0	>0	>0	Null	≤ 0	$\tilde{\Pi}_t(\cdot)\to\mu_{23}(\cdot)$	E	P	P
>0	>0	>0	>0	>0	≤ 0	>0	$\tilde{\Pi}_t(\cdot)\to\mu_{13}(\cdot)$	P	E	P
>0	>0	>0	>0	>0	>0	≤ 0	$\tilde{\Pi}_t(\cdot)\to\mu_{23}(\cdot)$	E	P	P
>0	>0	≤ 0	≤ 0	>0	Null	Null	$\tilde{\Pi}_t(\cdot)\to\pi^*(\cdot)$	P	P	P
>0	>0	>0	≤ 0	>0	>0	Null	$\tilde{\Pi}_t(\cdot)\to\pi^*(\cdot)$	P	P	P
>0	>0	≤ 0	>0	>0	Null	>0	$\tilde{\Pi}_t(\cdot)\to\pi^*(\cdot)$	P	P	P
>0	>0	>0	>0	>0	>0	>0	$\tilde{\Pi}_t(\cdot)\to\pi^*(\cdot)$	P	P	P

oped in [10], we get the discretization equation of system (1.1):

$$\begin{cases} x_1^{k+1} = x_1^k + \left(r_1 x_1^k(1-\frac{x_1^k}{K_1}) - \alpha x_1^k x_3^k\right)\Delta t + \sigma_1 x_1^k\sqrt{\Delta t}\xi_k + \frac{\sigma_1^2}{2}x_1^k(\xi_k^2-1)\Delta t, \\ x_2^{k+1} = x_2^k + \left(r_2 x_2^k(1-\frac{x_2^k}{K_2}) - \beta x_2^k x_3^k\right)\Delta t + \sigma_2 x_2^k\sqrt{\Delta t}\eta_k + \frac{\sigma_2^2}{2}x_2^k(\eta_k^2-1)\Delta t, \\ x_3^{k+1} = x_3^k + x_3^k\left(-\sigma + C_1\alpha x_1^k + C_2\beta x_2^k - \gamma x_3^k\right)\Delta t + \sigma_3 x_3^k\sqrt{\Delta t}\zeta_k + \frac{\sigma_3^2}{2}x_3^k(\zeta_k^2-1)\Delta t, \end{cases}$$

where the time increment $\Delta t > 0$, ξ_k, η_k, ζ_k are independent Gaussian random variables which follow the distribution $N(0,1)$ for $k = 1, 2, ..., n$.

4.1. Examples and numerical simulations for $r_i < \frac{\sigma_i^2}{2}, \quad i = 1, 2$

Example 4.1. If we let $r_1 = 0.25$, $r_2 = 0.2$ $K_1 = K_2 = 1.6$, $\alpha = \beta = 0.0875$, $C_1 = 0.73$, $C_2 = 0.75$, $\sigma = 0.098$ and $\gamma = 0.1$. The white noise coefficients $\sigma_1 = \sigma_2 = 0.8$ and $\sigma_3 = 0.1$. Then

$$\lambda_1(\delta^*) = r_1 - \frac{\sigma_1^2}{2} = -0.07 < 0, \quad \lambda_2(\delta^*) = r_2 - \frac{\sigma_2^2}{2} = -0.12 < 0.$$

From (I) in Theorem 3.1 it follows that $x_i(t)$ $i = 1, 2, 3$ converge to 0 almost surely. Fig.1 confirms this.

4.2. Numerical simulations for $r_1 < \frac{\sigma_1^2}{2}$ and $r_2 > \frac{\sigma_2^2}{2}$

Example 4.2. Case 1. Let $K_2 = 1$ and the other parameters are the same as those in Example 4.1. Choose the white noise coefficients $\sigma_1 = 0.8$, $\sigma_2 = \sigma_3 = 0.1$. By calculation, we have

$$\lambda_1(\delta^*) = r_1 - \frac{\sigma_1^2}{2} = -0.07 < 0, \quad \lambda_2(\delta^*) = r_2 - \frac{\sigma_2^2}{2} = 0.195 > 0,$$

$$\lambda_3(\mu_2) = \frac{C_2\beta K_2(r_2 - \frac{\sigma_2^2}{2})}{r_2} - \left(\sigma + \frac{\sigma_3^2}{2}\right) = -0.039 < 0.$$

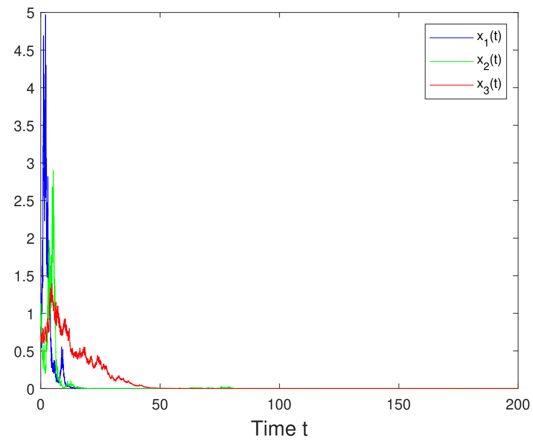

Figure 1. Numerical simulations for the solution of model (1.1) in Example 4.1 which shows that species $x_i(t)$ go extinct a.s., $i = 1, 2, 3$.

Then Case 1 in Theorem 3.1 shows that the occupation measure $\tilde{\Pi}_t(\cdot)$ converges almost surely to μ_2. In the meanwhile, $x_1(t)$ and $x_3(t)$ go to extinction exponentially, and $x_2(t)$ will be persistent almost surely, i.e

$$\lim_{t\to\infty} x_1(t) = 0, \quad \lim_{t\to\infty} \langle x_2(t)\rangle = 0.975, \quad \lim_{t\to\infty} x_3(t) = 0 \quad a.s.$$

From Fig.2 it follows that species $x_2(t)$ is persistent and other species are extinct.

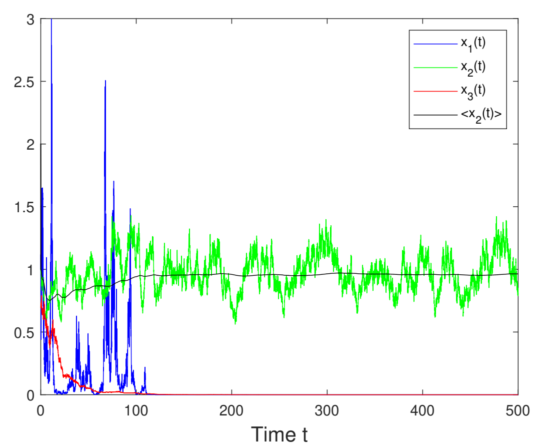

Figure 2. Numerical simulations for Case 1 in Example 4.2 which shows that only species $x_2(t)$ is persistent.

Case 2. Let $K_2 = 1.7$ and the other parameters are the same as those in Case1. And

$$\lambda_1(\delta^*) = r_1 - \frac{\sigma_1^2}{2} = -0.07 < 0, \quad \lambda_2(\delta^*) = r_2 - \frac{\sigma_2^2}{2} = 0.195 > 0,$$

$$\lambda_3(\mu_2) = \frac{C_2\beta K_2(r_2 - \frac{\sigma_2^2}{2})}{r_2} - \left(\sigma + \frac{\sigma_3^2}{2}\right) = 0.0058 > 0.$$

Then it is obvious that

$$\lambda_1(\mu_{23}) = r_1 - \frac{\sigma_1^2}{2} - \alpha \frac{C_2 \beta K_2 (r_2 - \frac{\sigma_2^2}{2}) - r_2(\sigma + \frac{\sigma_3^2}{2})}{\gamma r_2 + C_2 K_2 \beta^2} < 0.$$

Therefore the occupation measure $\tilde{\Pi}_t(\cdot)$ converges almost surely to μ_{23}. Case 2 in Theorem 3.1 shows that $x_1(t)$ goes extinct exponentially, and $x_i(t)$ $i = 2, 3$ will be persistent almost surely, and

$$\lim_{t\to\infty} x_1(t) = 0, \quad \lim_{t\to\infty} \langle x_2(t) \rangle = 1.6286, \quad \lim_{t\to\infty} \langle x_3(t) \rangle = 0.0388 \quad a.s.$$

From Fig.3 it follows that species $x_1(t)$ is extinct, $x_2(t)$ and $x_3(t)$ are persistent.

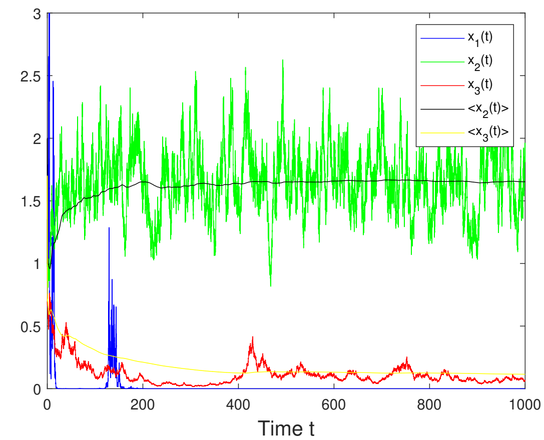

Figure 3. Numerical simulations for Case 2 in Example 4.2 which shows that species $x_2(t)$ and $x_3(t)$ are persistent while $x_1(t)$ is extinct almost surely.

4.3. Numerical simulations for $r_1 > \frac{\sigma_1^2}{2}$ and $r_2 < \frac{\sigma_2^2}{2}$

Example 4.3. Case 3. Let $K_1 = 1$ and the other parameters are the same as those in Example 4.1. Choose the white noise coefficients $\sigma_1 = \sigma_3 = 0.1$, $\sigma_2 = 0.8$, then

$$\lambda_1(\delta^*) = r_1 - \frac{\sigma_1^2}{2} = 0.245 > 0, \quad \lambda_2(\delta^*) = r_2 - \frac{\sigma_2^2}{2} = -0.12 < 0,$$
$$\lambda_3(\mu_1) = \frac{C_1 \alpha K_1 (r_1 - \frac{\sigma_1^2}{2})}{r_1} - \left(\sigma + \frac{\sigma_3^2}{2}\right) = -0.0404 < 0.$$

This means the occupation measure $\tilde{\Pi}_t(\cdot)$ converges almost surely to μ_1 which implies that $x_1(t)$ is persistent and $\lim_{t\to\infty}\langle x_1(t) \rangle = 0.98$, while $x_2(t)$ and $x_3(t)$ converges to 0 almost surely. Fig.4 confirms this.

Case 4. Let $K_1 = 1.8$ and the other parameters are the same as those in Case

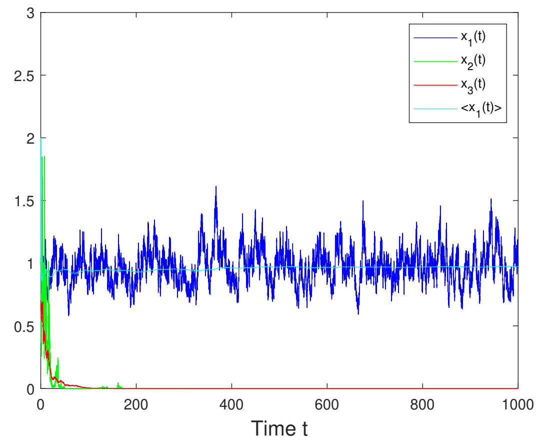

Figure 4. Numerical simulations for Case 3 in Example 4.3 which shows that only species $x_1(t)$ is persistent.

3 of Example 4.3. Then

$$\lambda_1(\delta^*) = r_1 - \frac{\sigma_1^2}{2} = 0.245 > 0, \quad \lambda_2(\delta^*) = r_2 - \frac{\sigma_2^2}{2} = -0.12 < 0,$$
$$\lambda_3(\mu_1) = \frac{C_1\alpha K_1(r_1 - \frac{\sigma_1^2}{2})}{r_1} - \left(\sigma + \frac{\sigma_3^2}{2}\right) = 0.0097 > 0,$$
$$\lambda_2(\mu_{13}) = r_2 - \frac{\sigma_2^2}{2} - \beta\frac{C_1\alpha K_1(r_1 - \frac{\sigma_1^2}{2}) - r_1(\sigma + \frac{\sigma_3^2}{2})}{\gamma r_1 + C_1 K_1 \alpha^2} < 0.$$

From Case 4 in Theorem 3.1 it follows that $\tilde{\Pi}_t(\cdot)$ converges almost surely to μ_{13}, which implies that $x_2(t)$ goes extinct and $x_1(t)$ and $x_3(t)$ are persistent. We also obtain

$$\lim_{t\to\infty} \langle x_1(t)\rangle = 1.9720, \quad \lim_{t\to\infty} x_2(t) = 0, \quad \lim_{t\to\infty} \langle x_3(t)\rangle = 0.0791.$$

From Fig.5 we can also see that $x_1(t)$ and $x_3(t)$ are persistent, and $x_2(t)$ is extinct.

4.4. Numerical simulations for $\lambda_i(\delta^*) > 0,\ i = 1, 2$

Example 4.4. Case 5. Let $K_1 = K_2 = 1$ and the other parameters are the same as those in Example 4.1. The white noise coefficients $\sigma_1 = \sigma_2 = 0.1$ and $\sigma_3 = 0.5$. By calculation, we have

$$\lambda_1(\delta^*) = r_1 - \frac{\sigma_1^2}{2} = 0.245 > 0, \quad \lambda_2(\delta^*) = r_2 - \frac{\sigma_2^2}{2} = 0.195 > 0,$$
$$\lambda_3(\mu_1) = \frac{C_1\alpha K_1(r_1 - \frac{\sigma_1^2}{2})}{r_1} - \left(\sigma + \frac{\sigma_3^2}{2}\right) = -0.0404 < 0,$$
$$\lambda_3(\mu_2) = \frac{C_2\beta K_2(r_2 - \frac{\sigma_2^2}{2})}{r_2} - \left(\sigma + \frac{\sigma_3^2}{2}\right) = -0.039 < 0,$$

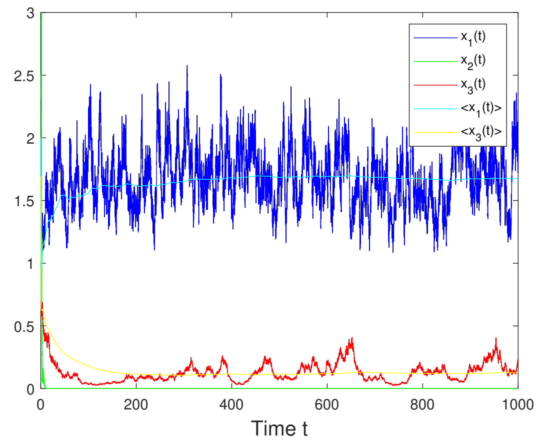

Figure 5. Numerical simulations for Case 4 in Example 4.3 which shows that species $x_1(t)$ and $x_3(t)$ are persistent, and $x_1(t)$ is extinct.

$$\lambda_3(\mu_{12}) = -\sigma - \frac{\sigma_3^2}{2} + C_1\alpha\frac{K_1(r_1 - \frac{\sigma_1^2}{2})}{r_1} + C_2\beta\frac{K_2(r_2 - \frac{\sigma_2^2}{2})}{r_2} = -0.0964 < 0.$$

Then Case 5 in Theorem 3.1 shows that $x_3(t)$ converges to 0 exponentially and $\tilde{\Pi}_t(\cdot)$ converges weakly to μ_{12} almost surely for any initial value $X(0) \in \mathbb{R}_+^{3,0}$. Moreover, we have

$$\lim_{t\to\infty}\langle x_1(t)\rangle = 0.98, \quad \lim_{t\to\infty}\langle x_2(t)\rangle = 0.975, \quad \lim_{t\to\infty} x_3(t) = 0 \quad a.s.$$

Fig.6 also shows that $x_1(t)$ and $x_2(t)$ are persistent, and $x_3(t)$ goes to extinct.

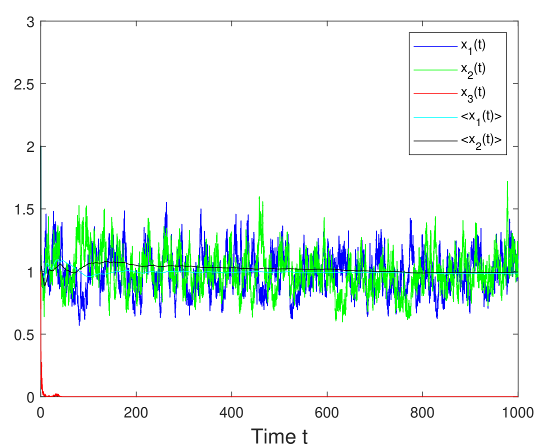

Figure 6. Numerical simulations for Case 5 in Example 4.4 which shows that species $x_1(t)$ and $x_2(t)$ are persistent, and $x_3(t)$ is extinct.

Case 6. Let $r_1 = 0.4$, $K_1 = 4$, $r_2 = 0.2$, $K_2 = 1$ and the other parameters are the same as those in Example 4.1. The white noise coefficients $\sigma_1 = \sigma_3 = 0.1$ and

$\sigma_2 = 0.5$. Then

$$\lambda_1(\delta^*) = r_1 - \frac{\sigma_1^2}{2} = 0.395 > 0, \quad \lambda_2(\delta^*) = r_2 - \frac{\sigma_2^2}{2} = 0.075 > 0,$$

$$\lambda_3(\mu_1) = \frac{C_1\alpha K_1(r_1 - \frac{\sigma_1^2}{2})}{r_1} - \left(\sigma + \frac{\sigma_3^2}{2}\right) = 0.1493 > 0,$$

$$\lambda_3(\mu_2) = \frac{C_2\beta K_2(r_2 - \frac{\sigma_2^2}{2})}{r_2} - \left(\sigma + \frac{\sigma_3^2}{2}\right) = -0.039 < 0,$$

$$\lambda_2(\mu_{13}) = r_2 - \frac{\sigma_2^2}{2} - \beta\frac{C_1\alpha K_1(r_1 - \frac{\sigma_1^2}{2}) - r_1(\sigma + \frac{\sigma_3^2}{2})}{\gamma r_1 + C_1 K_1 \alpha^2} = -0.0088 < 0.$$

Then $x_2(t)$ converges to 0 and $\tilde{\Pi}_t(\cdot)$ converges weakly to μ_{13} almost surely. Furthermore

$$\lim_{t\to\infty}\langle x_1(t)\rangle = 3.1120, \quad \lim_{t\to\infty} x_2(t) = 0, \quad \lim_{t\to\infty}\langle x_3(t)\rangle = 0.9578 \quad a.s.$$

Fig.7 confirms this.

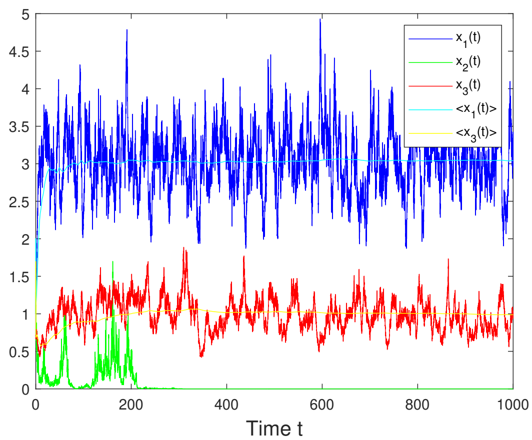

Figure 7. Numerical simulations for Case 6 in Example 4.4 which shows that species $x_1(t)$ and $x_3(t)$ are persistent, and $x_2(t)$ is extinct.

Case 7. Let $r_1 = 0.25$, $K_1 = 1$, $r_2 = 0.4$, $K_2 = 4$ and the other parameters are the same as those in Example 4.1. The white noise coefficients $\sigma_1 = 0.7$ and $\sigma_2 = \sigma_3 = 0.1$. Then

$$\lambda_1(\delta^*) = r_1 - \frac{\sigma_1^2}{2} = 0.005 > 0, \quad \lambda_2(\delta^*) = r_2 - \frac{\sigma_2^2}{2} = 0.395 > 0,$$

$$\lambda_3(\mu_1) = \frac{C_1\alpha K_1(r_1 - \frac{\sigma_1^2}{2})}{r_1} - \left(\sigma + \frac{\sigma_3^2}{2}\right) = -0.1017 < 0,$$

$$\lambda_3(\mu_2) = \frac{C_2\beta K_2(r_2 - \frac{\sigma_2^2}{2})}{r_2} - \left(\sigma + \frac{\sigma_3^2}{2}\right) = 0.1562 > 0,$$

$$\lambda_1(\mu_{23}) = r_1 - \frac{\sigma_1^2}{2} - \alpha\frac{C_2\beta K_2(r_2 - \sigma_2^2/2) - r_2(\sigma + \frac{\sigma_3^2}{2})}{\gamma r_2 + C_2 K_2 \beta^2} = -0.0818 < 0.$$

Then $x_1(t)$ converges to 0 and $\tilde{\Pi}_t(\cdot)$ converges weakly to μ_{23} almost surely. Furthermore

$$\lim_{t\to\infty} x_1(t) = 0, \quad \lim_{t\to\infty} \langle x_2(t)\rangle = 3.0817, \quad \lim_{t\to\infty} \langle x_3(t)\rangle = 0.9922 \quad a.s.$$

Fig.8 confirms this.

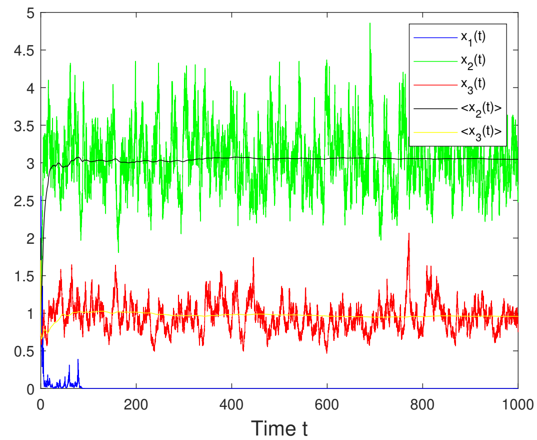

Figure 8. Numerical simulations for Case 7 in Example 4.4 which shows that species $x_1(t)$ is extinct, and $x_2(t)$, $x_3(t)$ are persistent.

Case 8. Let $r_1 = 0.4$, $K_1 = 4$, $\beta = 0.2$ and the other parameters are the same as those in Example 4.1. The white noise coefficients $\sigma_1 = \sigma_3 = 0.1$ and $\sigma_2 = 0.2$. By calculation, we have

$$\begin{aligned}
&\lambda_1(\delta^*) = r_1 - \frac{\sigma_1^2}{2} = 0.395 > 0, \quad \lambda_2(\delta^*) = r_2 - \frac{\sigma_2^2}{2} = 0.18 > 0,\\
&\lambda_3(\mu_1) = \frac{C_1\alpha K_1(r_1 - \frac{\sigma_1^2}{2})}{r_1} - \left(\sigma + \frac{\sigma_3^2}{2}\right) = 0.1493 > 0,\\
&\lambda_3(\mu_2) = \frac{C_2\beta K_2(r_2 - \frac{\sigma_2^2}{2})}{r_2} - \left(\sigma + \frac{\sigma_3^2}{2}\right) = 0.113 > 0,\\
&\lambda_1(\mu_{23}) = r_1 - \frac{\sigma_1^2}{2} - \alpha\frac{C_2\beta K_2(r_2 - \sigma_2^2/2) - r_2(\sigma + \frac{\sigma_3^2}{2})}{\gamma r_2 + C_2K_2\beta^2} = 0.3659 > 0,\\
&\lambda_2(\mu_{13}) = r_2 - \frac{\sigma_2^2}{2} - \beta\frac{C_1\alpha K_1(r_1 - \frac{\sigma_1^2}{2}) - r_1(\sigma + \frac{\sigma_3^2}{2})}{\gamma r_1 + C_1K_1\alpha^2} = -0.0116 < 0.
\end{aligned}$$

Then Case 8 in Theorem 3.1 implies that species $x_2(t)$ is extinct exponentially, $x_1(t)$ and $x_3(t)$ are persistent. We also obtain

$$\lim_{t\to\infty} \langle x_1(t)\rangle = 3.1120, \quad \lim_{t\to\infty} x_2(t) = 0, \quad \lim_{t\to\infty} \langle x_3(t)\rangle = 0.9578 \quad a.s.$$

Fig.9 confirms this.

Case 9. Let $r_2 = 0.4$, $K_2 = 4$, $\alpha = 0.2$ and the other parameters are the same as those in Example 4.1. The white noise coefficients $\sigma_1 = 0.4$ and $\sigma_2 = \sigma_3 = 0.1$.

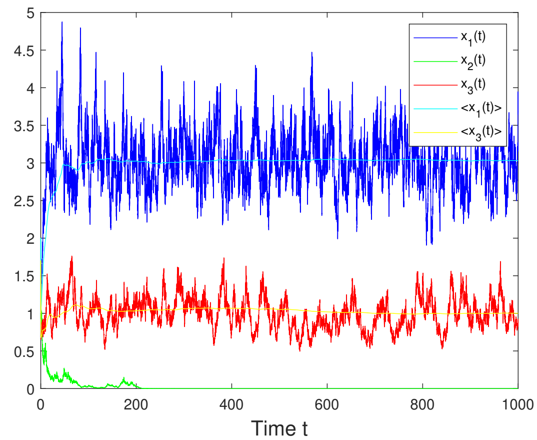

Figure 9. Numerical simulations for Case 8 in Example 4.4 which shows that species $x_2(t)$ is extinct, and $x_1(t)$, $x_3(t)$ are persistent.

By calculation, we have

$$\lambda_1(\delta^*) = r_1 - \frac{\sigma_1^2}{2} = 0.17 > 0, \quad \lambda_2(\delta^*) = r_2 - \frac{\sigma_2^2}{2} = 0.395 > 0,$$

$$\lambda_3(\mu_1) = \frac{C_1\alpha K_1(r_1 - \frac{\sigma_1^2}{2})}{r_1} - \left(\sigma + \frac{\sigma_3^2}{2}\right) = 0.0558 > 0,$$

$$\lambda_3(\mu_2) = \frac{C_2\beta K_2(r_2 - \frac{\sigma_2^2}{2})}{r_2} - \left(\sigma + \frac{\sigma_3^2}{2}\right) = 0.1562 > 0,$$

$$\lambda_1(\mu_{23}) = r_1 - \frac{\sigma_1^2}{2} - \alpha\frac{C_2\beta K_2(r_2 - \sigma_2^2/2) - r_2(\sigma + \frac{\sigma_3^2}{2})}{\gamma r_2 + C_2K_2\beta^2} = -0.0285 < 0,$$

$$\lambda_2(\mu_{13}) = r_2 - \frac{\sigma_2^2}{2} - \beta\frac{C_1\alpha K_1(r_1 - \frac{\sigma_1^2}{2}) - r_1(\sigma + \frac{\sigma_3^2}{2})}{\gamma r_1 + C_1K_1\alpha^2} = 0.3590 > 0.$$

Then Case 9 in Theorem 3.1 implies that species $x_1(t)$ is extinct exponentially, $x_2(t)$ and $x_3(t)$ are persistent. Fig.10 confirms this.

Case 10. Take the parameters as those in Example 4.1, and $\sigma_1 = \sigma_2 = \sigma_3 = 0.1$. Then

$$\lambda_1(\delta^*) = r_1 - \frac{\sigma_1^2}{2} = 0.245 > 0, \quad \lambda_2(\delta^*) = r_2 - \frac{\sigma_2^2}{2} = 0.195 > 0,$$

$$\lambda_3(\mu_1) = \frac{C_1\alpha K_1(r_1 - \frac{\sigma_1^2}{2})}{r_1} - \left(\sigma + \frac{\sigma_3^2}{2}\right) = -0.0028 < 0,$$

$$\lambda_3(\mu_2) = \frac{C_2\beta K_2(r_2 - \frac{\sigma_2^2}{2})}{r_2} - \left(\sigma + \frac{\sigma_3^2}{2}\right) = -0.0006 < 0,$$

$$\lambda_3(\mu_{12}) = -\sigma - \frac{\sigma_3^2}{2} + C_1\alpha\frac{K_1(r_1 - \frac{\sigma_1^2}{2})}{r_1} + C_2\beta\frac{K_2(r_2 - \frac{\sigma_2^2}{2})}{r_2} = 0.0995 > 0.$$

Hence the first condition of Case 10 in Theorem 3.1 are satisfied, which means that

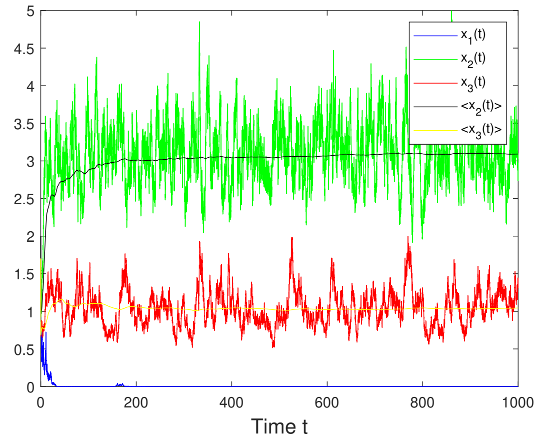

Figure 10. Numerical simulations for Case 9 in Example 4.4 which shows that species $x_1(t)$ is extinct, and $x_2(t)$, $x_3(t)$ are persistent.

$\mathbb{P}(t, X, \cdot)$ converges to an invariant probability measure π^* on $\mathbb{R}_+^{3,0}$, and

$$\lim_{t\to\infty} \langle x_1(t)\rangle = \frac{det(A^{(1)})}{det(A)} = 1.2613, \quad \lim_{t\to\infty} \langle x_2(t)\rangle = \frac{det(A^{(2)})}{det(A)} = 1.1766,$$
$$\lim_{t\to\infty} \langle x_3(t)\rangle = \frac{det(A^{(3)})}{det(A)} = 0.5478 \quad a.s.$$

That is to say, all the species $x_i(t)$, $i = 1, 2, 3$ are persistent. Fig.11 confirms this. Other cases can also be satisfied if suitable parameters are chosen, here we omit.

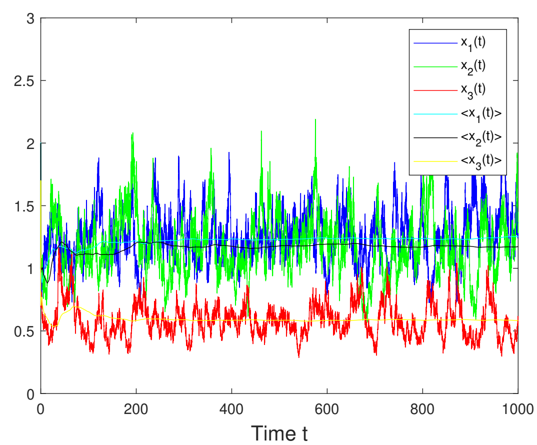

Figure 11. Numerical simulations for Case 10 in Example 4.4 which shows that species $x_i(t)$,$i = 1, 2, 3$ are persistent.

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