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EXISTENCE OF SOLUTIONS FOR A COUPLED SYSTEM OF CAPUTO-HADAMARD FRACTIONAL DIFFERENTIAL EQUATIONS WITH *P*-LAPLACIAN OPERATOR*

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Abstract In this paper, by using the Schauder fixed point theorem and Banach contraction mapping principle, the existence and uniqueness of solutions for a coupled system of Caputo-Hadamard fractional differential equations with p-Laplacian operator are established. As applications, two examples are given to illustrate the main results. The interesting point of this article is that the boundary value conditions contain integrals, and the approximate solutions are given by using the iterative method.

Keywords Coupled system, p-Laplacian operator, lterative method, simulation.

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1. Introduction

Recently, the integer order differential equations have been widely used in physics, chemistry, biology, engineering and other fields [4, 6, 15, 20, 25]. However, they also have some limitations in describing certain practical problems, such as the description of capacitance and inductance in physical, the melting of polymer materials in chemistry and so on. This has attracted more and more scholars to research fractional differential equations [7, 12, 13, 22]. Compared with single fractional differential equations, the conditions of coupled system are more complex and the research results are relatively few, see [1, 3] and their references. Therefore, it is meaningful to study the coupled system of fractional differential equations.

The Hadamard fractional derivative was introduced by Hadamard in 1892, its integral kernel contains a logarithm function of arbitrary exponent. In [1], Aljoudi et al. studied the following coupled system of Caputo-Hadamard fractional differential

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equations:

$$\begin{cases} ({}^CD^{\alpha}+\lambda^CD^{\alpha-1})u(t)=f\left(t,u(t),v(t),^CD^{\zeta}v(t)\right),\ 1<\alpha\leq 2, 0<\zeta\leq 1,\\ ({}^CD^{\beta}+\lambda^CD^{\beta-1})u(t)=f\left(t,u(t),^CD^{\bar{\zeta}}u(t),v(t)\right),\ 1<\beta\leq 2, 0<\bar{\zeta}\leq 1,\\ u(1)=0, a_1I^{\gamma_1}v(\eta_1)+b_1u(T)=K_1,\ \gamma_1>0, 1<\gamma_1<T,\\ v(1)=0, a_2I^{\gamma_2}u(\eta_2)+b_2v(T)=K_2,\ \gamma_2>0, 1<\gamma_2<T, \end{cases}$$

where $\lambda > 0$, ${}^CD^{(\cdot)}$ and $I^{(\cdot)}$ are the Caputo-Hadamard fractional derivative and Hadamard fractional integral, $f, g : [1, e] \times \mathbb{R}^3 \to \mathbb{R}$. The uniqueness and existence results are established by using Leary-Schauder alternative and contraction mapping principle.

As is known to all, the differential equations with p-Laplacian operate contain the general differential equations, moreover, it also has deep engineering and physical significance. For example, the study of the turbulent flow in a porous media, non-newtonian fluids and the spontaneous combustion theory of chemically active gases all involve p-Laplacian differential equations, see [8,16,23,24] and their references.

In [8], relying on the extension of Mawhin's continuation theorem due to Ge, Hu et al. considered the existence of solutions for a coupled system of fractional p-Laplacian equations:

$$\begin{cases} D^{\beta}_{0+}\phi_p\left(D^{\alpha}_{0+}u(t)\right)=f\left(t,v(t),D^{\delta}_{0+}v(t)\right),\ t\in(0,1),\\ D^{\gamma}_{0+}\phi_p\left(D^{\delta}_{0+}v(t)\right)=g\left(t,u(t),D^{\alpha}_{0+}u(t)\right),\ t\in(0,1),\\ D^{\alpha}_{0+}u(0)=D^{\alpha}_{0+}u(1)=D^{\delta}_{0+}v(0)=D^{\delta}_{0+}v(1)=0, \end{cases}$$

where $D^{(\cdot)}_{0+}$ are the standard Caputo fractional derivatives. $0 < \alpha, \beta, \delta, \gamma \leq 1$, $1 < \alpha+\beta < 2$, $1 < \gamma+\delta < 2$, and $f, g : [0, 1] \times \mathbb{R}^2 \to \mathbb{R}$ is continuous. They obtained sufficient conditions for the existence of at least one positive solutions.

In the literature mentioned above, the authors only obtained the existence of solutions from the theoretical aspect, yet, the shape of the solutions is rarely involved, see [2,9,17,19]. Therefore, it is quite necessary to give the numerical simulation and existence for positive solutions to fractional differential equations with p-Laplacian operator.

Motivated by above-mentioned ideas, we investigate the following Caputo-Hadamard fractional differential equations with p-Laplacian operator:

$$\begin{cases} {}^C_HD^{\theta_1}_{1+}\phi_p\left({}^C_HD^{\xi_1}_{1+}u(t)\right)=f\left(t,u(t),v(t)\right),\ 1<t<e,\\ {}^C_HD^{\theta_2}_{1+}\phi_p\left({}^C_HD^{\xi_2}_{1+}v(t)\right)=g\left(t,v(t),u(t)\right),\ 1<t<e,\\ u(1)=\phi_p\left({}^C_HD^{\xi_1}_{1+}u(1)\right)=\phi_p\left({}^C_HD^{\xi_1}_{1+}u(1)\right)'=0,\\ v(1)=\phi_q\left({}^C_HD^{\xi_2}_{1+}v(1)\right)=\phi_q\left({}^C_HD^{\xi_2}_{1+}v(1)\right)'=0,\\ u(e)=\int_1^e x(s)u(s)ds,\ v(e)=\int_1^e y(s)v(s)ds, \end{cases} \tag{1.1}$$

where $1 < \theta_1,\ \theta_2,\ \xi_1,\ \xi_2 \leq 2$, $\phi_p(s) = |s|^{p-2}s$, $1 < p \leq 2$, $\phi_p^{-1} = \phi_q$, $\frac{1}{p}+\frac{1}{q} = 1$. $f, g : [1, e] \times \mathbb{R}^3_+ \to \mathbb{R}_+$ are given functions, $x(s), y(s)$ are integrable on $[1, e]$,

${}^{C}_{H}D^{(\cdot)}_{1^+}$ are the standard Caputo-Hadamard fractional derivatives. The existence and uniqueness of solutions are established by using Schauder fixed point theorem and Banach contraction principle. In this paper, we obtain some new conclusions, and use the iterative method to simulate the examples, which further proves our results.

2. Preliminary

In this section, some basic definitions and lemmas are introduced to help us understand the main results and proofs in Section 3 and 4.

Definition 2.1 ([11]). The Hadamard fractional integral of order $\alpha > 0$ for a function $y : [1, +\infty) \to \mathbb{R}$ is defined as

$$ {}^{H}I^{\alpha}y(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\ln \frac{t}{s}\right)^{\alpha-1} y(s)\frac{ds}{s} $$

provided the integral exists.

Definition 2.2 ([11]). The Hadamard fractional derivative of order $\alpha > 0$ of a function $y : [1, +\infty) \to \mathbb{R}$ is given by

$$ {}^{H}D^{\alpha}y(t) = \frac{1}{\Gamma(n-\alpha)} \left(t\frac{d}{dt}\right)^n \int_1^t \left(\log \frac{t}{s}\right)^{n-\alpha-1} y(s)\frac{ds}{s}, $$

where $n-1 < \alpha < n, n = [\alpha]+1$, $[\alpha]$ is the integer of α and $\log(\cdot) = \log_e(\cdot)$.

Lemma 2.1 ([18]). *Let $n \in \mathbb{N}$, $n-1 < \alpha < n$, $\delta = t\frac{d}{dt}$ and $y \in AC^n_\delta[a,T]$, where*

$$ AC^n_\delta[a,T] = \left\{ y : [a,T] \to \mathbb{R} : \delta^{n-1}y(t) \in AC[a,T] \right\}. $$

(i) *if $\alpha \neq n$, the Caputo-Hadamard fractional derivative of order $\alpha > 0$ is defined as*

$$ {}^{C}_{H}D^{\alpha}_{a^+}y(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \left(\log \frac{t}{s}\right)^{n-\alpha-1} \delta^n y(s)\frac{ds}{s} = {}^{H}I^{n-\alpha}\delta^n y(t). $$

(ii) *if $\alpha = n$, the Caputo-Hadamard fractional derivative of order n is defined as*

$$ {}^{C}_{H}D^{n}_{a^+}y(t) = \delta^n y(t). $$

Lemma 2.2 ([10]). *Let $u \in C^n_\delta([a,T],\mathbb{R})$. Then*

$$ {}^{H}I^{\alpha}_{a^+}({}^{C}_{H}D^{\alpha}_{a^+})u(t) = u(t) - \sum_{j=0}^{n-1} c_j \left(\ln \frac{t}{a}\right)^j, $$

where $C^n_\delta([a,T],\mathbb{R}) = \left\{u : [a,T] \to \mathbb{R}; \delta^{n-1}u \in C([a,T],\mathbb{R})\right\}$, $c_j \in \mathbb{R}$.

Lemma 2.3 ([14]). *p-Laplacian operator has following basic properties:*

(1) *If $1 < p < 2$, $xy > 0$, and $|x|, |y| \geq m > 0$, then $|\phi_p(x) - \phi_p(y)| \leq (p-1)m^{p-2}|x-y|$.*

(2) *If $p \geq 2$, $|x|, |y| \leq N$, then $|\phi_p(x) - \phi_p(y)| \leq (p-1)N^{p-2}|x-y|$.*

Lemma 2.4 ([5], Schauder's fixed point theorem). *Let U is a nonempty convex subset of a Banach space X. Let $S : \Omega \to \Omega$ be a continuous mapping such that Ω is a relatively compact subset of X. Then, S has at least one fixed point in Ω.*

Lemma 2.5 ([5], Banach contraction mapping principle)**.** *Assume (X,ρ) be a nonempty complete metric space, $F\subseteq X$ is a closed set, $T:F\to F$ be a mapping. If there is a constant $k\in[0,1)$ such that*

$$\rho\left(Tx,Ty\right)\le k\rho(x,y).$$

Then T has a unique fixed point x^* and satisfies $Tx^*=x^*$, T is called the contraction mapping.

Lemma 2.6. *Let $h_1,h_2\in AC_\delta^n([1,e],R)$, the p-Laplacian fractional differential equation system*

$$\begin{cases} {}^C_HD^{\theta_1}_{1^+}\phi_p\left({}^C_HD^{\xi_1}_{1^+}u(t)\right)=h_1(t),\ 1<t<e,\ 1<\theta_1,\xi_1\le 2,\\ {}^C_HD^{\theta_2}_{1^+}\phi_p\left({}^C_HD^{\xi_2}_{1^+}v(t)\right)=h_2(t),\ 1<t<e,\ 1<\theta_2,\xi_2\le 2,\\ u(1)=\phi_p\left({}^C_HD^{\xi_1}_{1^+}u(1)\right)=\phi_p\left({}^C_HD^{\xi_1}_{1^+}u(1)\right)'=0,\\ v(1)=\phi_q\left({}^C_HD^{\xi_2}_{1^+}v(1)\right)=\phi_q\left({}^C_HD^{\xi_2}_{1^+}v(1)\right)'=0,\\ u(e)=\displaystyle\int_1^e x(s)u(s)ds,\ v(e)=\int_1^e y(s)v(s)ds \end{cases}$$

has a unique solution

$$\begin{aligned} u(t)=&\frac{1}{\Gamma(\xi_1)}\int_1^t\left(\ln\frac{t}{s}\right)^{\xi_1-1}\phi_q\left({}^HI^{\theta_1}h_1(\tau)\right)\frac{ds}{s}+\frac{\ln t}{\Gamma(\xi_1)\left(1-M_1\right)}\\ &\times\left\{\int_1^e x(s)\int_1^s\left(\ln\frac{s}{\tau}\right)^{\xi_1-1}\phi_q\left(\frac{1}{\Gamma(\theta_1)}\int_1^\tau\left(\ln\frac{\tau}{r}\right)^{\theta_1-1}h_1(r)\frac{dr}{r}\right)\frac{d\tau}{\tau}ds\right.\\ &\left.-\int_1^t\left(\ln\frac{t}{s}\right)^{\xi_1-1}\phi_q\left({}^HI^{\theta_1}h_1(e)\right)\frac{ds}{s}\right\}, \end{aligned}\tag{2.1}$$

and

$$\begin{aligned} v(t)=&\frac{1}{\Gamma(\xi_2)}\int_1^t\left(\ln\frac{t}{s}\right)^{\xi_2-1}\phi_q\left({}^HI^{\theta_2}h_2(\tau)\right)\frac{ds}{s}+\frac{\ln t}{\Gamma(\xi_2)\left(1-M_2\right)}\\ &\times\left\{\int_1^e x(s)\int_1^s\left(\ln\frac{s}{\tau}\right)^{\xi_2-1}\phi_q\left(\frac{1}{\Gamma(\theta_2)}\int_1^\tau\left(\log\frac{\tau}{r}\right)^{\theta_2-1}h_2(r)\frac{dr}{r}\right)\frac{d\tau}{\tau}ds\right.\\ &\left.-\int_1^t\left(\ln\frac{t}{s}\right)^{\xi_2-1}\phi_q\left({}^HI^{\theta_2}h_2(e)\right)\frac{ds}{s}\right\}, \end{aligned}\tag{2.2}$$

where

$$M_1=\int_1^e x(s)\ln(s)ds<1, M_2=\int_1^e y(s)\ln(s)ds<1.$$

Proof. According to Lemma 2.2, we can obtain

$$\phi_p\left({}^C_HD^{\xi_1}_{1^+}u(t)\right)=\ {}^HI^{\theta_1}h_1(t)+c_{u1}+c_{u2}\ln t,\tag{2.3}$$

and

$$\phi_p\left({}^C_HD^{\xi_2}_{1^+}v(t)\right)=\ {}^HI^{\theta_2}h_2(t)+c_{v1}+c_{v2}\ln t.\tag{2.4}$$

By the boundary conditions

$$\phi_p\left({}_H^C D_{1^+}^{\xi_1} u(1)\right) = \phi_q\left({}_H^C D_{1^+}^{\xi_2} u(1)\right)' = 0,$$

and

$$\phi_p\left({}_H^C D_{1^+}^{\xi_2} v(1)\right) = \phi_q\left({}_H^C D_{1^+}^{\xi_2} v(1)\right)' = 0,$$

we have $c_{u1} = c_{u2} = c_{v1} = c_{v2} = 0$.

Then

$${}_H^C D_{1^+}^{\xi_1} u(t) = \phi_q\left({}^H I^{\theta_1} h_1(t)\right),$$

and

$${}_H^C D_{1^+}^{\xi_2} v(t) = \phi_q\left({}^H I^{\theta_2} h_2(t)\right).$$

In a similar way, the equations (2.3) and (2.4) can be written as

$$u(t) = {}^H I^{\xi_1} \phi_q\left({}^H I^{\theta_1} h_1(t)\right) + c_{u3} + c_{u4} \ln t, \tag{2.5}$$

and

$$v(t) = {}^H I^{\xi_2} \phi_q\left({}^H I^{\theta_2} h_2(t)\right) + c_{v3} + c_{v4} \ln t. \tag{2.6}$$

Since $u(1) = v(1) = 0$, then $c_{u3}, c_{v3} = 0$.

Consider the boundary conditions

$$u(e) = \int_1^e x(s)u(s)ds, v(e) = \int_1^e y(s)v(s)ds.$$

We can obtain that

$$\begin{aligned} u(e) = {}^H I^{\xi_1} \phi_q\left({}^H I^{\theta_1} h_1(e)\right) + c_{u4} &= \int_1^e x(s)u(s)ds \\ &= \int_1^e x(s)\left({}^H I^{\xi_1} \phi_q\left({}^H I^{\theta_1} h_1(s)\right) + c_{u4} \ln s\right) ds, \end{aligned}$$

and

$$\begin{aligned} v(e) = {}^H I^{\xi_2} \phi_q\left({}^H I^{\theta_2} h_2(e)\right) + c_{v4} &= \int_1^e y(s)v(s)ds \\ &= \int_1^e y(s)\left({}^H I^{\xi_2} \phi_q\left({}^H I^{\theta_2} h_2(s)\right) + c_{v4} \ln s\right) ds. \end{aligned}$$

Therefore

$$\begin{aligned} c_{u4} =& \frac{1}{\Gamma(\xi_1)(1-M_1)} \times \left\{ \int_1^e x(s) \int_1^s \left(\ln \frac{s}{\tau}\right)^{\xi_1 - 1} \phi_q\left({}^H I^{\theta_1} h_1(\tau)\right) \frac{d\tau}{\tau} ds \right. \\ & \left. - \int_1^t \left(\ln \frac{t}{s}\right)^{\xi_1 - 1} \phi_q\left({}^H I^{\theta_1} h_1(e)\right) \frac{ds}{s} \right\}, \end{aligned} \tag{2.7}$$

and

$$\begin{aligned} c_{v4} =& \frac{1}{\Gamma(\xi_2)(1-M_2)} \times \left\{ \int_1^e y(s) \int_1^s \left(\ln \frac{s}{\tau}\right)^{\xi_2 - 1} \phi_q\left({}^H I^{\theta_2} h_2(\tau)\right) \frac{d\tau}{\tau} ds \right. \\ & \left. - \int_1^t \left(\ln \frac{t}{s}\right)^{\xi_2 - 1} \phi_q\left({}^H I^{\theta_2} h_2(e)\right) \frac{ds}{s} \right\}. \end{aligned} \tag{2.8}$$

Substituting (2.7), (2.8) into (2.5), (2.6), we get (2.1) and (2.2). □

3. Main results

In this section, we set a operator for equation (1.1), and then give some sufficient conditions for the existence and uniqueness of solutions. In particular, two examples are simulated by using the iterative method.

Let $X = \{u : u \in C([1, e], \mathbb{R})\}$, $Y = \{v : v \in C([1, e], \mathbb{R})\}$, be the spaces with the norm $\|u\|_X = \max_{t\in[0,1]} |u(t)|$, $\|v\|_Y = \max_{t\in[0,1]} |v(t)|$. Define the norm $\|(u, v)\|_{X\times Y} = \|u\|_X + \|v\|_Y$ for any $(u, v) \in X \times Y$. Obviously, $(X \times Y, \|\cdot\|)$ is a Banach space.

Define the operator $A : X \times Y \to X \times Y$ as follows:

$$A(u, v)(t) = (A_1u(t), A_2v(t)),$$

where

$$\begin{aligned} A_1u(t) =& \frac{1}{\Gamma(\xi_1)} \int_1^t \left(\ln \frac{t}{s}\right)^{\xi_1-1} \phi_q \left({}^H I^{\theta_1} f(\tau, u(\tau), v(\tau))\right) \frac{ds}{s} + \frac{\ln t}{\Gamma(\xi_1)\,(1 - M_1)} \\ &\times \left\{ \int_1^e x(s) \int_1^s \left(\ln \frac{s}{\tau}\right)^{\xi_1-1} \phi_q \left(\frac{1}{\Gamma(\theta_1)} \int_1^\tau \left(\ln \frac{\tau}{r}\right)^{\theta_1-1} \right.\right. \\ &\left.\left. f(r, u(r), v(r)) \frac{dr}{r} \right) \frac{d\tau}{\tau} ds - \int_1^t \left(\ln \frac{t}{s}\right)^{\xi_1-1} \phi_q \left({}^H I^{\theta_1} g(e)\right) \frac{ds}{s} \right\}, \end{aligned}$$

and

$$\begin{aligned} A_2v(t) =& \frac{1}{\Gamma(\xi_2)} \int_1^t \left(\ln \frac{t}{s}\right)^{\xi_2-1} \phi_q \left({}^H I^{\theta_2} g(\tau, u(\tau), v(\tau))\right) \frac{ds}{s} + \frac{\ln t}{\Gamma(\xi_2)\,(1 - M_2)} \\ &\times \left\{ \int_1^e y(s) \int_1^s \left(\ln \frac{s}{\tau}\right)^{\xi_2-1} \phi_q \left(\frac{1}{\Gamma(\theta_2)} \int_1^\tau \left(\ln \frac{\tau}{r}\right)^{\theta_2-1} \right.\right. \\ &\left.\left. g(r, u(r), v(r)) \frac{dr}{r} \right) \frac{d\tau}{\tau} ds - \int_1^t \left(\ln \frac{t}{s}\right)^{\xi_2-1} \phi_q \left({}^H I^{\theta_2} g(e)\right) \frac{ds}{s} \right\}. \end{aligned}$$

In order to establish main results, we need the following assumptions.

(H1) There exist two positive constants L_1, L_2 such that

$$\begin{aligned} L_1 &= \max_{t\in[1,e]} |f(t, u(t), v(t))|, \\ L_2 &= \max_{t\in[1,e]} |g(t, v(t), u(t))|. \end{aligned}$$

(H2) There exist positive functions $\varphi_1(t), \varphi_2(t)$ such that

$$\begin{aligned} |f(t, x, y, z) - f(t, x', y', z')| &\le \varphi_1(t) (|x - x'| + |y - y'| + |z - z'|), \\ |g(t, x, y, z) - g(t, x', y', z')| &\le \varphi_2(t) (|x - x'| + |y - y'| + |z - z'|), \end{aligned}$$

for all $t \in [1, e]$ and $u, x, y, z, u', x', y', z' \in \mathbb{R}$. Let $\alpha = \max_{t\in[1,e]} |f(t, 0, 0)|$, $\beta = \max_{t\in[1,e]} |g(t, 0, 0)|$.

Theorem 3.1. *Assume $f, g : [1, e] \times \mathbb{R}^3 \to \mathbb{R}$ are continuous functions, and* (H1) *hold, the coupled system of p-Laplacian fractional differential equation (*1.1) has at least one solution.

Proof. Let

$$\Omega_\omega = \{(u,v) : (u,v) \in X \times Y, \| (u,v) \|_{X\times Y} \le \omega\},$$

$\omega = \omega_1 + \omega_2$, where

$$\omega_1 = \frac{2 - M_1 + \int_1^e x(s)ds}{\Gamma(\xi_1+1)(1-M_1)} \phi_q \left(\frac{L_1}{\Gamma(\theta_1+1)} \right),$$

and

$$\omega_2 = \frac{2 - M_2 + \int_1^e y(s)ds}{\Gamma(\xi_2+1)(1-M_2)} \phi_q \left(\frac{L_2}{\Gamma(\theta_2+1)} \right).$$

Since f, g are continuous functions, then A_1, A_2 are also continuous, it implies that A is continuous. The proof has two steps.

1). $A : \Omega_\omega \to \Omega_\omega$ is uniformly bounded.

According to (H2), for any $(u,v) \in \Omega_\omega$, we have

$$\begin{aligned}
&\|A_1 u(t)\|_X \\
\le& \frac{1}{\Gamma(\xi_1)} \int_1^t \left(\ln \frac{t}{s} \right)^{\xi_1 - 1} \phi_q \left(\frac{L_1}{\Gamma(\theta_1)} \int_1^s \left(\ln \frac{s}{\tau} \right)^{\theta_1 - 1} \frac{d\tau}{\tau} \right) \frac{ds}{s} + \frac{\ln t}{\Gamma(\xi_1)(1-M_1)} \\
&\times \left\{ \int_1^e x(s) \int_1^s \left(\ln \frac{s}{\tau} \right)^{\xi_1 - 1} \phi_q \left(\frac{L_1}{\Gamma(\theta_1)} \int_1^\tau \left(\ln \frac{\tau}{r} \right)^{\theta_1 - 1} \frac{dr}{r} \right) \frac{d\tau}{\tau} ds \right. \\
&\left. + \int_1^t \left(\ln \frac{t}{s} \right)^{\xi_1 - 1} \phi_q \left(\frac{L_1}{\Gamma(\theta_1)} \int_1^e \left(\ln \frac{e}{\tau} \right)^{\theta_1 - 1} \frac{d\tau}{\tau} \right) \frac{ds}{s} \right\} \\
\le& \frac{1}{\Gamma(\xi_1)} \int_1^t \left(\ln \frac{t}{s} \right)^{\xi_1 - 1} \phi_q \left(\frac{L_1}{\Gamma(\theta_1+1)} \right) \frac{ds}{s} \\
&+ \frac{\ln t}{\Gamma(\xi_1)(1-M_1)} \times \left\{ \int_1^e x(s) \int_1^s \left(\ln \frac{s}{\tau} \right)^{\xi_1 - 1} \phi_q \left(\frac{L_1}{\Gamma(\theta_1+1)} \right) \frac{d\tau}{\tau} ds \right. \\
&\left. + \int_1^t \left(\ln \frac{t}{s} \right)^{\xi_1 - 1} \phi_q \left(\frac{L_1}{\Gamma(\theta_1+1)} \right) \frac{ds}{s} \right\} \\
\le& \frac{1}{\Gamma(\xi_1+1)} \phi_q \left(\frac{L_1}{\Gamma(\theta_1+1)} \right) + \frac{\int_1^e x(s) \phi_q \left(\frac{L_1}{\Gamma(\theta_1+1)} \right) ds + \phi_q \left(\frac{L_1}{\Gamma(\theta_1+1)} \right)}{\Gamma(\xi_1+1)(1-M_1)} \\
\le& \frac{2 - M_1 + \int_1^e x(s)ds}{\Gamma(\xi_1+1)(1-M_1)} \phi_q \left(\frac{L_1}{\Gamma(\theta_1+1)} \right) = \omega_1.
\end{aligned}$$

In a similar way, we can get that

$$\|A_2 v(t)\|_Y \le \frac{2 - M_2 + \int_1^e y(s)ds}{\Gamma(\xi_2+1)(1-M_2)} \phi_q \left(\frac{L_2}{\Gamma(\theta_2+1)} \right) = \omega_2.$$

Hence

$$\|A(u,v)(t)\|_{X\times Y} = \|A_1 u(t)\|_X + \|A_2 v(t)\|_Y \le \omega,$$

which implies that $A : \Omega_\omega \to \Omega_\omega$ is uniformly bounded.

2). A is equicontinuous.

Let $t_1, t_2 \in [1,e]$ and $t_1 < t_2$, we have

$$|A_1 u(t_2) - A_1 u(t_1)|$$

$$\leq \frac{1}{\Gamma(\xi_1)} \int_1^{t_1} \left(\left(\ln \frac{t_2}{s} \right)^{\xi_1 - 1} - \left(\ln \frac{t_1}{s} \right)^{\xi_1 - 1} \right) \phi_q \left(\frac{L_1}{\Gamma(\theta_1 + 1)} \right) \frac{ds}{s}$$
$$+ \frac{1}{\Gamma(\xi_1)} \int_{t_1}^{t_2} \left(\ln \frac{t_2}{s} \right)^{\xi_1 - 1} \phi_q \left(\frac{L_1}{\Gamma(\theta_1 + 1)} \right) \frac{ds}{s}$$
$$+ \frac{\ln t_2 - \ln t_1}{\Gamma(\xi_1)(1 - M_1)} \times \left\{ \int_1^e x(s) \int_1^s \left(\ln \frac{s}{\tau} \right)^{\xi_1 - 1} \phi_q \left(\frac{L_1}{\Gamma(\theta_1 + 1)} \right) \frac{d\tau}{\tau} ds \right.$$
$$+ \int_1^{t_1} \left(\left(\ln \frac{t_2}{s} \right)^{\xi_1 - 1} - \left(\ln \frac{t_1}{s} \right)^{\xi_1 - 1} \right) \phi_q \left(\frac{L_1}{\Gamma(\theta_1 + 1)} \right) \frac{ds}{s}$$
$$\left. + \int_{t_1}^{t_2} \left(\ln \frac{t_2}{s} \right)^{\xi_1 - 1} \phi_q \left(\frac{L_1}{\Gamma(\theta_1 + 1)} \right) \frac{ds}{s} \right\}.$$

Then, if $t_2 \to t_1$, we have $|A_1 u(t_2) - A_1 u(t_1)| \to 0$.

Similarly, if $t_2 \to t_1$, we can obtain that $|A_2 v(t_2) - A_2 v(t_1)| \to 0$, it shows that A is equicontinuous. By Arzela-Ascoli's theorem, the operator A is compact on Ω_ω.

Hence, all the conditions of Lemma 2.4 are satisfied, the coupled system of p-Laplacian fractional differential equation (1.1) has at least one solution. □

Example 3.1. Consider the following coupled system of p-Laplacian fractional differential equation

$$\begin{cases} {}^C_H D^{\frac{7}{4}}_{1^+} \phi_p \left({}^C_H D^{\frac{3}{2}}_{1^+} u(t) \right) = \dfrac{t^2}{20\pi} \left(1 + \dfrac{1}{1 + u(t)} + \dfrac{1}{1 + v(t)} \right), \ 1 \leq t \leq e, \\ {}^C_H D^{\frac{5}{3}}_{1^+} \phi_p \left({}^C_H D^{\frac{4}{3}}_{1^+} v(t) \right) = \dfrac{\sqrt{4t}}{3} \left(2 + \dfrac{5}{1 + u(t) + v(t)} \right), \ 1 \leq t \leq e, \\ u(1) = \phi_p \left({}^C_H D^{\frac{3}{2}}_{1^+} u(1) \right) = \phi_p \left({}^C_H D^{\frac{3}{2}}_{1^+} u(1) \right)' = 0, \\ v(1) = \phi_q \left({}^C_H D^{\frac{4}{3}}_{1^+} v(1) \right) = \phi_q \left({}^C_H D^{\frac{4}{3}}_{1^+} v(1) \right)' = 0, \\ u(e) = \displaystyle\int_1^e x(s) u(s) ds, \ v(e) = \int_1^e y(s) v(s) ds. \end{cases} \tag{3.1}$$

Let $\theta_1 = \frac{7}{4}$, $\theta_2 = \frac{5}{3}$, $\xi_1 = \frac{3}{2}$, $\xi_2 = \frac{4}{3}$, $x(s) = \frac{1}{2}$, $y(s) = \frac{1}{3}, p = q = 2$,

$$f(t, v(t), u(t)) = \frac{t^2}{20\pi} \left(1 + \frac{1}{1 + u(t)} + \frac{1}{1 + v(t)} \right) \leq \frac{3e^2}{20\pi},$$

and

$$g(t, v(t), u(t)) = \frac{\sqrt{4t}}{3} \left(2 + \frac{5}{1 + u(t) + v(t)} \right) \leq \frac{7\sqrt{4e}}{3}.$$

We get that $L_1 = \frac{3e^2}{20\pi}$, $L_2 = \frac{7\sqrt{4e}}{3}$, $M_1 = \frac{1}{2} \int_1^e log(s) ds = \frac{1}{2}$, $M_2 = \frac{1}{3} \int_1^e log(s) ds = \frac{1}{3}$, $\omega_1 \approx 0.7786$, $\omega_2 \approx 14.427$, $\omega \approx 15.2056$.

According to Theorem 3.1, the coupled system of p-Laplacian fractional differential equation (3.1) at least has one solution.

Now, we use the iterative method that proposed by Wei et al in [21] to simulate this process and give the iterative error.

Let

$$\nu(t) = f(t, u(t), v(t)), \ \nu_0(t) = f(t, 0, 0) = \frac{3t^2}{20\pi},$$

and

$$\mu(t) = g\left(t, v(t), u(t)\right),\ \mu_0(t) = g(t,0,0) = \frac{7\sqrt{4t}}{3}.$$

By Lemma 2.6, we have

$$\begin{aligned} u(t) =& \frac{1}{\Gamma(\frac{3}{2})}\int_1^t \left(\ln\frac{t}{s}\right)^{\frac{1}{2}} \phi_q\left({}^H I^{\frac{7}{4}}\nu(\tau)\right)\frac{ds}{s} + \frac{\ln t}{\frac{1}{2}\times\Gamma(\frac{3}{2})} \\ &\times\left\{\frac{1}{2}\int_1^e\int_1^s\left(\ln\frac{s}{\tau}\right)^{\frac{1}{2}}\phi_q\left(\frac{1}{\Gamma(\frac{7}{4})}\int_1^\tau\left(\ln\frac{\tau}{r}\right)^{\frac{3}{4}}\nu(r)\frac{dr}{r}\right)\frac{d\tau}{\tau}ds\right. \\ &\left.-\int_1^t\left(\ln\frac{t}{s}\right)^{\frac{1}{2}}\phi_q\left(\frac{1}{\Gamma(\frac{7}{4})}\int_1^e\left(\ln\frac{e}{\tau}\right)^{\frac{3}{4}}\nu(\tau)\frac{d\tau}{\tau}\right)\frac{ds}{s}\right\}, \end{aligned}$$

and

$$\begin{aligned} v(t) =& \frac{1}{\Gamma(\frac{4}{3})}\int_1^t \left(\ln\frac{t}{s}\right)^{\frac{1}{3}} \phi_q\left({}^H I^{\frac{5}{3}}\mu(\tau)\right)\frac{ds}{s} + \frac{\ln t}{\frac{2}{3}\times\Gamma(\frac{4}{3})} \\ &\times\left\{\frac{1}{3}\int_1^e\int_1^s\left(\ln\frac{s}{\tau}\right)^{\frac{1}{3}}\phi_q\left(\frac{1}{\Gamma(\frac{5}{3})}\int_1^\tau\left(\ln\frac{\tau}{r}\right)^{\frac{2}{3}}\mu(r)\frac{dr}{r}\right)\frac{d\tau}{\tau}ds\right. \\ &\left.-\int_1^t\left(\ln\frac{t}{s}\right)^{\frac{1}{3}}\phi_q\left(\frac{1}{\Gamma(\frac{5}{3})}\int_1^e\left(\ln\frac{e}{\tau}\right)^{\frac{2}{3}}\mu(\tau)\frac{d\tau}{\tau}\right)\frac{ds}{s}\right\}. \end{aligned}$$

The iteration formulas are

$$\nu_{k+1}(t) = f(t, u_k(t), v_k(t)) = \frac{t^2}{20\pi}\left(1 + \frac{1}{1+u_k(t)} + \frac{1}{1+v_k(t)}\right),$$

and

$$\mu_{k+1}(t) = g(t, v_k(t), u_k(t)) = \frac{\sqrt{4t}}{3}\left(2 + \frac{5}{1+u_k(t)+v_k(t)}\right).$$

The iterative process and iterative error values is given in Figure 1 and Table 1.

Table 1. iterative error values $E(u_k)$ and $E(v_k)$

k	2	5	8	12	15
$E(u_k)$	5.7776e-02	2.0016e-05	5.8581e-08	2.3974e-12	1.7764e-15
$E(v_k)$	2.8563e-01	1.4098e-04	2.8096e-07	7.8142e-12	9.7700e-15

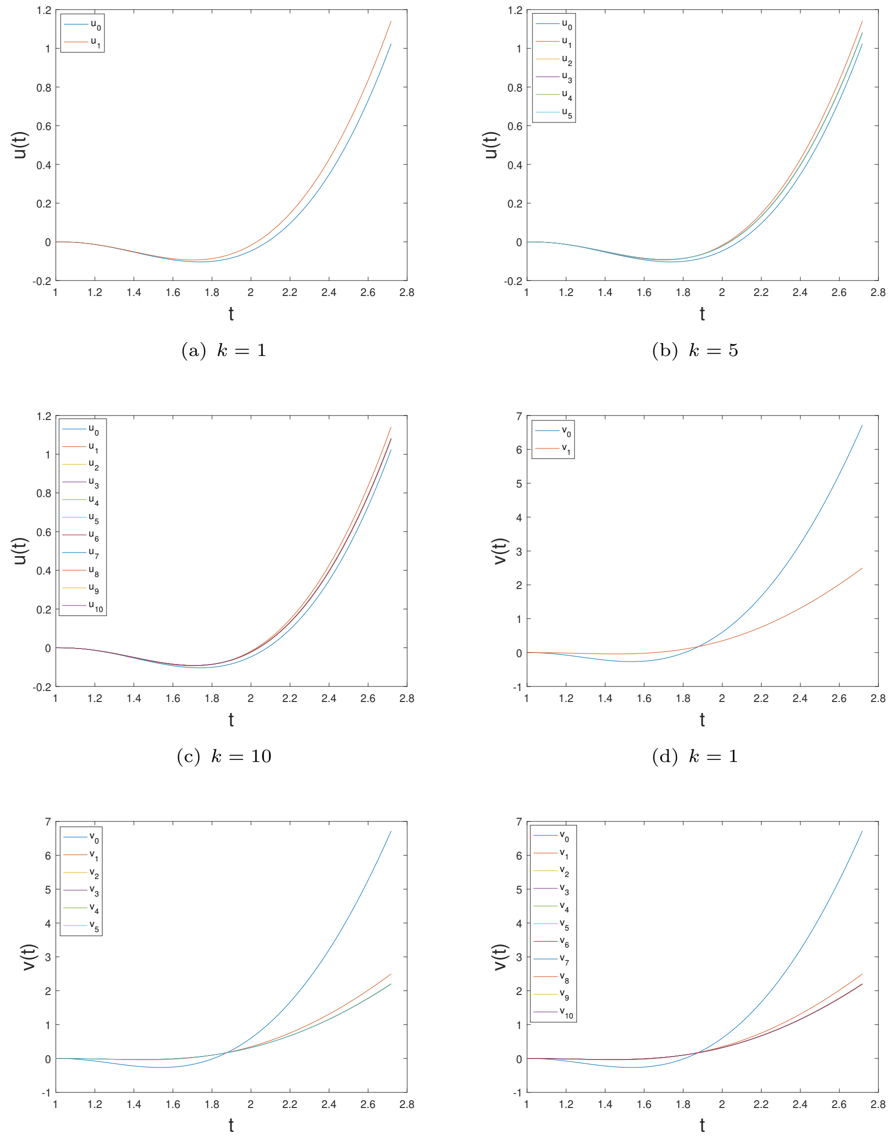
By Figure 1 and Table 1, we find the iterative process of u_k and v_k are convergent. Thus, the coupled system of p-Laplacian fractional differential equation (3.1) at least have one solution.

Theorem 3.2. *Assume $f, g : [1, e]\times\mathbb{R}^3 \to \mathbb{R}$ are continuous functions,* (H2) *hold, and there exists a constant ρ such that $\rho = \rho_1 + \rho_2 < 1$, where*

$$\rho_1 = \frac{(q-1)N_1^{q-2}\varphi_1 \times \left\{\int_1^e x(s)ds + 2 - M_1\right\}}{\Gamma(\xi_1+1)\Gamma(\theta_1+1)\left(1-M_1\right)},$$

and

$$\rho_2 = \frac{(q-1)N_2^{q-2}\varphi_2 \times \left\{\int_1^e y(s)ds + 2 - M_2\right\}}{\Gamma(\xi_2+1)\Gamma(\theta_2+1)\left(1-M_2\right)}.$$



(a) $k = 1$ (b) $k = 5$

(c) $k = 10$ (d) $k = 1$

(e) $k = 5$ (f) $k = 10$

Figure 1. iterative process u_k and v_k

the coupled system of p*-Laplacian fractional differential equation (*1.1) has a unique solution.

Proof. Let

$$\Omega_R = \{(u, v) : (u, v) \in X \times Y, \| (u, v) \|_{X \times Y} \leq R\},$$

where

$$R^{q-2} \leq 2\max\left\{\frac{(1-M_1)\Gamma(\xi_1+1)\Gamma^{q-1}(\theta_1+1)}{K_1^{q-1}\left(2-M_1+\int_1^e x(s)ds\right)}, \frac{(1-M_2)\Gamma(\xi_2+1)\Gamma^{q-1}(\theta_2+1)}{K_2^{q-1}\left(2-M_2+\int_1^e y(s)ds\right)}\right\}.$$

The proof has two steps.

1). $A\Omega_R \subseteq \Omega_R$.

By condition (H2), for any $(u,v) \in \Omega_R$, there are constants K_1, K_2 such that

$$\begin{aligned}|f(t,u(t),v(t))| \leq & |f(t,u(t),v(t)) - f(t,0,0)| + |f(t,0,0)| \\ \leq & \varphi_1(|u(t)|+|v(t)|) + \alpha \\ \leq & \varphi_1 R + \alpha \leq K_1 R,\end{aligned}$$

and

$$\begin{aligned}|g(t,v(t),u(t))| \leq & |g(t,v(t),u(t)) - g(t,0,0)| + |g(t,0,0)| \\ \leq & \varphi_2(|v(t)|+|u(t)|) + \beta \\ \leq & \varphi_2 R + \beta \leq K_2 R.\end{aligned}$$

Consequently, we can obtain that

$$|A_1 u(t)| \leq \frac{2-M_1+\int_1^e x(s)ds}{\Gamma(\xi_1+1)(1-M_1)}\phi_q\left(\frac{K_1 R}{\Gamma(\theta_1+1)}\right),$$

which shows that

$$\|A_1 u(t)\|_X \leq \frac{2-M_1+\int_1^e x(s)ds}{\Gamma(\xi_1+1)(1-M_1)}\phi_q\left(\frac{K_1 R}{\Gamma(\theta_1+1)}\right) \leq \frac{R}{2}.$$

Furthermore

$$\|A_2 v(t)\|_Y \leq \frac{2-M_2+\int_1^e y(s)ds}{\Gamma(\xi_2+1)(1-M_2)}\phi_q\left(\frac{K_2 R}{\Gamma(\theta_2+1)}\right) \leq \frac{R}{2}.$$

Hence

$$\|A(u,v)(t)\|_{X\times Y} = \|A_1 u(t)\|_X + \|A_2 v(t)\|_Y \leq R,$$

it implies that $A\Omega_R \subseteq \Omega_R$.

2). A is a contraction mapping.

Let

$$\begin{aligned}N_1 \geq & \left|\frac{1}{\Gamma(\theta_1)}\int_1^e \left(\ln\frac{e}{s}\right)^{\theta_1-1} f(t,u(t),v(t))\frac{ds}{s}\right|, \\ N_2 \geq & \left|\frac{1}{\Gamma(\theta_2)}\int_1^e \left(\ln\frac{e}{s}\right)^{\theta_2-1} g(t,v(t),u(t))\frac{ds}{s}\right|.\end{aligned}$$

Since $1 < p \leq 2$, then $q \geq 2$. By Lemma 2.3 and (H2), for any $(u,v) \in \Omega_R$, $t \in [1,e]$, we have

$$\begin{aligned}&\|A_1(u,v)(t) - A_1(u',v')(t))\|_X \\ \leq & \frac{1}{\Gamma(\xi_1)}\int_1^t \left(\ln\frac{t}{s}\right)^{\xi_1-1}\left|\phi_q\left(\frac{1}{\Gamma(\theta_1)}\int_1^s \left(\ln\frac{s}{\tau}\right)^{\theta_1-1} f(\tau,u(\tau),v(\tau))\frac{d\tau}{\tau}\right)\right.\end{aligned}$$

$$
\begin{aligned}
&- \phi_q\left(\frac{1}{\Gamma(\theta_1)}\int_1^s\left(\ln\frac{s}{\tau}\right)^{\theta_1-1} f\left(\tau,u'(\tau),v'(\tau)\right)\frac{d\tau}{\tau}\right)\Bigg|\frac{ds}{s} + \frac{\ln t}{\Gamma(\xi_1)\left(1-M_1\right)}\\
&\times\left\{\int_1^e x(s)\int_1^s\left(\ln\frac{s}{\tau}\right)^{\xi_1-1}\Bigg|\phi_q\left(\frac{1}{\Gamma(\theta_1)}\int_1^\tau\left(\ln\frac{\tau}{r}\right)^{\theta_1-1} f\left(r,u(r),v(r)\right)\frac{dr}{r}\right)\right.\\
&- \phi_q\left(\frac{1}{\Gamma(\theta_1)}\int_1^\tau\left(\ln\frac{\tau}{r}\right)^{\theta_1-1} f\left(r,u'(r),v'(r)\right)\frac{dr}{r}\right)\Bigg|\frac{d\tau}{\tau}ds\\
&+\int_1^t\left(\ln\frac{t}{s}\right)^{\xi_1-1}\Bigg|\phi_q\left(\frac{1}{\Gamma(\theta_1)}\int_1^e\left(\ln\frac{e}{\tau}\right)^{\theta_1-1} f\left(\tau,u(\tau),v(\tau)\right)\frac{d\tau}{\tau}\right)\\
&\left.- \phi_q\left(\frac{1}{\Gamma(\theta_1)}\int_1^e\left(\ln\frac{e}{\tau}\right)^{\theta_1-1} f\left(\tau,u'(\tau),v'(\tau)\right)\frac{d\tau}{\tau}\right)\Bigg|\frac{ds}{s}\right\}\\
\le&\frac{(q-1)N_1^{q-2}}{\Gamma(\xi_1)\Gamma(\theta_1+1)}\int_1^t\left(\ln\frac{t}{s}\right)^{\xi_1-1}\varphi_1\left(|u-u'|+|v-v'|\right)\frac{ds}{s}\\
&+\frac{(q-1)N_1^{q-2}}{\Gamma(\xi_1)\Gamma(\theta_1+1)\left(1-M_1\right)}\times\left\{\int_1^e x(s)\int_1^s\left(\ln\frac{s}{\tau}\right)^{\xi_1-1}\varphi_1(|u-u'|\right.\\
&\left.+|v-v'|)\frac{d\tau}{\tau}ds+\int_1^t\left(\ln\frac{t}{s}\right)^{\xi_1-1}\varphi_1\left(|u-u'|+|v-v'|\right)\frac{ds}{s}\right\}\\
\le&\frac{(q-1)N_1^{q-2}\varphi_1\|(u,v)-(u',v')\|}{\Gamma(\xi_1+1)\Gamma(\theta_1+1)}\\
&+\frac{(q-1)N_1^{q-2}\varphi_1\|(u,v)-(u',v')\|\times\left\{\int_1^e x(s)ds+1\right\}}{\Gamma(\xi_1+1)\Gamma(\theta_1+1)\left(1-M_1\right)}\\
=&\frac{(q-1)N_1^{q-2}\varphi_1\times\left\{\int_1^e x(s)ds+2-M_1\right\}}{\Gamma(\xi_1+1)\Gamma(\theta_1+1)\left(1-M_1\right)}\|(u,v)-(u',v')\|\\
=&\rho_1\|(u,v)-(u',v')\|.
\end{aligned}
$$

In a similar process, we get that

$$
\begin{aligned}
&\|A_2(u,v)(t)-A_2(u',v')(t))\|_Y\\
\le&\frac{(q-1)N_2^{q-2}\varphi_2\times\left\{\int_1^e y(s)ds+2-M_2\right\}}{\Gamma(\xi_2+1)\Gamma(\theta_2+1)\left(1-M_2\right)}\|(u,v)-(u',v')\|\\
=&\rho_2\|(u,v)-(u',v')\|.
\end{aligned}
$$

Then

$$
\begin{aligned}
\|A(u,v)(t)-A(u',v')(t)\| \le&\rho_1\|(u,v)-(u',v')\|+\rho_2\|(u,v)-(u',v')\|\\
=&\rho\|(u,v)-(u',v')\|.
\end{aligned}
$$

Hence A is a contraction mapping. By using the Banach contraction principle, the coupled system of p-Laplacian fractional differential equation (1.1) has a unique solution. □

Example 3.2. Consider the following coupled system of p-Laplacian fractional differential equation

$$\begin{cases} {}^C_H D^{\frac{5}{4}}_{1^+}\phi_p\left({}^C_H D^{\frac{3}{2}}_{1^+}u(t)\right) = \dfrac{t^2}{40}(1+u(t)+v(t)),\ 1\le t\le e,\\ {}^C_H D^{\frac{3}{2}}_{1^+}\phi_p\left({}^C_H D^{\frac{11}{10}}_{1^+}v(t)\right) = \dfrac{\sqrt{t}}{10}(1+u(t)+v(t)),\ 1\le t\le e,\\ u(1) = \phi_p\left({}^C_H D^{\frac{3}{2}}_{1^+}u(1)\right) = \phi_p\left({}^C_H D^{\frac{3}{2}}_{1^+}u(1)\right)' = 0,\\ v(1) = \phi_q\left({}^C_H D^{\frac{11}{10}}_{1^+}v(1)\right) = \phi_q\left({}^C_H D^{\frac{11}{10}}_{1^+}v(1)\right)' = 0,\\ u(e) = \displaystyle\int_1^e x(s)u(s)ds,\ v(e) = \int_1^e y(s)v(s)ds. \end{cases} \tag{3.2}$$

Where $\theta_1 = \frac{5}{4},\ \theta_2 = \frac{3}{2},\ \xi_1 = \frac{3}{2},\ \xi_2 = \frac{11}{10},\ x(s) = \frac{1}{50},\ y(s) = \frac{1}{100}, p = q = 2,$ $M_1 = \frac{1}{50},\ M_2 = \frac{1}{100}$.

We have

$$|f(t,u,v) - f(t,u',v')| \le \frac{e^2}{40}\left(|u-u'| + |v-v'|\right),$$

and

$$|g(t,u,v) - g(t,u',v')| \le \frac{\sqrt{e}}{10}\left(|u-u'| + |v-v'|\right),$$

$\varphi_1 = \frac{e^2}{40},\ \varphi_2 = \frac{\sqrt{e}}{10},\ \rho_1 \approx 0.2521,\ \rho_2 \approx 0.2403,\ \rho \approx 0.4915 < 1.$

Thus, all the conditions of Theorem 3.2 are satisfied, it show that the coupled system of p-Laplacian fractional differential equation (3.2) has a unique solution.

Now, we use the iterative method to simulate the example, and give the iterative process and iterative error.

Let

$$\nu(t) = f\left(t, u(t), v(t)\right),\ \nu_0(t) = f(t,0,0) = \frac{t^2}{40},$$

and

$$\mu(t) = g\left(t, v(t), u(t)\right),\ \mu_0(t) = g(t,0,0) = \frac{\sqrt{t}}{10}.$$

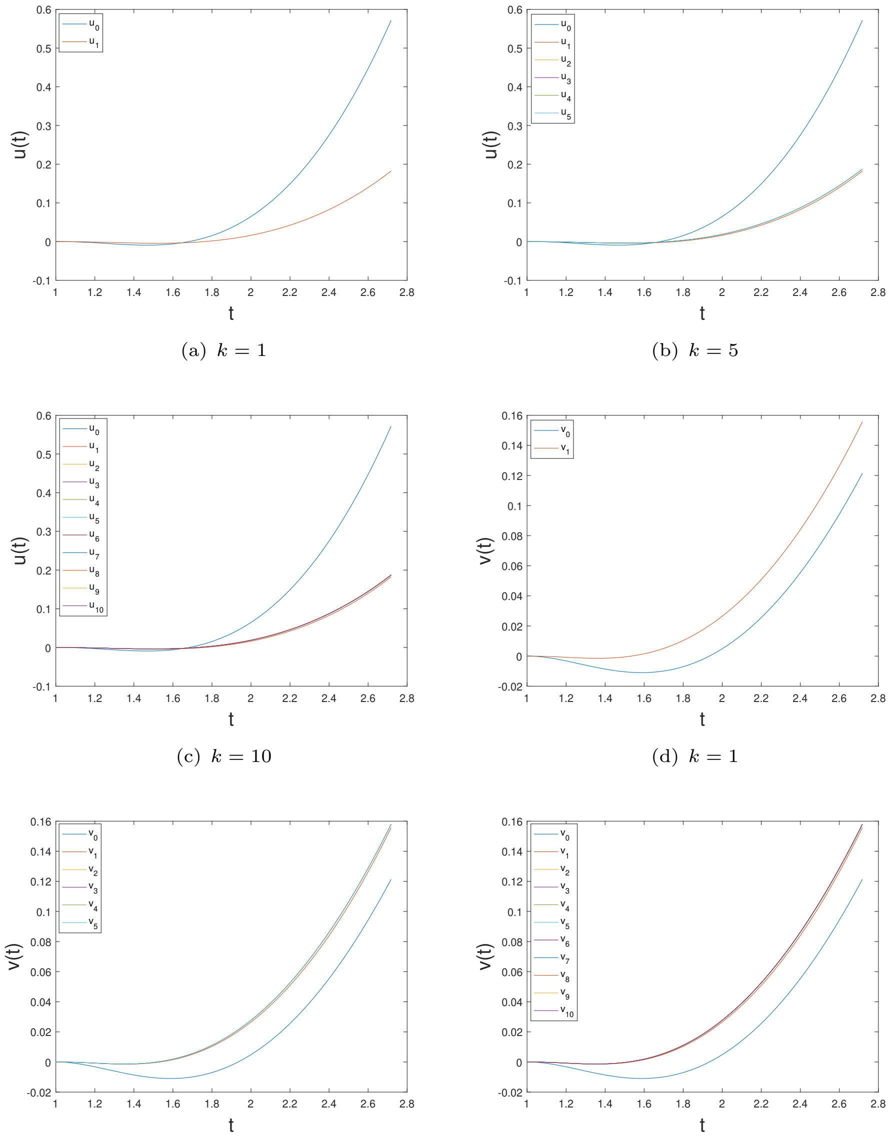
By Lemma 2.6, we have

$$\begin{aligned} u(t) =& \frac{1}{\Gamma(\frac{3}{2})}\int_1^t \left(\ln\frac{t}{s}\right)^{\frac{1}{2}} \phi_q\left({}^H I^{\frac{5}{4}}\nu(\tau)\right)\frac{ds}{s} + \frac{\ln t}{\frac{1}{2}\times\Gamma(\frac{3}{2})}\\ &\times\left\{\frac{1}{2}\int_1^e\int_1^s \left(\ln\frac{s}{\tau}\right)^{\frac{1}{2}}\phi_q\left(\frac{1}{\Gamma(\frac{5}{4})}\int_1^\tau\left(\log\frac{\tau}{r}\right)^{\frac{1}{4}}\nu(r)\frac{dr}{r}\right)\frac{d\tau}{\tau}ds\right.\\ &\left.-\int_1^t\left(\log\frac{t}{s}\right)^{\frac{1}{2}}\phi_q\left(\frac{1}{\Gamma(\frac{5}{4})}\int_1^e\left(\ln\frac{e}{\tau}\right)^{\frac{1}{4}}\nu(\tau)\frac{d\tau}{\tau}\right)\frac{ds}{s}\right\}, \end{aligned}$$

and

$$\begin{aligned} v(t) =& \frac{1}{\Gamma(\frac{11}{10})}\int_1^t \left(\ln\frac{t}{s}\right)^{\frac{1}{10}} \phi_q\left({}^H I^{\frac{3}{2}}\mu(\tau)\right)\frac{ds}{s} + \frac{\ln t}{\frac{2}{3}\times\Gamma(\frac{11}{10})}\\ &\times\left\{\frac{1}{3}\int_1^e\int_1^s \left(\log\frac{s}{\tau}\right)^{\frac{1}{10}}\phi_q\left(\frac{1}{\Gamma(\frac{3}{2})}\int_1^\tau\left(\ln\frac{\tau}{r}\right)^{\frac{1}{2}}\mu(r)\frac{dr}{r}\right)\frac{d\tau}{\tau}ds\right. \end{aligned}$$

$$- \int_1^t \left(\ln \frac{t}{s} \right)^{\frac{1}{10}} \phi_q \left(\frac{1}{\Gamma(\frac{3}{2})} \int_1^e \left(\ln \frac{e}{\tau} \right)^{\frac{1}{2}} \mu(\tau) \frac{d\tau}{\tau} \right) \frac{ds}{s} \Bigg\}.$$



(a) $k = 1$ (b) $k = 5$

(c) $k = 10$ (d) $k = 1$

(e) $k = 5$ (f) $k = 10$

Figure 2. iterative process u_k and v_k

The iteration formulas are

$$\nu_{k+1}(t) = f(t, u_k(t), v_k(t)) = \frac{t^2}{40} (1 + u_k(t) + v_k(t)),$$

and

$$\mu_{k+1}(t) = g(t, v_k(t), u_k(t)) = \frac{\sqrt{t}}{10}(1 + u_k(t) + v_k(t)).$$

The iterative process and iterative error values is given in Figure 2 and Table 2.

Table 2. iterative error values $E(u_k)$ and $E(v_k)$

k	2	4	6	8	10
$E(u_k)$	5.1460e-03	7.4105e-06	6.8046e-09	7.6688e-12	8.8818e-15
$E(v_k)$	2.4443e-03	3.9906e-06	3.0372e-09	4.2119e-12	3.9968e-15

By Figure 2 and Table 2, we find the iterative process of u_k and v_k are convergent. Thus, the coupled system of p-Laplacian fractional differential equation (3.2) at least have one solution.

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