Journal of Applied Analysis and Computation
Volume 12, Number 5, October 2022, 1901–1921
Website:http://www.jaac-online.com
DOI:10.11948/20210385

ON SOLUTIONS OF INFINITE SYSTEMS OF INTEGRAL EQUATIONS COORDINATEWISE CONVERGING AT INFINITY

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Abstract In the paper we are going to prove the existence of solutions of an infinite system of nonlinear quadratic integral equations of Volterra-Hammerstein type. Those solutions are continuous and bounded functions defined on the real half-axis $\mathbb{R}_+$ and created by function sequences which are coordinatewise converging to proper limits at infinity. Considerations of the paper are located in the Banach space consisting of functions defined, continuous and bounded on $\mathbb{R}_+$ with values in the space of real bounded sequences. The main tool applied in the paper is the technique of measures of noncompactness.

Keywords Function sequence, Banach space, infinite system of integral equations, measure of noncompactness, fixed point theorem.

MSC(2010) 45G15, 47H08.

1. Introduction

Integral equations play an important role in the description of several events appearing in the real world. Namely, they can be encountered in applications to exact sciences (mathematics, physics, astronomy, thermodynamics, hydrodynamics, propagation of waves) and also in biology, economics etc. We refer to monographs [9, 12, 17, 20, 21], where various aspects of applications of integral equations are discussed.

In the paper we focus on the discussion of infinite systems of integral equations which recently have turned a great attention of a lot of researchers working in nonlinear functional analysis and in numerous applications in various fields. Infinite systems of integral equations can be applied when we consider some problems described by partial differential equations (cf. [11, 20]). Then, we can express such a problem with help of a nonlinear integral equation, where an unknown function depends on two (or more) variables. Obviously, such an integral equation is very hard to solve. In order to facilitate the desired solvability of such an equation we can apply the method of an infinite system of integral equations, where an unknown function is represented by a sequence of functions depending on one variable.

As the second example of the appearance of infinite system of integral equations we can consider the infinite system of differential equations associated with the birth

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and death stochastic process (cf. [11, 13, 14]). Obviously, in some models of those stochastic processes we obtain infinite systems of nonlinear differential equations. Equivalently, we can investigate infinite systems of nonlinear integral equations in place of infinite systems of differential equations. It is worthwhile mentioning that infinite systems of differential and integral equations can be encountered in the theory of boundary value problems for ordinary differential equations and in the theory of integral equations of fractional orders (cf. [10, 19], for example).

Investigations of infinite systems of integral equations are connected with the representation of solutions of those systems in the form of function sequences defined on an interval I. In the case when the interval I is bounded the problem of the existence of solutions is not very complicated although it is not easy (cf. [6, 19]). However, when we are looking for function sequences being solutions of an infinite system of integral equations defined on an unbounded interval (for example, on $\mathbb{R}_+ = [0, \infty)$), the problem starts to be very hard and complicated.
Up to now only a few papers have been published on solutions of infinite systems of integral equations which are defined on the real half-axis $\mathbb{R}_+$ (cf. [3, 4, 7, 8]). The investigations of such solutions require to construct appropriate tools which enable us to apply a fixed point theorem being convenient in the considered situation. It turnes out that we can apply as the required tool the technique of suitable measures of noncompactness constructed in the space of functions defined, continuous and bounded on the interval $\mathbb{R}_+$ with values in the sequence space, for example, in the spaces c_0, l_1 or l_∞. Taking into account the expected generality of obtained results, the sequence space l_∞ seems to be the most suitable for our purposes. Such a direction of investigations was initiated in the papers [4, 7].

In the present paper we continue and extend that direction of investigations applying a measure of noncompactness which promises to obtain the most interesting and useful results concerning solutions of infinite systems of integral equations in question. Namely, we prove that under suitable assumptions there exists a solution of considered infinite system of integral equations formed by a function sequence $(x_n(t))$ defined on the interval $\mathbb{R}_+$ such that every coordinate $x_n = x_n(t)$ tends to a proper limit at infinity. Moreover, the sequence formed by those proper limits is an element of the sequence space l_∞.

As we pointed out earlier the results of the present paper extend essentially those obtained in the mentioned papers [3, 4, 7, 8].

2. Notation, definitions and auxiliary facts

This section is devoted to establish the notation used in the paper and to provide definitions of the concepts creating the basis of the study conducted in the paper. Moreover, we recall also some results needed in our investigations.

In what follows we denote by $\mathbb{R}$ the set of real numbers and by $\mathbb{N}$ the set of natural numbers. We put $\mathbb{R}_+$ to denote the interval $[0, \infty)$. Further, assume that E is a Banach space with the norm $\|\cdot\|_E$ and the zero element θ. We will denote by $B(x, r)$ the closed ball centered at x and with radius r. We write B_r to denote the ball $B(\theta, r)$. The standard algebraic operations on subsets X, Y of the Banach space E will be denoted by $X + Y$ and λX, for $\lambda \in \mathbb{R}$. Apart from this the symbol $\overline{X}$ stands for the closure of the set X while $\operatorname{Conv} X$ denotes the closed convex hull of the set X.

Next, let us denote by $\mathfrak{M}_E$ the family of all nonempty and bounded subsets of E and by $\mathfrak{N}_E$ its subfamily consisting of all relatively compact sets.
The following definition forms the basis of our study (cf. [5,6]).

Definition 2.1. A function $\mu : \mathfrak{M}_E \to \mathbb{R}_+$ is said to be a measure of noncompactness in the space E if it satisfies the following conditions:

(i) The family $\ker \mu = \{X \in \mathfrak{M}_E : \mu(X) = 0\}$ is nonempty and $\ker \mu \subset \mathfrak{N}_E$.

(ii) $X \subset Y \Rightarrow \mu(X) \leq \mu(Y)$.

(iii) $\mu(\overline{X}) = \mu(X)$.

(iv) $\mu(\mathrm{Conv} X) = \mu(X)$.

(v) $\mu(\lambda X + (1-\lambda)Y) \leq \lambda\, \mu(X) + (1-\lambda)\, \mu(Y)$ for $\lambda \in [0,1]$.

(vi) If (X_n) is a sequence of closed sets from $\mathfrak{M}_E$ such that $X_{n+1} \subset X_n$ for $n = 1, 2, \ldots$ and if $\lim\limits_{n\to\infty} \mu(X_n) = 0$ then the set $X_\infty = \bigcap\limits_{n=1}^{\infty} X_n$ is nonempty.

The family $\ker \mu$ from axiom (i) is called the kernel of the measure of noncompactness μ. Let us notice that the intersection set X_∞ defined in axiom (vi) is an element of the family $\ker \mu$. Indeed, it follows immediately from the inclusion $X_\infty \subset X_n$ for $n = 1, 2, \ldots$. Hence we obtain that $\mu(X_\infty) = 0$ and consequently, $X_\infty \in \ker \mu$. This simple remark will be crucial in applications.

Further assume that μ is a measure of noncompactness in the space E. The measure μ is called sublinear [5] provided it satisfies the following additional conditions:

(vii) $\mu(X + Y) \leq \mu(X) + \mu(Y)$.

(viii) $\mu(\lambda X) = |\lambda|\, \mu(X)$ for $\lambda \in \mathbb{R}$.

If the measure of noncompactness μ satisfies the condition

(ix) $\mu(X \cup Y) = \max\{ \mu(X), \mu(Y)\}$

then it is referred to as the measure with the maximum property. Moreover, if $\ker \mu = \mathfrak{N}_E$ we say that μ is full. If μ is sublinear and full measure of noncompactness with the maximum property then it is called to be regular.

From historical point of view the first measure of noncompactness was defined in 1930 by K. Kuratowski [18]. However, the most important and useful measure of noncompactness is the so-called Hausdorff measure of noncompactness defined in [15,16] by the formula

$$\chi(X) = \inf\{\varepsilon > 0 : X \text{ has a finite } \varepsilon\text{-net in } E\},$$

where $X \in \mathfrak{M}_E$. Indeed, χ is a regular measure of noncompactness and in some Banach spaces like $C([a,b])$, c_0 and l_p we can give formulas expressing χ in connection with the structure of the mentioned Banach spaces [1,2,5].

It is worthwhile mentioning that in a lot of Banach spaces we are not able to give formulas for the Hausdorff measure of noncompactness χ. Even more, we are not in a position to provide formulas for full measures of noncompactness [5,6]. Therefore, in such a situation we restrict ourselves to measures of noncompactness in the sense of Definition 2.1 which are not full. In the sequel of the paper we will consider such a measure of noncompactness.

Now, we recall a useful fixed point theorem of Darbo type involved the concept of measure of noncompactness (cf. [5]).

Theorem 2.1. *Assume that μ is a given measure of noncompactness in a Banach space E and Ω is a nonempty, bounded, closed and convex subset of E. Let $Q : \Omega \to \Omega$ be a continuous operator such that there exists a constant $k \in [0,1)$ for which $\mu(QX) \leq k\mu(X)$ for an arbitrary nonempty subset X of Ω. Then there exists at least one fixed point of the operator Q in the set Ω.*

Remark 2.1. It can be shown that the set of all fixed points of the operator Q belongs to the family $\ker\mu$ [5]. This simple observation is very essential in characterization of solutions of considered operator equation which are proved with help of Theorem 2.1.

Further on, we are going to decsribe a measure of noncompactness used in considerations of this paper. To this end let us first assume that E is a given Banach space with the norm $\|\cdot\|_E$ and μ is a measure of noncompactness in the space E. Consider the Banach space $BC(\mathbb{R}_+, E)$ consisting of all functions $x : \mathbb{R}_+ \to E$ which are continuous and bounded on the interval $\mathbb{R}_+$. The space $BC(\mathbb{R}_+, E)$ will be furnished with the standard supremum norm

$$\|x\|_\infty = \sup\{\|x(t)\|_E : t \in \mathbb{R}_+\}.$$

Next, take an arbitrary nonempty and bounded subset X of the space $BC(\mathbb{R}_+, E)$. Fix $x \in X$, $\varepsilon > 0$ and define the modulus of the uniform continuity of the function x (cf. [4]) by putting

$$\omega^\infty(x,\varepsilon) = \sup\big\{\|x(t) - x(s)\|_E : t, s \in \mathbb{R}_+, |t-s| \leq \varepsilon\big\}.$$

Observe that $\lim\limits_{\varepsilon\to 0} \omega^\infty(x,\varepsilon) = 0$ if and only if the function x is uniformly continuous on the interval $\mathbb{R}_+$.
Further, let us define the following quantities

$$\begin{aligned} &\omega^\infty(X,\varepsilon) = \sup\{\omega^\infty(x,\varepsilon) : x \in X\}, \\ &\omega_0^\infty(X) = \lim_{\varepsilon\to 0} \omega^\infty(X,\varepsilon). \end{aligned} \tag{2.1}$$

Now, let us consider the function $\overline{\mu}_\infty$ defined on the family $\mathfrak{M}_{BC(\mathbb{R}_+,E)}$ by the formula

$$\overline{\mu}_\infty(X) = \lim_{T\to\infty} \overline{\mu}_T(X), \tag{2.2}$$

where $\overline{\mu}_T(X)$ is defined in the following way

$$\overline{\mu}_T(X) = \sup\big\{\mu(X(t)) : t \in [0,T]\big\}$$

for any fixed $T > 0$.

Further, for a given $T > 0$ let us put

$$b_T(X) = \sup_{x\in X}\Big\{\sup\big\{\|x(t) - x(s)\|_E : t, s \geq T\big\}\Big\}.$$

Next, let us define the following quantity

$$b_\infty(X) = \lim_{T\to\infty} b_T(X). \tag{2.3}$$

Finally, linking quantities defined by (2.1), (2.2) and (2.3), we can consider the function μ_b defined on the family $\mathfrak{M}_{BC(\mathbb{R}_+,E)}$ in the following way (cf. [4])

$$\mu_b(X) = \omega_0^\infty(X) + \overline{\mu}_\infty(X) + b_\infty(X). \tag{2.4}$$

It can be shown that the function μ_b is a measure of noncompactness in the space $BC(\mathbb{R}_+, E)$ (cf. [4]). The kernel $\ker \mu_b$ of the measure μ_b consists of all nonempty and bounded subsets of the space $BC(\mathbb{R}_+, E)$ such that functions from X are uniformly continuous and equicontinuous (equivalently, functions from X are equicontinuous on $\mathbb{R}_+$) and tend to limits (being elements of E) at infinity with the same rate. Apart from this, all cross-sections $X(t) = \{x(t) : x \in X\}$ of the set X belong to the kernel $\ker \mu$ of the measure of noncompactness μ in the Banach space E (cf. [4]). The measure μ_b is not full and has the maximum property. If the measure μ is sublinear in the space E then the measure μ_b defined by (2.4) is also sublinear [4].

Let us mention that in the similar way as above we may define other measures of noncompactness in the space $BC(\mathbb{R}_+, E)$ (see [4]).

Taking into account our further purposes we will consider as the Banach space E the sequence space l_∞ equipped with the standard supremum norm.

Thus, in what follows we consider the Banach space $BC(\mathbb{R}_+, l_\infty)$ consisting of functions $x : \mathbb{R}_+ \to l_\infty$ being continuous and bounded on $\mathbb{R}_+$. If $x \in BC(\mathbb{R}_+, l_\infty)$ then we can write this function in the form

$$x(t) = (x_n(t)) = (x_1(t), x_2(t), \ldots)$$

for $t \in \mathbb{R}_+$, where the sequence $(x_n(t))$ is an element of the space l_∞ for any fixed t. The norm of the function $x = x(t) = (x_n(t))$ is defined by the equality

$$\|x\|_\infty = \sup\{\|x(t)\|_{l_\infty} : t \in \mathbb{R}_+\} = \sup_{t\in\mathbb{R}_+} \big\{ \sup\{|x_n(t)| : n = 1, 2, \ldots\}\big\}.$$

In our further considerations the space $BC(\mathbb{R}_+, l_\infty)$ will be denoted by BC_∞.

Now, we can express the formula for the measure of noncompactness defined by (2.4) in the Banach space BC_∞, provided the measure of noncompactness in the space l_∞ is defined in the following way [5]

$$\mu^1(X) = \lim_{n\to\infty} \Big\{ \sup_{x=(x_i)\in X} \big\{ \sup\{|x_k| : k \geq n\}\big\}\Big\} \tag{2.5}$$

for $X \in \mathfrak{M}_{l_\infty}$. In this case the component $\overline{\mu}_\infty$ defined by (2.2) will be denoted by $\overline{\mu}^1_\infty$.

Thus, our measure of noncompactness μ_b defined by (2.4) will be denoted by μ^1_b and is defined as a particular case of (2.4) by the following formula

$$\mu^1_b(X) = \omega^\infty_0(X) + \overline{\mu}^1_\infty(X) + b_\infty(X), \tag{2.6}$$

where the components on the right hand side of formula (2.6) are defined in the following way [4]:

$$\omega^\infty_0(X) = \lim_{\varepsilon\to 0} \Big\{ \sup_{x\in X} \big\{ \sup\{\sup_{n\in\mathbb{N}} |x_n(t) - x_n(s)| : t, s \in \mathbb{R}_+, |t - s| \leq \varepsilon\}\big\}\Big\}, \tag{2.7}$$

$$\overline{\mu}^1_\infty(X) = \lim_{T\to\infty} \left\{ \sup_{t\in[0,T]} \left\{ \lim_{n\to\infty} \Big\{ \sup_{x\in X} \big\{ \sup\{|x_k(t)| : k \geq n\}\big\}\Big\}\right\}\right\}, \tag{2.8}$$

$$b_\infty(X) = \lim_{T\to\infty} \Big\{ \sup_{x\in X} \big\{ \sup\{\sup_{n\in\mathbb{N}} |x_n(t) - x_n(s)| : t, s \geq T\}\big\}\Big\}. \tag{2.9}$$

Remark 2.2. Let us notice that formula (2.2) for $\overline{\mu}_\infty$ can be simplified in the following way

$$\overline{\mu}_\infty(X) = \sup\{\mu(X(t)) : t \in \mathbb{R}_+\}$$

for $X \in \mathfrak{M}_{BC(\mathbb{R}_+,E)}$. For the proof we refer to [7].

Thus, the quantity $\overline{\mu}^1_\infty(X)$ defined by (2.8) can be expressed as follows

$$\overline{\mu}^1_\infty(X) = \sup_{t\geq 0}\left\{\lim_{n\to\infty}\left\{\sup_{x\in X}\left\{\sup\{|x_k(t)| : k \geq n\}\right\}\right\}\right\} \tag{2.10}$$

for $X \in \mathfrak{M}_{BC_\infty}$.

Remark 2.3. Let us observe that the kernel $\ker \mu^1_b$ of the measure of noncompactness μ^1_b defined by formula (2.6) can be described as the family of all sets $X \in \mathfrak{M}_{BC_\infty}$ such that functions $x = x(t) = (x_n(t))$ from X are equicontinuous on $\mathbb{R}_+$ and tend coordinatewise to proper limits at infinity i.e., for any $n \in \mathbb{N}$ there exists a number $g_n \in \mathbb{R}$ such that $\lim\limits_{t\to\infty} x_n(t) = g_n$. Obviously, the sequence $g = (g_n)$ is an element of the space l_∞. Apart from this let us notice that the functions of the sequence $(x_n(t))$ tend to limits (g_n) with the same rate. Additionally, let us mention that all cross-sections $X(t)$ of the set X belong to the kernel $\ker \mu^1$ of the measure μ^1 defined by (2.5) being the family of some relatively compact subsets of the space l_∞ (cf. [5]).

3. Main result

In this section we will investigate the infinite system of the quadratic integral equations of Volterra-Hammerstein type of the following form

$$x_n(t) = a_n(t) + f_n(t, x_1(t), x_2(t), \ldots)\int_0^t k_n(t,\tau)g_n(\tau, x_1(\tau), x_2(\tau), \ldots)d\tau, \tag{3.1}$$

where $t \in \mathbb{R}_+$ and $n = 1, 2, \ldots$.

Our aim is to show that the infinite system of integral equations (3.1) has a solution $x(t) = (x_n(t))$ in the space $BC_\infty = BC(\mathbb{R}_+, l_\infty)$ such that there exists a limit $\lim\limits_{t\to\infty} x_n(t)$. Obviously that limit is an element of the space l_∞. As we pointed out in Section 2, Remark 2.3, the functions of the sequence $(x_n(t))$ tend coordinatewise to proper limits at infinity (with the same rate). Our considerations are located in the mentioned Banach space BC_∞ discussed previously in details. Apart from this let us point out that in our study solutions of infinite system (3.1) we will use the measure of noncompactness $\mu^1_b(X)$ expressed by formula (2.6) given in the previous section.

In what follows we formulate the assumptions under which the infinite system (3.1) will be investigated.

(i) The sequence $(a_n(t))$ is an element of the space BC_∞ such that there exists the proper limit $\lim\limits_{t\to\infty} a_n(t)$ uniformly with respect to $n \in \mathbb{N}$ i.e., the following condition of the Cauchy type is satisfied

$$\forall_{\varepsilon>0}\, \exists_{T>0}\, \forall_{t,s\geq T}\, \forall_{n\in\mathbb{N}}\ |a_n(t) - a_n(s)| \leq \varepsilon.$$

Moreover, $\lim\limits_{n\to\infty} a_n(t) = 0$ for any $t \in \mathbb{R}_+$.

(ii) The functions $k_n(t,\tau) = k_n : \mathbb{R}_+^2 \to \mathbb{R}$ are continuous on the set $\mathbb{R}_+^2$ ($n = 1,2,\ldots$). Moreover, the functions $t \to k_n(t,\tau)$ are equicontinuous on the set $\mathbb{R}_+$ uniformly with respect to $\tau \in \mathbb{R}_+$ i.e., the following condition is satisfied

$$\forall_{\varepsilon>0}\, \exists_{\delta>0}\, \forall_{n\in\mathbb{N}}\, \forall_{\tau\in\mathbb{R}_+}\, \forall_{t_1,t_2\in\mathbb{R}_+}\ \left[|t_2 - t_1| \le \delta \Rightarrow |k_n(t_2,\tau) - k_n(t_1,\tau)| \le \varepsilon\right].$$

(iii) There exists a constant $K_1 > 0$ such that $\int_0^t |k_n(t,\tau)|d\tau \le K_1$ for $t \in \mathbb{R}_+$ and $n = 1,2,\ldots$. Moreover, $\lim_{t\to\infty} \int_0^t |k_n(t,\tau)|d\tau = 0$ uniformly with respect to $n \in \mathbb{N}$ i.e., the following condition is satisfied

$$\forall_{\varepsilon>0}\, \exists_{T>0}\, \forall_{t\ge T}\, \forall_{n\in\mathbb{N}} \int_0^t |k_n(t,\tau)|d\tau \le \varepsilon.$$

(iv) The sequence $(k_n(t,\tau))$ is equibounded on $\mathbb{R}_+^2$ i.e., there exists a constant $K_2 > 0$ such that $|k_n(t,\tau)| \le K_2$ for $t,\tau \in \mathbb{R}_+$ and $n = 1,2,\ldots$.

(v) The function f_n is defined on the set $\mathbb{R}_+ \times \mathbb{R}^\infty$ and takes real values for $n = 1,2,\ldots$. Moreover, the function $t \to f_n(t,x_1,x_2,\ldots)$ is uniformly continuous on $\mathbb{R}_+$ locally uniformly with respect to $x = (x_n) \in l_\infty$ and uniformly with respect to $n \in \mathbb{N}$ i.e., the following condition is satisfied

$$\begin{gathered}\forall_{\varepsilon>0}\, \forall_{r>0}\, \exists_{\delta>0}\, \forall_{x=(x_i)\in l_\infty, \|x\|_{l_\infty}\le r}\, \forall_{n\in\mathbb{N}}\, \forall_{t,s\in\mathbb{R}_+} \\ \big[\,|t-s| \le \delta \Rightarrow |f_n(t,x_1,x_2,\ldots) - f_n(s,x_1,x_2,\ldots)| \le \varepsilon\big].\end{gathered}$$

(vi) The function sequence $(\overline{f}_n)$ defined by the equality $\overline{f}_n(t) = |f_n(t,0,0,\ldots)|$ (for $t \in \mathbb{R}_+$ and $n = 1,2,\ldots$) is bounded on $\mathbb{R}_+$ and $\lim_{n\to\infty} \overline{f}_n(t) = 0$ for any $t \in \mathbb{R}_+$.

(vii) For each $r > 0$ there exists a proper limit $\lim_{t\to\infty} f_n(t,x_1,x_2,\ldots)$ uniformly with respect to $x \in l_\infty$ such that $\|x\|_{l_\infty} \le r$ and $n \in \mathbb{N}$ i.e., the following condition is satisfied

$$\forall_{\varepsilon>0}\, \forall_{r>0}\, \exists_{T>0}\, \forall_{t,s\ge T}\, \forall_{x\in l_\infty, \|x\|_{l_\infty}\le r}\, \forall_{n\in\mathbb{N}}\ \big|f_n(t,x) - f_n(s,x)\big| \le \varepsilon.$$

(viii) There exists a function $m : \mathbb{R}_+ \to \mathbb{R}_+$ such that m is nondecreasing on $\mathbb{R}_+$, continuous at 0 and the following inequality is satisfied

$$\big|f_n(t,x_1,x_2,\ldots) - f_n(t,y_1,y_2,\ldots)\big| \le m(r)\sup\{|x_i - y_i| : i \ge n\}$$

for any $r > 0$, for $x = (x_i)$, $y = (y_i) \in l_\infty$ such that $\|x\|_{l_\infty} \le r$, $\|y\|_{l_\infty} \le r$ for all $t \in \mathbb{R}_+$ and $n = 1,2,\ldots$.

(ix) The function g_n is defined on the set $\mathbb{R}_+ \times \mathbb{R}^\infty$ and takes real values for $n = 1,2,\ldots$. Moreover, the operator g defined on the set $\mathbb{R}_+ \times l_\infty$ by the formula

$$(gx)(t) = (g_n(t,x)) = (g_1(t,x), g_2(t,x), \ldots)$$

transforms the set $\mathbb{R}_+ \times l_\infty$ into l_∞ and is such that the family of functions $\{(gx)(t)\}_{t\in\mathbb{R}_+}$ is equicontinuous on the space l_∞ i.e., for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\|(gy)(t) - (gx)(t)\|_{l_\infty} \le \varepsilon$$

for any $t \in \mathbb{R}_+$ and for all $x, y \in l_\infty$ such that $\|x - y\|_{l_\infty} \le \delta$.

(x) The operator g defined in assumption (ix) is bounded on the set $\mathbb{R}_+ \times l_\infty$. More precisely, there exists a positive constant $\overline{G}$ such that $\|(gx)(t)\|_{l_\infty} \leq \overline{G}$ for any $x \in l_\infty$ and $t \in \mathbb{R}_+$.

(xi) There exists a positive solution r_0 of the inequality

$$A + \overline{F}\,\overline{G}K_1 + \overline{G}K_1 r\, m(r) \leq r$$

such that $\overline{G}K_1 m(r_0) < 1$, where the constants $\overline{G}$, K_1 were defined above and the constants A, $\overline{F}$ are defined in the following way

$$A = \sup\{|a_n(t)| : t \in \mathbb{R}_+, n = 1, 2, \ldots\},$$
$$\overline{F} = \sup\{\overline{f}_n(t) : t \in \mathbb{R}_+, n = 1, 2, \ldots\}.$$

Now, we formulate remarks and lemmas concerning some components involved in infinite system (3.1).

Remark 3.1. Observe that in view of assumptions (i) and (vi) the constants A and F defined above are finite.

Remark 3.2. The function sequence $(\overline{f}_n)$ from assumption (vi) is an element of the space BC_∞.

In order to prove the above assertion let us first notice that in view of assumption (vi) the sequence $(\overline{f}_n(t))$ is an element of the space l_∞ for any fixed $t \in \mathbb{R}_+$. Now, we show that the function $\overline{f} : \mathbb{R}_+ \to l_\infty$ defined by the equality $\overline{f}(t) = (\overline{f}_n(t))$, is continuous on $\mathbb{R}_+$.

To prove this fact let us fix arbitrarily $\varepsilon > 0$. Then, for arbitrary $t, s \in \mathbb{R}_+$ we have

$$\begin{aligned} |\overline{f}_n(t) - \overline{f}_n(s)| &= \big||f_n(t,0,0,\ldots)| - |f_n(s,0,0,\ldots)|\big| \\ &\leq |f_n(t,0,0,\ldots) - f_n(s,0,0,\ldots)|, \end{aligned}$$

for any $n = 1, 2, \ldots$. Hence, in view of assumption (v) we can choose a number $\delta > 0$ such that for any $n \in \mathbb{N}$ and for arbitrary $t, s \in \mathbb{R}_+$ such that $|t - s| \leq \delta$, we have

$$|\overline{f}_n(t) - \overline{f}_n(s)| \leq \varepsilon.$$

This implies that $\|\overline{f}(t) - \overline{f}(s)\|_{l_\infty} \leq \varepsilon$ for $t, s \in \mathbb{R}_+$, $|t - s| \leq \delta$. Thus the sequence $\overline{f}(t) = (\overline{f}_n(t))$ is an element of the space BC_∞.

Lemma 3.1. *Let the function $x(t) = (x_n(t))$ be an element of the space BC_∞. Then the sequence (x_n) is equibounded and locally equicontinuous on $\mathbb{R}_+$.*

The proof can be conducted in the same way as the proof of Lemma 4.1 in [3] and is therefore omitted.

Lemma 3.2. *Let $x(t) = (x_n(t))$ be an element of the space BC_∞ such that there exists a proper limit $\lim_{t\to\infty} x_n(t)$ uniformly with respect to $n \in \mathbb{N}$ i.e., the following condition is satisfied*

$$\forall_{\varepsilon>0}\, \exists_{T>0}\, \forall_{t,s\geq T}\, \forall_{n\in\mathbb{N}}\; |x_n(t) - x_n(s)| \leq \varepsilon$$

(cf. assumption (i)). Then the sequence (x_n) is equibounded and equicontinuous on $\mathbb{R}_+$.

Proof. The equiboundedness of the sequence (x_n) on $\mathbb{R}_+$ follows immediately from Lemma 3.1. In order to prove the equicontinuity of the sequence (x_n) on $\mathbb{R}_+$ let us fix $\varepsilon > 0$. Then, in view of the assumption imposed in our lemma we can find a number $T > 0$ such that $|x_n(t) - x_n(s)| \leq \frac{\varepsilon}{2}$ for $t, s \geq T$ and for $n = 1, 2, \ldots$. On the other hand, in virtue of Lemma 3.1 we infer that the sequence (x_n) is equicontinuous on the interval $[0, T]$. This means that we can find a number $\delta > 0$ such that $|x_n(t_2) - x_n(t_1)| \leq \frac{\varepsilon}{2}$ for $t_1, t_2 \in [0, T]$ such that $|t_2 - t_1| \leq \delta$ and for all $n = 1, 2, \ldots$

Now, let us take arbitrary numbers $t_1, t_2 \in \mathbb{R}_+$ such that $|t_2 - t_1| \leq \delta$. Without loss of generality we can assume that $t_1 < t_2$.

If $t_1, t_2 \in [0, T]$ then, according to the above established fact we have that

$$|x_n(t_2) - x_n(t_1)| \leq \frac{\varepsilon}{2}$$

for $n = 1, 2, \ldots$.

If $t_1, t_2 \geq T$ then in view of the above made choice of the number T we have that

$$|x_n(t_2) - x_n(t_1)| \leq \frac{\varepsilon}{2}.$$

Further, let us assume that $t_1 < T \leq t_2$. Then, for an arbitrarily fixed $n \in \mathbb{N}$, taking into account the above established facts we get

$$|x_n(t_2) - x_n(t_1)| \leq |x_n(t_2) - x_n(T)| + |x_n(T) - x_n(t_1)| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

This shows that the sequence (x_n) is equicontinuous on the interval $\mathbb{R}_+$. □

Now, we are prepared to formulate the main result of this paper.

Theorem 3.1. *Under assumptions (i) - (xi) the infinite system of integral equations* (3.1) *has at least one solution $x(t) = (x_n(t))$ in the space $BC_\infty = BC(\mathbb{R}_+, l_\infty)$ which is uniformly continuous on $\mathbb{R}_+$ and tends at infinity to a limit being an element of the space l_∞.*

Proof. At the beginning we define three operators F, V, Q on the space BC_∞ in the following way:

$$\begin{aligned}
(Fx)(t) &= \big((F_n x)(t)\big) = \Big(f_n\big(t, x(t)\big)\Big) = \Big(f_n\big(t, x_1(t), x_2(t), \ldots\big)\Big), \\
(Vx)(t) &= \big((V_n x)(t)\big) = \Big(\int_0^t k_n(t,\tau) g_n\big(\tau, x_1(\tau), x_2(\tau), \ldots\big) d\tau\Big), \\
(Qx)(t) &= \big((Q_n x)(t)\big) = \big(a_n(t) + (F_n x)(t)\,(V_n x)(t)\big).
\end{aligned}$$

We start with showing that the operator F transforms the space BC_∞ into itself. To this end fix the function $x = x(t) = (x_n(t)) \in BC_\infty$. Then, in view of assumptions (viii) and (vi), for an arbitrary fixed $n \in \mathbb{N}$, we get

$$\begin{aligned}
|(F_n x)(t)| &\leq |f_n(t, x_1(t), x_2(t), \ldots) - f_n(t, 0, 0)| + |f_n(t, 0, 0, \ldots)| \\
&\leq m(\|x(t)\|_{l_\infty}) \sup\big\{|x_i(t)| : i \geq n\big\} + \overline{f}_n(t). \qquad (3.2)
\end{aligned}$$

Hence, we obtain the inequality

$$|(F_n x)(t)| \leq \overline{F} + m\big(\|x(t)\|_{l_\infty}\big)\,\|x(t)\|_{l_\infty},$$

which implies the following estimate

$$\|Fx\|_{BC_\infty} \leq \overline{F} + m(\|x\|_{BC_\infty})\|x\|_{BC_\infty} \tag{3.3}$$

for any $x \in BC_\infty$. This estimate shows that the function Fx is bounded on $\mathbb{R}_+$.

To prove the continuity of the function Fx on the interval $\mathbb{R}_+$, let us fix $\varepsilon > 0$. Then, on the basis of assumption (v) we can find a number $\delta = \delta(\varepsilon, \|x\|_{l_\infty}) > 0$ such that for $t, s \in \mathbb{R}_+$ with $|t - s| \leq \delta$ the following inequality holds

$$|f_n(t, x_1, x_2, \ldots) - f_n(s, x_1, x_2, \ldots)| \leq \varepsilon.$$

This implies that

$$\|(Fx)(t) - (Fx)(s)\|_{l_\infty} \leq \varepsilon$$

for all $t, s \in \mathbb{R}_+$ such that $|t-s| \leq \delta$. This means that the function Fx is continuous (even uniformly continuous) on $\mathbb{R}_+$. Hence we infer that the operator F transforms the space BC_∞ into itself.

In what follows we show that the operator V acts from the space BC_∞ into itself.

To this end, similarly as above, let as fix a function $x = x(t) = (x_n(t))$ belonging to the space BC_∞. Next, take an arbitrary number $t \in \mathbb{R}_+$. Then, for a fixed natural number n, in view of assumptions (iii) and (x), we obtain

$$\begin{aligned} \big|(V_n x)(t)\big| &\leq \int_0^t \big|k_n(t,\tau)\big|\big|g_n(\tau, x_1(\tau), x_2(\tau), \ldots)\big| d\tau \leq \int_0^t \big|k_n(t,\tau)\big|\overline{G} d\tau \\ &= \overline{G} \int_0^t \big|k_n(t,\tau)\big| d\tau \leq \overline{G} K_1. \end{aligned} \tag{3.4}$$

From the above estimate we conclude that the function Vx is bounded on the interval $\mathbb{R}_+$.

In order to prove the continuity of the function Vx on $\mathbb{R}_+$, let us fix $\varepsilon > 0$. Then, for arbitrarily fixed numbers $t_1, t_2 \in \mathbb{R}_+$ such that $|t_2 - t_1| \leq \delta$, in virtue of assumptions (ii), (iv) and (x) (assuming additionally that $t_1 < t_2$), we derive the following estimates

$$\begin{aligned} &\big|(V_n x)(t_2) - (V_n x)(t_1)\big| \\ &\leq \Big| \int_0^{t_2} k_n(t_2,\tau) g_n(\tau, x_1(\tau), x_2(\tau), \ldots) d\tau - \int_0^{t_2} k_n(t_1,\tau) g_n(\tau, x_1(\tau), x_2(\tau), \ldots) d\tau \Big| \\ &\quad + \Big| \int_0^{t_2} k_n(t_1,\tau) g_n(\tau, x_1(\tau), x_2(\tau), \ldots) d\tau - \int_0^{t_1} k_n(t_1,\tau) g_n(\tau, x_1(\tau), x_2(\tau), \ldots) d\tau \Big| \\ &\leq \int_0^{t_2} \big|k_n(t_2,\tau) - k_n(t_1,\tau)\big|\big|g_n(\tau, x_1(\tau), x_2(\tau), \ldots)\big| d\tau \\ &\quad + \int_{t_1}^{t_2} \big|k_n(t_1,\tau)\big|\big|g_n(\tau, x_1(\tau), x_2(\tau), \ldots)\big| d\tau \\ &\leq \int_0^{t_2} \omega_k(\delta)\big|g_n(\tau, x_1(\tau), x_2(\tau), \ldots)\big| d\tau + \int_{t_1}^{t_2} K_2 \big|g_n(\tau, x_1(\tau), x_2(\tau), \ldots)\big| d\tau, \end{aligned} \tag{3.5}$$

where K_2 is a constant from assumption (iv) and $\omega_k(\delta)$ denotes a common modulus of continuity of the sequence of functions $t \to k_n(t, \tau)$ on the interval $\mathbb{R}_+$ (according to assumption (ii)). Obviously we have that $\omega_k(\delta) \to 0$ as $\delta \to 0$.

Now, keeping in mind estimate (3.5) and assumption (x), we obtain

$$\big|(V_n x)(t_2) - (V_n x)(t_1)\big| \leq \overline{G}\omega_k(\delta) + K_2\overline{G}\delta. \tag{3.6}$$

Hence we deduce that the function Vx is continuous on the interval $\mathbb{R}_+$. Finally, linking the boundedness of the function Vx with its continuity on $\mathbb{R}_+$ we conclude that the operator V transforms the space BC_∞ into itself.

Further on, taking into account the fact that the space $BC_\infty = BC(\mathbb{R}_+, l_\infty)$ forms a Banach algebra with respect to the coordinatewise multiplication of functions sequences and keeping in mind the definition of the operator Q as well as assumption (i) we infer that for an arbitrarily fixed function $x = x(t) \in BC_\infty$ the function $(Qx)(t) = \big((Q_n x)(t)\big) = \big(a_n(t) + (F_n x)(t)\,(V_n x)(t)\big)$ acts from the interval $\mathbb{R}_+$ into the space l_∞. Indeed, in view of the fact that $\big((F_n x)(t)\big) \in l_\infty$ for each $t \in \mathbb{R}_+$ and in the light of estimate (3.4) we get

$$\big|(Q_n x)(t)\big| \leq \big|a_n(t)\big| + \overline{G}K_1\big|(F_n x)(t)\big|.$$

Hence, applying (3.2) we infer that $(Qx)(t) = \big((Q_n x)(t)\big) \in l_\infty$ for any $t \in \mathbb{R}_+$.

Next, let us observe that the continuity of the function Qx on $\mathbb{R}_+$ is an immediate consequence of the fact that both the function Fx and the function Vx are continuous on $\mathbb{R}_+$. Similarly we can also derive that the function Qx is bounded on the interval $\mathbb{R}_+$. In fact, in order to justify this assertion it is only sufficient to apply assumption (i) and the representation of the operator Q given at the beginning of this proof.

Finally, let us notice that gathering all the above established properties of the function Qx we deduce that the operator Q transforms the space BC_∞ into itself.

Further on, let us note that based on estimates (3.3) and (3.4), for arbitrarily fixed $n \in \mathbb{N}$ and $t \in \mathbb{R}_+$, we get

$$\begin{aligned}\big|(Q_n x)(t)\big| &\leq \big|a_n(t)\big| + \big|(F_n x)(t)\big|\big|(V_n x)(t)\big| \\ &\leq A + \big[\overline{F} + m\big(\|x\|_{BC_\infty}\big)\|x\|_{BC_\infty}\big]\overline{G}K_1 \\ &\leq A + \overline{F}\,\overline{G}K_1 + \overline{G}K_1 m\big(\|x\|_{BC_\infty}\big)\|x\|_{BC_\infty}.\end{aligned}$$

From the above estimate and assumption (xi) we conclude that there exists a number $r_0 > 0$ such that the operator Q transforms the ball B_{r_0} into itself.

In what follows we intend to show that the operator Q is continuous on the ball B_{r_0}. Keeping in mind the representation of the operator Q mentioned previously we see that it is sufficient to show the continuity of the operators F and V, separately. To this end let us fix $\varepsilon > 0$ and $x \in B_{r_0}$. Next, take an arbitrary point $y \in B_{r_0}$ such that $\|x - y\|_{BC_\infty} \leq \varepsilon$. Then, for a fixed $t \in \mathbb{R}_+$, in view of assumption (viii), we get

$$\begin{aligned}\big\|(Fx)(t) - (Fy)(t)\big\|_{l_\infty} &\leq m(r_0)\sup\big\{|x_i - y_i| : i \geq n\big\} \\ &\leq m(r_0)\|x - y\|_{l_\infty} \\ &\leq \varepsilon m(r_0).\end{aligned}$$

This shows that the operator F is continuous on the ball B_{r_0}.

To prove the continuity of the operator V on the ball B_{r_0} let us consider the function $\delta = \delta(\varepsilon)$ defined in the following way

$$\delta(\varepsilon) = \sup\Big\{\big|g_n(t,x) - g_n(t,y)\big| : x, y \in l_\infty, \|x - y\|_{l_\infty} \leq \varepsilon, t \in \mathbb{R}_+, n \in \mathbb{N}\Big\}.$$

Then, in view of assumption (ix) we have $\delta(\varepsilon) \to 0$ as $\varepsilon \to 0$. Now, taking $x, y \in B_{r_0}$ such that $\|x - y\|_{BC_\infty} \le \varepsilon$ and $t \in \mathbb{R}_+$ and fixing $n \in \mathbb{N}$ we obtain

$$\begin{aligned}
&\big|(V_n x)(t) - (V_n y)(t)\big| \\
&\le \int_0^t \big|k_n(t,\tau)\big|\big|g_n(\tau, x_1(\tau), x_2(\tau), \dots) - g_n(\tau, y_1(\tau), y_2(\tau), \dots)\big| d\tau \\
&\le \int_0^t \big|k_n(t,\tau)\big| d\tau \delta(\varepsilon) \\
&\le K_1 \delta(\varepsilon).
\end{aligned}$$

This implies the estimate

$$\|Vx - Vy\|_{BC_\infty} \le K_1 \delta(\varepsilon).$$

Thus we see that the operator V is continuous on the ball B_{r_0}.

Finally, in the light of the above mentioned statement we conclude that the operator Q is continuous on B_{r_0}.

Further, let us fix an arbitrary number $\varepsilon > 0$ and choose a number $\delta = \delta(\varepsilon, r_0) > 0$ according to assumption (v). Next, take a nonempty subset X of the ball B_{r_0}. Assume that $x \in X$. Then, for arbitrarily fixed $n \in \mathbb{N}$ and $t, s \in \mathbb{R}_+$ with $|t - s| \le \delta$, in view of assumptions (v) and (vii), we obtain

$$\begin{aligned}
\big|(F_n x)(t) - (F_n x)(s)\big| &= \big|f_n(t, x_1(t), x_2(t), \dots) - f_n(s, x_1(s), x_2(s), \dots)\big| \\
&\le \big|f_n(t, x_1(t), x_2(t), \dots) - f_n(s, x_1(t), x_2(t), \dots)\big| \\
&\quad + \big|f_n(s, x_1(t), x_2(t), \dots) - f_n(s, x_1(s), x_2(s), \dots)\big| \\
&\le \varepsilon + m(r_0) \sup\Big\{\big|x_i(t) - x_i(s)\big| : i \ge n\Big\} \\
&\le \varepsilon + m(r_0) \sup\Big\{\big|x_i(t) - x_i(s)\big| : i \in \mathbb{N}\Big\} \\
&\le \varepsilon + m(r_0)\omega^\infty(x, \delta).
\end{aligned}$$

The above estimate implies the following one

$$\omega^\infty(Fx, \varepsilon) \le \varepsilon + m(r_0)\omega^\infty(x, \delta). \tag{3.7}$$

Further, similarly as above, let us fix $\varepsilon > 0$ and choose a number $\delta > 0$ according to assumption (ii) (we may choose the common number δ with respect to assumptions (ii) and (v)). Next, fix $n \in \mathbb{N}$ and $t, s \in \mathbb{R}_+$ (say, $s < t$) such that $|t - s| = t - s \le \delta$. Then, repeating the reasoning conducted in order to obtain estimate (3.6), in view of that estimate, we get

$$\big|(V_n x)(t) - (V_n x)(s)\big| \le \overline{G}\omega_k(\delta) + K_2\overline{G}\delta,$$

where K_2 is a constant appearing in assumption (iv) and $\omega_k(\delta)$ denotes the above introduced common modulus of continuity of the function sequence $t \to k_n(t, \tau)$ on the interval $\mathbb{R}_+$ (recall that $\omega_k(\delta) \to 0$ as $\delta \to 0$).

Hence, we derive the following estimate

$$\omega^\infty(Vx, \varepsilon) \le \overline{G}\omega_k(\delta) + \overline{G}K_2\delta. \tag{3.8}$$

Further on, keeping in mind the representation of the operator Q, for an arbitrary function $x \in X$ and for arbitrary $t, s \in \mathbb{R}_+$, we have

$$\begin{aligned}\big\|(Qx)(t)-(Qx)(s)\big\|_{l_\infty} \leq \big\|a(t)-a(s)\big\|_{l_\infty} \\ + \big\|(Vx)(t)\big\|_{l_\infty}\big\|(Fx)(t)-(Fx)(s)\big\|_{l_\infty} \\ + \big\|(Fx)(s)\big\|_{l_\infty}\big\|(Vx)(t)-(Vx)(s)\big\|_{l_\infty},\end{aligned}$$

where $a(t) = \big(a_n(t)\big)$.

Next, fix $\varepsilon > 0$ and assume that $|t-s| \leq \varepsilon$. Then, from the above inequality and estimates (3.7), (3.8), (3.3) and (3.4), we get

$$\begin{aligned}\omega^\infty(Qx,\varepsilon) &\leq \omega^\infty(a,\varepsilon) + \overline{G}K_1\omega^\infty(Fx,\varepsilon) \\ &\quad + \big(\overline{F} + r_0 m(r_0)\big)\big(\overline{G}\omega_k(\varepsilon) + \overline{G}K_2\varepsilon\big) \\ &\leq \omega^\infty(a,\varepsilon) + \overline{G}K_1 m(r_0)\omega^\infty(x,\varepsilon) + \overline{G}K_1\varepsilon \\ &\quad + \big(\overline{F} + r_0 m(r_0)\big)\big(\overline{G}\omega_k(\varepsilon) + \overline{G}K_2\varepsilon\big).\end{aligned}$$

Now, in view of Lemma 3.2 we infer that $\omega^\infty(a,\varepsilon) \to 0$ as $\varepsilon \to 0$. Next, taking into account that $\omega_k(\varepsilon) \to 0$ as $\varepsilon \to 0$, from the above obtained estimate we deduce the following inequality

$$\omega_0^\infty(QX) \leq \overline{G}K_1 m(r_0)\omega_0^\infty(X). \tag{3.9}$$

In what follows we will investigate the behaviour of the operator Q with respect to the second term $\overline{\mu}^1_\infty$ (cf. formula (2.8)) of the measure of noncompactness μ_b^1 defined by (2.6). To this end take a nonempty subset X of the ball B_{r_0} and choose an element $x = x(t) \in X$. Further, fix a natural number n and $T > 0$. Then, for an arbitrarily fixed number $t \in [0,T]$, in virtue of the representation of the operator Q and estimates (3.2) and (3.4), we get

$$\begin{aligned}\big|(Q_n x)(t)\big| &\leq |a_n(t)| + \big|f_n(t, x_1(t), x_2(t), \ldots)\big| \int_0^t \big|k_n(t,\tau)\big|\big|g_n(\tau, x_1(\tau), x_2(\tau), \ldots)\big| d\tau \\ &\leq |a_n(t)| + \Big[\overline{f}_n(t) + m\big(\|x(t)\|_{l_\infty}\big) \sup\big\{\big|x_i(t)\big| : i \geq n\big\}\Big]\overline{G}K_1.\end{aligned}$$

Now, taking supremum over $x \in X$, from the above estimate we obtain

$$\sup_{x\in X}\big|(Q_n x)(t)\big| \leq |a_n(t)| + \overline{G}K_1\left[\overline{f}_n(t) + m(r_0)\sup_{x\in X}\Big\{\sup\big\{\big|x_i(t)\big| : i \geq n\big\}\Big\}\right].$$

Hence, in view of assumptions (i) and (vi), we derive the following inequality

$$\lim_{n\to\infty}\Big\{\sup_{x\in X}\big|(Q_n x)(t)\big|\Big\} \leq \overline{G}K_1 m(r_0)\left\{\lim_{n\to\infty}\Big\{\sup_{x\in X}\Big\{\sup\big\{\big|x_i(t)\big| : i \geq n\big\}\Big\}\Big\}\right\}.$$

Finally, if we take supremum over $t \in [0,T]$ on both sides of the above inequality and if we pass with $T \to \infty$, in view of formula (2.8) we have

$$\overline{\mu}^1_\infty(QX) \leq \overline{G}K_1 m(r_0)\overline{\mu}^1_\infty(X). \tag{3.10}$$

Now, we proceed to the study the behaviour of the operator Q with respect to the quantity $b_\infty = b_\infty(X)$ defined by (2.9) which creates the last component of the measure of noncompactness μ_b^1 (cf. formula (2.6)).

Thus, take a nonempty subset X of the ball B_{r_0} and an arbitrary number $T > 0$. Next, fix numbers t, s such that $t, s \geq T$ and $n \in \mathbb{N}$. Then, for an arbitrarily fixed function $x \in X$ we obtain

$$
\begin{aligned}
&\big|(Q_n x)(t) - (Q_n x)(s)\big| \\
\leq&\big|a_n(t) - a_n(s)\big| + \Big|f_n(t, x_1(t), x_2(t), \ldots)\int_0^t k_n(t,\tau) g_n(\tau, x_1(\tau), x_2(\tau), \ldots) d\tau \\
&- f_n(s, x_1(s), x_2(s), \ldots)\int_0^s k_n(s,\tau) g_n(\tau, x_1(\tau), x_2(\tau), \ldots) d\tau\Big| \\
\leq&\big|a_n(t) - a_n(s)\big| + \Big|f_n(t, x_1(t), x_2(t), \ldots)\int_0^t k_n(t,\tau) g_n(\tau, x_1(\tau), x_2(\tau), \ldots) d\tau \\
&- f_n(s, x_1(s), x_2(s), \ldots)\int_0^t k_n(t,\tau) g_n(\tau, x_1(\tau), x_2(\tau), \ldots) d\tau\Big| \\
&+ \Big|f_n(s, x_1(s), x_2(s), \ldots)\int_0^t k_n(t,\tau) g_n(\tau, x_1(\tau), x_2(\tau), \ldots) d\tau \\
&- f_n(s, x_1(s), x_2(s), \ldots)\int_0^s k_n(s,\tau) g_n(\tau, x_1(\tau), x_2(\tau), \ldots) d\tau\Big| \\
\leq&\big|a_n(t) - a_n(s)\big| + \big|f_n(t, x_1(t), x_2(t), \ldots) - f_n(s, x_1(s), x_2(s), \ldots)\big| \\
&\times \int_0^t \big|k_n(t,\tau)\big|\big|g_n(\tau, x_1(\tau), x_2(\tau), \ldots)\big| d\tau \\
&+ \big|f_n(s, x_1(s), x_2(s), \ldots)\big|\Big|\int_0^t k_n(t,\tau) g_n(\tau, x_1(\tau), x_2(\tau), \ldots) d\tau \\
&- \int_0^s k_n(s,\tau) g_n(\tau, x_1(\tau), x_2(\tau), \ldots) d\tau\Big| \\
\leq&\big|a_n(t) - a_n(s)\big| + \Big[\big|f_n(t, x_1(t), x_2(t), \ldots) - f_n(s, x_1(t), x_2(t), \ldots)\big| \\
&+ \big|f_n(s, x_1(t), x_2(t), \ldots) - f_n(s, x_1(s), x_2(s), \ldots)\big|\Big] \\
&\times \int_0^t \overline{G}\,\big|k_n(t,\tau)\big| d\tau + \Big[\overline{f}_n(s) + m\big(\|x(s)\|_{l_\infty}\big)\sup\Big\{\big|x_i(s)\big| : i \geq n\Big\}\Big] \\
&\times \Big\{\int_0^t \big|k_n(t,\tau)\big|\big|g_n(\tau, x_1(\tau), x_2(\tau), \ldots)\big| d\tau \\
&+ \int_0^s \big|k_n(s,\tau)\big|\big|g_n(\tau, x_1(\tau), x_2(\tau), \ldots)\big| d\tau\Big\} \\
\leq&\big|a_n(t) - a_n(s)\big| + \Big[\Omega_{r_0}(f, T) + m(r_0)\sup\Big\{\big|x_i(t) - x_i(s)\big| : i \geq n\Big\}\Big]\overline{G}K_1 \\
&+ \big[\overline{F} + r_0 m(r_0)\big]\Big\{\int_0^t \big|k_n(t,\tau)\big|\overline{G} d\tau + \int_0^s \big|k_n(s,\tau)\big|\overline{G} d\tau\Big\},
\end{aligned}
\tag{3.11}
$$

where we denoted

$$\Omega_{r_0}(f, T) = \sup\Big\{\big|f_n(t, x_1, x_2, \ldots) - f_n(s, x_1, x_2, \ldots)\big| : t, s \geq T, x = (x_i) \in B_{r_0}, n \in \mathbb{N}\Big\}.$$

Observe that $\lim_{T\to\infty} \Omega_{r_0}(f, T) = 0$, in view of assumption (vii).

Further, from estimate (3.11), for $t, s \geq T$ and for $n \in \mathbb{N}$ we obtain

$$\big|(Q_n x)(t) - (Q_n x)(s)\big| \leq \big|a_n(t) - a_n(s)\big|$$

$$+\left[\Omega_{r_0}(f,T)+m(r_0)\sup\left\{\left|x_i(t)-x_i(s)\right|:i\geq n\right\}\right]\overline{G}K_1$$
$$+\left[\overline{F}+r_0m(r_0)\right]\overline{G}\Big\{\int_0^t\left|k_n(t,\tau)\right|d\tau+\int_0^s\left|k_n(s,\tau)\right|d\tau\Big\}.$$

Now, keeping in mind the above estimate, assumptions (i) and (iii) and the above established facts, in view of formula (2.9) we derive the following inequality

$$b_\infty(QX)\leq\overline{G}K_1m(r_0)b_\infty(X). \tag{3.12}$$

Finally, linking estimates (3.9), (3.10), (3.12) and taking into account formula (2.6), we obtain the following inequality for an arbitrary nonempty subset X of the ball B_{r_0}:

$$\mu_b^1(QX)\leq\overline{G}K_1m(r_0)\mu_b^1(X).$$

Hence, combining the fact that the operator Q maps continuously the ball B_{r_0} into itself, assumption (xi) and Theorem 2.1 we infer that the infinite system of Volterra-Hammerstein integral equations (3.1) has at least one solution $x=x(t)$ in the space $BC_\infty=BC(\mathbb{R}_+,l_\infty)$ which belongs to the ball B_{r_0} and is uniformly continuous on the interval $\mathbb{R}_+$.

Moreover, since the mentioned solution $x=x(t)$ of infinite system (3.1) belongs to the kernel $\ker\mu_b^1$ we conclude that there exists a limit $\lim\limits_{t\to\infty}x(t)$ in the space l_∞ i.e., there exists an element $g=(g_n)\in l_\infty$ such that $\lim\limits_{t\to\infty}x(t)=g$. Equivalently this means that if we write $x(t)=(x_n(t))$ then for any fixed $n\in\mathbb{N}$ there exists a proper limit $\lim\limits_{n\to\infty}x_n(t)\ (=g_n)$ (cf. Remark 2.3). Other words this means that the solution $x=x(t)=(x_n(t))$ is coordinatewise converging at infinity. The proof is complete. □

4. An example

In this section we provide an example which illustrates the existence result concerning the infinite system of integral equations (3.1) and contained in Theorem 3.1.

Namely, we will consider the infinite system of integral equations having the form

$$\begin{aligned}x_n(t)=&te^{-2nt}+\left(\frac{x_n^2(t)+1}{n+t}+\frac{x_{n+1}^2(t)}{2n+t}+\frac{x_{n+2}(t)+2}{n^2+t}\right)\\&\times\int_0^te^{-\gamma(t+n)\tau}\arctan\left(\frac{x_1(\tau)+x_n(\tau)+x_{n+1}(\tau)}{n+t+\beta}\right)d\tau,\end{aligned} \tag{4.1}$$

where $t\in\mathbb{R}_+$, $n=1,2,\ldots$ and $\beta>0$, $\gamma>0$ are some constants. Observe that infinite system (4.1) is a particular case of system (3.1) if we put

$$a_n(t)=te^{-2nt}, \tag{4.2}$$
$$f_n(t,x_1,x_2,\ldots)=\frac{x_n^2+1}{n+t}+\frac{x_{n+1}^2}{2n+t}+\frac{x_{n+2}+2}{n^2+t}, \tag{4.3}$$
$$k_n(t,\tau)=e^{-\gamma(t+n)\tau}, \tag{4.4}$$

$$g_n(t, x_1, x_2, \ldots) = \arctan\left(\frac{x_1 + x_n + x_{n+1}}{n + t + \beta}\right) \tag{4.5}$$

for $n = 1, 2, \ldots$ and $t \in \mathbb{R}_+$.

In what follows we are going to show that the infinite system of integral equations (4.1) has a solution $x = x(t) = (x_n(t))$ in the Banach space $BC_\infty = BC(\mathbb{R}_+, l_\infty)$ which is coordinatewise converging at infinity in the sense of Remark 2.3. To this end we will apply Theorem 3.1 i.e., we show that functions defined by formulas (4.2)-(4.5) satisfy assumptions (i)-(xi) of Theorem 3.1.

First, let us notice that the function $a_n(t)$ defined by (4.2) satisfies the Lipschitz condition with the constant $L = 1 + e^{-1}$ for $n = 1, 2, \ldots$. We omit elementary details of the proof.

Hence we infer that the function $a(t) = (a_n(t))$ is an element of the space BC_∞ and satisfies the Cauchy condition indicated in assumption (i). Moreover, in view of the inequality

$$a_n(t) = te^{-2nt} \leq \frac{1}{2n} e^{-1}$$

we infer that $\lim_{n\to\infty} a_n(t) = 0$ for any $t \in \mathbb{R}_+$. Apart from this we have that

$$|a_n(t)| \leq \frac{1}{2} e^{-1}$$

for all $t \in \mathbb{R}_+$ and $n = 1, 2, \ldots$. Thus the function sequence $(a_n(t))$ is equibounded on $\mathbb{R}_+$. Further, we deduce that

$$A = \sup\left\{|a_n(t)| : t \in \mathbb{R}_+, n = 1, 2, \ldots\right\} = \frac{1}{2} e^{-1}.$$

Summing up we conclude that assumption (i) is satisfied.

Now, consider the function $k_n(t, \tau)$ defined by (4.4). Obviously, the function k_n is continuous on the set $\mathbb{R}_+^2$ for any $n = 1, 2, \ldots$. It is easily seen that

$$\frac{\partial k_n}{\partial t} = -\gamma\tau e^{-\gamma(t+n)\tau} = -\gamma\tau e^{-\gamma n\tau} e^{-\gamma t\tau} \tag{4.6}$$

for $n = 1, 2, \ldots$. It is easy to check that if we consider the function $z_n(\tau) = \tau e^{-\gamma n\tau}$ then $z_n(\tau) \leq \frac{1}{\gamma n} \leq \frac{1}{\gamma}$ for any $\tau \in \mathbb{R}_+$ and $n = 1, 2, \ldots$. Hence and in view of (4.6) we deduce that the partial derivative $\frac{\partial k_n}{\partial t}$ is bounded i.e.,

$$\left|\frac{\partial k_n(t, \tau)}{\partial t}\right| \leq 1$$

for $t, \tau \in \mathbb{R}_+$ and for $n = 1, 2, \ldots$. Hence it follows that the function $t \to k_n(t, \tau)$ satisfies the Lipschitz condition on the set $\mathbb{R}_+$ uniformly with respect to $\tau \in \mathbb{R}_+$ and $n = 1, 2, \ldots$. Thus the function sequence $(k_n(t, \tau))$ satisfies assumption (ii).

Next, for arbitrarily fixed $n \in \mathbb{N}$ we get

$$\int_0^t \left|k_n(t, \tau)\right| d\tau = \int_0^t e^{-\gamma(t+n)\tau} d\tau = \frac{1}{\gamma(t+n)}(1 - e^{-\gamma(t+n)t}) \leq \frac{1}{\gamma(t+n)}.$$

Hence we see that

$$\lim_{t\to\infty} \int_0^t \left|k_n(t, \tau)\right| d\tau = 0$$

uniformly with respect to $t \in \mathbb{R}_+$ and $n = 1, 2, \ldots$. Moreover, we have that

$$\int_0^t \left|k_n(t,\tau)\right| d\tau \leq \frac{1}{\gamma}$$

for $t \in \mathbb{R}_+$ and $n = 1, 2, \ldots$. Thus, assumption (iii) is satisfied with the constant $K_1 = \frac{1}{\gamma}$.

To verify assumption (iv) let us observe that

$$\left|k_n(t,\tau)\right| = e^{-\gamma(t+n)\tau} \leq 1$$

for $t, \tau \in \mathbb{R}_+$ and $n = 1, 2, \ldots$. Hence we infer that assumption (iv) is satisfied with the constant $K_2 = 1$.

Further on we show that the function $f_n = f_n(t, x_1, x_2, \ldots)$ verifies assumption (v) ($n = 1, 2, \ldots$). To this end fix $\varepsilon > 0$, $r > 0$ and take an arbitrary element $x = (x_i) \in l_\infty$ such that $\|x\|_{l_\infty} \leq r$. Then, for arbitrary chosen numbers $t, s \in \mathbb{R}_+$ and $n \in \mathbb{N}$ we obtain:

$$\begin{aligned}
&\left|f_n(t,x_1,x_2,\ldots) - f_n(s,x_1,x_2,\ldots)\right| \\
\leq &\left|\frac{x_n^2+1}{n+t} - \frac{x_n^2+1}{n+s}\right| + \left|\frac{x_{n+1}^2}{2n+t} - \frac{x_{n+1}^2}{2n+s}\right| + \left|\frac{x_{n+2}+2}{n^2+t} - \frac{x_{n+2}+2}{n^2+s}\right| \\
\leq &\frac{|sx_n^2 - tx_n^2 + s - t|}{(n+t)(n+s)} + x_{n+1}^2 \frac{|t-s|}{(2n+t)(2n+s)} + \frac{|sx_{n+2} - tx_{n+2} + 2s - 2t|}{(n^2+t)(n^2+s)} \\
\leq &\frac{x_n^2|t-s| + |t-s|}{(n+t)(n+s)} + x_{n+1}^2 \frac{|t-s|}{(2n+t)(2n+s)} + \frac{|x_{n+2}||t-s| + 2|t-s|}{(n^2+t)(n^2+s)} \\
\leq &\frac{r^2|t-s| + |t-s|}{(1+t)(1+s)} + r^2 \frac{|t-s|}{(2+t)(2+s)} + \frac{r|t-s| + 2|t-s|}{(1+t)(1+s)} \\
\leq &(r^2+1)|t-s| + r^2|t-s| + (r+2)|t-s| \\
= &(2r^2 + r + 3)|t-s|.
\end{aligned}$$

From the above estimate we conclude that assumption (v) is satisfied.

Now, let us observe that

$$\overline{f}_n(t) = \left|f_n(t,0,0,\ldots)\right| = \frac{1}{n+t} + \frac{2}{n^2+t}.$$

Hence we infer that $\lim_{n\to\infty} \overline{f}_n(t) = 0$ for any $t \in \mathbb{R}_+$.

Moreover, we have the following estimate

$$\overline{f}_n(t) \leq \frac{1}{1+t} + \frac{2}{1+t} \leq 1 + 2 = 3$$

for any $t \in \mathbb{R}_+$ and $n = 1, 2, \ldots$.

Hence we conclude that the function sequence $(\overline{f}_n)$ satisfies assumption (vi). Moreover, we may accept that $\overline{F} = 3$, where the constant $\overline{F}$ is defined in assumption (xi).

In order to verify assumption (vii) let us fix an arbitrary number $r > 0$. Take $\varepsilon > 0$ and $x \in l_\infty$ such that $\|x\|_{l_\infty} \leq r$ and choose an arbitrary number $T > 0$. Then, for $t, s \in \mathbb{R}_+$ such that $t, s \geq T$ and for an arbitrarily fixed natural number n, we obtain

$$\left|f_n(t,x) - f_n(s,x)\right|$$

$$
\begin{aligned}
&\leq \left|\frac{x_n^2+1}{n+t}-\frac{x_n^2+1}{n+s}\right|+\left|\frac{x_{n+1}^2}{2n+t}-\frac{x_{n+1}^2}{2n+s}\right|+\left|\frac{x_{n+2}+2}{n^2+t}-\frac{x_{n+2}+2}{n^2+s}\right|\\
&\leq \frac{|x_n^2(s-t)+(s-t)|}{(n+t)(n+s)}+x_{n+1}^2\frac{|t-s|}{(2n+t)(2n+s)}+\frac{|x_{n+2}(s-t)+2(s-t)|}{(n^2+t)(n^2+s)}\\
&\leq (x_n^2+1)\frac{t+s}{(n+t)(n+s)}+x_{n+1}^2\frac{t+s}{(2n+t)(2n+s)}+(|x_{n+2}|+2)\frac{t+s}{(n^2+t)(n^2+s)}\\
&\leq (r^2+1)\left[\frac{t}{(n+t)(n+s)}+\frac{s}{(n+t)(n+s)}\right]+r^2\left[\frac{t}{(2n+t)(2n+s)}+\frac{s}{(2n+t)(2n+s)}\right]\\
&\quad+(r+2)\left[\frac{t}{(n^2+t)(n^2+s)}+\frac{s}{(n^2+t)(n^2+s)}\right]\\
&\leq (r^2+1)\left(\frac{1}{n+s}+\frac{1}{n+t}\right)+r^2\left(\frac{1}{2n+s}+\frac{1}{2n+t}\right)+(r+2)\left(\frac{1}{n^2+s}+\frac{1}{n^2+t}\right)\\
&\leq (r^2+1)\frac{2}{1+T}+r^2\frac{2}{2+T}+(r+2)\frac{2}{1+T}\\
&\leq (r^2+1)\frac{2}{T+1}+r^2\frac{2}{T+1}+(r+2)\frac{2}{T+1}\\
&= (2r^2+r+3)\frac{2}{T+1}.
\end{aligned}
$$

From the above obtained estimate we infer that assumption (vii) is satisfied.

In what follows let us fix $r>0$ and $n\in\mathbb{N}$. Next, take $x=(x_i)$, $y=(y_i)\in l_\infty$ such that $\|x\|_{l_\infty}\leq r$, $\|y\|_{l_\infty}\leq r$. Then we get

$$
\begin{aligned}
&\big|f_n(t,x_1,x_2,\ldots)-f_n(t,y_1,y_2,\ldots)\big|\\
&\leq \left|\frac{x_n^2+1}{n+t}-\frac{y_n^2+1}{n+t}\right|+\left|\frac{x_{n+1}^2}{2n+t}-\frac{y_{n+1}^2}{2n+t}\right|+\left|\frac{x_{n+2}+2}{n^2+t}-\frac{y_{n+2}+2}{n^2+t}\right|\\
&\leq \frac{1}{n+t}|x_n^2-y_n^2|+\frac{1}{2n+t}|x_{n+1}^2-y_{n+1}^2|+\frac{1}{n+t}|x_{n+2}-y_{n+2}|\\
&\leq \frac{1}{1+t}|x_n-y_n||x_n+y_n|+\frac{1}{2+t}|x_{n+1}-y_{n+1}||x_{n+1}+y_{n+1}|+\frac{1}{1+t}|x_{n+2}-y_{n+2}|\\
&\leq \frac{1}{1+t}|x_n-y_n|\big(|x_n|+|y_n|\big)+\frac{1}{1+t}|x_{n+1}-y_{n+1}|\big(|x_{n+1}|+|y_{n+1}|\big)+\frac{1}{1+t}|x_{n+2}-y_{n+2}|\\
&\leq \frac{1}{1+t}\Big[2r|x_n-y_n|+2r|x_{n+1}-y_{n+1}|+|x_{n+2}-y_{n+2}|\Big]\\
&\leq (4r+1)\max\big\{|x_i-y_i|:i=n,n+1,n+2\big\}\\
&\leq (4r+1)\sup\big\{|x_i-y_i|:i\geq n\big\}.
\end{aligned}
$$

Thus we see that assumption (viii) is satisfied with the function m having the form $m(r)=4r+1$.

Now, keeping in mind formula (4.5) we are going to check assumption (ix). To this end fix $t\in\mathbb{R}_+$, $n\in\mathbb{N}$ and take $x=(x_i)$, $y=(y_i)\in l_\infty$. Then we get

$$
\begin{aligned}
\big|g_n(t,y)-g_n(t,x)\big| &= \big|g_n(t,y_1,y_2,\ldots)-g_n(t,x_1,x_2,\ldots)\big|\\
&= \left|\arctan\left(\frac{y_1+y_n+y_{n+1}}{n+t+\beta}\right)-\arctan\left(\frac{x_1+x_n+x_{n+1}}{n+t+\beta}\right)\right|\\
&\leq \left|\frac{y_1+y_n+y_{n+1}}{n+t+\beta}-\frac{x_1+x_n+x_{n+1}}{n+t+\beta}\right|
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{1}{n+t+\beta}\big(|y_1-x_1|+|y_n-x_n|+|y_{n+1}-x_{n+1}|\big)\\
&\leq \frac{3}{n+t+\beta}\max\big\{|y_1-x_1|,|y_n-x_n|,|y_{n+1}-x_{n+1}|\big\}\\
&\leq 3\sup\big\{|y_n-x_n| : n\in\mathbb{N}\big\}\\
&= 3\|y-x\|_{l_\infty}.
\end{aligned}
$$

Hence we see that the function $g=(gx)(t)$ satisfies assumption (ix).

Next, let us observe that for arbitrarily fixed $t\in\mathbb{R}_+$, $n\in\mathbb{N}$ and $x\in l_\infty$ we have

$$
|g_n(t,x)| = |g_n(t,x_1,x_2,\ldots)| = \left|\arctan\Big(\frac{x_1+x_n+x_{n+1}}{n+t+\beta}\Big)\right| \leq \frac{\pi}{2}.
$$

Obviously this implies that $\|(gx)(t)\|_{l_\infty}\leq\frac{\pi}{2}$ for any $x\in l_\infty$ and $t\in\mathbb{R}_+$. Thus the operator g is bounded on the set $\mathbb{R}_+\times l_\infty$ and we can accept that $\overline{G}=\frac{\pi}{2}$, where $\overline{G}$ is the constant appearing in assumption (x).

Finally, we are going to verify assumption (xi). To this end let us consider the inequality from that assumption. Indeed, taking into account all constants A, $\overline{F}$, $\overline{G}$, K_1 established above and keeping in mind that the function $m=m(r)$ has the form $m(r)=4r+1$, we obtain that the mentioned inequality has the form

$$
\frac{1}{2}e^{-1}+3\frac{\pi}{2}\frac{1}{\gamma}+\frac{\pi}{2\gamma}r(4r+1)\leq r.
$$

Equivalently, we get the inequality

$$
\frac{1}{2}\gamma e^{-1}+\frac{3\pi}{2}+\frac{\pi r}{2}(4r+1)\leq\gamma r,
$$

which can be written in the form

$$
2\pi r^2+(\frac{\pi}{2}-\gamma)r+\frac{3}{2}\pi+\frac{1}{2}e^{-1}\gamma\leq 0. \tag{4.7}
$$

It is easy to check that the above inequality has a positive solution for suitable value of the parameter γ. For example, for $\gamma=5\pi$ the number $r_0=\frac{9}{8}$ is a solution of inequality (4.7).

Observe that if $r_0>0$ is a solution of inequality (4.7) (equivalently: $r_0>0$ is a solution of the first inequality from assumption (xi)) then we have that

$$
\overline{G}K_1m(r_0)<\frac{1}{r_0}\big[A+\overline{F}\,\overline{G}K_1+\overline{G}K_1r_0m(r_0)\big]\leq\frac{r_0}{r_0}=1.
$$

Hence we infer that the second part of assumption (xi) of Theorem 3.1 is also satisfied.

Thus, in view of Theorem 3.1 we conclude that infinite system of integral equations (4.1) has a solution $x(t)$ in the Banach space BC_∞ which is coordinatewise converging at infinity.

Remark 4.1. Notice that the constant β appearing in infinite system (4.1) (cf. formula (4.5)) has no influence on conditions ensuring the solvability of that infinite system.

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