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NONLINEAR STABILITY OF BREATHER SOLUTIONS TO THE MODIFIED KDV-SINE-GORDON EQUATION*

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Abstract In this work, we are concerned with the modified KdV-sine-Gordon (mKdV-sG) equation. Breather solutions of the mKdV-sG equation are derived via using simplified Hirota's bilinear method. Moreover, we construct a new Lyapunov functional to present nonlinear stability of breather solutions to the mKdV-sG equation.

Keywords Nonlinear stability, the mKdV-sG equation, breather solutions, a new Lyapunov functional.

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1. Introduction

In this paper, we study nonlinear stablity of breather solutions to the following modified KdV-sine-Gordon (mKdV-sG) equation

$$u_{xt} + a(\frac{3}{2}u_x^2 u_{xx} + u_{xxxx}) = b\sin u, \tag{1.1}$$

with a, b are nonlinear parameters. In [16], the mKdV-sG equation was first found integrable by Konno, Kameyama and Sanuki when studying the nonlinear wave propagation in an infinite one-dimensional mon-atomic lattice in which the anharmonic potential competes with the dislocation potential and which can be solved by the inverse scattering transform method. Many physically meaningful systems, such as the modified Korteweg-de Vries (mKdV) equation [1], the Sine-Gordon (sG) equation [34,35], and the negative order modified Korteweg-de Vries (nmKdV) equation [2, 10, 29], are associated to mKdV-sG equation through reciprocal transformations. Moreover, those equations admit breather solutions which are known to describe a kind of wave solutions [11, 24, 31, 42].

In recent years, most of the works concentrated for exploring N-soliton solutions, multiple complex soliton solutions, N-periodic wave solutions and then studying the numerical analysis of solition solutions. Such as, the N-soliton solutions were discussed in [5,17] for the mKdV-sG equation by two kinds of bilinear forms. Infinitely

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many nonlocal symmetries and nonlocal conservation laws for the mKdV-sG equation were presented by Liang [19]. The complex simplified Hirota's forms and Lie symmetry analysis for multiple real and complex soliton solutions and the numerical evaluations of periodic wave solutions via a numerical method to the mKdV-sG equation have been studied in [36]. Some recent work on soliton solutions and some numerical results can be found in [12, 20, 22, 23, 26, 27, 37]. In this paper, we will study the breather solutions and their nonlinear stability by using simplified Hirota's bilinear method and constructing a new Lyapunov functional to the mKdV-sG equation.

Breather solutions which were found earlier in numerical simulations [6–9, 30] as long lifetime structures which show up spontaneously. We call breather a time periodic solution which is localized in space or equivalently which decays to zero at infinite distance. In a physical setting, breather solutions have been considered in completely integrable equations such as the sine-Gordon equation [4, 32] and the modified Korteweg-de Vries (mKdV) equation [1, 38]. So far, the first breather-type solution of the nonlinear Schrödinger equation was found over 40 years ago by Kuznetsov [18] and Ma [25], followed by the discovery of the Peregrine breather a few years later [28]. Alejo [1, 3] studied the nonlinear stability of breather solutions to the mKdV equation and the Gardner equation. Afterwards, Wang *et al.* considered the results of the coupled modified Korteweg-de Vries (cmKdV) equation and the negative order modified Korteweg-de Vries (nmKdV) equation [39, 40]. Based on the previous studies, the objectives of this work will be on extending our results in [39] to conduct breather solutions and their nonlinear stability of to the mKdV-sG equation.

The paper is organized as follows. In Section 2, we get stability tests via computing the generalized Weinstein conditions for the mKdV-sG breather solutions. According to the conservation laws, we get variational characterization of breather solutions in Section 3. Section 4 is devoted to the discussions of spectral properties through the analysis of the spectral stability . In Section 5, we present the proof of our main Theorem. The conclusions are summarized in Section 6.

2. Generalized Weinstein conditions

With variable transformation

$$u = 2i \ln \frac{f^*}{f}, \tag{2.1}$$

the mKdV-sG equation (1.1) reduce to the following bilinear form

$$D_x^2 f \cdot f^* = 0, \tag{2.2}$$

$$(D_x D_t + a D_x^4) f \cdot f = \frac{1}{2} b(f^2 - f^*), \tag{2.3}$$

where the D operator [13] is defined by

$$\begin{aligned} & D_t^m D_x^n a(t,x) \cdot b(t,x) \\ = & \frac{\partial^m}{\partial s^m} \frac{\partial^n}{\partial y^n} a(t+s, x+y) b(t-s, x-y)|_{s=0,y=0}, \\ & m, n = 0, 1, 2, \cdots . \end{aligned} \tag{2.4}$$

and f^* means the conjugate of f. Set $f = F + iG$, then $f^* = F - iG$, the bilinear equations (2.2)-(2.3) can be transformed to

$$D_x^2(F \cdot F + G \cdot G) = 0, \tag{2.5}$$
$$(D_x D_t + aD_x^4)(F \cdot F - G \cdot G) = 0, \tag{2.6}$$
$$(D_x D_t + aD_x^4 - b)F \cdot G = 0. \tag{2.7}$$

By solving the bilinear mKdV-sG equation (2.5)-(2.7), we can get the breather solutions

$$G = \exp(p_1 x - (ap_1^3 - \frac{b}{p_1})t + \exp(p_2 x - (ap_2^3 - \frac{b}{p_2})t),$$
$$F = 1 - \frac{(p_1 - p_2)^2}{(p_1 + p_2)^2} \exp[(p_1 + p_2)x - (ap_1^3 + ap_2^3 - \frac{b}{p_1} - \frac{b}{p_2})t],$$

where $p_1 = p + iq$, $p_2 = p - iq$ and p, q are real. The transformation between u and F, G is

$$u = 4 \arctan \frac{G}{F}.$$

With some calculations, we can write the breather solutions as

$$u = 4 \arctan(|\frac{p}{q}| \cdot \frac{\cos\{qx + [a(3p^2 q - q^3) + \frac{bq}{p^2+q^2}]t\}}{\cosh\{(px - [a(p^3 - 3pq^2) - \frac{bq}{p^2+q^2}]t + \ln|\frac{q}{p}|\}}.$$

Without loss of generality, in this paper, we choose $p,\ q > 0$, we can get the mKdV-sG breather solutions

$$B = 4\left[\arctan\left(\frac{p}{q} \cdot \frac{\cos(qy_1)}{\cosh(py_2)}\right)\right], \tag{2.8}$$

with $y_1 = x + [a(3p^2 - q^2) + \frac{b}{p^2+q^2}]t + x_1$, $y_2 = x - [a(p^2 - 3q^2) - \frac{b}{p^2+q^2}]t + \frac{\ln\frac{q}{p}}{p} + x_2$. Note that from the formula B, we have

$$B(t, x; x_1, x_2) = B(t - t_0, x - x_0), \tag{2.9}$$

and

$$B(t, x; x_1 + \frac{k\pi}{q}, x_2) = (-1)^k B(t, x; x_1, x_2), \tag{2.10}$$

where $t_0 = \frac{x_1 - x_2}{4(p^2+q^2)}$, $x_0 = \frac{\alpha x_2 - \beta x_1}{4(p^2+q^2)}$, $\alpha = a(3p^2 - q^2) + \frac{b}{p^2+q^2}$, $\beta = a(3q^2 - p^2) + \frac{b}{p^2+q^2}$, which reveal the mKdV-sG breather solutions are invariant under space and time translations and periodic for the first translation parameter.

In [14], Hu *et al.* presented the first three conservation quantities of the mKdV-sG equation by further numerical experiments

$$M_1[u] = \int_{\mathbb{R}} u_x^2 dx, \tag{2.11}$$
$$M_2[u] = \int_{\mathbb{R}} (u_x^2 - 4u_{xx}^2) dx, \tag{2.12}$$
$$M_3[u] = \int_{\mathbb{R}} (u_x^6 - 20u_x^2 u_{xx}^2 + 8u_{xxx}^2) dx. \tag{2.13}$$

With some calculations, breather solutions (2.8) can be transformed into

$$B_x = -4p\mathrm{sech}(py_2)\left[\frac{\sin(qy_1)+\frac{p}{q}\cos(qy_1)\tanh(py_2)}{1+\frac{p^2}{q^2\cos^2(qy_1)\mathrm{sech}^2(py_2)}}\right], \tag{2.14}$$

then, after some simplifications, we have

$$\begin{aligned} B_x^2 =& 16p^2\mathrm{sech}^2(py_2)\left[\frac{\sin(qy_1)+\frac{p}{q}\cos(qy_1)\tanh(py_2)}{1+\frac{p^2}{q^2\cos^2(qy_1)\mathrm{sech}^2(py_2)}}\right]^2 \\ =& 16p^2q^2\left[\frac{q^2\sin^2(qy_1)\cosh^2(py_2)+p^2\cos^2(qy_1)\sinh^2(py_2)}{p^2\cos^2(qy_1)+q^2\cosh^2(py_2)}\right] \\ &+ 16p^2q^2\frac{2pq\sin(qy_1)\cos(qy_1\sinh(py_2)\cosh(py_2)}{p^2\cos^2(qy_1)+q^2\cosh^2(py_2)}. \end{aligned} \tag{2.15}$$

In view of double angle formulas,

$$\begin{aligned} \Omega :=& \int_{-\infty}^{x} B_x^2(t,s;x_1,x_2)ds \\ =& \frac{8p\left[p^2+q^2-pq\sin(2qy_1)+p^2\cos(2qy_1)+q^2\sinh(2py_2)+q^2\cosh(2py_2)\right]}{p^2+q^2+q_2\cosh(2py_2)+p^2\cos(2qy_1)}. \end{aligned} \tag{2.16}$$

As $x \longrightarrow +\infty$, we have

$$M_1[B] = \int_{\mathbb{R}} B_x^2 dx = 16p. \tag{2.17}$$

Then, integrating by parts for (1.1) in space, one has

$$\partial_t\Omega + \frac{a}{4}(B_x^4 - 4B_{xx}^2) = 2b - 2b\cos B, \tag{2.18}$$

by integrating for (2.18) in space, when $x \longrightarrow +\infty$, we have

$$\begin{aligned} M_2[B] &= \int_{\mathbb{R}} B_x^4 dx - \int_{\mathbb{R}} 4B_{xx}^2 dx \\ &= -\frac{4}{a}\left[\int_{\mathbb{R}} 2b(1-\cos B)dx - \int_{\mathbb{R}} \partial_t\Omega dx\right] \\ &= -\frac{4}{a}[\frac{8bp}{p^2+q^2} + 16p\beta]. \end{aligned} \tag{2.19}$$

These two results show the explicit dependence of the mass (2.11) and the energy (2.12) on the scaling parameters $p,\ q$. Let $\partial_p B = \wedge_p B$, $\partial_q B = \wedge_q B$, we obtain

$$\partial_p M_1[B] = 8\int_{\mathbb{R}} B_x \wedge_p B_x dx = 16 > 0,$$

$$\partial_q M_1[B] = 8\int_{\mathbb{R}} B_x \wedge_q B_x dx = 0,$$

which leads to the generalized Weinstein condition [41] for any breather solutions $B_{p,q}$ of the mKdV-sG equation (1.1).

3. Variational characterization of breather solutions

In this section, according to the conservation quantities (2.11), (2.12) and (2.13), we introduce a new Lyapunov function for the mKdV-sG equation (1.1), well defined in the H^3-topology, as follows:

$$\begin{aligned}H[u](t) :=& M_3[u](t) - 4(p^2-q^2)M_2[u](t) + 8(p^2+q^2)^2 M_1[u](t)\\ =& \int_{\mathbb{R}} (u_x^6 - 20u_x^2u_{xx}^2 + 8u_{xxx}^2)dx - 4(p^2-q^2)\int_{\mathbb{R}}(u_x^4 - 4u_{xx}^2)dx\\ &+ 8(p^2+q^2)^2\int_{\mathbb{R}} u_x^2 dx, \end{aligned}\tag{3.1}$$

with p and q are scaling parameters. Clearly, $H[u]$ is a real-valued conserved quantity. Moreover, one has the following result.

Proposition 3.1. *Let $B = u$ be any mKdV-sG breather solution, then for any $z \in H^3(\mathbb{R})$ with sufficiently small H^3-norm, we have*

$$H[B+z] = H[B] + \frac{1}{2}Q[z] + N[z],\tag{3.2}$$

where $Q[z]$, $N[z]$ are fixed later. Moreover, $\mid N[z] \mid \leq K\|z\|^3_{H^3(\mathbb{R})}$ for some constant K.

Proof. By computing $H[B+z]$ directly, we have the following decomposition

$$H[B+z] = H[B] - 16\int_{\mathbb{R}} G[B](t)z dx + \frac{1}{2}Q[z] + N[z],\tag{3.3}$$

where

$$\begin{aligned}G[B] =& B_{(6x)} - 2(p^2-q^2)(2B_{(4x)} + 3B_x^2B_{xx}) + (p^2+q^2)^2B_{xx}\\ &+ \frac{15}{8}B_x^4B_{xx} + \frac{5}{2}B_{xx}^3 + 10B_xB_{xx}B_{xxx} + \frac{5}{2}B_x^2B_{(4x)},\end{aligned}\tag{3.4}$$

$$\begin{aligned}Q[z] =& 4\int_{\mathbb{R}} z_{xxx}^2 dx + 8(p^2-q^2)\int_{\mathbb{R}} z_{xx}^2dx + 4(p^2+q^2)^2\int_{\mathbb{R}} z_x^2 dx\\ &- 40\int_{\mathbb{R}} B_xB_{xx}z_xz_{xx}dx - 10\int_{\mathbb{R}} B_{xx}^2z_x^2dx - 10\int_{\mathbb{R}} B_x^2z_{xx}^2dx\\ &- 12(p^2-q^2)\int_{\mathbb{R}} B_x^2z_x^2dx + \frac{15}{2}\int_{\mathbb{R}} B_x^4z_x^2dx,\end{aligned}\tag{3.5}$$

and

$$\begin{aligned}N[z] =& 20\int_{\mathbb{R}} B_x^3z_x^3dx - 16(p^2-q^2)\int_{\mathbb{R}} B_xz_x^3dx + 15\int_{\mathbb{R}} B_x^2z_x^4dx\\ &- 4(p^2-q^2)\int_{\mathbb{R}} z_x^4dx + 6\int_{\mathbb{R}} B_xz_x^5dx + \int_{\mathbb{R}} z_x^6dx\\ &- 40\int_{\mathbb{R}} B_xz_xz_{xx}^2dx - 40\int_{\mathbb{R}} B_{xx}z_x^2z_{xx}dx - 20\int_{\mathbb{R}} z_x^2z_{xx}^2dx.\end{aligned}\tag{3.6}$$

Note that, from direct estimates for $N[z]$, we have $N[z] = O\left(\|z(t)\|^3_{H^3(\mathbb{R})}\right)$, which means

$$\mid N[z] \mid \leq K\|z\|^3_{H^3(\mathbb{R})}.$$

From (1.1) and (2.18), we have

$$\begin{aligned}G[B] =&\frac{(b\sin B - B_{xt} - \frac{3}{2}aB_x^2B_{2x})_{2x}}{a} - \frac{4(p^2-q^2)}{a}(b\sin B - B_{xt})\\&+(p^2+q^2)^2B_{xx} + \frac{15}{8}B_x^4B_{xx} + \frac{5}{2}B_{xx}^3 + 10B_xB_{xx}B_{xxx} + \frac{5}{2}B_x^2B_{(4x)}\\=&\frac{1}{a}[B_{xxxt} + b\cos BB_{xx} - b\sin BB_x^2] - \frac{4(p^2-q^2)}{a}(b\sin B - B_{xt})\\&+(p^2+q^2)^2B_{xx} + \frac{15}{8}B_x^4B_{xx} - \frac{1}{2}B_{xx}^3 + B_xB_{xx}B_{xxx} - \frac{1}{2}B_x^2B_{(4x)}\\=&0,\end{aligned} \tag{3.7}$$

this implies Proposition 3.1 is proved. □

4. Spectral analysis

Let $z \in H^6(\mathbb{R})$, B be any breather solution of equation(1.1), with shift parameters x_1, x_2. From the definition of $Q[z]$, we introduce the following operator:

$$\begin{aligned}L[z](x,t) =& -4z_{(6x)} + 8(p^2-q^2)z_{(4x)} - 4(p^2+q^2)^2z_{xx} + 30B_{xx}^2z_{xx}\\&+20B_{xx}B_{xxx}z_x - 30B_x^4z_{xx} + 20B_xB_{xxx}z_{xx} - 10B_x^2z_{(4x)}\\&+12(p^2-q^2)(2B_xB_{xx}z_x + B_x^2z_{xx}) - \frac{15}{2}B_x^4z_{xx}.\end{aligned} \tag{4.1}$$

Clearly, $Q[z] := \int_{\mathbb{R}} zL[z]dx$. Moreover, let $z, \omega \in H^6(\mathbb{R})$, we have

$$\begin{aligned}&\int_{\mathbb{R}} \omega L[z](x,t)dx\\=&\int_{\mathbb{R}}[4z_{(3x)}\omega_{(3x)} + 8(p^2-q^2)z_{xx}\omega_{xx} + 4(p^2+q^2)^2z_x\omega_x]dx\\&+\int_{\mathbb{R}}[30B_{xx}^2z_x\omega_x + 30B_x^4z_x\omega_x - 10B_{xx}^2z_{xx}\omega_{xx} + 40B_xB_{(3x)}z_x\omega_x]dx\\&+\int_{\mathbb{R}}[40B_xB_{xx}z_x\omega_{xx} + 150B_x^3B_{xx}z_x\omega]dx + 12(p^2-q^2)\int_{\mathbb{R}}B_x^2z_x\omega_xdx\\&+\frac{15}{2}\int_{\mathbb{R}}B_x^4z_x\omega_xdx = \int_{\mathbb{R}}zL[\omega](x,t)dx,\end{aligned} \tag{4.2}$$

which leads to $L^* = L$ in $H^6(\mathbb{R})$. Therefore, it is easy to see that $D(L^*)$ can be identified with $D(L) = H^6(\mathbb{R})$.

In the following, we have the following results.

Proposition 4.1. *Let $\partial_{x_1}B = B_1$, $\partial_{x_2}B = B_2$. Set a Wronskian matrix of B_1, B_2*

$$W[B_1, B_2](t,x) =: \begin{bmatrix} B_1 & B_2 \\ (B_1)_x & (B_2)_x \end{bmatrix}(x,t), \tag{4.3}$$

then we can get

$$\sum_{x\in\mathbb{R}} \dim\ker W[B_1,B_2](t,x) = \dim\ker W[B_1,B_2](t; y_0 - \beta t - x_0) = 1, \tag{4.4}$$

for any breather solution of equation(1.1).

Proof. At first, by direct computing yields

$$\begin{aligned} & detW[B_1, B_2](t,x) \\ &= B_1(B_2)_x - B_2(B_1)_x \\ &= (p\sin 2qy_2 - q\sinh 2py_1)\cdot\frac{4p^3q^3[p^2(1+\cos 2qy_1)+q^2((1+\cosh 2py_2)]}{p^2\cos^2(qy_1)+q^2\cosh^2(py_2)}. \end{aligned} \tag{4.5}$$

Define

$$g(y_2) = g_{t,p,q,\widetilde{x}_2} = q\sinh(2py_2) - p\sin(2q\widetilde{y}_2), \tag{4.6}$$

where $\widetilde{y}_2 = y_2 + (\alpha-\beta)t + \widetilde{x}_2,\ \widetilde{x}_2 = (x_1 - x_2 - \frac{\ln\frac{q}{p}}{p}) \in \mathbb{R},\ \alpha - \beta = 4(p^2 - q^2)$.
In fact, for $y_2 \in \mathbb{R}$, when $|\sinh 2py_2| > \frac{p}{q}$, $g(y_2)$ has no root. Moreover, by a simple argument, we can find that there exists $R_0 = R_0(p,q) > 0$ such as $g(y_2) > 0$ in the case of $y_2 > R_0$ and $g(y_2) < 0$ in the case of $y_2 < -R_0$. Combining the condition of $g(y_2)$ is continuous, we get $g(y_2)$ has a root $y_0 = y_0(t,p,q,\widetilde{x}_2) \in [-R_0, R_0]$. If $y_2 \neq 0$, we can deduce that

$$g^{'}(y_2) = 2pq\left(\cosh(2py_2) - \cos(2q\widetilde{y}_2)\right). \tag{4.7}$$

This result means that if $y_0 \neq 0$, we have

$$\sum_{x\in\mathbb{R}} \dim\ker W[B_1,B_2](t,x) = \dim\ker W[B_1,B_2](t; y_0 - \beta t - x_0) = 1,$$

since B_1 or $(B_1)_x$ are not zero at the same point. This completes Proposition 4.1. □

Proposition 4.2. *Let $B_1 = \partial_{x_1}B,\ B_2 = \partial_{x_2}B,\ \partial_p B = \wedge_p B,\ \partial_q B = \wedge_q B$. Then the operator L has unique negative eigenvalue, counting multiplicity, namely $-\lambda_0^2$. Moreover, we have*

$$\ker L = Span\left\{B_1(t,.,x_1,x_2), B_2(t,.,x_1,x_2)\right\},$$

and

$$Q[z] \geq -c_{p,q}\|z\|^2_{H^3(\mathbb{R})},$$

where $a_{p,q}$ is a positive constant only depending on $p,\ q$, for any breather solution B of equation(1.1).

Proof. Firstly, by (3.7), we have $\partial_{x_1}G[B] = \partial_{x_2}G[B] = 0$, it follows

$$L[B_1](t;x_1,x_2) = L[B_2](t;x_1,x_2) = 0. \tag{4.8}$$

Moreover, from (3.7), we also have after derivation with respect to p and q,

$$\begin{aligned} L(\wedge_p B) &= 4p[2B_{(4x)} + 3B_x^2 B_{xx} - (p^2+q^2)B_{xx}], \\ L(\wedge_q B) &= -4q[2B_{(4x)} + 3B_x^2 B_{xx} + (p^2+q^2)B_{xx}]. \end{aligned}$$

In view of (2.17) and (2.19),

$$\begin{aligned}\int_{\mathbb{R}} \wedge_p BL(\wedge_p B)dx &= -p\partial_p M_2 + 2p(p^2+q^2)\partial_p M_1 \\ &= 32p(p^2+q^2) + \frac{32}{a}[\frac{3b(q^2-p^2}{a(p^2+q^2)^2} + 2p(p^2+3q^2)] > 0,\end{aligned} \tag{4.9}$$

$$\int_{\mathbb{R}} \wedge_q BL(\wedge_q B)dx = q\partial_q M_2 + 2q(p^2+q^2)\partial_q M_1 = -\frac{64}{a}[6 + \frac{1}{(p^2+q^2)^2}] < 0. \tag{4.10}$$

Then, from the identity of the operator L, it is easy to see L is a compact perturbation of its constant coefficients operator L_0

$$L_0[z](x,t) := z_{(6x)} - 2(p^2-q^2)z_{(4x)} + (p^2+q^2)^2 z_{xx}, \tag{4.11}$$

by the Weyl Theorem on continuous Spectrum. Clearly,

$$e^{\pm px}\cos(qx),\ e^{\pm px}\sin(qx),\ p, q > 0,$$

are base functions of the null space for L_0. Therefore, the kernel of the operator $L|_{H^6(\mathbb{R})}$ is spanned by at most two $L^2 - functions$ in positive times $[0, +\infty)$, among the above four functions. Which proves $\ker L = Span\{B_1(t,.,x_1,x_2), B_2(t,.,x_1,x_2)\}$.

Finally, combining the above results and (4.4) by the uniqueness criterium (cf. [15, 21]), we get the negative eigenvalue of L is unique, namely, $-\lambda_0^2$. Then, let $B_0 := \frac{p(\wedge_q B)+q(\wedge_p B)}{8pq(p^2+q^2)}$, we have $L[B_0] = -B_{xx}$. Additionally,

$$\int_{\mathbb{R}} B_0 B_{xx} dx = -\int_{\mathbb{R}} B_0 L[B_0]dx = -Q[B_0] = \frac{1}{p(p^2+q^2)} > 0,$$

we find that B_0 is a negative direction of Q. Now, let B_{-1} be an eigenfunction to $-\lambda_0^2$. Remembering the form $Q[z] = \int_{\mathbb{R}} zL[z]dx$, and $Q[B_1] = Q[B_2] = 0$. Thanks to $\wedge_p B$ is a positive direction for Q. Moreover, the form Q is bounded below, that is

$$Q[z] \geq -a_{p,q}\|z\|^2_{H^3(\mathbb{R})}.$$

□

Lemma 4.1. *Let B be any breather solution of equation(1.1), and B_1, B_2 be the kernels of the operator L. Assume that there exists $\eta_0(p,q)$, for any $z \in H^3(\mathbb{R})$, satisfying*

$$\int_{\mathbb{R}} B_1 z dx = \int_{\mathbb{R}} B_2 z dx = 0, \tag{4.12}$$

one has

$$Q[z] \geq \eta_0\|z\|^2_{H^3(\mathbb{R})} - \frac{1}{\eta_0}(\int_{\mathbb{R}} B_x z_x dx)^2. \tag{4.13}$$

Proof. From Proposition 4.2, (4.12) and the additional orthogonality condition $\int_{\mathbb{R}} B_x z_x dx = 0$, we decompose z and B_0 in span (B_{-1}, B_1, B_2), as follows,

$$z = \widetilde{z} + mB_{-1},\ B_0 = b_0 + nB_{-1} + rB_1 + sB_2,\ m, n, r, s \in \mathbb{R},$$

with

$$\int_{\mathbb{R}} \widetilde{z}B_{-1}dx = \int_{\mathbb{R}} \widetilde{z}B_1dx = \int_{\mathbb{R}} \widetilde{z}B_2dx = \int_{\mathbb{R}} b_0B_{-1}dx = \int_{\mathbb{R}} b_0B_1dx = \int_{\mathbb{R}} b_0B_2dx = 0.$$

In addition that

$$\int_{\mathbb{R}} B_{-1}B_1dx = \int_{\mathbb{R}} B_{-1}B_2dx = 0,$$

one has

$$Q[z] = \int_{\mathbb{R}} (L[\widetilde{z}] - m\lambda_0^2 B_{-1})(\widetilde{z} + mB_{-1})dx = Q[\widetilde{z}] - m^2\lambda_0^2. \tag{4.14}$$

In view of $L[B_0] = -B_{xx}$,

$$\begin{aligned} 0 &= \int_{\mathbb{R}} B_x z_x dx = \int_{\mathbb{R}} zL[B_0]dx = \int_{\mathbb{R}} L[\widetilde{z} + mB_{-1}]B_0dx \\ &= -\int_{\mathbb{R}} (m\lambda_0^2 B_{-1} - L[\widetilde{z}])(b_0 + nB_{-1} + rB_1 + sB_2)dx \\ &= L[\widetilde{z}]b_0 - mn\lambda_0^2, \end{aligned} \tag{4.15}$$

and similarly,

$$\int_{\mathbb{R}} B_0B_{xx}dx = -\int_{\mathbb{R}} B_0L[B_0]dx = -\int_{\mathbb{R}} (b_0 + nB_1)(L[b_0] - n\lambda_0^2)dx = -Q[b_0] + n^2\lambda_0^2. \tag{4.16}$$

Inserting (4.15) and (4.16) into (4.14) yields

$$Q[z] = Q[\widetilde{z}] - \frac{(\int_{\mathbb{R}} L[\widetilde{z}]b_0dx)^2}{\int_{\mathbb{R}} B_0B_{xx}dx + Q[b_0]}. \tag{4.17}$$

If $\widetilde{z} = \lambda b_0$, $\lambda \neq 0$, we have

$$\frac{(\int_{\mathbb{R}} L[\widetilde{z}]b_0dx)^2}{\int_{\mathbb{R}} B_0B_{xx}dx + Q[b_0]} \leq CQ[\widetilde{z}],\ 0 < C < 1.$$

Therefore, for some $C_1 > 0$, we can deduce that

$$\begin{aligned} Q[z] &\geq (1 - C_1)Q[\widetilde{z}] \geq \frac{1}{2}(1 - C_1)Q[\widetilde{z}] + (1 - C_1)m^2\lambda_0^2 \\ &\geq \frac{1}{C_1}(2\|\widetilde{z}\|^2_{H^3(\mathbb{R})} + 2m^2\|B_{-1}\|^2_{H^3(\mathbb{R})}) \geq \frac{1}{C_1}\|z\|^2_{H^3(\mathbb{R})}, \end{aligned} \tag{4.18}$$

this result show that our Lemma 4.1. □

5. Nonlinear stability of mKdV-sG breathers

In this section, our main result can be summarized as the following Theorem.

Theorem 5.1. *Let $p,\ q>0$, $u(x,t)|_{t=0}=u_0\in H^3(\mathbb{R})$, if there exists a η_0 depending on $p,\ q$ such that*

$$(Rt1)\qquad \|u_0-B_{p,q}(0,0;0,0)\|_{H^3(\mathbb{R})}\le\eta,\ \forall\eta\in(0,\eta_0).$$

Then there exist $x_1(t),\ x_2(t)$ such that the solution $u(x,t)$ of equation (1.1) with $u(x,t)|_{t=0}=u_0$ satisfies

$$(Rt2)\qquad \sup_{t\in\mathbb{R}}\|u(t)-B_{p,q}(x,t;x_1(t),x_2(t))\|_{H^3(\mathbb{R})}\le K\eta,$$

with

$$(Rt3)\qquad \sup_{t\in\mathbb{R}}(|x_1^{'}(t)|+|x_2^{'}(t)|)\le KK_1\eta,$$

for some constants $K>0,\ K_1>0$.

Proof of Theorem 5.1. Considering that $u_0\in H^3(\mathbb{R})$ satisfies (Rt1) and $u\in C(\mathbb{R};H^3(\mathbb{R}))$ is the corresponding solution of equation (1.1) with $u(0)=u_0$. By contradiction, assume that a maximal time of stability T_0, namely

$$\begin{aligned}T_0:=\sup_{t\in\mathbb{R}}\{T>0\mid &\text{ for all } t\in[0,T_0],\ \text{there exist } \widetilde{x}_1(t),\ \widetilde{x}_2(t)\in\mathbb{R}\ \text{ such that}\\ &\sup_{t\in[0,T_0]}\|u(t)-B(t,x;\widetilde{x}_1(t),\widetilde{x}_2(t))\|_{H^3(\mathbb{R})}\le K^*\eta\},\end{aligned}\tag{5.1}$$

is finite, with $K_1^*>2$, $\eta\in(0,\eta_0)$ is fixed in Theorem 5.1. Our main interesting is to consider positive times, since the negative time case is analogous, we omit it. Therefore, if we can prove $T_0=+\infty$ such that T_0 is contradict with (5.1), we complete the proof. In fact, for all $t\in[0,T_0]$, by taking η_0 smaller, if Theorem 5.1 is satisfied if only if the following results are satisfied

$$(Rt3)\qquad \|Z(0)\|_{H^3(\mathbb{R})}\le K_1\eta,$$

$$(Rt3)\qquad \int_{\mathbb{R}}B_1(t;x_1(t),x_2(t))z(t)dx=\int_{\mathbb{R}}B_2(t;x_1(t),x_2(t))z(t)dx=0,$$

$$(Rt3)\qquad \|z(t)\|_{H^3(\mathbb{R})}+|x_1^{'}(t)|+|x_2^{'}(t)|\le K_1K^*\eta,\qquad \text{with } z(t):=u(t)-B(t).$$

Firstly, it is clear that from (Rt1), we can get (Rt4). In the following, according to the Implicit Function Theorem. Set

$$J_i(u(t),x_1,x_2):=\int_{\mathbb{R}}(u(t,x)-B(t,x_i,x_1,x_2))B_idx,\ i=1,2,$$

one has

$$\partial_{x_k}J_i(u(t),x_1,x_2)|_{(B(t),0,0)}=-\int_{\mathbb{R}}B_k(t,x_i,0,0)B_i(t,x_i,0,0)dx.$$

Then, let J be the 2×2 matrix with components $J_{i,k}:=(\partial_{x_k}J_i)_{i,k=1,2}$, we have

$$\det J=-\left[\int_{\mathbb{R}}B_1^2dx\int_{\mathbb{R}}B_2^2dx-(\int_{\mathbb{R}}B_1B_2dx)^2\right](t;0,0).$$

Since B_1 and B_2 are not parallel in the same time point, we have $\det J\neq0$, which leads to $(Rt5)$.

Finally, in view of (2.9),

$$B(t, x; x_1, x_2) = B(t - t_0(x), x - x_0(t)),$$

it follows

$$\begin{aligned} H[B(t, \cdot; x_1(t), x_2(t))] &= H[B(t - t_0(t), \cdot - x_0(t); 0, 0)] \\ &= H[B(t - t_0(t), \cdot; 0, 0)] \\ &= H[B(\cdot, \cdot; 0, 0)](t - t_0(t)). \end{aligned}$$

Since $\partial_t H[B(t, \cdot; x_1(t), x_2(t))] = H^{'}[B(\cdot, \cdot; 0, 0)](t - t_0(t)) \times (1 - t_0^{'}(t)) \equiv 0$, it follows that $H[B](t) = H[B](0)$ is constant in time. Combinging (3.1) and (3.2), we have

$$\begin{aligned} Q[z](t) &\leq Q[z](0) + K_1\|z(t)\|^3_{H^3(\mathbb{R})} + K_1\|z(0)\|^3_{H^3(\mathbb{R})} \\ &\leq +K_1\|z(0)\|^2_{H^3(\mathbb{R})} + K_1\|z(t)\|^3_{H^3(\mathbb{R})}. \end{aligned} \tag{5.2}$$

By Lemma 4.1, we obtain

$$\begin{aligned} \|z(t)\|^2_{H^3(\mathbb{R})} &\leq K_1\|z(0)\|^2_{H^3(\mathbb{R})} + K_1\|z(t)\|^3_{H^3(\mathbb{R})} + K_1|\int_{\mathbb{R}} B_x(t)z_x(t)|^2 \\ &= K_1\eta^2 + K_1(K^*)^3\eta^3 + K_1K_1|\int_{\mathbb{R}} B_x(t)z_x(t)|^2. \end{aligned} \tag{5.3}$$

After expanding $u = B + z$ in the conservation of mass (2.11), we get

$$\begin{aligned} |\int_{\mathbb{R}} B_x(t)z_x(t)| &\leq K_1|\int_{\mathbb{R}} B_x(0)z_x(0)| + K_1\|z(0)\|^2_{H^3(\mathbb{R})} + K_1\|z(t)\|^2_{H^3(\mathbb{R})} \\ &\leq K_1(\eta + (K^*)^2\eta^2), \text{for all } t \in [0, T_0]. \end{aligned} \tag{5.4}$$

Replacing (5.4) into (5.3), we have

$$\|z(t)\|^2_{H^3(\mathbb{R})} \leq K_1\eta^2(1 + (K^*)^2\eta^3) \leq \frac{1}{2}(K^*)^2\eta^2,$$

by taking K^* large enough, which contradicts assumption (5.1). □

6. Conclusion

In this paper, we have presented some details of nonlinear stability of breather solutions for the mKdV-sG equation. Our conclusion is that mKdV-sG breathers are nonlinear stable at the H^3-level of regularity. When waves transmission some point, it can be translate to breathers. As we explained above, the introduction of a new Lyapunov functional for which breather solutions are local minimizers at the essential point. Moreover, this functional controls the perturbation terms and the instability directions that appear during of the dynamics.

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