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EXISTENCE AND UNIQUENESS OF SOLUTIONS TO THE ULTRAPARABOLIC HAMILTON-JACOBI EQUATION

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Abstract We demonstrate the existence and uniqueness of solutions to the ultra-parabolic Hamilton-Jacobi equation. These solutions maintain the probabilistic interpretation of being the maximal expectation of a controlled ultra-diffusion process relative to a discounted performance criteria.

Keywords Ultraparabolic Hamilton-Jacobi equation, ultradiffusion process, optimal control.

MSC(2010) 49L12, 35K70, 93E20.

1. Introduction

The classification of partial differential equations as ultraparabolic appears to be due to Il'in [11], although the problem clearly predates his involvement*. In terms of applications, we will simply note that historically an interest in ultraparabolic equations arose relative to works by Kolmogorov [14, 15] and Uhlenbeck and Ornstein [25] in connection with Brownian motion in phase space, Chandrasekhar [2] with respect to the theory of boundary layers, and by Marshak [22] via linearization of the kinetic Boltzmann equations. Recent surveys may be found in Akhmetov *et al.* [1] and Lanconelli *et al.* [17]. In general, ultraparabolic equations do not possess properties that are fundamental to parabolic equations such as a strong maximum principle, interior *a priori* estimates, and so forth.

Chronologically, we note that the first theoretical investigations of ultraparabolic operators were by Piskunov [23] (classical) and Lions [18] (weak), who affected analysis along temporal *characteristics* and so were able to derive existence and uniqueness results through the surjectiveness of the transformation, albeit with the strict caveat that the speed of propagation vary spatially. Genčev [8] utilized *elliptic* regularization on bounded domains, but was unable to provide any distinct interpretation of the temporal derivatives. Il'in [11] employed the method of fundamental solutions, obtaining classical regularity for the Cauchy problem. Vladimirov and Drožžinov [26] extended Il'in's approach by using a convolution of the fundamental solution in order to obtain weak solutions on unbounded domains, but require the data to remain constant. Recently, Marcozzi [19] established the well-posedness and Galerkin approximation of the generalized solution (strong and weak) to the

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*The prefix *ultra-* meaning "beyond," in which case, *ultraparabolic* means *beyond* parabolic, and *ultradiffusion* means *beyond* diffusion.

linear ultraparbolic terminal value problem using energic techniques; these results were extended in [20] to unbounded domains, likewise applicable here.

Some more general recent studies include: Tersenov [24], wherein the Cauchy problem for a quasilinear ultraparabolic equation is considered; Han and Mo [9], who present generalized asymptotic solutions to a class of nonlinear singularly perturbed ultraparabolic equations; Kozhanov and Kosheleva [16], who analyze the solvability of linear inverse problems for ultraparabolic equations with unknown coefficients depending only on the spatial variables; Khoa et al. [13], who consider a uniqueness result for an age-dependent reaction-diffusion equation; and Ivasyshen and Pasichnyk [12], who obtain the fundamental solution for an ultraparabolic equation with infinitely increasing coefficients.

In this paper, we demonstrate the existence and uniqueness of regular solutions to the ultraparabolic Hamilton-Jacobi equation on bounded domains. In particular, the results are readily extended to higher dimensions in both space and time. The paper is organized as follows. In section 2, we derive a positivity result which is decisive for the existence result. In section 3, we consider the existence of solutions to the ultraparabolic Hamilton-Jacobi equation, which utilizes the existence of the optimizer of the Hamiltonian, the well-posedness of the linear terminal value ultraparabolic problem, as well as regularity. Section 4 presents the probabilistic interpretation of the solution to the ultraparabolic Hamilton-Jacobi equation as the discounted expectation of a controlled ultradiffusion process. This result in turn leads immediately to the uniqueness of the solution to the associated ultraparabolic Hamilton-Jacobi equation. A summary of the results is presented in section 5.

2. Positivity Result

We define

$$V = \overset{\circ}{W}{}_2^1(0,X) \quad \text{and} \quad H = L_2(0,X)\,,$$

in which case "$V \subseteq H \subset V^*$" is an *evolution triple*. Let $\mathcal{Q} = (0,T)\times(0,\Theta)\times(0,X)$, for positive finite T, Θ, and X, and $\mathcal{O}_{T,\Theta} = (0,T)\times(0,\Theta)$, $\mathcal{O}_{\Theta,X} = (0,\Theta)\times(0,X)$, and $\mathcal{O}_{T,X} = (0,T)\times(0,X)$. For $\mathcal{X} = L_2(\mathcal{O}_{T,\Theta};V)$ and $\mathcal{X}^* = L_2(\mathcal{O}_{T,\Theta};V^*)$, let

$$W = W_2^1(\mathcal{O}_{T,\Theta};V,H) = \{u \in \mathcal{X} \,:\, \nabla_{\boldsymbol{t}}(u) \in \mathcal{X}^* \times \mathcal{X}^*\}\,,$$

where $\boldsymbol{t} = (t,\vartheta)$ and $\nabla_{\boldsymbol{t}}(u) = (\partial u/\partial t, \partial u/\partial\vartheta)$, which we equip with the norm

$$\|u\|_W = \|u\|_{\mathcal{X}} + \|\partial u/\partial t\|_{\mathcal{X}^*} + \|\partial u/\partial\vartheta\|_{\mathcal{X}^*}\,.$$

We consider the *ultraparabolic* terminal-boundary value problem for $u \in W$ satisfying the evolutionary equation

$$-\frac{\partial u}{\partial t} - \frac{\partial(bu)}{\partial\vartheta} + A(t,\vartheta)\,u = f \;\; a.e. \text{ on } \;\mathcal{Q} \tag{2.1a}$$

subject to the terminal conditions

$$u(T,\vartheta,x) = \upsilon(\vartheta,x) \;\; a.e. \text{ on } \;\mathcal{O}_{\Theta,X} \tag{2.1b}$$

$$u(t,\Theta,x) = 0 \;\; a.e. \text{ on } \;\mathcal{O}_{T,X} \tag{2.1c}$$

and boundary conditions

$$u(t,\vartheta,0) = u(t,\vartheta,X) = 0 \;\; a.e. \text{ on } \;\mathcal{O}_{T,\Theta}\,, \tag{2.1d}$$

where

$$A(t,\vartheta)\,u = -\frac{\partial}{\partial x}\left(a_2\frac{\partial u}{\partial x}\right) + a_1\frac{\partial u}{\partial x} + a_0\,u\,,$$

for given

$$a_0\,,\; a_1\,,\; a_2 \in L_\infty(\mathcal{Q})\,, \tag{2.2a}$$
$$b > 0\,;\; b \in L_\infty(\mathcal{Q})\,;\; b^{-1}, \partial b/\partial\vartheta \in L_\infty(\mathcal{Q})\,, \tag{2.2b}$$
$$\upsilon \in L_2(0,\Theta;H), \tag{2.2c}$$
$$f \in L_2(\mathcal{O}_{T,\Theta};V^*)\,, \tag{2.2d}$$

and

$$a_2 \geq \alpha > 0\,, \tag{2.2e}$$

for some α.

The *generalized evolution problem* associated with (2.1) is: for given data satisfying (2.2), find $u \in W$ such that

$$-\frac{\partial}{\partial t}\,(u(t,\vartheta)|v)_H - \frac{\partial}{\partial\vartheta}\,(b(t,\vartheta)\,u(t,\vartheta)|v)_H + a\,(t,\vartheta;u(t,\vartheta),v) = \langle f(t,\vartheta),v\rangle_V\,, \tag{2.3a}$$

for all $v \in V$ and almost all $(t,\vartheta) \in \mathcal{O}_{T,\Theta}$,

$$u(T,\vartheta,x) = \upsilon(\vartheta,x) \;\; a.e. \text{ on } \;\mathcal{O}_{\Theta,X} \tag{2.3b}$$

and

$$u(t,\Theta,x) = 0 \;\; a.e. \text{ on } \;\mathcal{O}_{T,X}\,, \tag{2.3c}$$

where

$$a\,(t,\vartheta;u,v) = \int_0^X a_2\,\frac{\partial u}{\partial x}\,\frac{\partial v}{\partial x}\,dx + \int_0^X a_1\,\frac{\partial u}{\partial x}\,v\,dx + \int_0^X a_0\,u\,v\,dx\,,$$

for all $u,v \in V$ and $(t,\vartheta) \in \mathcal{O}_{T,\Theta}$, $(u|v)_H$ is the inner product on the Hilbert space H, and $\langle f,v\rangle_V$ is the value of the linear functional $f \in V^*$ at $v \in V$.

We consider

(A) The mapping $v \to v^- = \sup(-v,0)$ is Lipschitz continuous from $V \to V$.

The condition (A) is satisfied in the case $V = \overset{\circ}{W}{}_2^1\,(0,X)$ by the Sobolev Embedding theorem.

Lemma 2.1. *Suppose* (A) *and let* $u \in W$*, then*

$$\begin{aligned}
&2\int_{\mathcal{O}_{T,\Theta}}\left[\left(-\frac{\partial u}{\partial t},u^-\right) + \left(-\frac{\partial(bu)}{\partial\vartheta},u^-\right)\right]d\mathcal{O}\\
&= \int_0^\Theta \|u^-(T,\theta)\|_H^2\,d\theta - \int_0^\Theta \|u^-(0,\theta)\|_H^2\,d\theta\\
&\quad + \int_0^T \|\sqrt{b(\tau,\Theta)}\,u^-(\tau,\Theta)\|_H^2\,d\tau - \int_0^T \|\sqrt{b(\tau,0)}\,u^-(\tau,0)\|_H^2\,d\tau
\end{aligned}$$

and

$$u^- \in L^2(\mathcal{O}_{T,\Theta};V) \cap C^0\left(\overline{\mathcal{O}}_{T,\Theta}\,;\,H\right)\,.$$

Proof. We first show that the mapping $u \to u^-$ is continuous (actually Lipschitz continuous) from $W(\mathcal{O}_{T,\Theta};V) \to L^2(\mathcal{O}_{T,\Theta};V) \cap C^0\left(\overline{\mathcal{O}}_{T,\Theta}\,;\,H\right)$. To this end, if u, $v \in W(\mathcal{O}_{T,\Theta};V)$, we have

$$|u^-(t,\vartheta,x) - v^-(t,\vartheta,x)| \le |u(t,\vartheta,x) - v(t,\vartheta,x)|$$

in which case

$$\|u^-(t,\vartheta) - v^-(t,\vartheta)\|_H \le \|u(t,\vartheta) - v(t,\vartheta)\|_H\,,$$

where $\|\cdot\|_H = |\cdot|$, such that

$$\|u^- - v^-\|_{C^0\left(\overline{\mathcal{O}}_{T,\Theta}\,;\,H\right)} \le \|u-v\|_{C^0\left(\overline{\mathcal{O}}_{T,\Theta}\,;\,H\right)} \le C\|u-v\|_{W(\mathcal{O}_{T,\Theta};V)}\,.$$

Similarly,

$$\|u^-(t,\vartheta) - v^-(t,\vartheta)\|_{L^2(\mathcal{O}_{T,\Theta};V)} \le C\,\|u(t,\vartheta) - v(t,\vartheta)\|_{L^2(\mathcal{O}_{T,\Theta};V)}\,,$$

from which the continuity of the mapping follows.

Let u_j be a sequence of functions in $C^1\left(\overline{\mathcal{O}}_{T,\Theta}\,;\,V\right)$, then by Green's Theorem applied to (2.3)

$$\begin{aligned} &2\int_{\mathcal{O}_{T,\Theta}} \left[\left(-\frac{\partial u_j}{\partial t}, u_j^-\right) + \left(-\frac{\partial (b u_j)}{\partial \vartheta}, u_j^-\right)\right] d\mathcal{O} \\ &= \int_0^\Theta \|u_j^-(T,\theta)\|_H^2\, d\theta - \int_0^\Theta \|u_j^-(0,\theta)\|_H^2\, d\theta \\ &\quad + \int_0^T \|\sqrt{b(\tau,\Theta)}\, u_j^-(\tau,\Theta)\|_H^2\, d\tau - \int_0^T \|\sqrt{b(\tau,0)}\, u_j^-(\tau,0)\|_H^2\, d\tau\,, \end{aligned}$$

in which case the result follows by proceeding to the limit and by the continuity above. $\square$

For $f \in L^2(\mathcal{O}_{T,\Theta};V^*)$, we say that f is non-negative (i.e. $f \ge 0$) if

$$f \ge 0 \Leftrightarrow \int_{\mathcal{O}_{T,\Theta}} (f,v)\, d\mathcal{O} \ge 0\,,\ \ \forall v \in L^2(\mathcal{O}_{T,\Theta};V)\ \ni\ v \ge 0\,. \tag{2.4}$$

For a solution of (2.3), we have the following:

Theorem 2.1. Positivity. *Let $f \in L^2(\mathcal{O}_{T,\Theta};V^*)$ and $\overline{u} \in L^2(0,\Theta;H)$. We assume* (A)*, $f \ge 0$, and $\overline{u} \ge 0$, then $u^- = 0$.*

Proof. From the weak form (2.3), it follows that

$$\int_{\mathcal{O}_{T,\Theta}} \left[\left(-\frac{\partial u}{\partial t}, v\right) + \left(-\frac{\partial (bu)}{\partial \vartheta}, v\right) + a(t,\vartheta;u,v)\right] d\mathcal{O} = \int_{\mathcal{O}_{T,\Theta}} (f,v)\, d\mathcal{O}\,,$$

for all $v \in L^2(\mathcal{O}_{T,\Theta};V)$. Setting $v = u^-$ in the above and using the Lemma 2.1, we have that

$$\begin{aligned} &\int_{\mathcal{O}_{T,\Theta}} a(t,\vartheta;u^-,u^-)\, d\mathcal{O} + \int_{\mathcal{O}_{T,\Theta}} (f,v)\, d\mathcal{O} \\ &+ \int_0^\Theta \|u^-(0,\theta)\|_H^2\, d\theta - \int_0^\Theta \|u^-(T,\theta)\|_H^2\, d\theta \end{aligned}$$

$$+\int_0^T \|\sqrt{b(\tau,0)}\,u^-(\tau,0)\|_H^2\,d\tau - \int_0^T \|\sqrt{b(\tau,\Theta)}\,u^-(\tau,\Theta)\|_H^2\,d\tau = 0\,.$$

Since $u^-(T,\vartheta)=0$ if $\overline{u}\geq 0$, then $u^-(0,\Theta)=0$, and from (2.4) we have

$$\int_{\mathcal{O}_{T,\Theta}} (f,\overline{u})\,d\mathcal{O}\geq 0\,,$$

we deduce that

$$\int_{\mathcal{O}_{T,\Theta}} a(t,\vartheta;u^-,u^-)\,d\mathcal{O}\leq 0\,,$$

in which case $u^-=0$. □

3. Hamilton-Jacobi Equation

We demonstrate the existence of a solution to the ultraparabolic Hamilton-Jacobi equation. To begin, we define the Hamiltonian and show that it may be optimized. To this end, let $\mathcal{V}\subset\mathbb{R}$ compact, and $f(t,\vartheta,x,\nu)$, $g(t,\vartheta,x,\nu)$, $c(t,\vartheta,x,\nu)$, and $\Psi(t,\vartheta,x,\nu)$ map $\overline{\mathcal{Q}}\times\mathcal{V}\to\mathbb{R}$. Moreover, we suppose that

$$f\,,\ g\,,\ c\,,\ \Psi\ \text{ are uniformly continuous with respect to }\ \nu \tag{3.1a}$$

and for fixed $\nu\in\mathcal{V}$,

$$f(\cdot,\nu)\,,\ g(\cdot,\nu)\,,\ c(\cdot,\nu)\,,\ \Psi(\cdot,\nu)\in C^1(\overline{\mathcal{Q}})\,. \tag{3.1b}$$

Let

$$L(t,\vartheta,x,u,p,q\,;\nu)=\Psi+c\,u+f\,p+g\,q\,, \tag{3.2}$$

for $(u,p,q)\in\mathbb{R}^3$. Given (3.1), we note that L attains its maximum for fixed t, ϑ, x, u, p, and q. We define the *Hamiltonian* such that

$$H(t,\vartheta,x,u,p,q)=\max_{\nu\in\mathcal{V}}\ L(t,\vartheta,x,u,p,q\,;\nu)\,. \tag{3.3}$$

On account of (3.1), we note that H is measurable,

$$|H(t,\vartheta,x,u,p,q)|\leq C\left(h(t,\vartheta,x)+|u|+|p|+|q|\right)\,, \tag{3.4a}$$

where $h(t,\vartheta,x)\in L(\mathcal{Q})$ and

$$\begin{aligned}&|H(t,\vartheta,x,u_1,p_1,q_1)-H(t,\vartheta,x,u_2,p_2,q_2)|\\ \leq &C\left(|u_1-u_2|+|p_1-p_2|+|q_1-q_2|\right).\end{aligned} \tag{3.4b}$$

Lemma 3.1. *We suppose (3.1); there exists an optimizer of the Hamiltonian, which is measurably dependent on the data.*

Proof. We set $z=(t,\vartheta,x,u,p,q)$. Given (3.1), the function L is continuous with respect to z. We let

$$D=\{\ (z\,;\nu)\mid L=H\ \}\,,$$

in which case D is closed and, in particular, σ-compact, i.e. $D=D_1\cup D_2\cup\cdots$, where D_1, D_2, … are compact. Let

$$D^z=\{v\mid (z\,;v)\in D\ \}\,,$$

and

$$\Delta = \{ \ z \mid D^z \neq \emptyset \ \} .$$

We have that $\Delta = \overline{\mathcal{Q}} \times \mathbb{R}^3$, from (3.3). It follows then from the theorem for the existence of measurable sections of multivalued mappings that there exists a measurable mapping $\nu = \nu(z)$, such that

$$(z; v(z)) \in D ,$$

for a.e. $z \in D$ (e.g. [3], Theorem 1.14, [6], Appendix B, Lemma B). □

In addition to (3.1), we take

$$a_2(t, \vartheta, x) \in C^2 \left([0, T] \times \mathbb{R}^2 \right) \tag{3.5}$$

such that

$$a_2 \geq \alpha > 0 , \tag{3.6}$$

for given α, and

$$b > 0 . \tag{3.7}$$

sufficiently large. Let $\overline{u}(x, \vartheta)$ be such that

$$\overline{u} \in W^{1,2,p}(\mathcal{O}_{\Theta,X}) \cap L^p \left((0, \Theta); W_0^{1,p}(0, X) \right) , \tag{3.8}$$

for $p > 3/2$, where

$$W^{1,2,p}(\mathcal{O}_{\Theta,X}) = \left\{ \ u \in L^p(\mathcal{O}_{\Theta,X}) \mid \frac{\partial u}{\partial \vartheta} \, , \ \frac{\partial u}{\partial x} \, , \ \frac{\partial^2 u}{\partial x^2} \in L^p(\mathcal{O}_{\Theta,X}) \ \right\} .$$

Let

$$\mathcal{W}^{1,2,p}(\mathcal{Q}) = \left\{ u \in L^p(\mathcal{Q}) \mid \frac{\partial u}{\partial t}, \frac{\partial u}{\partial \vartheta}, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2} \in L^p(\mathcal{Q}) \ \right\} .$$

We seek a solution $u \in \mathcal{W}^{1,2,p}(\mathcal{Q})$, for p sufficiently large, of the *Hamilton-Jacobi* equation†

$$-\frac{\partial u}{\partial t} - b \frac{\partial u}{\partial \vartheta} - a_2 \, \frac{\partial^2 u}{\partial x^2} - H(t, \vartheta, x, u, \partial u / \partial \vartheta, \partial u / \partial x) = 0 \ \text{ in } \ \mathcal{Q} \tag{3.9a}$$

subject to the terminal conditions

$$u(T, \vartheta, x) = \overline{u}(\vartheta, x) \ \text{ in } \ \mathcal{O}_{\Theta,X} \tag{3.9b}$$

and

$$u(t, \Theta, x) = 0 \ \text{ in } \ \mathcal{O}_{T,X} \tag{3.9c}$$

and boundary conditions

$$u(t, \vartheta, 0) = u(t, \vartheta, X) = 0 \ \text{ in } \ \mathcal{O}_{T,\Theta} . \tag{3.9d}$$

Noting the terminal conditions (3.9b) and (3.9c), we remark that the variables t and ϑ are *temporal* in nature, whereas by (3.9d), the variable x is *spatial*.

†Significantly, the variational approach to ultraparabolic operators offers sufficient resolution of the spatial derivatives as to provide for constructive approximations (cf. [17]), which contrasts with the weaker viscosity interpretation, where additional regularity is required in order to obtain a constructive approximation of the solution (cf. [7]).

Theorem 3.1. Existence. *Suppose (3.1), (3.5)-(3.8); then a solution* u of *(3.9) exists with*

$$u \in W^{1,2,p}(\mathcal{Q})\,. \tag{3.10}$$

Proof. We consider a sequence of functions u^0, u^1,... defined as follows: let $u^0 \in W^{1,2,p}(\mathcal{Q})$ satisfy

$$-\frac{\partial u^0}{\partial t} - b\frac{\partial u^0}{\partial \vartheta} + A(t,\vartheta)\, u^0 = 0 \;\; a.e. \text{ in } \; \mathcal{Q}\,, \tag{3.11a}$$

$$u^0(T,\vartheta,x) = \overline{u}(\vartheta,x) \;\; a.e. \text{ in } \; \mathcal{O}_{\Theta,X}\,, \tag{3.11b}$$

$$u^0(t,\Theta,x) = 0 \;\; a.e. \text{ in } \; \mathcal{O}_{T,X}\,, \tag{3.11c}$$

$$u^0(t,\vartheta,0) = u(t,\vartheta,X) = 0 \;\; a.e. \text{ in } \; \mathcal{O}_{T,\Theta}\,, \tag{3.11d}$$

where

$$A(t,\vartheta)u = -a_2\,\frac{\partial^2 u}{\partial x^2}\,.$$

Having defined $u^n \in W^{1,2,p}(\mathcal{Q})$, we denote by $\nu^n(x,\vartheta,t)$ a measurable function with values in $\mathcal{V}$, such that

$$H(t,\vartheta,x,u^n,\partial u^n/\partial\vartheta,\partial u^n/\partial x) = L(t,\vartheta,x,u^n,\partial u^n/\partial\vartheta,\partial u^n/\partial x,\nu^n)\,. \tag{3.12}$$

We then define u^{n+1} such that

$$\begin{aligned}
&-\frac{\partial u^{n+1}}{\partial t} - b\frac{\partial u^{n+1}}{\partial \vartheta} + A(t,\vartheta)\, u^{n+1} \\
=&\Psi(x,\vartheta,t,\nu^n) + f(t,\vartheta,x,\nu^n)\frac{\partial u^{n+1}}{\partial\vartheta} + g(t,\vartheta,x,\nu^n)\frac{\partial u^{n+1}}{\partial x} \\
&+ c(t,\vartheta,x,\nu^n)\, u^{n+1} \;\; a.e. \text{ in } \; \mathcal{Q},
\end{aligned} \tag{3.13a}$$

$$u^{n+1}(T,\vartheta,x) = \overline{u}(\vartheta,x) \;\; a.e. \text{ in } \; \mathcal{O}_{\Theta,X}\,, \tag{3.13b}$$

$$u^{n+1}(t,\Theta,x) = 0 \;\; a.e. \text{ in } \; \mathcal{O}_{T,X}\,, \tag{3.13c}$$

$$u^{n+1}(t,\vartheta,0) = u^{n+1}(t,\vartheta,X) = 0 \;\; a.e. \text{ in } \; \mathcal{O}_{T,\Theta}\,. \tag{3.13d}$$

We note that (3.11) and (3.13) are well-posed (cf. [19] for the p=2 case, and [20], Appendix A for $p \neq 2$), in which case the sequence $\{u^n\}$ contained in $W^{1,2,p}(\mathcal{Q})$ is uniquely defined such that

$$\|u^{n+1}\|_{W^{1,2,p}(\mathcal{Q})} \le C\,, \tag{3.14}$$

as f, g, c and Ψ are bounded (independent of n).

In addition, we have the estimate

$$\begin{aligned}
&-\frac{\partial u^n}{\partial t} - b\frac{\partial u^n}{\partial \vartheta} + A(t,\vartheta)\, u^n \\
=&\Psi(t,\vartheta,x,\nu^{n-1}) + f(t,\vartheta,x,\nu^{n-1})\frac{\partial u^n}{\partial\vartheta} + g(t,\vartheta,x,\nu^{n-1})\frac{\partial u^n}{\partial x} + c(t,\vartheta,x,\nu^{n-1})\, u^n \\
\le&\Psi(t,\vartheta,x,\nu^n) + f(t,\vartheta,x,\nu^n)\frac{\partial u^n}{\partial\vartheta} + g(t,\vartheta,x,\nu^n)\frac{\partial u^n}{\partial x} + c(t,\vartheta,x,\nu^n)\, u^n \\
=&H(t,\vartheta,x,u,\partial u^n/\partial\vartheta,\partial u^n/\partial x,\nu^n)
\end{aligned}$$

on account of (3.12), in which case, from (3.13),

$$
\begin{aligned}
&- \frac{\partial}{\partial t}\left(u^{n+1}-u^{n}\right) - \frac{\partial}{\partial \vartheta}\left(bu^{n+1}-bu^{n}\right) + A(t,\vartheta)\left(u^{n+1}-u^{n}\right)\\
\geq & f(t,\vartheta,x,\nu^{n})\frac{\partial\left(u^{n+1}-u^{n}\right)}{\partial\vartheta} + g(t,\vartheta,x,\nu^{n})\frac{\partial\left(u^{n+1}-u^{n}\right)}{\partial x}\\
&+ c(t,\vartheta,x,\nu^{n})\left(u^{n+1}-u^{n}\right) \quad a.e. \text{ in } \mathcal{Q}\,,
\end{aligned}
$$

as well as

$$
\begin{aligned}
&\left(u^{n+1}-u^{n}\right)(T,\vartheta,x)=0 \;\; a.e. \text{ in } \mathcal{O}_{\Theta,X}\,,\\
&\left(u^{n+1}-u^{n}\right)(t,\Theta,x)=0) \;\; a.e. \text{ in } \mathcal{O}_{\Theta,X}\,,\\
&\left(u^{n+1}-u^{n}\right)(t,\vartheta,0)=u^{n}(t,\vartheta,X)=0 \;\; a.e. \text{ in } \mathcal{O}_{T,\Theta}\,.
\end{aligned}
$$

From the positivity result Theorem 2.1, we obtain

$$
-\left(u^{n+1}-u^{n}\right) \leq 0\,.
$$

Since f and $\overline{u}$ are bounded, and taking into account (3.14), it follows that

$$
u^{n} \uparrow u \;\text{ and }\; u^{n} \rightharpoonup u \;\text{ in }\; W^{1,2,p}(\mathcal{Q}) \tag{3.15a}
$$

and moreover

$$
u^{n} \to u \;\text{ in }\; W^{1,0,p}(\mathcal{Q})\,, \tag{3.15b}
$$

through compactness[‡].

For $\nu \in \mathcal{V}$, we have

$$
\begin{aligned}
&- \frac{\partial u^{n}}{\partial t} - \frac{\partial\, bu^{n}}{\partial \vartheta} + A(t,\vartheta)\,u^{n} - \Psi(t,\vartheta,x,\nu)\\
&- f(t,\vartheta,x,\nu)\frac{\partial u^{n}}{\partial\vartheta} - g(t,\vartheta,x,\nu)\frac{\partial u^{n}}{\partial x} - c(t,\vartheta,x,\nu)\,u^{n}\\
\geq & - \frac{\partial u^{n}}{\partial t} - \frac{\partial\, bu^{n}}{\partial \vartheta} + A(t,\vartheta)\,u^{n} - \Psi(x,\vartheta,t,\nu^{n})\\
&- f(t,\vartheta,x,\nu^{n})\frac{\partial u^{n}}{\partial\vartheta} - g(t,\vartheta,x,\nu^{n})\frac{\partial u^{n}}{\partial x} - c(t,\vartheta,x,\nu^{n})\,u^{n}\\
= & - \frac{\partial}{\partial t}\left(u^{n}-u^{n+1}\right) - \frac{\partial}{\partial \vartheta}\left(bu^{n}-bu^{n-1}\right) + A(t,\vartheta)\left(u^{n}-u^{n+1}\right)\\
&- f(t,\vartheta,x,\nu^{n})\frac{\partial\left(u^{n}-u^{n+1}\right)}{\partial\vartheta} - g(t,\vartheta,x,\nu^{n})\frac{\partial\left(u^{n}-u^{n+1}\right)}{\partial x}\\
&- c(t,\vartheta,x,\nu^{n})\left(u^{n}-u^{n+1}\right) \quad a.e. \text{ in } \mathcal{Q}\,,
\end{aligned} \tag{3.16}
$$

such that the right hand side converges weakly to 0 in $L^{p}(\mathcal{Q})$, by virtue of (3.15a). Proceeding to the weak limit in (3.16), we obtain

$$
\begin{aligned}
&- \frac{\partial u}{\partial t} - \frac{\partial\, bu}{\partial \vartheta} + A(t,\vartheta)\,u - \Psi(t,\vartheta,x,\nu)\\
&- f(t,\vartheta,x,\nu)\frac{\partial u}{\partial\vartheta} - g(t,\vartheta,x,\nu)\frac{\partial u}{\partial x} - c(t,\vartheta,x,\nu)\,u \geq 0
\end{aligned}
$$

[‡]The notation $u^{n} \uparrow u$ means u_{n} is monotonically increasing and convergent to u.

or

$$-\frac{\partial u}{\partial t}-\frac{\partial\, bu}{\partial \vartheta}+A(t,\vartheta)\,u \geq H(t,\vartheta,x,u,\partial u/\partial\vartheta,\partial u/\partial x,\nu)\ \ a.e.\ \text{ in }\ \mathcal{Q}\,, \tag{3.17}$$

for all $\nu \in \mathcal{V}$.

Conversely, we have

$$\begin{aligned}
&-\frac{\partial u}{\partial t}-\frac{\partial\, bu}{\partial \vartheta}+A(t,\vartheta)\,u-H(t,\vartheta,x,u,\partial u/\partial\vartheta,\partial u/\partial x)\\
\leq &-\frac{\partial u}{\partial t}-\frac{\partial\, bu}{\partial \vartheta}+A(t,\vartheta)\,u-\Psi(t,\vartheta,x,\nu)\\
&-f(t,\vartheta,x,\nu)\frac{\partial u}{\partial \vartheta}-g(t,\vartheta,x,\nu)\frac{\partial u}{\partial x}-c(t,\vartheta,x,\nu)\,u\\
= &-\frac{\partial}{\partial t}\left(u-u^{n+1}\right)-\frac{\partial}{\partial \vartheta}\left(bu-bu^{n+1}\right)+A(t,\vartheta)\left(u-u^{n+1}\right)\\
&-f(t,\vartheta,x,\nu)\frac{\partial\left(u-u^{n+1}\right)}{\partial\vartheta}-g(t,\vartheta,x,\nu)\frac{\partial\left(u-u^{n+1}\right)}{\partial x}\\
&-c(t,\vartheta,x,\nu)\,\left(u-u^{n+1}\right)+\Psi(t,\vartheta,x,\nu^{n})-\Psi(t,\vartheta,x,\nu)
\end{aligned} \tag{3.18}$$

such that the right hand side of (3.18) converges to 0 weakly in $L^p(\mathcal{Q})$. We have then

$$-\frac{\partial u}{\partial t}-\frac{\partial\, bu}{\partial \vartheta}+A(t,\vartheta)\,u \leq H(t,\vartheta,x,u,\partial u/\partial\vartheta,\partial u/\partial x)\ \ a.e.\ \text{ in }\ \mathcal{Q}\,,$$

which together with (3.17), shows that u is a solution of the Hamilton-Jacobi equation. □

4. Probabilistic Interpretation

The ultradiffusion[§] process is defined; we demonstrate that there exists an optimal strategy which maximizes a given expected discounted performance criteria. In turn, this optimized expectation is characterized as the unique generalized solution to the ultraparabolic Hamilton-Jacobi equation. We remark that due to the nature of the ultradiffusion, we are able to admit target sets into the analysis.

A *strategy* is a measurable mapping $t,\vartheta,x \to \nu(t,\vartheta,x)$ from $[0,T]\times\mathcal{O}_{\Theta,X}\to\mathcal{V}$. For $(\vartheta,x)\notin\mathcal{O}_{\Theta,X}$, we define $\nu(t,\vartheta,x)$ in an arbitrary manner. We write $\nu(t)=\nu(t,\vartheta,x)$ for brevity. Let $(\Omega,\mathcal{F}_t,\mathcal{F}_t^s)$ denote the canonical space, $X(t,\omega)=\omega(t)$, for $\omega(t)\in\Omega$, and $P\equiv P^{t\vartheta x}$ the unique measure on $(\Omega,\mathcal{F}_t)$ such that for $s>t$,

$$\begin{aligned}
d\Theta(s)&=\left[b+f(s,\Theta(s),X(s),\nu(s))\right]ds,\\
dX(s)&=\sigma(s,\Theta(s),X(s))\,dw(s)
\end{aligned}$$

[§]Consistency is the key idea in utilizing the *ultra* prefix. That is, the expectation of a non-degenerate diffusion process is associated with a non-degenerate diffusion (parabolic) initial/boundary value problem, whereas that for an ultradiffusion process is associated with an ultradiffusion (ultraparabolic) initial/boundary value problem. Note that ultradiffusion operators are not degenerate parabolic (cf. [4,5]) and as such, "ultradiffusion" appears to be a more appropriate description than "degenerate" diffusion. The description is likewise consistent with the ultradiffusion usage employed in hierarchical systems (cf. [9]), as ultradiffusion processes are locally isomorphic to a parameterized diffusion process (cf. [16]).

for $\sigma(t,\vartheta,x) := \sqrt{2\,a_2}$, where $(\Theta(t),X(t)) = (\vartheta,x)$ and $w(s)$ is a Wiener process.

For a given strategy v, we write

$$\begin{aligned}
f_\nu(s) &= f(s,\Theta(s),X(s),\nu(s)),\\
g_\nu(s) &= g(s,\Theta(s),X(s),\nu(s)),\\
c_\nu(s) &= c(s,\Theta(s),X(s),\nu(s)),\\
\Psi_\nu(s) &= \Psi(s,\Theta(s),X(s),\nu(s))
\end{aligned}$$

and

$$\sigma(s) = \sigma(s,\Theta(s),X(s))$$

for $s \geq t$. On $(\Omega,\mathcal{F}_t)$, we define for $s \leq T$ the martingale

$$M(s) = \exp\left[-\int_t^s \sigma^{-1}(\lambda)\,g_\nu(\lambda)\,dw(\lambda) - \frac{1}{2}\int_t^s [\sigma^{-1}(\lambda)\,g_\nu(\lambda)]^2\,d\lambda\right],$$

with respect to $\mathcal{F}_t^s$ and the measure $P^{t\vartheta x}$, as well as the measure $Q_\nu^{t\theta x}$, given by the Girsanov transformation $P^{t\vartheta x} \to Q_\nu^{t\theta x}$, such that

$$dQ_\nu^{t\theta x}(\omega) = M_T(\omega)\,dP^{t\vartheta x}(\omega)\,,$$

on $\mathcal{F}_t$, for $\omega \in \Omega$. Endowing $(\Omega,\mathcal{F}_t)$ with the measure $Q_v^{x,\theta,t}$, the process $(\Theta(t),X(t))$ is a solution of

$$\begin{aligned}
d\Theta(s) &= (b + f_\nu(s))\,ds, && (4.1a)\\
dX(s) &= g_\nu(s)\,ds + \sigma(s)\,d\widetilde{w}(s), && (4.1b)\\
\Theta(t) &= \vartheta\,, && (4.1c)\\
X(t) &= x\,, && (4.1d)
\end{aligned}$$

for $s > t$, where $\widetilde{w}(s)$ is a standardized Wiener process and a $\mathcal{F}_t^s$ martingale. We denote by $\tau = \tau_{t\vartheta x}$ the exit time from $\mathcal{O}_{\Theta,X}$ of the process $(\Theta(t),X(t))$. For every strategy $\nu(t,\vartheta,x)$, we define the *performance index* such that

$$\begin{aligned}
J^{t\vartheta x}(T;\nu) =&\mathbb{E}^{Q_v^{t\vartheta x}}\left\{\int_t^{T\wedge\tau} \Psi_\nu(s)\exp\left(-\int_t^s c_\nu(\lambda)\,d\lambda\right)\,ds\right.\\
&\left. + \overline{u}(\Theta(T),X(T))\exp\left(-\int_t^T c_\nu(\lambda)\,d\lambda\right)\,\chi_{T\leq\tau}\right\}, \qquad (4.2)
\end{aligned}$$

which is a measure of the expected discounted system performance. The optimization problem is to maximize the performance index over all controls $\nu \in \mathcal{V}$.

Relative to the ultradiffusion process (4.1),we may characterize (4.2) in terms of the solution to the ultraparabolic Hamilton-Jacobi equation (3.9), thereby establishing the existence of an optimal control strategy.

Theorem 4.1. Uniqueness. *We suppose (3.1), (3.5)-(3.8), then there exists a unique solution $u \in W^{1,2,p}(\mathcal{Q})$ to (3.9), such that*

$$u(t,\vartheta,x) = \max_{\nu(t,\vartheta,x)\in\mathcal{V}} J^{t\vartheta x}(T;\nu) = J^{t\vartheta x}(T;\hat{\nu})\,, \qquad (4.3)$$

for some optimal strategy $\hat{\nu}$.

Proof. We put

$$z(t) = \exp\left\{-\int_t^T c_\nu(s, \vartheta(s), x(s))\, ds\right\}$$

then

$$\frac{dz}{dt} = c_\nu(t, \vartheta(t), x(t))\, z$$

such that

$$z(T) = 1\,.$$

Let u be any solution of (3.9). Applying the generalized Ito's formula to the functional

$$\Phi(t, \vartheta, x, z) = u(t, \vartheta, x)\, z$$

relative to (4.1), we obtain

$$\begin{aligned}
&u(T\wedge\tau, \Theta(T\wedge\tau), X(T\wedge\tau)) \exp\left(-\int_t^{T\wedge\tau} c_\nu(s)\, ds\right) - u(t, \Theta(t), X(t)) \\
&= \int_t^{T\wedge\tau} \left(\frac{\partial u}{\partial t} + b\frac{\partial u}{\partial \vartheta} + a_2\frac{\partial^2 u}{\partial x^2} + f_\nu \frac{\partial u}{\partial \vartheta} + g_\nu \frac{\partial u}{\partial x} + c_\nu u\right)(s, \Theta(s), X(s)) \\
&\quad \exp\left(-\int_s^{T\wedge\tau} c_\nu(\lambda)\, d\lambda\right)\, ds + \int_t^{T\wedge\tau} \sigma\, \frac{\partial u}{\partial x} \exp\left(-\int_s^{T\wedge\tau} c_\nu(\lambda)\, d\lambda\right) d\widetilde{w}(s)
\end{aligned}$$

for all $t \in [0, T]$. Upon taking the expectation, it follows that

$$\begin{aligned}
&u(t, \vartheta, x) \\
&= \mathbb{E}^{Q_\nu^{t\theta x}} \left\{ \int_t^{T\wedge\tau} \left(-\frac{\partial u}{\partial t} - b\frac{\partial u}{\partial \vartheta} - a_2 \frac{\partial^2 u}{\partial x^2} \right.\right. \\
&\quad \left.\left. - f_\nu(s)\frac{\partial u}{\partial \vartheta} - g_\nu(s)\frac{\partial u}{\partial x} - c_\nu(s) u \right)(s, \Theta(s), X(s)) \exp\left(-\int_s^{T\wedge\tau} c_\nu(\lambda)\, d\lambda\right)\, ds \right\} \\
&\quad + \mathbb{E}^{Q_\nu^{t\theta x}} \left\{ u(T\wedge\tau, \Theta(T\wedge\tau), X(T\wedge\tau)) \exp\left(-\int_t^{T\wedge\tau} c_\nu(s)\, ds\right) \right\}.
\end{aligned} \tag{4.4}$$

However, since u is a solution to (2.9), we also have

$$\begin{aligned}
-\frac{\partial u}{\partial t} - b\frac{\partial u}{\partial \vartheta} - a_2\frac{\partial^2 u}{\partial x^2} &= H(t, \vartheta, x, u, \partial u/\partial\vartheta, \partial u/\partial x) \\
&= \Psi_{\hat\nu} + f_{\hat\nu}\frac{\partial u}{\partial \vartheta} + g_{\hat\nu}\frac{\partial u}{\partial x} + c_{\hat\nu} u\,,
\end{aligned} \tag{4.5}$$

from (3.3) and Lemma 3.1, for some optimal strategy $\hat\nu \in \mathcal{V}$. Taking $\vartheta = \Theta(s)$ and $x = X(s)$ in (4.5) and substituting the result into (4.4) with $\nu = \hat\nu$, we obtain (4.3). Since any solution of (3.9) has the form (4.3), uniqueness follows. $\square$

Remark 3.1. *Removing the positivity constraint on velocity.* In (4.1a), we require at least $b + f(s) \geq 2\epsilon > 0$. For any function f, we may define $f^\pm(s) = \max(\pm f(s), 0)$, then $f(s) = f^+(s) - f^-(s)$ and

$$d\Theta^+(s) = [\epsilon + f^+(s)]\, ds$$

$$d\Theta^-(s) = [\epsilon + f^-(s)]\, ds$$

where now $\Theta(s) = \Theta^+(s) - \Theta^-(s)$. In practice, letting $\epsilon = 0$ suffices on account of regularity.

A *process* (Θ, v) for the control problem (4.1) consists of a control function $\nu \in \mathcal{V}$ and a temporal state *trajectory* $\Theta(t)$, for all $t \in [0,T]$, which is a solution *a.e.* of the stochastic differential equation (4.1a). The process (Θ, v) is said to be *feasible* if the trajectory Θ satisfies the endpoint contraints $\Theta(t) \in C(t) \subseteq [0,\Theta]$ and $\Theta(T) \in C(T) \subseteq [0,\Theta]$. Let $\mathcal{F}(\vartheta, t)$ denote the set of all feasible processes¶. In particular, with the inclusion of target sets, we may restate Proposition 2 as follows:

Corollary 3.1. *We suppose* (3.1), (3.5)-(3.8); *if there exists an optimal control* $\hat{\nu} \in \mathcal{F}(\vartheta, t)$, *then there exists a unique solution* $u \in W^{1,2,p}(\mathcal{Q})$ *to* (3.9) *such that*

$$u(t,\vartheta,x) = \max_{\nu \in \mathcal{F}(\vartheta,t)} J^{t\vartheta x}(T;\nu) = J^{t\vartheta x}(T;\hat{\nu})\,. \tag{4.6}$$

Remark 3.2. *Control activation.* With respect to (4.1), it is possible to activate the control at a time $t \leq \tau < T$ by introducing the target

$$\Theta(\tau) = 0\,,$$

in which case $\Theta(t) = 0$ for all $t < \tau$ on account of the positivity of the velocity. Alternately, we may allow the control to be active only when $\Theta(t)$ stays within a certain bound, for example, the target may be

$$\Theta(T) \leq \overline{\Theta}\,,$$

for some $\overline{\Theta} > 0$, in which case the control remains active as long as $\Theta(t)$ stays below the prescribed bound.

Remark 3.3. *Unifying notation.* Given that the control of ultradiffusion processes generalize both deterministic and stochastic optimal control, we may interpret (3.6) as a general optimal control problem:‖

$$\begin{aligned}
&J^{t\vartheta x}(T;\nu) = \max!\\
&d\Theta(s) = (b + f_\nu(s))\, ds\\
&dX(s) = g_\nu(s)\, ds + \sigma(s)\, d\widetilde{w}(s)\\
&(\Theta(t), X(t)) = (\vartheta, x) \in \overline{\mathcal{O}}_{\Theta,X}\\
&h_i^{(1)}(t_i, \Theta(t_i)) \leq 0\\
&h_j^{(2)}(t_j, \Theta(t_j)) = 0\\
&\nu(s) \in \mathcal{V}
\end{aligned}$$

for all $s \in (t,T)$, $t_i \in [t,T]$, $h_i^{(1)} : [t,T] \times [0,\infty) \to \mathbb{R}$ $(i = 1,2,\ldots,I)$, and $t_j \in [t,T]$, $h_j^{(2)} : [t,T] \times [0,\infty) \to \mathbb{R}$ $(j = 1,2,\ldots,J)$.

¶We may apply other inter-temporal constraints as well.

‖The *maximum problem* is $\max_u F(u) = \alpha$ with the shorthand notation $F(u) = \max!$.

5. Conclusion

From a deterministic viewpoint, we have demonstrated the existence and uniqueness of regular solutions to ultraparabolic Hamilton-Jacobi equations. As this was accomplished via a probabilistic interpretation of solutions via a stochastic optimal control problem, from a stochastic point of view then, we have likewise demonstrated the existence of an optimal control strategy which maximizes the expectation of a discounted performance criteria relative to a defined controlled ultradiffusion process. Moreover, we have obtain the interesting consequence of allowing target sets to exist within the stochastic control framework.

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