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EXPLICIT SOLUTIONS FOR THE CONFORMABLE REGULARIZED LONG WAVE BURGER'S EQUATION

Shahid Ali[1] and Ahmad Javid[1,†]

Abstract In this paper, a dynamical analysis of the conformable regularized long-wave burgers equation is carried out with help of improved tan $\left(\frac{\phi(\eta)}{2}\right)$-expansion method. Fractional complex transform converts a nonlinear fractional differential equation in an ordinary differential form which resulted into a number of exact solutions like exponential function solutions, hyperbolic function solutions, trigonometric function solutions and rational function solutions. The constarint conditions are also given for each solution. The physical profiles of proposed solutions are portrayed by 3D and 2D graphs as well as the influence of fractional parameter is also studied for some solutions. Our proposed results showed that improved tan $\left(\frac{\phi(\eta)}{2}\right)$-expansion method is reliable method to solve the nonlinear equation in mathematical physics.

Keywords Improved tan $\left(\frac{\phi(\eta)}{2}\right)$-expansion approach, conformable regularized long wave Burgers equation, travelling wave solutions, periodic solutions.

MSC(2010) 78A60, 35Q51, 35Q55.

1. Introduction

The study of various nonlinear partial differential equations helps investigators to recognize composite natural wonders and non linear fractional differential equation has been elaborated by their many requisition in different regions of applied mathematics, physics, fluid mechanics and plasma physics [1, 2, 19, 20, 26]. We focus here on the conformable regularized long wave burgers equation.

The benifits of the conformable derivatives allow the essential situations to apply these methods to fractional nonlinear partial derivatives in spite of many constraints. In this study, we investigate the conformable regularized long wave burgers equation [10], which is given as

$$D_t^{\alpha} u(x,t) + pu_x + quu_x + su_{xxt} = 0, t > 0, 0 < \alpha \leq 1.$$

Where p, q and s are arbitrary constants. $D_t^{\alpha} u(x,t)$ is the conformable derrivative. The integer order structure emerge in the paper [10] to explain shallow water waves spread in a waterway space. Many significant assets covering presence, individuality, and finiteness of the solutions of many problems linked to regularized long wave burgers equation are described in this paper. Zhao and Xuan [35] verify

†The corresponding author.
Email: ahmadjavid001@gmail.com, ahmad.javid@sns.nust.edu.pk(A. Javid)
[1]Department of Mathematics, School of Natural Sciences (SNS), National University of Sciences and Technology (NUST), Islamabad, Pakistan

the presence of solutions of regularized long wave burgers equation. The monotone and vibrating kink type waves are explained by Zhou and Liu [36]. Kaya gives many accurate solutions to some intensive initial value problems for the regularized long wave burgers equation by assistance of adomian decomposition method [11]. Many hyperbolic and trignometric analytic solutions are found by using expansion methods [30].

For the last few years, many analytic methods had been invented and implemented for finding exact solutions [3,4,12,21–25]. Hirota analyzed different collision of soliton solution of KdV equation [8]. Ablowitz and Clarkson explained solitons in non linear evolution equation and inverse scattering transform [5]. Zhou implemented homogenous balance method to find exact solutions of nonlinear equations in mathematical physics [32]. Feng implemented first integral method to analyze Burgers-KdV equation [7]. Wazwaz implemented tanh method for finding exact wave solutions of nonlinear equations [33]. Korteweg-deVries Burgers (KdVB) equation has been discussed by Saeed et al. in [27] and analyzed using tangent hyperbolic method to see ion acoustic waves in relativistic plasma. Shah et al [28] studied electrons, positrons and hot ions in three-component relativistic system with $\frac{G'}{G}$ method. Manafian and Lakestani applied $\tan\left(\frac{\phi(\eta)}{2}\right)$-expansion method on Biswas-Milovic equation for Kerr law nonlinearity and investigated optical soliton solutions for the Gerdjikov-Ivanov model [16, 17]. There are many requisition of the $\tan\left(\frac{\phi(\eta)}{2}\right)$-expansion method . Manafian and Zinati [18] implemented the $\tan\left(\frac{\phi(\eta)}{2}\right)$-expansion method for exponential function, hyperbolic function, trignometric and rational function solutions of some nonlinear fractional physical models as time fractional Burgers equation, time fractional biological population model, space-time fractional Fokas equation, space time fractional Whitham-Broer-Kaup equation and time fractional Cahn-Hilliard equation. Ugurlu et al. [31] analyzed exact solutions like trignometric functions, exponential function, hyperbolic function of potiential Korteweg- De Vries equation and $(3+1)$ dimensional surface water wave equation with $\tan\left(\frac{\phi(\eta)}{2}\right)$-expansion method. Khan et al [31] implemented this method on (2+1) dimensional kadomtsev-petviashvili-benjamin-bona-mahony wave equation for exact solutions. Bekir et al. [13] applied this method for analytic solutions of $(2+1)$dimensional Zoomeron , the Duffing and the symmetric-regularized long wave (SRLW) equation. Rezazadeh et al. [29] applied new auxiliary equation approach for fractional resonant Schrodinger equation. Rezazadeh et al. [14] investigated solitons of $(2+1)$ dimensional Burgers- Huxley equation using different techniques. Rezazadeh et al. discussed the dynamical behaviour of exact solutions for a $(2+1)$ dimensional bogoyavlenskii coupled system [15]. Rezazadeh et al. [34] found numerical solutions of time fractional zakharov-kuznetsov equation by transform decomposition method. Rezazadeh et al. found solitary wave solutions for conformable klein-gorden equation with quantic nonlinearity [9].
The conformable derrivative was explained in [6]. This operator is easy , logical and effective explanation of fractional derrivative for order $\gamma \in (0,1]$. The conformable derrivative of order $\gamma \in (0,1]$ is explained by given definition [6]:

$$D_t^{\gamma} f(t) = \lim_{\sigma \to 0} \frac{f(t+\sigma t^{1-\gamma}) - f(t)}{\sigma}, f:(0,\infty) \to R.$$

Many charecterization of cd are given in [5, 7, 32]
(a) $D_t^{\gamma} t^{\zeta} = \zeta t^{\zeta-\gamma},\ \forall \gamma \in R,$
(b) $D_t^{\gamma}(fg) = f_t D^{\gamma} g + g_t D^{\gamma} f,$

(c) $D_t^\gamma(fog) = t^{1-\gamma}g'(t)f'(t)$,
(d) $D_t^\gamma(\frac{f}{g}) = \frac{g_t D^\gamma f - f_t D^\gamma g}{g^2}$.
These derrivatives are easy to apply. Recently, there are many researchers which used conformable form for fractional calculations [23, 24].

In this study, we analyzed new travelling wave solutions of conformable regularized long wave burgers equation. We used $\tan\left(\frac{\phi(\eta)}{2}\right)$-expansion method with fractional complex transform. All the solutions are putting into back given equation and verified true.

2. Description of improved $\tan\left(\frac{\phi(\eta)}{2}\right)$-expansion Method

This segment contains short explanation of improved $\tan\left(\frac{\phi(\eta)}{2}\right)$-expansion Method.

Step 1.1: Let having nonlinear partial differential equation:

$$L(p, p_x, p_{xx}, ..., D_t^\alpha p, ...) = 0, \tag{2.1}$$

Eq. (2.1) decreases to an ODE

$$Q(p(\eta), \mu p'(\eta), \mu^2 p''(\eta), ..., -\mu v p'(\eta), ...) = 0. \tag{2.2}$$

Using the transform $p(x,t) = p(\eta)$, $\eta = \mu(x - \frac{vt^\alpha}{\alpha})$, where μ, v are arbitrary constants.

Step 1.2: Assume that the Eq. (2.2) has a solution:

$$p(\eta) = V(\phi) = \sum_{w=0}^{L} N_w \left[a + \tan\left(\frac{\phi(\eta)}{2}\right)\right]^w + \sum_{w=1}^{L} C_w \left[a + \tan\left(\frac{\phi(\eta)}{2}\right)\right]^{-w}, \tag{2.3}$$

where $N_L \neq 0$, $C_L \neq 0$, and $\phi = \phi(\eta)$ assure the given ordinary differential equation:

$$\phi'(\eta) = m\sin(\phi(\eta)) + n\cos(\phi(\eta)) + r. \tag{2.4}$$

Coming suitable solutions of equation Eq. (2.4) will given as:

Family 1.11: If $\sigma = m^2 + n^2 - r^2 < 0$ and $n - r \neq 0$, then

$$\phi(\eta) = 2\ \tan^{-1}\left[\frac{m}{n-r} - \frac{\sqrt{-\sigma}}{n-r}\tan\left(\frac{\sqrt{-\sigma}}{2}\hat{\eta}\right)\right]. \tag{2.5}$$

Family 1.12: If $\sigma = m^2 + n^2 - r^2 > 0$ and $n - r \neq 0$, then

$$\phi(\eta) = 2\ \tan^{-1}\left[\frac{m}{n-r} + \frac{\sqrt{\sigma}}{n-r}\tanh\left(\frac{\sqrt{\sigma}}{2}\hat{\eta}\right)\right]. \tag{2.6}$$

Family 1.13: If $\sigma = m^2 + n^2 - r^2 > 0$, $n \neq 0$ and $r = 0$, then

$$\phi(\eta) = 2\ \tan^{-1}\left[\frac{m}{n} + \frac{\sqrt{n^2+m^2}}{n}\tanh\left(\frac{\sqrt{n^2+m^2}}{2}\hat{\eta}\right)\right]. \tag{2.7}$$

Family 1.14: If $\sigma = m^2 + n^2 - r^2 < 0$, $r \neq 0$ and $n = 0$, then

$$\phi(\eta) = 2\ \tan^{-1}\left[-\frac{m}{n} + \frac{\sqrt{r^2 - m^2}}{r}\tan\left(\frac{\sqrt{r^2 - m^2}}{2}\hat{\eta}\right)\right]. \tag{2.8}$$

Family 1.15: If $\sigma = m^2 + n^2 - r^2 > 0$, $n - r \neq 0$ and $m = 0$, then

$$\phi(\eta) = 2\ \tan^{-1}\left[\sqrt{\frac{n + r}{n - r}}\tanh\left(\frac{\sqrt{n^2 - r^2}}{2}\hat{\eta}\right)\right]. \tag{2.9}$$

Family 1.16: If $m = 0$ and $r = 0$, then

$$\phi(\eta) = \tan^{-1}\left[\frac{e^{2n\hat{\eta}} - 1}{e^{2n\hat{\eta}} + 1}, \frac{2e^{n\hat{\eta}}}{e^{2n\hat{\eta}} + 1}\right]. \tag{2.10}$$

Family 1.17: If $n = 0$ and $r = 0$, then

$$\phi(\eta) = \tan^{-1}\left[\frac{2e^{m\hat{\eta}}}{e^{2m\hat{\eta}} + 1}, \frac{e^{2m\hat{\eta}} - 1}{e^{m\hat{\eta}} + 1}\right]. \tag{2.11}$$

Family 1.18: If $m^2 + n^2 = r^2$, then

$$\phi(\eta) = -2\ \tan^{-1}\left[\frac{(n + r)(m\hat{\eta} + 2)}{m^2\hat{\eta}}\right]. \tag{2.12}$$

Family 1.19: If $m = n = r = km$, then

$$\phi(\eta) = 2\ \tan^{-1}\left[e^{km\hat{\eta}} - 1\right]. \tag{2.13}$$

Family 2.00: If $m = r = km$ and $n = -km$, then

$$\phi(\eta) = -2\ \tan^{-1}\left[\frac{e^{km\hat{\eta}}}{-1 + e^{km\hat{\eta}}}\right]. \tag{2.14}$$

Family 2.11: If $r = m$, then

$$\phi(\eta) = -2\ \tan^{-1}\left[\frac{(m + n)e^{n\hat{\eta}} - 1}{(m - n)e^{n\hat{\eta}} - 1}\right]. \tag{2.15}$$

Family 2.12: If $m = r$, then

$$\phi(\eta) = 2\ \tan^{-1}\left[\frac{(n + r)e^{n\hat{\eta}} + 1}{(n - r)e^{n\hat{\eta}} - 1}\right]. \tag{2.16}$$

Family 2.13: If $r = -m$, then

$$\phi(\eta) = 2\ \tan^{-1}\left[\frac{e^{n\hat{\eta}} + n - x}{e^{n\hat{\eta}} - n - m}\right]. \tag{2.17}$$

Family 2.14: If $n = -r$, then

$$\phi(\eta) = -2\ \tan^{-1}\left[\frac{me^{m\hat{\eta}}}{re^{m\hat{\eta}} - 1}\right]. \tag{2.18}$$

Family 2.15: If $n = 0$ and $m = r$, then

$$\phi(\eta) = -2\ \tan^{-1}\left[\frac{r\hat{\eta}+2}{r\hat{\eta}}\right]. \tag{2.19}$$

Family 2.16: If $m = 0$ and $n = r$, then

$$\phi(\eta) = 2\ \tan^{-1}\left[r\hat{\eta}\right]. \tag{2.20}$$

Family 2.17: If $m = 0$ and $n = -r$, then

$$\phi(\eta) = -2\ \tan^{-1}\left[\frac{1}{r\hat{\eta}}\right]. \tag{2.21}$$

Family 2.18: If $m = 0$ and $n = 0$, then

$$\phi(\eta) = r\hat{\eta}. \tag{2.22}$$

Family 2.19: If $n = r$, then

$$\phi(\eta) = 2\ \tan^{-1}\left[\frac{e^{m\hat{\eta}}-r}{m}\right], \tag{2.23}$$

where $\hat{\eta} = \eta + S$, $N_w(w = 0, 1, 2, ..., L)$, $C_w(w = 1, 2, ..., L)$, m, n and r are constants to be estimated. For finding L , we compare highest order derivative with highest order nonlinear term.
Step 1.3: Replacing Eq. (2.3) into Eq. (2.2) with value of L from step 2. Clarifying same powers of $\tan\left(\frac{\phi(\eta)}{2}\right)$, $\cot\left(\frac{\phi(\eta)}{2}\right)$ and gathering coefficients. Taking each coefficient to zero, system of equations is ottained.
Step 1.4: Equations attained in step 3 are simplified to estimate constants N_0, N_1, N_2, $\dots$, N_M, C_1, C_2, $\dots$, C_M, v, μ. Then these values are putting in Eq. (2.3) to get solutions.

3. Exact solutions along $\tan\left(\frac{\phi(\eta)}{2}\right)$ expansion method

Here, we implement improved $\tan\left(\frac{\phi(\eta)}{2}\right)$ expansion method to find travelling wave solutions of the time fractional regularized long wave-burgers equation, which is given as

$$D_t^{\alpha}u(x,t) + pu_x + quu_x + su_{xxt} = 0,$$

α is a parameter explaining the fractional time derivative and $0 < \alpha \leq 1$.

To obtain travelling wave solution , by using the transform $u(x, t) = p(\eta)$, $\eta = \mu(x - \frac{vt^{\alpha}}{\alpha})$ and integrating once, above equation converted in given below nonlinear ordinary differential equation:

$$-vp(\eta) + pp(\eta) + \frac{q}{2}p(\eta)^2 - sv\mu^2 p(\eta)'' = 0, \tag{3.1}$$

Here p, v, q and s are arbitrary constants. compare highest order linear term with nonlinear highest order degree, get $L = 2$. And solution for $a = 0$ of Eq. (3.1) develop

$$p(\eta) = N_0 + N_1\left[\tan\left(\frac{\phi(\eta)}{2}\right)\right] + N_2\left[\tan\left(\frac{\phi(\eta)}{2}\right)\right] + C_1\left[\tan\left(\frac{\phi(\eta)}{2}\right)\right]^{-1}$$

$$+ C_2\left[\tan\left(\frac{\phi(\eta)}{2}\right)\right]^{-2}. \tag{3.2}$$

Putting Eq. (3.2) by Eq. (2.4) into Eq. (3.1) and collecting values of same power of $\tan\left(\frac{\phi(\eta)}{2}\right)$, Comparing each coefficient of each polynomial to zero, system of equations is attained as given below:

$$\left(\tan\left(\frac{\phi(\eta)}{2}\right)\right)^4 : \frac{1}{2}\, q{N_2}^2 + 3\, sv\mu^2 N_2 nr - \frac{3}{2}\, sv\mu^2 N_2 n^2 - \frac{3}{2}\, sv\mu^2 N_2 r^2 = 0,$$

$$\left(\tan\left(\frac{\phi(\eta)}{2}\right)\right)^3 : qN_1N_2 + sv\mu^2 N_1 nr + 5\, sv\mu^2 N_2 mn - 5\, sv\mu^2 N_2 mr - \frac{1}{2}\, sv\mu^2 N_1 n^2 - \frac{1}{2}\, sv\mu^2 N_1 r^2 = 0,$$

$$\left(\tan\left(\frac{\phi(\eta)}{2}\right)\right)^2 : qN_0N_2 + \frac{1}{2}\, q{N_1}^2 + pN_2 - vN_2 + \frac{3}{2}\, sv\mu^2 N_1 mn - \frac{3}{2}\, sv\mu^2 N_1 mr - 4\, sv\mu^2 N_2 m^2 + 2\, sv\mu^2 N_2 n^2 - 2\, sv\mu^2 N_2 r^2 = 0,$$

$$\left(\tan\left(\frac{\phi(\eta)}{2}\right)\right)^1 : qN_0N_1 + qN_2C_1 - vN_1 + pN_1 - 3\, sv\mu^2 N_2 mn - 3\, sv\mu^2 N_2 mr - \frac{1}{2}\, sv\mu^2 N_1 r^2 - sv\mu^2 N_1 m^2 + \frac{1}{2}\, sv\mu^2 N_1 n^2 = 0,$$

$$\left(\tan\left(\frac{\phi(\eta)}{2}\right)\right)^0 : \frac{1}{2}\, q{N_0}^2 + qN_1C_1 + qN_2N_2 - \frac{1}{2}\, sv\mu^2 N_2 n^2 - \frac{1}{2}\, sv\mu^2 N_2 r^2 - \frac{1}{2}\, sv\mu^2 C_2 n^2 - \frac{1}{2}\, sv\mu^2 C_2 r^2 - vA_0 + pN_0 - sv\mu^2 N_2 nr + \frac{1}{2}\, sv\mu^2 C_1 mn - \frac{1}{2}\, sv\mu^2 C_1 mr + sv\mu^2 C_2 nr - \frac{1}{2}\, sv\mu^2 N_1 mn - \frac{1}{2}\, sv\mu^2 N_1 mr = 0,$$

$$\left(\cot\left(\frac{\phi(\eta)}{2}\right)\right)^1 : qN_1C_2 + qN_0C_1 - vC_1 + pC_1 + 3\, sv\mu^2 C_2 mn - 3\, sv\mu^2 C_2 mr - sv\mu^2 C_1 m^2 + \frac{1}{2}\, sv\mu^2 C_1 n^2 - \frac{1}{2}\, sv\mu^2 C_1 r^2 = 0,$$

$$\left(\cot\left(\frac{\phi(\eta)}{2}\right)\right)^2 : qN_0C_2 + \frac{1}{2}\, q{C_1}^2 + pC_2 - vC_2 - \frac{3}{2}\, sv\mu^2 C_1 mn - \frac{3}{2}\, sv\mu^2 C_1 mr - 4sv\mu^2 C_2 m^2 + 2sv\mu^2 C_2 n^2 - 2\, sv\mu^2 C_2 r^2 = 0,$$

$$\left(\cot\left(\frac{\phi(\eta)}{2}\right)\right)^3 : qC_1C_2 - sv\mu^2 C_1 nr - 5\, sv\mu^2 C_2 mn - 5\, sv\mu^2 C_2 mr - \frac{1}{2}\, sv\mu^2 C_1 n^2 - \frac{1}{2}\, sv\mu^2 C_1 r^2 = 0,$$

$$\left(\cot\left(\frac{\phi(\eta)}{2}\right)\right)^4 : \frac{1}{2}\, q{C_2}^2 - 3\, sv\mu^2 C_2 nr - \frac{3}{2}\, sv\mu^2 C_2 n^2 - \frac{3}{2}\, sv\mu^2 C_2 r^2 = 0.$$

Where m, n and r are constants. Using Eq. (3.2) and the value of constants given in set 1.1, families 1.11, 1.12, 1.15, 1.16 and 1.18 can be written as:
Set 3.1:

$$m = m, \quad n = n, \quad r = r, \quad \mu = \mu, \quad C_1 = 0, \quad C_2 = 0,$$

$$v=\frac{p}{m^2\mu^2 s+\mu^2 n^2 s-\mu^2 r^2 s+1},\ N_0=-3\,\frac{p\mu^2 s\left(n^2-r^2\right)}{\left(m^2\mu^2 s+\mu^2 n^2 s-\mu^2 r^2 s+1\right)q},$$
$$N_1=-6\,\frac{m\mu^2 sp\left(n-r\right)}{\left(m^2\mu^2 s+\mu^2 n^2 s-\mu^2 r^2 s+1\right)q},\ N_2=3\,\frac{sp\mu^2\left(n^2-2\,nr+r^2\right)}{\left(m^2\mu^2 s+\mu^2 n^2 s-\mu^2 r^2 s+1\right)q},$$

where m, n and r are random constants. Using Eq. (3.2) and value of constants given in Set 3.1, families 1.11, 1.12, 1.15, 1.16 and 1.18 can be written as:

$$p_1(\eta)=3\,\frac{\left(\sqrt{-m^2-n^2+r^2}\tan\left(\frac{1}{2}\sqrt{-m^2-n^2+r^2}\left(\eta+S\right)\right)-m\right)^2 p\mu^2 s}{\left(m^2\mu^2 s+\mu^2 n^2 s-\mu^2 r^2 s+1\right)q}, \tag{3.3}$$

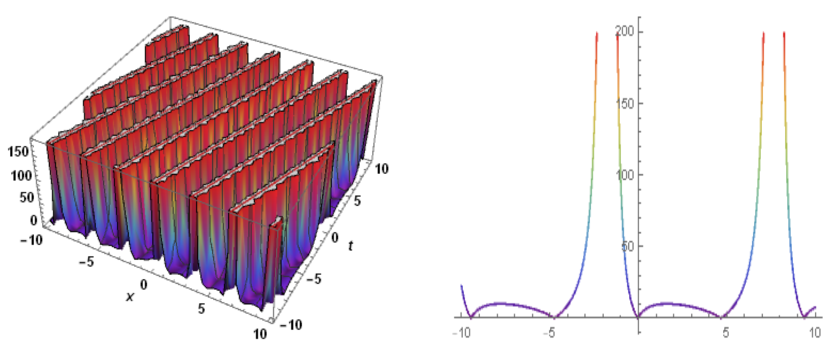
$$p_2(\eta)=3\,\frac{\left(\sqrt{m^2+n^2-r^2}\tan\left(\frac{1}{2}\sqrt{m^2+n^2-r^2}\left(\eta+S\right)\right)+m\right)^2 p\mu^2 s}{\left(m^2\mu^2 s+\mu^2 n^2 s-\mu^2 r^2 s+1\right)q} \tag{3.4}$$

$$\begin{aligned}p_3(\eta)=&-3\,\frac{p\mu^2 s\left(n^2-r^2\right)}{\left(\mu^2 n^2 s-\mu^2 r^2 s+1\right)q}\\&+3\,\frac{p\mu^2 s\left(n^2-2\,nr+r^2\right)\left(n+r\right)\left(\tanh\left(\frac{1}{2}\sqrt{n^2-r^2}\left(\eta+S\right)\right)\right)^2}{\left(n-r\right)\left(\mu^2 n^2 s-\mu^2 r^2 s+1\right)q},\end{aligned} \tag{3.5}$$

$$\begin{aligned}p_4(\eta)=&-3\,\frac{p\mu^2 sn^2}{\left(\mu^2 n^2 s+1\right)q}\\&+3\frac{p\mu^2 sn^2}{\left(\mu^2 n^2 s+1\right)q}\left(\tan\left(\frac{1}{2}\ \arctan\left(\frac{e^{2\,n(\eta+S)}-1}{e^{2\,n(\eta+S)}+1},2\,\frac{e^{n(\eta+S)}}{e^{2\,n(\eta+S)}+1}\right)\right)\right)^2,\end{aligned} \tag{3.6}$$

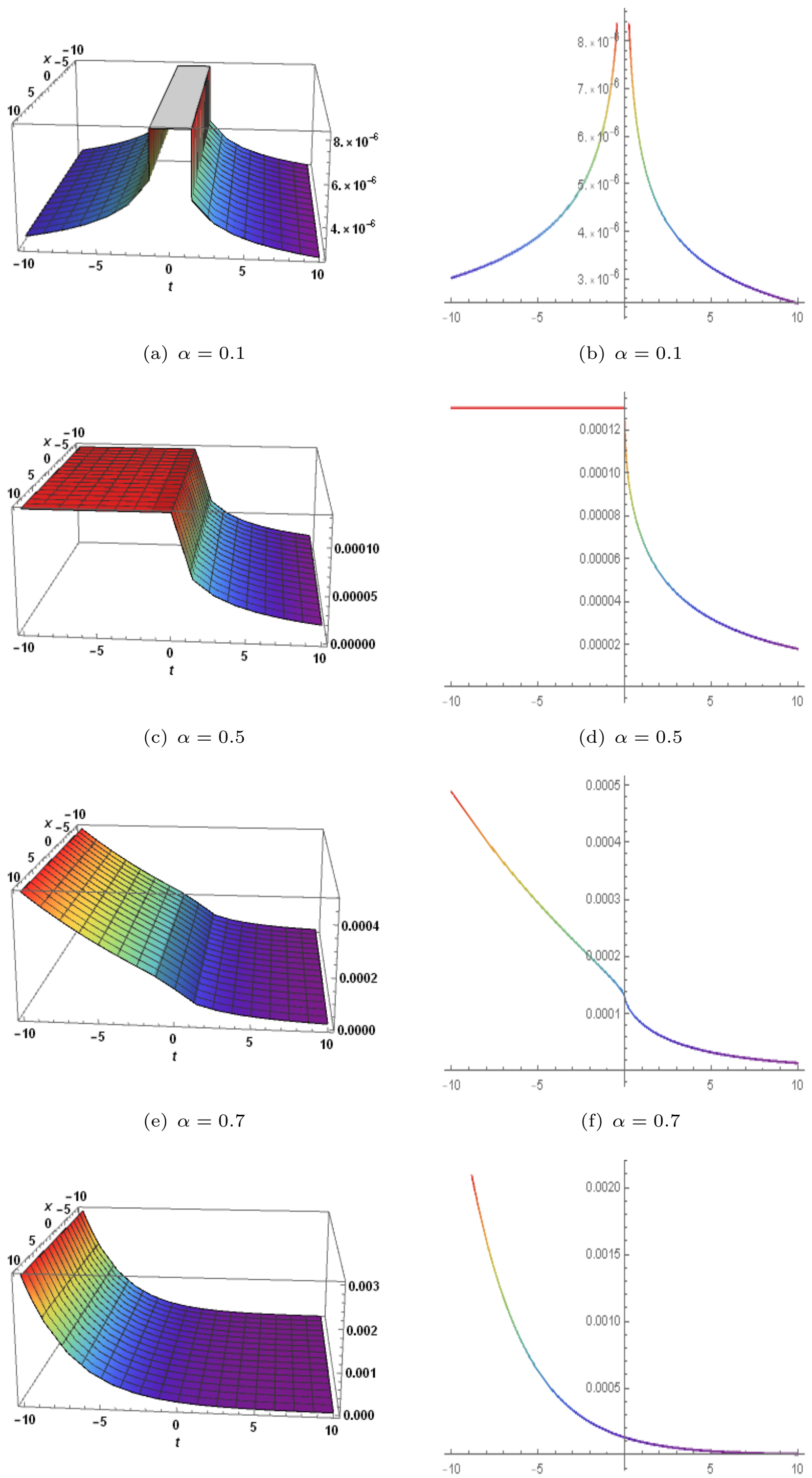
$$\begin{aligned}p_5(\eta)=&-3\,\frac{p\mu^2 s\left(n^2-r^2\right)}{\left(m^2\mu^2 s+\mu^2 n^2 s-\mu^2 r^2 s+1\right)q}\\&-6\,\frac{m\mu^2 sp\left(m\left(\eta+S\right)+2\right)}{\left(\eta+S\right)\left(m^2\mu^2 s+\mu^2 n^2 s-\mu^2 r^2 s+1\right)q}\\&+3\,\frac{p\mu^2 s\left(n^2-2\,nr+r^2\right)\left(m\left(\eta+S\right)+2\right)^2}{\left(n-r\right)^2\left(\eta+S\right)^2\left(m^2\mu^2 s+\mu^2 n^2 s-\mu^2 r^2 s+1\right)q}.\end{aligned} \tag{3.7}$$

Figure 1 highlights the $3D$ and $2D$ wave profiles of the solution $p_1(\eta)$ with given set of parameters.



(a) (b)

Figure 1. 3D and corresponding 2D graph of $p_1(\eta)$ with $p=-1,\ \mu=1,\ s=1,\ S=1,\ q=1,\ \alpha=1,\ m=2,\ n=1,\ r=-3$.



(a) $\alpha = 0.1$ (b) $\alpha = 0.1$

(c) $\alpha = 0.5$ (d) $\alpha = 0.5$

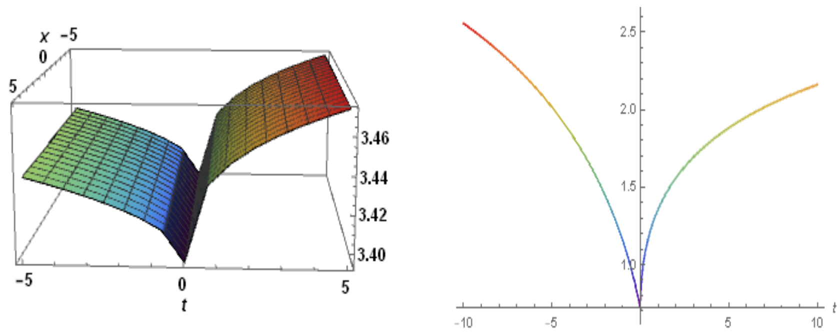
(e) $\alpha = 0.7$ (f) $\alpha = 0.7$

(g) $\alpha = 1$ (h) $\alpha = 1$

Figure 2. 3D and corresponding 2D graphs of $p_3(\eta)$ with different values of fractional parameter α along side parameters $p = -1,\ \mu = 1,\ s = 1,\ S = 1,\ q = 1,\ m = 0,\ n = 3,\ r = -1$.

Figure 2 highlights the $3D$ and corresponding $2D$ wave profiles of the solution $p_3(\eta)$ with given set of parameters. The changes in physical profiles of the waves are influenced by different values of fractional parameter α.

Figure 3 shows the $3D$ and $2D$ graphs of $p_5(\eta)$ for given set of parameters.



(a) (b)

Figure 3. 3D and corresponding 2D graphs of $p_5(\eta)$ with different values of fractional parameter α along side parameters $p=-1,\ \mu=1,\ s=1,\ S=1,\ q=1,\ m=1,\ n=3,\ r=2$

Set 3.2:

$$m=m,\quad n=n,\quad r=r,\quad \mu=\mu,\quad C_1=0,\quad C_2=0,$$
$$v=-\frac{p}{m^2\mu^2s+\mu^2n^2s-\mu^2r^2s-1},$$
$$N_0=-\frac{p\mu^2s\left(2\,m^2-n^2+r^2\right)}{(m^2\mu^2s+\mu^2n^2s-\mu^2r^2s-1)\,q},$$
$$N_1=6\,\frac{m\mu^2sp\,(n-r)}{(m^2\mu^2s+\mu^2n^2s-\mu^2r^2s-1)\,q},$$
$$N_2=-3\,\frac{sp\mu^2\left(n^2-2\,nr+r^2\right)}{(m^2\mu^2s+\mu^2n^2s-\mu^2r^2s-1)\,q},$$

where m, n and r are random constants. Using Eq. (3.2) and value of constants given in Set 3.2, families 1.11, 1.12, 1.15, 1.16 and 1.18 can be written as:

$$p_6(\eta)=-\,3\,\frac{\left(\sqrt{-m^2-n^2+r^2}\tan\left(\frac{1}{2}\sqrt{-m^2-n^2+r^2}\,(\eta+S)\right)-m\right)^2 sp\mu^2}{(m^2\mu^2s+\mu^2n^2s-\mu^2r^2s-1)\,q}, \tag{3.8}$$

$$\begin{aligned}p_7(\eta)=&-\frac{p\mu^2s\left(2\,m^2-n^2+r^2\right)}{(m^2\mu^2s+\mu^2n^2s-\mu^2r^2s-1)\,q}\\&+6\frac{m\mu^2sp\,(n-r)}{(m^2\mu^2s+\mu^2n^2s-\mu^2r^2s-1)\,q}\\&\times\left(\frac{m}{n-r}+\frac{\sqrt{m^2+n^2-r^2}\tan\left(\frac{1}{2}\sqrt{m^2+n^2-r^2}\,(\eta+S)\right)}{n-r}\right)\\&-3\frac{p\mu^2s\left(n^2-2\,nr+r^2\right)}{(m^2\mu^2s+\mu^2n^2s-\mu^2r^2s-1)\,q}\end{aligned}$$

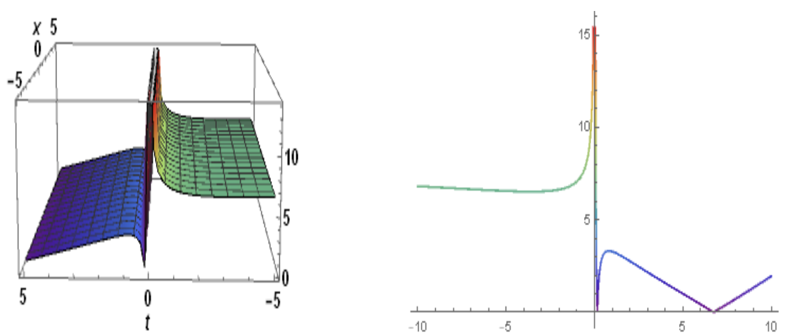
$$\times\left(\frac{m}{n-r}+\frac{\sqrt{m^2+n^2-r^2}\tan\left(\frac{1}{2}\sqrt{m^2+n^2-r^2}\,(\eta+S)\right)}{n-r}\right)^2,$$

$$p_8(\eta)=-\frac{p\mu^2 s\left(-n^2+r^2\right)}{\left(\mu^2n^2s-\mu^2r^2s-1\right)q}-3\,\frac{p\mu^2s\left(n^2-2\,nr+r^2\right)(n+r)\left(\tanh\left(\frac{1}{2}\sqrt{n^2-r^2}\,(\eta+S)\right)\right)^2}{(n-r)\left(\mu^2n^2s-\mu^2r^2s-1\right)q}, \tag{3.9}$$

$$p_9(\xi)=\frac{p\mu^2sn^2}{\left(\mu^2n^2s-1\right)q}-3\frac{p\mu^2sn^2}{\left(\mu^2n^2s-1\right)q}\left(\tan\left(\frac{1}{2}\ \arctan\left(\frac{e^{2\,n(\eta+S)}-1}{e^{2\,n(\eta+S)}+1},2\,\frac{e^{n(\eta+S)}}{e^{2\,n(\eta+S)}+1}\right)\right)\right)^2\times\frac{p\mu^2sn^2}{\left(\mu^2n^2s-1\right)q}, \tag{3.10}$$

$$p_{10}(\eta)=-\frac{p\mu^2s\left(2\,m^2-n^2+r^2\right)}{\left(m^2\mu^2s+\mu^2n^2s-\mu^2r^2s-1\right)q}+6\,\frac{m\mu^2sp\,(m\,(\eta+S)+2)}{(\eta+C)\left(m^2\mu^2s+\mu^2n^2s-\mu^2r^2s-1\right)q}-3\,\frac{p\mu^2s\left(n^2-2\,nr+r^2\right)(m\,(\eta+S)+2)^2}{(n-r)^2\,(\eta+S)^2\left(m^2\mu^2s+\mu^2n^2s-\mu^2r^2s-1\right)q}. \tag{3.11}$$

Figure 4 shows the $3D$ and $2D$ graphs of $p_7(\eta)$ for given set of parameters.



(a) (b)

Figure 4. 3D and corresponding 2D graphs depict the solution of $p_7(\eta)$ with $p=-1,\ \mu=1,\ s=1,\ \alpha=0.2,\ S=1,\ q=1,\ m=2,\ n=1,\ r=-1$.

Set 3.3:

$$m=0,\quad n=n,\quad r=r,$$
$$v=-\frac{p}{4\,\mu^2n^2s-4\,\mu^2r^2s-1},$$
$$\mu=\mu,\quad N_1=0,\quad C_1=0,$$
$$N_0=-2\,\frac{p\mu^2s\left(n^2-r^2\right)}{\left(4\,\mu^2n^2s-4\,\mu^2r^2s-1\right)q},$$

$$N_2 = -3\,\frac{sp\mu^2\left(n^2 - 2\,nr + r^2\right)}{\left(4\,\mu^2 n^2 s - 4\,\mu^2 r^2 s - 1\right)q},$$
$$C_2 = -3\,\frac{sp\mu^2\left(n^2 + 2\,nr + r^2\right)}{\left(4\,\mu^2 n^2 s - 4\,\mu^2 r^2 s - 1\right)q},$$

where n and r are random constants. Using Eq. (3.2) and values of constants given in Set 3.3, families 2.15, 2.16, 2.11 and 2.13 can be written as:

$$\begin{aligned} p_{11}(\eta) = &-2\,\frac{p\mu^2 s\left(n^2 - r^2\right)}{\left(4\,\mu^2 n^2 s - 4\,\mu^2 r^2 s - 1\right)q} \\ &-3\,\frac{p\mu^2 s\left(n^2 - 2\,nr + r^2\right)(n+r)\left(\tanh\left(\frac{1}{2}\sqrt{n^2 - r^2}\,(\eta + S)\right)\right)^2}{(n-r)\left(4\,\mu^2 n^2 s - 4\,\mu^2 r^2 s - 1\right)q} \\ &-3\,\frac{sp\mu^2\left(n^2 + 2\,nr + r^2\right)(n-r)}{\left(4\,\mu^2 n^2 s - 4\,\mu^2 r^2 s - 1\right)q\,(n+r)\left(\tanh\left(\frac{1}{2}\sqrt{n^2 - r^2}\,(\eta + S)\right)\right)^2}, \end{aligned} \tag{3.12}$$

$$\begin{aligned} p_{12}(\eta) = &-2\,\frac{p\mu^2 s n^2}{\left(4\,\mu^2 n^2 s - 1\right)q} \\ &-3\,\frac{p\mu^2 s n^2}{\left(4\,\mu^2 n^2 s - 1\right)q}\left(\tan\left(\frac{1}{2}\,\arctan\left(\frac{e^{2\,n(\eta+S)} - 1}{e^{2\,n(\eta+S)} + 1}, 2\,\frac{e^{n(\eta+S)}}{e^{2\,n(\eta+S)} + 1}\right)\right)\right)^2 \\ &-3\,\frac{p\mu^2 s n^2}{\left(4\,\mu^2 n^2 s - 1\right)q}\left(\tan\left(\frac{1}{2}\,\arctan\left(\frac{e^{2\,n(\eta+S)} - 1}{e^{2\,n(\eta+S)} + 1}, 2\,\frac{e^{n(\eta+S)}}{e^{2\,n(\eta+S)} + 1}\right)\right)\right)^{-2}, \end{aligned} \tag{3.13}$$

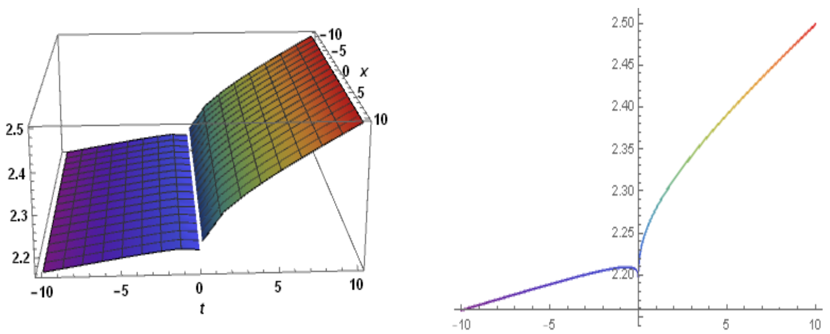
$$\begin{aligned} p_{13}(\eta) = &-2\,\frac{p\mu^2 s n^2}{\left(4\,\mu^2 n^2 s - 1\right)q} - 3\,\frac{p\mu^2 s n^2\left(ne^{n(\eta+S)} - 1\right)^2}{\left(-ne^{n(\eta+S)} - 1\right)^2\left(4\,\mu^2 n^2 s - 1\right)q} \\ &-3\,\frac{p\mu^2 s n^2\left(-ne^{n(\eta+S)} - 1\right)^2}{\left(4\,\mu^2 n^2 s - 1\right)q\left(ne^{n(\xi+S)} - 1\right)^2}, \end{aligned} \tag{3.14}$$

$$\begin{aligned} p_{14}(\eta) = &-2\,\frac{p\mu^2 s n^2}{\left(4\,\mu^2 n^2 s - 1\right)q} - 3\,\frac{p\mu^2 s n^2\left(e^{n(\eta+S)} + n\right)^2}{\left(e^{n(\eta+S)} - n\right)^2\left(4\,\mu^2 n^2 s - 1\right)q} \\ &-3\,\frac{p\mu^2 s n^2\left(e^{n(\eta+S)} - n\right)^2}{\left(4\,\mu^2 n^2 s - 1\right)q\left(e^{n(\eta+S)} + n\right)^2}. \end{aligned} \tag{3.15}$$

Figure 5 shows the $3D$ and $2D$ graphs of $p_{13}(\eta)$ for given set of parameters.

Set 3.4:

$$m = 0, \quad n = n, \quad r = r, \quad \mu = \mu,$$
$$v = \frac{p}{4\,\mu^2 n^2 s - 4\,\mu^2 r^2 s + 1}, \quad N_0 = -6\,\frac{p\mu^2 s\left(n^2 - r^2\right)}{\left(4\,\mu^2 n^2 s - 4\,\mu^2 r^2 s + 1\right)q},$$
$$N_1 = 0, \quad N_2 = 3\,\frac{sp\mu^2\left(n^2 - 2\,nr + r^2\right)}{\left(4\,\mu^2 n^2 s - 4\,\mu^2 r^2 s + 1\right)q},$$
$$C_1 = 0, \quad C_2 = 3\,\frac{sp\mu^2\left(n^2 + 2\,nr + r^2\right)}{\left(4\,\mu^2 n^2 s - 4\,\mu^2 r^2 s + 1\right)q},$$



(a) (b)

Figure 5. 3D and corresponding 2D graphs depict the solution of $p_{13}(\eta)$ with $p = -1,\ \mu = 1,\ s = 1,\ \alpha = 0.3,\ S = 1,\ q = 1,\ m = 0,\ n = 2,\ r = 0$.

where n and r are random constants. Using Eq. (3.2) and values of constants given in Set 3.4, families 2.15, 2.16, 2.11 and 2.13 can be written as:

$$p_{15}(\eta) = -6\,\frac{p\mu^2 s\left(n^2 - r^2\right)}{\left(4\,\mu^2 n^2 s - 4\,\mu^2 r^2 s + 1\right) q} + 3\,\frac{p\mu^2 s\left(n^2 - 2\,nr + r^2\right)(n + r)\left(\tanh\left(\frac{1}{2}\sqrt{n^2 - r^2}\,(\eta + S)\right)\right)^2}{(n - r)\left(4\,\mu^2 n^2 s - 4\,\mu^2 r^2 s + 1\right) q} + 3\,\frac{sp\mu^2\left(n^2 + 2\,nr + r^2\right)(n - r)}{\left(4\,\mu^2 n^2 s - 4\,\mu^2 r^2 s + 1\right) q\,(n + r)\left(\tanh\left(\frac{1}{2}\sqrt{n^2 - r^2}\,(\eta + S)\right)\right)^2}, \tag{3.16}$$

$$p_{16}(\eta) = -6\,\frac{p\mu^2 sn^2}{\left(4\,\mu^2 n^2 s + 1\right) q} + 3\,\frac{p\mu^2 sn^2}{\left(4\,\mu^2 n^2 s + 1\right) q}\left(\tan\left(\frac{1}{2}\ \arctan\left(\frac{e^{2\,n(\eta+S)} - 1}{e^{2\,n(\eta+S)} + 1}, 2\,\frac{e^{n(\eta+S)}}{e^{2\,n(\xi+S)} + 1}\right)\right)\right)^2 + 3\,\frac{p\mu^2 sn^2}{\left(4\,\mu^2 n^2 s + 1\right) q}\left(\tan\left(\frac{1}{2}\ \arctan\left(\frac{e^{2\,n(\eta+S)} - 1}{e^{2\,n(\eta+S)} + 1}, 2\,\frac{e^{n(\eta+S)}}{e^{2\,n(\eta+S)} + 1}\right)\right)\right)^{-2}, \tag{3.17}$$

$$p_{17}(\eta) = -6\,\frac{p\mu^2 sn^2}{\left(4\,\mu^2 n^2 s + 1\right) q} + 3\,\frac{p\mu^2 sn^2\left(ne^{n(\eta+S)} - 1\right)^2}{\left(-ne^{n(\eta+S)} - 1\right)^2\left(4\,\mu^2 n^2 s + 1\right) q} + 3\,\frac{p\mu^2 sn^2\left(-ne^{n(\eta+S)} - 1\right)^2}{\left(4\,\mu^2 n^2 s + 1\right) q\left(ne^{n(\eta+S)} - 1\right)^2}, \tag{3.18}$$

$$p_{18}(\eta) = -6\,\frac{p\mu^2 sn^2}{\left(4\,\mu^2 n^2 s + 1\right) q} + 3\,\frac{p\mu^2 sn^2\left(e^{n(\eta+S)} + n\right)^2}{\left(e^{n(\eta+S)} - n\right)^2\left(4\,\mu^2 n^2 s + 1\right) q} + 3\,\frac{p\mu^2 sn^2\left(e^{n(\eta+S)} - n\right)^2}{\left(4\,\mu^2 n^2 s + 1\right) q\left(e^{n(\eta+S)} + n\right)^2}. \tag{3.19}$$

4. Results and Discussion with Concluding Remarks

In this work, we applied improved tan $\left(\frac{\phi(\eta)}{2}\right)$-expansion method (ITEM) for exact solutions of conformable regularized long wave burgers equation. Exact traveling wave solutions for conformable regularized long wave burgers equation are presented in 4 different sets in which each contains four to five solutions which include dark, singular and other traveling wave solutions. The physical features of some of these solutions are highlighted in 3D and corresponding 2D plots with appropriate choice of parameteric values satisfying constraint conditions. In particular, Figure 2 portrays the solution $p_3(\eta)$ with values of fractional parameter $\alpha = 0.1,\ \alpha = 0.5,\ \alpha = 0.7$ and $\alpha = 1$. It is observed that the fractional parameter influences the shape dynamics of solutions as in Figure 2. It is quite evident that, the ITEM is a reliable technique towards the study of fractional nonlinear evolution equations arising in different scientific regimes. To the best of our knowledge, the results reported in this manuscript are new and have not been reported before and can open a new window towards the comprehension of fractional differential equations.

Conflict of interest. The authors declare that there is no conflict of interest.

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