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THE GENERALIZED LYAPUNOV FUNCTION AS AO'S POTENTIAL FUNCTION: EXISTENCE IN DIMENSIONS 1 AND 2

Haoyu Wang[1,†], Wenqing Hu[2], Xiaoliang Gan[3] and Ping Ao[4]

Abstract By using Ao's decomposition for stochastic dynamical systems, a new notion of potential function has been introduced by Ao and his collaborators recently. We show that this potential function agrees with the *generalized Lyapunov function* of the deterministic part of the stochastic dynamical system. We further prove the existence of Ao's potential function in dimensions 1 and 2 via the solution theory of first-order partial differential equations. Our framework reveals the equivalence between Ao's potential function and Lyapunov function, the latter being one of the most significant central notions in dynamical systems. Using this equivalence, our existence proof can also be interpreted as the proof of existence of Lyapunov function for a general dynamical system.

Keywords Stochastic dynamical system, the generalized Lyapunov function, Ao's decomposition, potential function.

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1. Introduction

Stability has been one of the major research directions for dynamical systems. By stability, we mean the study of whether a dynamical system can maintain a predetermined state under various accidental or continued interference, without swinging or unrest [19]. An unstable system, with small interference can not work normally; and with big interference can bring disaster and even devastating consequences, such as social unrest, financial crisis, power grid collapse, or plane crash. Due to its role in applications, the study of stability has become a very important research topic. A fundamental tool used to investigate the local stability of equilibrium points in dynamical systems [30,33] is the Lyapunov function, which was originally proposed in the doctoral thesis "General problems of motion stability" [29] by the Russian mathematician Alexander Mikhimovich Lyapunov in 1892. In his doctoral thesis, Lyapunov not only strictly defined the notion of stability, but also gave a novel method – the so called second method which can be directly applied to

†The corresponding author. Email:wanghaoyu042@163.com(H. Wang)
[1]Department of Mathematics, 688 Yingbin Avenue, Zhejiang Normal University, 321004 Jinhua, China
[2]Department of Mathematics and Statistics, 202 Rolla Building, Missouri University of Science and Technology, 65401 Rolla, Missouri, USA
[3]School of Mathematics and Computing Science, 1 Jinji Road, Guilin University of Electronic Technology, 541004 Guilin, China
[4]Department of physics, 99 Shangda Road, Shanghai University, 200444 Shanghai, China

study stability. He was inspired by the energy function in physics and influenced by famous mathematician Poincaré's topography system, and he generalized the classical notion of energy to his more abstract notion of Lyapunov function. The greatest advantage of Lyapunov function lies in that one can use it to capture the stability of the dynamical system without facing the usual difficulty of solving differential equations. Although the Lyapunov function can be explicitly constructed for many known systems, until now, there has not been a general method for explicitly constructing these Lyapunov functions in nonlinear dynamical systems. This has been regarded as a hard problem by many well-known mathematicians in related fields [13, 21, 26, 33, 37]. It is also interesting to note that Krasovskii also held the idea that proving the existence of Lyapunov function might carry with a constructive method, but this method has not been found [24].

Recently, Ao and his collaborators proposed a new paradigm for stochastic dynamical systems, or rather dynamical systems perturbed by stochastic noise, called the Ao's decomposition (see [2, 3, 25, 42, 43]). In Ao's decomposition, a potential function is found [42, 43] which results in the consistency of stable points between stochastic dynamical system and the corresponding deterministic dynamical system. This potential function has been unifying some previous notions such as energy landscape or energy potential proposed by Wright [41] and later Waddington [40]. It enables qualitative analysis of complex dynamical behaviors far from equilibrium point (e.g., multi-steady states and periodic attractors), which are ubiquitous in real systems [9,35] but are beyond the scope of the classical Lyapunov functions. It is also helpful in understanding a highly complex stochastic multi-steady system, as it is necessary to compare the relative stability of different attractors [39], account for the transition rates between neighboring steady states induced by noise [10, 11, 16, 17], and form an intuitive picture that reveals the essential mechanism underlying the complex system [3]. It is also found that Ao's potential function has been widely applied in many fields such as physics, chemistry and biology: In physics, it is closely related to the non-equilibrium thermodynamic framework [2]; In chemistry, it provides useful explanations for protein folding [22]; In biology, it has been used to explore basic problems in evolution such as studying the robustness, adaptability and efficiency of real biological networks [38, 45].

Inspired by famous mathematician Poincaré's global idea of studying dynamic system as a whole [6], here we introduce the notion of a *generalized Lyapunov function* as a natural generalization of the classical Lyapunov function. It is defined on the whole state space and its time derivative does not increase which has an infimum at the attractor. Furthermode, we show that our generalized Lyapunov function is equivalent to the potential function in Ao's decomposition, and can serve as the Lyapunov function for the deterministic part of the stochastic dynamical system considered under the Ao's decomposition framework (see [26, 43]).

We further investigate the problem of the existence of Ao's potential function, or equivalently our generalized Lyapunov function, debating the point of view by some researchers who insist that it doesn't exist [31, 32, 36]. Our method relies on the solution theory of first-order partial differential equations, as we reduce the problem to proving the existence of solutions for a first-order quasilinear partial differential equations. When dimension is 1, the solutions must exist, so that the generalized Lyapunov function exists on the whole real interval. Furthermore, it can be constructed explicitly. When dimension is 2, we firstly prove the local existence of implicit solutions by first-integral method. To prove the existence of explicit solutions, we consider the corresponding Cauchy problem. Under the assumption that the determinant of the coefficients of the equation is not zero at the initial parameter, the existence of explicit solutions is proved by the existence theorem of implicit functions. The generalized Lyapunov function globally exists for 2 dimensional case except at some singularities. We will study the case further when dimension is 3 or higher. As a byproduct, our proof of the existence reveals the fundamental structure of general stochastic dynamical systems: it can be divided into four important components – divergence, curl, the gradient of the generalized Lyapunov function and white Gaussian noise.

This paper is organized as follows. In Section 2 we introduce our generalized Lyapunov function and compare it with the Ao's potential function in Ao's decomposition theory. In Section 3 we reduce the problem of the existence of the generalized Lyapunov function to the existence of the solution of a first-order system of partial differential equations. In Section 4 we show the existence of solutions of the PDE system introduced in Section 3. In Section 5 we make a few conclusions.

2. The generalized Lyapunov function and Ao's potential function

2.1. The definition of the generalized Lyapunov function

Let us first restrict ourselves to smooth dynamics, but the results we present here can be directly extended to more general cases. Consider a smooth dynamical system given by

$$\dot{q} = f(q). \tag{2.1}$$

Let q^* be an equilibrium point for the system and let $L : U \to \mathbb{R}$ be a continuously differentiable function defined on an open neighborhood U of the point q^*. Then, L is called a conventional Lyapunov function if it satisfies the following two conditions:

(i) $\dot{L}(q) = \left.\dfrac{\mathrm{d}L}{\mathrm{d}t}\right|_{q'} \leq 0, \quad \text{for all } q' \in U;$

(ii) $L(q^*) = 0$ and $L(q) > 0, \quad \text{if } q \neq q^*.$

Obviously the above definition is local. Inspired by Poincaré's general idea for studying dynamical systems [6], we introduce the generalized Lyapunov function as a natural generalization of its classical version.

Definition 2.1 (The generalized Lyapunov function)**.** Let $\varphi : U \to \mathbb{R}$ be a continuously differentiable scalar function, where U is a set representing the whole state space. Then φ is called a *generalized Lyapunov function* if it satisfies the following conditions:

(i) $\dot{\varphi}(q) = \left.\dfrac{\mathrm{d}\varphi}{\mathrm{d}t}\right|_{q''} \leq 0, \quad \text{for all } q'' \text{ belongs to } U\ ;$

(ii) (Condition on attractors) $\nabla\varphi(q^*) = 0$ for all points q^* in the attraction domain.

It is easy to see that the generalized Lyapunov function has dynamical equivalence to classical Lyapunov function near equilibrium point [26,43]. Later we will see that this is consistent with the fact that Ao's decomposition would enable stochastic dynamical behaviours correspond well to dynamical behaviours near equilibrium point [42]. Beyond that, the generalized Lyapunov function also enables qualitative analysis of complex dynamical behaviors far from equilibrium point (e.g., multi-steady states and periodic attractors), which are ubiquitous in real systems. For this natural generalization, the value of the function should decrease along the trajectories, reaching a local minimum point at attractors and keeping constant on it. Due to all these reasons, we see that the criteria (i) in Definition 2.1 keeps the same as the criteria (i) for the conventional Lyapunov function. However, in the attraction domain, we take a weaker form of the criteria (ii) from the conventional Lyapunov function for the criteria (ii) of the generalized Lyapunov function, in that we only require the gradient of the function to be 0 instead of having exact minimum.

Comparing with classical Lyapunov function, the generalized Lyapunov function(the potential function) provides intuitive and global landscapes which is easier for researchers to observe. Classical Lyapunov function can only solve single steady-state problems locally, however, real dynamical systems are complex and usually have more than one steady state. Therefore, the generalized Lyapunov function has a wider range of applications in real dynamical systems. For example, Hu and Xu studied the phenomenon of multi-stable chaotic attractors existing in generalized synchronization for a driving and response system named Rössler system [18]. Angeli and Sontag studied the emergence of multi-stability and hysteresis in those monotone input/output systems that arise, under positive feedback, starting from monotone systems with well-defined steady-state responses [1]. Liu and You studied multi-stability, existence of almost periodic solutions of a class of recurrent neural networks with bounded activation functions and all criteria they proposed can be easily extended to fit many concrete forms of neural networks such as Hopfield neural networks, or cellular neural networks, etc. [28]. The generalized Lyapunov function has provided a more general and more unified perspective for researchers to investigate different types of dynamical systems. In addition, our definition of generalized Lyapunov function is similar to Conley and other researchers¡¯ work (see [7, 15]) when considering discrete dynamical system. It must be noted that our theoretical framework is totally different from the stochastic Lyapunov function (see [4]) under Itô's process perspective. This is because our generalized Lyapunov function serves as the Ao's potential function of a stochastic dynamical system, and it corresponds to the Lyapunov function of the deterministic part of the underlying stochastic dynamical system. This relation will be revealed in the next subsection.

2.2. Ao's decomposition theory and Ao's potential function

Let us first consider the Langevin equation [12, 20]. Here we view it as a Markov process and we use the physicists' notation for the noise, and we can write this equation in the form

$$\dot{q} = f(q,t) + \zeta(q,t). \tag{2.2}$$

The equation (2.2) is discussed in n-dimensional real Euclidean space. Here the state variable $q = [q_1(t), q_2(t), \cdots, q_n(t)]^\tau \in \mathbb{R}^n$ is a function of time t. The function f is in general nonlinear, and we assume that $f(q,t)$ is an infinitely differentiable smooth function. The noise $\zeta(q,t)$ is a function of the state variable q and the time variable t, which is almost everywhere nondifferentiable. Let's consider the case that $\zeta(q,t)$

is an n-dimensional white Gaussian noise with zero mean:

$$\langle \zeta(q,t) \rangle = 0, \tag{2.3}$$

and the covariance

$$\langle \zeta(q,t)\zeta^\tau(q,t') \rangle = 2D(q)\delta(t-t'). \tag{2.4}$$

Here we use the standard notations in physics literature. The superscript τ denotes the transpose of a vector, $\delta(t-t')$ is the Dirac delta function, $\langle\cdot\rangle$ indicates the average over the noise distribution, and the diffusion matrix $D(q)$ is a symmetric positive semi–definite matrix.

Recently, a new formulation of (2.2), named the Ao's decomposition, has been introduced in a series of works by Ao and collaborators [2, 3, 25, 42, 43]. Following the idea of Ao's decomposition, we act a matrix operator $S(q(t)) + A(q(t))$ on both sides of the equation (2.2), and we expect that the equation (2.2) can formally be transformed into the following equation

$$[S(q(t)) + A(q(t))]\dot{q} = -\nabla\varphi(q(t)) + \xi(q,t), \tag{2.5}$$

where the matrix $S(q(t))$ is a symmetric positive semi-definite matrix (which we call the "friction matrix"), the matrix $A(q(t))$ is an anti–symmetric matrix (which we call the "Lorenz matrix"), $\varphi(q)$ is a real and single valued function of $q_1, q_2, \cdots, q_n$, and $\xi(q,t)$ is an n dimension white Gaussian noise with zero mean

$$\langle \xi(q,t) \rangle = 0, \tag{2.6}$$

and the covariance

$$\langle \xi(q,t)\xi^\tau(q,t') \rangle = 2S(q)\delta(t-t'). \tag{2.7}$$

Equation (2.4) and equation (2.7) are expressions of the fluctuation-dissipation theorem (see [5, pp.387]), in which $D(q)$ and $S(q)$ reflect "dissipation", the covariance structures $\langle \zeta(q,t)\zeta^\tau(q,t') \rangle$ and $\langle \xi(q,t)\xi^\tau(q,t') \rangle$ reflect "fluctuation".

Definition 2.2 (Ao's decomposition and Ao's potential function)**.** The way we write the Langevin equation (2.2) into the equation (2.5) is called *Ao's decomposition.* The function $\varphi(q)$ is called *Ao's potential function*(see [2]).

Our main result of this paper can be summarized as the following

Theorem 2.1 (Summary of Main Reuslt)**.** *We have*

(a) Ao's decomposition (2.5) implies the Langevin equation (2.2).

(b) The Ao's potential function $\varphi(q)$ is just a king of Lyapunov function of the deterministic system $\dot{q} = f(q)$, which is the deterministic part of the stochastic system (2.2).

(c) Under general circumstances, in dimensions 1 and 2, the Langevin equation (2.2) also implies Ao's decomposition (2.5), and therefore Ao's potential function or the generalized Lyapunov function exists.

Proof. Statements (a), (b) will be shown in Section 3. Statement (c) will be shown in Sections 3 and 4. □

3. Reduction of the problem into PDEs

We first look at part (a) of Theorem 2.1, that (2.5) implies (2.2). To this end, we assume that the function matrix $[S(q) + A(q)]$ is invertible. In a straightforward way, equation (2.5) can be transformed into the equation

$$\dot{q} = -[S(q(t)) + A(q(t))]^{-1}\nabla\varphi(q(t)) + \zeta(q,t),$$

where $\zeta(q,t)$ is a noise that takes the form $\zeta(q,t) = [S(q)+A(q)]^{-1}\xi(q,t)$. To match (2.2), we can then set $f(q) = -[S(q) + A(q)]^{-1}\nabla\varphi(q)$. Combining with the explicit representation of $\zeta(q,t)$ in terms of $S(q), A(q)$ and $\xi(q,t)$, as well as (2.7), we can calculate

$$\begin{aligned}\langle\zeta(q,t)\rangle &= \langle[S(q) + A(q)]^{-1}\xi(q,t)\rangle \\ &= [S(q) + A(q)]^{-1}\langle\xi(q,t)\rangle \\ &= 0,\end{aligned}$$

and

$$\begin{aligned}\langle\zeta(q,t)\zeta^{\tau}(q,t')\rangle &= \langle[S(q) + A(q)]^{-1}\xi(q,t)\xi^{\tau}(q,t')[S(q) + A(q)]^{-\tau}\rangle \\ &= [S(q) + A(q)]^{-1}\langle\xi(q,t)\xi^{\tau}(q,t')\rangle[S(q) + A(q)]^{-\tau} \\ &= 2[S(q) + A(q)]^{-1}S(q)[S(q) + A(q)]^{-\tau}\delta(t-t').\end{aligned}$$

Comparing the above two calculations with (2.3), (2.7), we obtain

$$D(q) = [S(q) + A(q)]^{-1}S(q)[S(q) + A(q)]^{-\tau},$$

which gives an explicit representation of $D(q)$. As long as $D(q)$ is obtained, we can construct $\zeta(t)$ in (2.2). Therefore, we have proved that (2.5) implies (2.2).

We then demonstrate our rationale how we show that (2.2) implies (2.5). In fact, transforming from (2.2) to (2.5) requires much more efforts. In this case, we need to obtain $S(q)$, $A(q)$ and $\varphi(q)$ from general dynamics (2.2). Within this section, we propose a heuristic inference. Although this inference is not a rigorous mathematical proof, but it can lead to a reformulation of the problem into PDEs. The argument will be made rigorous in Section 4 where we use PDE methods to demonstrated the existence. The main idea of this heuristic inference is that the equations (2.2) and (2.5) can describe the same dynamical behaviors in $\mathbb{R}^n$. Hence, we may replace $\dot{q}$ in (2.5) by the right hand side of equation (2.2), and we obtain

$$[S(q(t)) + A(q(t))][f(q(t)) + \zeta(q(t),t)] = -\nabla\varphi(q(t)) + \xi(q(t),t).$$

Regarding t as a parameter in $q(t)$, the above equation can be briefly written as

$$[S(q) + A(q)][f(q) + \zeta(q,t)] = -\nabla\varphi(q) + \xi(q,t).$$

The above equation contains a deterministic part that is differentiable up to arbitrary order, and a random part that is nondifferentiable almost everywhere. From the physical point of view, the two kinds of noises $\zeta(q,t)$ and $\xi(q,t)$ have the same source. Inspired by this, we may assume that we can establish the following decomposition

$$[S(q) + A(q)]f(q) = -\nabla\varphi(q), \tag{3.1}$$

$$[S(q) + A(q)]\zeta(q,t) = \xi(q,t). \tag{3.2}$$

This subjective decompositions (3.1), (3.2) are the key to understand Ao's decomposition. Actually, if we combine (2.4) and (2.7), together with (3.2), we will obtain

$$\begin{aligned}\langle \xi(q,t)\xi^\tau(q,t')\rangle &= \langle [S(q)+A(q)]\zeta(q,t)\zeta^\tau(q,t')[S(q)+A(q)]^\tau\rangle \\ &= 2[S(q)+A(q)]D(q)[S(q)+A(q)]^\tau\delta(t-t') \\ &= 2[S(q)+A(q)]D(q)[S(q)-A(q)]\delta(t-t') \\ &= 2S(q)\delta(t-t'),\end{aligned}$$

which implies

$$[S(q)+A(q)]D(q)[S(q)-A(q)] = S(q). \tag{3.3}$$

It is worth mentioning that from the physical point of view, equation (3.3) is a generalized Einstein relation in more than one dimension. Equations (3.1) and (3.3) are regarded as the key relations in Ao's decomposition theory, which reveal the relations between $f \leftrightarrow \varphi$ and $\zeta \leftrightarrow \xi$.

From equation (3.3) we have

$$\begin{aligned}D(q) &= [S(q)+A(q)]^{-1}\cdot\frac{1}{2}\left\{[S(q)+A(q)]+[S(q)-A(q)]\right\}\cdot[S(q)-A(q)]^{-1} \\ &= \frac{1}{2}\left\{[S(q)-A(q)]^{-1}+[S(q)+A(q)]^{-1}\right\} \\ &= \frac{1}{2}\left\{[S(q)+A(q)]^{-\tau}+[S(q)+A(q)]^{-1}\right\},\end{aligned}$$

where the symmetric part of $[S(q)+A(q)]^{-1}$ is $\frac{1}{2}[(S(q)+A(q))^{-1}+(S(q)+A(q))^{-\tau}]$. It's the diffusion matrix $D(q)$ defined in equation (2.4). Hence, we can rewrite the identity

$$[S(q)+A(q)][S(q)+A(q)]^{-1} = I$$

as

$$[S(q)+A(q)][D(q)+Q(q)] = I. \tag{3.4}$$

where the matrix $Q(q)$ is anti–symmetric unknown matrix function and I is the identity matrix. Substituting equation (3.4) into equation (3.1), we obtain

$$[D(q)+Q(q)]^{-1}f(q) = -\nabla\varphi(q). \tag{3.5}$$

From equation (3.1), we notice that if $\dot{q} = f(q^*) = 0$, then $\nabla\varphi(q^*) = 0$. Moreover, by (3.1), we have

$$\frac{d}{dt}\varphi(q) = \dot{q}^\tau\nabla\varphi(q) = -\dot{q}^\tau[S(q)+A(q)]\dot{q} = -\dot{q}^\tau S(q)\dot{q} \le 0.$$

Thus we find that $\varphi(q)$ satisfies $\dot{\varphi}(q) \le 0$ for all $q \in \mathbb{R}^n$. Hence, by Definition 2.1 we obtain that $\varphi(q)$ is a generalized Lyapunov function (compare with [43]). So this proved part (b) of Theorem 2.1.

Assuming (2.2) holds true, (2.4) is given, and $D(q)$ is known. Then in order to obtain (2.5), we only have to show that there exists an anti-symmetric matrix $Q(q)$

and a potential function $\varphi(q)$ that satisfy (3.5). As long as this is proved, it means that we have demonstrated the fact that (2.2) implies (2.5). And as a by-product, taking into account part (b) of Theorem 2.1, we have shown the existence of $\varphi(q)$ as a generalized Lyapunov function. So this will settle part (c) of Theorem 2.1. Here we reduce the existence of $Q(q)$ to a PDE problem, and we address the solution theory to this PDE problem in Section 4.

Assuming basic integrability conditions on $f(q), D(q)$ and $Q(q)$, by the classical Helmholtz-Weyl decomposition we see that it suffices to show that the curl part of the vector field $[D(q)+Q(q)]^{-1}f(q)$ vanishes. That is

$$\nabla \times \{[D(q)+Q(q)]^{-1}f(q)\} = 0. \tag{3.6}$$

We notice that (3.6) is a family of $\frac{n(n-1)}{2}$ first-order quasi-linear partial differential equations for the coefficients of $Q(q)$ in (3.5). Correspondingly, if the solution $Q(q)$ to the partial differential equation (3.6) exists, the generalized Lyapunov function $\varphi(q)$ exists, simply by the classical Helmholtz-Weyl decomposition.

We also notice that the above heuristic inference at the level of mathematical rigor, is actually saying that (3.6) is a *sufficient* but *not necessary* condition for (2.2)⇒(2.5). In fact, if (3.6) holds, then by the Helmholtz-Weyl decomposition there exists a function $\varphi = \varphi(q)$ such that (3.5) holds, with the anti-symmetric matrix $Q(q)$ from (3.6). Moreover, with $D(q)$ from (2.4) and $Q(q)$ from (3.6) at hand, we can construct the matrix $S(q)+A(q) = [D(q)+Q(q)]^{-1}$, where the matrix $S(q)$ is its symmetric part and $A(q)$ is its anti-symmetric part. The so-constructed matrices $S(q)$ and $A(q)$ will satisfy (3.1) and (3.3). Thus we can construct the noise $\xi(q,t)$ from (3.2), which together with (3.1) imply that we can construct (2.5) from (2.2).

Our problem has now been reduced to proving the existence of solution $Q(q)$ to (3.6), which is a first order PDE system. The rest of the paper is dedicated to the investigation of this first order PDE, in particular in dimensions as low as 1 and 2.

4. Existence of solutions to first–order quasilinear partial differential equations in dimensions 1 and 2

In Section 3, we have explained that proving the existence of the generalized Lyapunov function is equivalent to proving the existence of solutions to equation (3.6). As we have pointed out in Section 3, the equation (3.6) is a first-order partial differential equation with the unknown anti-symmetric matrix function $Q(x)$ to be solved.

In general, it's likely to happen that the solution of quasi-linear partial differential equation does not exist. This might even be the case for linear PDEs. Actually, in 1957, H. Lewy constructed a linear equation with no singularity while surprisingly there is no solution everywhere [27]. The equation he proposed has the form

$$Lu = F, \tag{4.1}$$

where L is a linear differential operator on function $u(x,y,z)$ defined by $Lu = -u_x - iu_y + 2i(x+iy)u_x$, and function F is an appropriately selected function belonging to class C^∞.

The above discussion shows that in fact, proving the existence of solutions to equations (3.6) can be difficult and complex. Below we'd like to discuss the special cases when the problem dimension is 1 and 2. We note that for linear case, the generalized Lyapunov function always exists (see [25]), so we only consider nonlinear case later.

4.1. Dimension $n = 1$

In this case, we note that $q = q_1$. We assume $D(q) = d(q_1)$, where d is an known function obtained by equation (2.4). The antisymmetric matrix $Q(q) = 0$. Thus, equation (3.6) can be transformed into the following form

$$\frac{\partial}{\partial q_1} \times \left[\frac{f(q_1)}{d(q_1)}\right] = 0. \tag{4.2}$$

When $\vec{a}, \vec{b}$ are one dimensional vectors, the angle θ between them is 0 or 2π. According to vector cross product rule, in this case we have

$$\vec{a} \times \vec{b} = |\vec{a}| \cdot |\vec{b}| \sin\theta. \tag{4.3}$$

Therefore, the equation (4.2) holds true. Furthermore, the generalized Lyapunov function can be constructed explicitly as

$$\varphi(q_1) = \int \frac{f(q_1)}{d(q_1)} dq_1. \tag{4.4}$$

4.2. Dimension $n = 2$

In this case, $q = (q_1, q_2)$. Let us assume that the anti-symmetric matrix $Q(q)$ is given by

$$Q(q) = \begin{pmatrix} 0 & Q_{12}(q) \\ -Q_{12}(q) & 0 \end{pmatrix},$$

for some unknown function $Q_{12}(q)$. The matrix $D(q)$ is known by equation (2.4), we can assume that

$$D(q) = \begin{pmatrix} d_1(q) & d_3(q) \\ d_3(q) & d_2(q) \end{pmatrix}.$$

Note that in dimension 2 we have $\nabla \times \boldsymbol{v} = \dfrac{\partial v_2}{\partial q_1} - \dfrac{\partial v_1}{\partial q_2}$ for $\boldsymbol{v} = (v_1, v_2)$. By direct calculation, from equation (3.6) we can obtain

$$w_1(q_1, q_2, Q_{12})\frac{\partial Q_{12}}{\partial q_1} + w_2(q_1, q_2, Q_{12})\frac{\partial Q_{12}}{\partial q_2} = w_3(q_1, q_2, Q_{12}), \tag{4.5}$$

in which

$$w_1(q_1, q_2, Q_{12}) = \frac{f_1 Q_{12}^2 - 2(d_3 f_1 - d_1 f_2)Q_{12} + f_1(d_3^2 - d_1 d_2)}{(d_1 d_2 + Q_{12}^2 - d_3^2)^2},$$

$$w_2(q_1,q_2,Q_{12}) = \frac{f_2Q_{12}^2 + 2d_3f_2Q_{12} - (d_1d_2+d_3^2)f_2}{(d_1d_2+Q_{12}^2-d_3^2)^2},$$

$$\begin{aligned} w_3(q_1,q_2,Q_{12}) =& \frac{(Q_{12}f_2+d_3f_2)(d_1^{(2)}d_2+d_1d_2^{(2)}-2d_3^{(2)}d_3)(d_1^{(1)}d_2+d_1d_2^{(1)}-2d_3)}{(d_1d_2+Q_{12}^2-d_3^2)^2} \\ &- \frac{(Q_{12}f_1-d_3f_1+d_1f_2)(d_1^{(1)}d_2+d_1d_2^{(1)}-2d_3)}{(d_1d_2+Q_{12}^2-d_3^2)^2} \\ &+ \frac{Q_{12}(f_1^{(1)}-f_2^{(2)})-d_3(f_1^{(1)}+f_2^{(2)})+f_2d_3^{(2)}-d_3^{(1)}f_1}{d_1d_2+Q_{12}^2-d_3^2} \\ &+ \frac{d_1^{(1)}f_2+d_1f_2^{(1)}}{d_1d_2+Q_{12}^2-d_3^2} + f_1 - \frac{f_1^{(2)}}{d_1}. \end{aligned}$$

In the above, the superscripts $^{(1)}$ and $^{(2)}$ denote the partial derivative concerning the variable q_1 and q_2 respectively, $d_1(q), d_2(q)$ and $d_3(q)$ are known functions. According to the definition of the generalized Lyapunov function, we only consider smooth function f, so w_1, w_2, w_3 are C^1 smooth functions of q_1, q_2, Q_{12}. Note that $d_1d_3 - d_2^2 > 0$ due to positive definiteness of $D(q)$. Under these circumstances, we want to prove the existence of solutions to equation (4.5).

First of all, we consider a first order system of ordinary differential equations with two unknown functions, which takes the form

$$\begin{cases} \dfrac{\mathrm{d}y_1}{\mathrm{d}x} = f_1(x,y_1,y_2), \\ \dfrac{\mathrm{d}y_2}{\mathrm{d}x} = f_2(x,y_1,y_2). \end{cases} \tag{4.6}$$

Note that in the following statement of definitions and lemmas, we use notation x, y_1, y_2, relatively, in the statement in proof of theorem which explained the main ideas of the proof, we use notation q_1, q_2, Q_{12} in regards to the equations itself.

We will make use of the following Lemmas regarding the existence of the solution to the general first-order system of PDEs (4.6). The proofs of these lemmas can be found in [14].

Lemma 4.1. *For $p_0 = (x_0, y_1^0, y_2^0) \in G$, there exists a neighbourhood $G_0 \subset G$ of p_0, subject to that the equations (4.6) have two independent first integrals in G_0.*

Lemma 4.2. *The equation (4.6) has at most two independent first integrals.*

Lemma 4.3. *Suppose $\Psi_1(x,y_1,y_2) = c_1, \Psi_2(x,y_1,y_2) = c_2$ are two independent first integrals of differential equations (4.6) in area G. Then any first integrals of differential equations (4.6) in area G,*

$$V(x,y_1,y_2) = C,$$

can be expressed by equations $\Psi_1(x,y_1,y_2) = c_1, \Psi_2(x,y_1,y_2) = c_2$ which reads

$$V(x,y_1,y_2) = h[\Psi_1(x,y_1,y_2), \Psi_2(x,y_1,y_2)], \tag{4.7}$$

where $h[*,*]$ is a continuously differentiable function.

Lemma 4.4. *Suppose $\Phi_1(x,y_1,y_2) = c_1, \Phi_2(x,y_1,y_2) = c_2$ are two known independent first integrals of differential equations (4.6). Then we can obtain the general* *solution of equation (4.6) in region G as:*

$$y_1 = \Phi_1(x, C_1, C_2),\ y_2 = \Phi_2(x, C_1, C_2), \tag{4.8}$$

where C_1, C_2 are arbitrary constants, and these general solutions express all solutions of equations (4.6) in G.

By making use of the above Lemmas and the first integral method [14], we prove the existence of the solution to the equation (4.5) in the following theorem.

Theorem 4.1. *If $V(q_1, q_2, Q_{12}) = C$ is an implicit solution of equation (4.5), such that $\frac{\partial V}{\partial Q_{12}} \neq 0$, then the solution to equation (4.5) exists in the region G_0 which is the neighbourhood of $q_0 = (q_1^0, q_2^0, Q_{12}^0) \in G$.*

Proof. We claim that the problem of solving equation (4.5) is equivalent to the problem of solving a linear homogeneous equation. Assume $V(q_1, q_2, Q_{12}) = C$ is some implicit solution of (4.5). By the assumption in Theorem 4.1, we can utilize the implicit function differentiation rule to obtain

$$\frac{\partial Q_{12}}{\partial q_i} = -\frac{\frac{\partial V}{\partial q_i}}{\frac{\partial V}{\partial Q_{12}}}, i = 1, 2. \tag{4.9}$$

Then we substitute equation (4.9) into equation (4.5) to get

$$w_1(q_1, q_2, Q_{12})\frac{\partial V}{\partial q_1} + w_2(q_1, q_2, Q_{12})\frac{\partial V}{\partial q_2} + w_3(q_1, q_2, Q_{12})\frac{\partial V}{\partial Q_{12}} = 0, \tag{4.10}$$

where V is viewed as function of q_1, q_2, Q_{12}. Thus we see that the equation (4.5) is transformed into first-order homogeneous linear partial differential equation (4.10) about the unknown function $V(q_1, q_2, Q_{12})$.

To prove the reverse statement, assume that the function $V(q_1, q_2, Q_{12})$ is the solution of the equation (4.9). By the assumption in Theorem 4.1, we know that $\frac{\partial V}{\partial Q_{12}} \neq 0$. Thus we can rearrange (4.9) and make use of (4.9), to obtain that

$$Q_{12} = Q_{12}(q_1, q_2)$$

is the solution of the equation (4.5). So we have proved the equivalence of (4.5) and (4.9).

The above guarantees that we just need to prove the existence of solution to (4.9). To this end we consider the characteristic equation corresponding to the equation (4.9) along the fixed characteristic line, which has the form

$$\frac{dq_1}{w_1(q_1, q_2, Q_{12})} = \frac{dq_2}{w_2(q_1, q_2, Q_{12})} = \frac{dQ_{12}}{w_3(q_1, q_2, Q_{12})}. \tag{4.11}$$

The above is equivalent to the following ordinary differential equations in two dimensions:

$$\begin{cases} \dfrac{\mathrm{d}q_2}{\mathrm{d}q_1} = \dfrac{w_2(q_1, q_2, Q_{12})}{w_1(q_1, q_2, Q_{12})}, \\ \dfrac{\mathrm{d}Q_{12}}{\mathrm{d}q_1} = \dfrac{w_3(q_1, q_2, Q_{12})}{w_1(q_1, q_2, Q_{12})}. \end{cases} \tag{4.12}$$

According to Lemmas 4.1, 4.2 and 4.3, assume that $q_0 = (q_1^0, q_2^0, Q_{12}^0) \in G$, there exists a neighborhood $G_0 \subset G$ of q_0, such that ordinary differential equations (4.12) has only two independent first integral in region G_0, denoted as $\Psi_1(q_1, q_2, Q_{12}) = C_1, \Psi_2(q_1, q_2, Q_{12}) = C_2$. Then according to Lemma 4.4, the general solution of the equation (4.9) can be written as

$$V(q_1, q_2, Q_{12}) = \Phi(\Psi_1(q_1, q_2, Q_{12}), \Psi_2(q_1, q_2, Q_{12})), \tag{4.13}$$

where Φ is a continuous differential function. It must be noted that this solution is an implicit solution. □

In order to seek for an explicit solution, we consider the corresponding Cauchy problem of the equation (4.5). We formulae the Cauchy problem as the following system of equations:

$$\begin{cases} w_1(q_1, q_2, Q_{12})\dfrac{\partial Q_{12}}{\partial q_1} + w_2(q_1, q_2, Q_{12})\dfrac{\partial Q_{12}}{\partial q_2} = w_3(q_1, q_2, Q_{12}), \\ Q_{12}(g(s), h(s)) = l(s). \end{cases} \tag{4.14}$$

In the above equation (4.14), it is assumed that there is a given curve Γ in the space (q_1, q_2, z), whose parametric equation is

$$q_1 = g(s), q_2 = h(s), z = l(s). \tag{4.15}$$

From (4.14), we aim at finding a solution of the form $Q_{12}(q_1, q_2)$, such that along Γ this function satisfies the condition

$$l(s) = Q_{12}(g(s), h(s)). \tag{4.16}$$

Then we want to prove the existence of solution to the Cauchy problem of equation (4.5) in the neighborhood of Γ. Because Γ can be covered by several finite open arcs on itself, we only need to prove the existence of solution near each open arc. Before we state the next Theorem, we would like to first propose the following

Assumption 4.1. *We assume that*

(i) *The functions $g(s), h(s), l(s)$ of the curve Γ defined in the neighbourhood of s_0 are C^1. We let $P_0 = (q_1^0, q_2^0, z_0) = (g(s_0), h(s_0), l(s_0))$;*

(ii) *In a neighbourhood of the point P_0, the coefficients of equation (4.5) w_1, w_2, w_3 are C^1;*

(iii) *At the point $(s_0, 0)$, we have*

$$\begin{vmatrix} g'(s_0) & h'(s_0) \\ w_1(q_1^0, q_2^0, z_0) & w_2(q_1^0, q_2^0, z_0) \end{vmatrix} \neq 0. \tag{4.17}$$

Based on the above assumption, we have

Theorem 4.2. *Under Assumption 4.1 (i)-(iii), the solution to the equation*

$$\begin{cases} w_1(q_1, q_2, Q_{12})\dfrac{\partial Q_{12}}{\partial q_1} + w_2(q_1, q_2, Q_{12})\dfrac{\partial Q_{12}}{\partial q_2} = w_3(q_1, q_2, Q_{12}), \\ Q_{12}(g(s), h(s)) = l(s), \end{cases} \tag{4.18}$$

exists in a neighbourhood of parameter $s = s_0$. This local solution can be extended to a global solution by excluding some singularities [8, 23, 34].

Proof. Intuitively, the integral surface $z = u(q_1, q_2)$ across Γ consists of characteristic curves of any point on Γ. Assume for every s near s_0, the local solutions of the characteristic equation takes the parametric form

$$q_1 = Q^1(s,t),\ q_2 = Q^2(s,t),\ Q_{12} = Q^{12}(s,t). \tag{4.19}$$

So we only need to find solution whose values are correspondingly $g(s), h(s), l(s)$ for (4.19) when $t = 0$ and satisfying the characteristic differential equation (4.11). Obviously, the functions Q^1, Q^2, Q^{12} about s, t satisfy

$$\frac{\partial Q^1}{\partial t} = a(Q^1, Q^2, Q^{12}),\ \frac{\partial Q^2}{\partial t} = b(Q^1, Q^2, Q^{12}),\ \frac{\partial Q^{12}}{\partial t} = c(Q^1, Q^2, Q^{12}) \tag{4.20}$$

and initial value condition

$$Q^1(s,0) = g(s),\ Q^2(s,0) = h(s),\ Q^{12}(s,0) = l(s). \tag{4.21}$$

From existence and uniqueness theorem of solutions of ordinary differential equations and continuous dependence theorem on parameters of solutions of ordinary differential equations, we can obtain there exist unique functions $Q^1(s,t), Q^2(s,t)$, $Q^{12}(s,t)$ is C^1 in a neighbourhood of the point $(s_0, 0)$ and satisfies equations (4.20) and (4.21). By Assumption 4.1 part (i) and initial value condition (4.21), we have

$$q_1^0 = Q^1(s_0,0),\ q_2^0 = Q_2(s_0,0). \tag{4.22}$$

Utilizing (4.20) and (4.21) and Assumption 4.1 part (iii), we can easily obtain

$$\begin{vmatrix} Q_s^1(s_0,0) & Q_s^2(s_0,0) \\ Q_t^1(s_0,0) & Q_t^2(s_0,0) \end{vmatrix} \neq 0. \tag{4.23}$$

Therefore, by the existence theorem of implicit function, in a neighbourhood of (x_0, y_0), we can deduce from

$$q_1 = Q^1(s,t),\ q_2 = Q^2(s,t) \tag{4.24}$$

to find reverse solution s, t, written as

$$s = S(q_1, q_2),\ t = T(q_1, q_2). \tag{4.25}$$

Then (4.19) can represent a surface $\Sigma : Q_{12} = u(q_1, q_2)$ expressed by parameters s, t. The function u defined by

$$Q_{12} = u(q_1, q_2) = Q^{12}(S(q_1,q_2), T(q_1,q_2)) \tag{4.26}$$

is the explicit representation of surface Σ. By statement mentioned before, the assumption (iii) guarantee that (4.19) locally represents a surface $\Sigma : Q_{12} = u(q_1, q_2)$. Futhermore, from parameter expression (4.19) we can easily obtain Σ is the integral surface. Therefore, u is the solution of equation (4.5). □

If condition (4.17) is not satisfied, it would lead to sigularities. If the determinant in (4.17) is equal to 0, it will lead to the contradiction with existence and uniqueness of solution. To see it, from equation $h(s) = u(f(s), g(s))$ and (4.5), at $s = s_0, q_1 = g(s_0), q_2 = h(s_0)$,

$$w_2 g' - w_1 h' = 0, l' = g' \frac{\partial Q_{12}}{\partial q_1} + h' \frac{\partial Q_{12}}{\partial q_2}, w_3 = w_1 \frac{\partial Q_{12}}{\partial q_1} + w_2 \frac{\partial Q_{12}}{\partial q_2}. \tag{4.27}$$

Therefore,

$$w_2 l' - w_3 h' = 0, w_1 l' - w_3 g' = 0, \tag{4.28}$$

This equation means g', h', l' is in proportion to w_1, w_2, w_3. Because Γ hasn't the characteristic direction at the point $(s_0, 0)$, the solution doesn't exist.

Discussion about Assumption 4.1.

It is obvious that (i) and (ii) in Assumption 4.1 always hold. For (iii), Zhu et al. are the first to propose the existence of Lyapunov function in dynamical system with limit cycle [44]. They consider the stochastic dynamical system

$$\begin{cases} \dot{q_1} = -q_2 + q_1(1 - q_1^2 - q_2^2) + \xi_1(t), \\ \dot{q_2} = q_1 + q_2(1 - q_1^2 - q_2^2) + \xi_2(t), \end{cases} \tag{4.29}$$

where friction matrix and transverse matrix are $S(q) = \frac{(1-q_1^2-q_2^2)^2}{(1-q_1^2-q_2^2)^2+1} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $A(q) = \frac{1-q_1^2-q_2^2}{(1-q_1^2-q_2^2)^2+1} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, respectively, and

$$D(q) + Q(q) = (S(q) + A(q))^{-1} = \begin{pmatrix} 1 & \frac{1}{1-q_1^2-q_2^2} \\ -\frac{1}{1-q_1^2-q_2^2} & 1 \end{pmatrix}. \tag{4.30}$$

Therefore, the diffusion matrix defined before is $D(q) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, the antisymmetric matrix defined before is $Q(q) = \begin{pmatrix} 0 & \frac{1}{1-q_1^2-q_2^2} \\ -\frac{1}{1-q_1^2-q_2^2} & 0 \end{pmatrix}$, so the solution of PDE (4.5) is $Q_{12}(q) = \frac{1}{1-q_1^2-q_2^2}$. It's very interesting that the limit cycle $q_1^2 + q_2^2 = 1$ exactly corresponds to the non-existence of solution by calculation

$$\begin{aligned} w_1(q_1, q_2, Q_{12}) &= \frac{q_2}{(1 - q_1^2 - q_2^2)^2} + \frac{q_1}{1 - q_1^2 - q_2^2} + q_2 + q_1(1 - q_1^2 - q_2^2), \\ w_2(q_1, q_2, Q_{12}) &= \frac{-q_1}{(1 - q_1^2 - q_2^2)^2} + \frac{q_2}{1 - q_1^2 - q_2^2} + q_1 + q_2(1 - q_1^2 - q_2^2). \end{aligned} \tag{4.31}$$

Then utilizing equation (4.17), we find another special case

$$\begin{aligned} &g'(s_0)[\frac{-q_1}{(1 - q_1^2 - q_2^2)^2} + \frac{q_2}{1 - q_1^2 - q_2^2} + q_1 + q_2(1 - q_1^2 - q_2^2)] \\ &= h'(s_0)[\frac{q_2}{(1 - q_1^2 - q_2^2)^2} + \frac{q_1}{1 - q_1^2 - q_2^2} + q_2 + q_1(1 - q_1^2 - q_2^2)], \end{aligned} \tag{4.32}$$

illustrating that the solution does not exist. Except for these two special cases, the solution exists locally.

5. Summary

In this paper, we introduced the notion of a generalized Lyapunov function and we demonstrated that it is equivalent to Ao's potential function $\varphi(q)$ in Ao's decomposition theory. We find that when dimension varies, the existence interval of the generalized Lyapunov function and corresponding restricted condition varies. By using results from the solution theory of first-order partial differential equations, we show that the generalized Lyapunov function globally exists in dimension 2 excluding some singularities. When the dimension is 3 and more than 3, the existence problem deserves further study and we leave it to a future work. Our proof reveals an essential structure of general stochastic dynamical system - divergence (provided by $D(q)$), curl (provided by $Q(q)$), the gradient of generalized Lyapunov function $\nabla\varphi(q)$ and white Gaussian noise $\xi(q,t)$.

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