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A FILLED PENALTY FUNCTION METHOD FOR SOLVING CONSTRAINED OPTIMIZATION PROBLEMS*

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Abstract An important method to solve constrained optimization problem is to approach the optimal solution of constrained optimization problem gradually by sequential unconstrained optimization method, namely penalty function method. And the filling function method is one of the effective methods to solve the global optimal problem. In this paper, a class of augmented Lagrangian objective filled penalty functions are defined to solve non-convex constraint optimization problems, the authors call it filled penalty function method. The theoretical properties of these functions, such as exactness, smoothness, global convergence, are discussed. On this basis, a local optimization algorithm and an approximate global optimization algorithm with corresponding examples are given for solving constrained optimization problems.

Keywords Filled penalty function method, non-convex constrained optimization problems, globally optimal point, convergence.

MSC(2010) 90C30

1. Introduction

Consider the following non-convex constraint optimization problem

$$(P) \qquad \begin{aligned} &\min \ f(x) \\ &s.t. \ \ g_i(x) \le 0, \quad i \in I, \end{aligned}$$

where $f : R^n \to R$, $g_i : R^n \to R, i \in I$ are assumed to be continuously differentiable, $I = \{1, 2, \ldots, m\}$ is a finite set of integers, $X = \{x \in R^n | g_i(x) \le 0, i \in I\}$ is the feasible set of (P).

Penalty function method is a prevailing method to find locally optimal solutions of constrained optimization problems(see [1, 4–9, 18, 24]). Its main idea is to transfer a constrained optimization problem into an unconstrained optimization problem. In most cases, for the traditional interior point penalty function and exterior point penalty function, the optimal solutions of constrained optimization problem can

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be obtained only when the penalty parameter of the traditional penalty function approaches infinity. Based on traditional penalty functions, Hestenes(1969) and Powell(1969) first replaced the objective function of equality constrained optimization problem with Lagrangian function, so as to obtain the augmented Lagrangian penalty function. Later, Rockafeller(1973) extended it to inequality constrained optimization problems and established the augmented Lagrangian penalty function for constrained optimization problems.

Augmented Lagrangian penalty function is composed of two part: Lagrangian function with a Lagrangian parameter and penalty function with a penalty parameter. An augmented Lagrangian method using the exponential penalty function was proposed in Echebest etc [2]

$$L(x, \mu, \rho) = f(x) + \sum_{i=1}^{m} \frac{\mu_i}{\rho}(e^{\rho g_i(x)} - 1)$$

where $\rho > 0$, $\mu \in R_+^m$. The boundedness of the penalty parameters is proved under classical conditions.

In recent years, many articles ([10, 12–14, 25]) on objective penalty functions have been published. Meng etc [15] defined an objective penalty function

$$E(x, M) = Q(f(x) - M) + \sum_{i=1}^{m} P(g_i(x)),$$

where $M \in R$ is the objective penalty parameter, $Q(t)$ and $P(t)$ are continuous differentiable functions with its own properties: $Q(t) > 0$ and $P(t) > 0$ are monotonically increasing for $t > 0$ and $Q(t) = P(t) = 0$ for $t \leq 0$. They obtained the conclusion that the optimal point of this penalty problem is the optimal point of problem (P).

After word, Zheng etc [26] proposed an augmented Lagrangian penalty function

$$L_M(x, u, v) = Q(f(x) - M) + u^T G(x) + v^T H(x), \quad x \in R^n, u, v \in R_+^m,$$

where $M \in R$ is the objective parameter, u and v are Lagrangian parameter and penalty parameter respectively. They showed the exactness of the augmented Lagrangian function and presented an algorithm to find the locally optimal point to problem (P).

In most cases, the problem (P) has more than one locally optimal value. At present, it is difficult to use the traditional deterministic optimization algorithm to determine the global optimal value. The main reasons are as follows: first, there is no conditions to judge whether the current optimal point is the globally optimal solution; the other is that it is difficult to find a feasible descending direction when the traditional algorithm obtains the locally optimal solution.

Many researchers utilize the method of filled functions to solve global optimization problems(see [3, 11, 16, 17, 20]). The main idea is to construct a filled function based on one of the locally optimal solutions of the original problem and then minimize the filled function so as to find the locally best point of the original problem which is better than the current locally best point. Then repeat this process to find a better locally optimal solutions than the current one.

Let x_l^* be one of the locally optimal points of problem (P), a class of filled functions has been defined in Wang etc [21]

$$T_1(x, x_l^*, \tau) = \frac{\phi(\tau[f(x) - f(x_l^*) + h])}{\|x - x_l^*\|},$$

where $\tau \geq 1$ and $h > 0$ are parameters, the function $\phi(\cdot)$ satisfies: $\phi(0) = 0$; for any $t \in [-t_1, \infty)$, $\phi'(t) > 0$ (where $t_1 \geq 0$); $\lim\limits_{t \to +\infty} \frac{t\phi'(t)}{\phi(t)} = 0$. The authors found another locally optimal point $\overline{x}$ of (P) by minimizing $T_1(x, x_l^*, \tau)$ and the new locally optimal point $\overline{x}$ satisfies that $f(\overline{x}) < f(x_l^*)$. Then, let point $\overline{x}$ be the new start point to find the next locally optimal point.

In this paper, a class of augmented Lagrangian objective penalty functions is introduced to find the locally optimal solutions for inequality constrained optimization problems. Based on the locally optimal points, a new class of augmented Lagrangian objective filled penalty functions is proposed to find an approximately global minimizers of the non-convex constrained optimization problems. A local search method and a global search method based on these functions are presented respectively. Meanwhile, the convergence theorems based on these two algorithms are proved. Finally, numerical experiments are listed to explain the rationality of the two optimization algorithms.

2. Augmented Lagrangian objective penalty function method

In this section, we establish a class of augmented Lagrangian objective penalty function to solve local optimization problems, and then a local search method based on these functions is proposed. In order to analyze the properties of these functions, we make the following assumption.

Assumption 2.1. $\lim\limits_{\|x\| \to +\infty} f(x) = +\infty$.

Let $L(P)$ be the set of locally optimal points of problem (P) and $G(P)$ be the set of globally optimal points of problem (P). According to Assumption 2.1, a closed bounded domain Y such that $G(P) \subseteq Y$ can be found and we only need to consider the following optimization problem

$$(EP) \qquad \begin{aligned} \min\ & f(x) \\ s.t.\ & g_i(x) \leq 0, \quad i \in I, \\ & x \in Y. \end{aligned}$$

Let $G(EP)$ be the set of globally optimal points of problem (EP), $G(EP) = G(P)$ holds.

Two classes of continuously differentiable functions $Q(\cdot)$ and $P(\cdot)$ are given.

$Q(t)$ is a continuously differentiable function satisfying

(1) $Q'(t) > 0$, $t \in (0, +\infty)$;

(2) $Q(t) = 0$ for per $t \leq 0$;

(3) $\lim_{t\to+\infty} Q(t) = +\infty$.

$P(t)$ is a continuously differentiable function satisfying

(1) $P'(t) > 0$, $t \in (0, +\infty)$;

(2) $P(t) = 0$ for per $t \leq 0$;

(3) $\lim\limits_{t\to+\infty} \dfrac{P(t)}{t} = 1$.

For example, $Q(t)=\max\{0,t\}^3$, $P(t)=\begin{cases} t+e^{-t}-1, & t\geq 0 \\ 0, & t<0; \end{cases}$ and $P(t)=\begin{cases} \sqrt{t^2+4}-2, & t\geq 0 \\ 0, & t<0. \end{cases}$

From the properties of $P(\cdot)$, it is easy to obtain the following results

(4) $\gamma P(\frac{t}{\gamma}) \geq P(t), \gamma > 0, t \in R$;

(5) $\lim\limits_{\gamma\to 0^+} \gamma P(\frac{t}{\gamma}) = t^+$, where $\gamma > 0$, $t^+ = \max\{t, 0\}$.

Based on these two functions, the augmented Lagrangian objective penalty function can be defined as

$$L(x, M, u, \beta) = Q(f(x) - M) + \sum_{i=1}^{m} \max\{u_i g_i(x), 0\}^3 + \beta\gamma \sum_{i=1}^{m} P(\frac{g_i(x)}{\gamma}),$$

where $M \in R$ is the objective parameter, $u \in R_+^m$ is the Lagrangian parameter, $\beta > 0$ is the penalty parameter, and $\gamma > 0$ satisfies $\beta\gamma > 1$. Then we can solve this unconstrained penalty optimization problem

$$(LOP) \quad \min_{x\in Y} L(x, M, u, \beta).$$

Definition 2.1. Let x_M^* be an optimal point of (LOP). If there is an $M^{'} < 0$ such that x_M^* is an optimal point for (P) for all $M \leq M^{'}$, then $L(x, M, u, \beta)$ is an exact Lagrangian objective penalty function and M is an exact value of Lagrangian objective penalty parameter.

Theorem 2.1. *Let x_l^* be an optimal point of (P) and x_M^* be an optimal point of (LOP). If x_M^* is feasible for (P) and $M \leq f(x_l^*)$, x_M^* is an optimal point of (P).*

Proof. Since x_M^* is feasible for (P) and $M \leq f(x_l^*)$, there is $M \leq f(x_l^*) \leq f(x_M^*)$. This implies that $0 \leq f(x_l^*) - M$ and $0 \leq f(x_M^*) - f(x^*) \leq f(x_M^*) - M$.

It follows from x_M^* is an optimal point for (LOP) that

$$Q(f(x_M^*) - M) \leq Q(f(x_l^*) - M).$$

Thus, $0 \leq f(x_M^*) - M \leq f(x_l^*) - M$, which implies that $f(x_M^*) \leq f(x_l^*)$, i.e., x_M^* is an optimal point of (P). □

Theorem 2.2. *Let x_M^* be a locally optimal point of (LOP) and $x^* \in G(P)$. The following statements hold*

(i) If $L(x_M^*, M, u, \beta) = 0$, then x_M^* is feasible to (P) and $f(x^*) < f(x_M^*) \leq M$.

(ii) *If $L(x_M^*, M, u, \beta) > 0$ and x_M^* is not feasible to (P), then $M < f(x)$, $f(x_M^*) < f(x)$ for all $x \in N(x_M^*) \cap X$, where $N(x_M^*)$ is a neighborhood of x_M^*.*

(iii) *If $L(x_M^*, M, u, \beta) > 0$ and x_M^* is feasible to (P), then x_M^* is a locally optimal point of (P).*

Proof. (i) The conclusion is obvious from the definition of $L(x_M^*, M, u, \beta)$.

(ii) For all $x \in N(x_M^*) \cap X$, there is

$$0 < L(x_M^*, M, u, \beta) \le L(x, M, u, \beta) = Q(f(x) - M).$$

From the definition of $Q(\cdot)$, we obtain $M < f(x)$. If $f(x_M^*) \le M$, then $f(x_M^*) \le f(x)$. If $f(x_M^*) > M$, there is

$$0 < Q(f(x_M^*) - M) < L(x_M^*, M, u, \beta) \le L(x, M, u, \beta) = Q(f(x) - M).$$

Thus, $f(x_M^*) < f(x)$ for all $x \in N(x_M^*) \cap X$.

(iii) Since $L(x_M^*, M, u, \beta) > 0$ and x_M^* is feasible to (P),

$$0 < Q(f(x_M^*) - M) = L(x_M^*, M, u, \beta) \le L(x_M^*, M, u, \beta) = Q(f(x) - M), \forall x \in N(x_M^*) \cap X.$$

It implies that $f(x_M^*) \le f(x)$ for all $x \in N(x_M^*) \cap X$. Hence, x_M^* is a locally optimal point of (P). $\square$

Theorem 2.2 shows that the local minimizers of (LOP) is also the local minimizers of (P) under some conditions. Based on this theorem, an algorithm can be introduced to compute the locally optimal points of (P), which is called Augmented Lagrangian Objective Penalty Function Algorithm (ALOP Algorithm, for short).

Let the victor-valued function $\mathbf{g}(x) = [g_1(x), \cdots, g_m(x)]^T$,

$$\|\mathbf{g}(x)\|_\infty = \max\{|g_1(x)|, \cdots, |g_m(x)|\}.$$

ALOP Algorithm.

Step 1: Choose $x_0 \in Y$, $0 < \varepsilon < 1$, $N > 1$, $p < 1$. Given $u_{i,1} > 1, i \in I$, $\beta_1 > 1$, $\gamma_1 < 1$ (there is $\beta_1\gamma_1 > 1$), $a_1 < \min_{x\in X} f(x) < b_1$, $M_1 = \dfrac{a_1 + b_1}{2}$, let $k := 1$.

Step 2: Solve $\min_{x\in Y} L(x, M_k, u_k, \beta_k)$ starting at x_{k-1}, let x_k be its locally optimal solution.

Step 3: If $L(x, M_k, u_k, \beta_k) = 0$, go to Step 4; otherwise, go to Step 5.

Step 4: Let $a_{k+1} = a_k$, $b_{k+1} = M_k$, $M_{k+1} = \dfrac{a_{k+1} + b_{k+1}}{2}$, $\gamma_{k+1} = p\gamma_k$, $k := k+1$, go to Step 2.

Step 5: If x_k is not feasible to (P), go to Step 6; otherwise, go to Step 7.

Step 6: Let $a_{k+1} = M_k$, $b_{k+1} = b_k$, $M_{k+1} = \dfrac{a_{k+1} + b_{k+1}}{2}$. If $\|\mathbf{g}(x_k)\|_\infty > \|\mathbf{g}(x_{k-1})\|_\infty$ or $\|\mathbf{g}(x_{k-1})\|_\infty \ge \|\mathbf{g}(x_k)\|_\infty > \dfrac{1}{4}\|\mathbf{g}(x_{k-1})\|_\infty$, $u_{k+1} = Nu_k$, $\beta_{k+1} = N\beta_k$, $\gamma_{k+1} = p\gamma_k$, $k := k+1$, and go to Step 2; if $\|\mathbf{g}(x_k)\|_\infty \le \dfrac{1}{4}\|\mathbf{g}(x_{k-1})\|_\infty$, $u_{k+1} = Nu_k$, $\beta_{k+1} = \beta_k$, $\gamma_{k+1} = p\gamma_k$, $k := k+1$, and go to Step 2.

Step 7: Stop and x_k is a locally optimal point of (P).

Remark 2.1. In Step 1, it is assumed that one can always obtain $a_1 < \min_{x\in X} f(x)$ and any efficient methods available can be used in Step 2. In the process of algorithm iteration, we modify the penalty parameter and Lagrangian parameter alternately based on the following considerations.

(1) If $\|\mathbf{g}(x_k)\|_\infty > \|\mathbf{g}(x_{k-1})\|_\infty$ or $\|\mathbf{g}(x_{k-1})\|_\infty \geq \|\mathbf{g}(x_k)\|_\infty > \frac{1}{4}\|\mathbf{g}(x_{k-1})\|_\infty$, the current iteration point has a tendency to deviate from the feasible region or the trend of current iteration point approaching the feasible region is not obvious. It can be obtained that both cases are related to unadjusted the penalty parameter.

(2) If $\|\mathbf{g}(x_k)\|_\infty \leq \frac{1}{4}\|\mathbf{g}(x_{k-1})\|_\infty$, the current iteration point approaches the feasible region significantly, this indicates that the penalty parameter does not need to be adjusted in the next iteration, but only Lagrangian parameter needs to be adjusted.

Theorem 2.3. *Let $\{x_k\}$ be the sequence generated by ALOP Algorithm, suppose that the sequence $\{L(x_k, M_k, u_k, \beta_k)\}$ is bounded.*

(i) If $\{x_k\}(k = 1, 2, \ldots, \overline{k})$ is a finite sequence (i.e., the ALOP Algorithm stops at the $\overline{k}$th iteration), then $x_{\overline{k}}$ is a local minimizer of (P).

(ii) If $\{x_k\}$ is an infinite sequence, then $\{x_k\}$ is bounded and for any limit point x_l^* of it, there exists $t_i \geq 0$, $i \in I$, such that

$$\begin{cases} \nabla f(x_l^*) + \sum_{i\in I} t_i \nabla g_i(x_l^*) = 0, \\ g_i(x_l^*) \leq 0, \quad t_i g_i(x_l^*) = 0. \end{cases}$$

Proof. (i) If the ALOP Algorithm stops at the $\overline{k}$th iteration and step 7 occurs, from (iii) in Theorem 2.2, $x_{\overline{k}}$ is a local minimizer of (P).

(ii) From Theorem 8 in Meng etc [15], we obtain the sequence $\{a_k\}$ increases to a^* and $\{b_k\}$ decreases to b^*, $M_k \to M^*$ with

$$a_k < M_k = \frac{a_k + b_k}{2} < b_k, \quad b_{k+1} - a_{k+1} = \frac{b_k - a_k}{2}, \ k = 1, 2, \ldots.$$

Obviously, $a^* = b^* = M^*$. Since $\{L(x_k, M_k, u_k, \beta_k)\}$ is bounded, so there is $B > 0$ such that

$$Q(f(x_k) - M_k) \leq L(x_k, M_k, u_k, \beta_k) \leq B, k = 1, 2, \ldots.$$

Thus, $\{x_k\}$ is bounded. Without loss of generality, suppose that $x_k \to x_l^*$.

Form the definition of $L(x_k, M_k, u_k, \beta_k)$, we obtain $\nabla L(x_k, M_k, u_k, \beta_k) = 0$, i.e.,

$$Q'(f(x_k) - M_k)\nabla f(x_k) + 3\sum_{i=1}^{m} \max\{u_{i,k} g_i(x_k), 0\}^2 u_{i,k} \nabla g_i(x_k)$$
$$+\beta_k \sum_{i=1}^{m} P'(\frac{g_i(x_k)}{\gamma_k})\nabla g_i(x_k) = 0$$

for $k = 1, 2, \ldots$. This is equivalent to

$$Q'(f(x_k) - M_k)\nabla f(x_k) + \sum_{i=1}^{m}(3u_{i,k} \max\{u_{i,k} g_i(x_k), 0\}^2 + \beta_k P'(\frac{g_i(x_k)}{\gamma_k}))\nabla g_i(x_k) = 0.$$

Let

$$c_k = 1 + 3u_{i,k} \max\{u_{i,k} g_i(x_k), 0\}^2 + \beta_k P'(\frac{g_i(x_k)}{\gamma_k}),$$

then $c_k > 0$, and it follows that

$$\frac{1}{c_k}Q^{'}(f(x_k) - M_k)\nabla f(x_k)$$
$$+\sum_{i=1}^{m}\frac{3u_{i,k}\max\{u_{i,k}g_i(x_k),0\}^2 + \beta_k P'(\frac{g_i(x_k)}{\gamma_k})}{c_k}\nabla g_i(x_k) = 0.$$

Let

$$\eta_k = \frac{1}{c_k};\quad \lambda_k = \frac{1}{c_k}Q'(f(x_k) - M_k);\quad \omega_{i,k} = 0, i \in I_1 = \{i|g_i(x_k) \le 0, i \in I\};$$
$$\omega_{i,k} = \frac{3u_{i,k}^3 g_i(x_k)^2 + \beta_k P'(\frac{g_i(x_k)}{\gamma_k})}{c_k}, i \in I_2 = \{i|g_i(x_k) > 0, i \in I\}.$$

For any k and $i \in I$, we have

$$\eta_k + \sum_{i\in I}\omega_{i,k} = 1;\quad \omega_{i,k} \ge 0.$$

Note that $\eta_k \to \eta \in (0,1), \omega_{i,k} \to \omega_i \in [0,1)$, $M_k \to M^*$ as $k \to \infty$, $\forall i \in I$. Since x_k is bounded, $f(x)$ and $Q(\cdot)$ are continuous differentiable functions, then $Q(f(x_k) - M_k) \to Q(f(x_l^*) - M^*)$ and $\lambda_k \to \lambda > 0$. Thus,

$$\lambda\nabla f(x_l^*) + \sum_{i\in I}\omega_i\nabla g_i(x_l^*) = 0,$$

which implies

$$\nabla f(x_l^*) + \sum_{i\in I}\frac{\omega_i}{\lambda}\nabla g_i(x_l^*) = 0.$$

Let $t_i = \dfrac{\omega_i}{\lambda} \ge 0$, we obtain

$$\nabla f(x_l^*) + \sum_{i\in I}t_i\nabla g_i(x_l^*) = 0.$$

Since $\{L(x_k, M_k, u_k, \beta_k)\}$ is bounded, so there is $C > 0$ such that

$$\beta_k\gamma_k\sum_{i=1}^{m}P(\frac{g_i(x_k)}{\gamma_k}) \le L(x_k, M_k, u_k, \beta_k) \le C, k = 1, 2, \ldots.$$

Let $k \to \infty$, $\gamma_k \to 0$, $\beta_k\gamma_k > 1$, we obtain $g_i(x_k) \le 0$, i.e., $g_i(x_l^*) \le 0$. Meantime, there is $\omega_{i,k} = 0$, $t_i = 0$ where $i \in I_1$. Hence, $t_i g_i(x^*) = 0$ holds. □

From now on, we introduce a class of augmented Lagrangian objective penalty functions and give an algorithm to find local optimal points of (P). Based on the locally optimal points, we can propose a class of augmented Lagrangian objective filled penalty functions to get the approximate global minimizers of (P).

3. Augmented Lagrangian objective filled penalty function method

In order to find an approximate global optimum point, we utilize the method of filled functions to introduce an augmented Lagrangian objective filled penalty function. The new objective filled penalty function might allow one to escape from the current locally optimal point to a better local optimal point. Based on this new filled penalty function, a global search method is presented and a convergence theorem for this algorithm is given.

Definition 3.1. Let x_l^* be one of the locally optimal points of (P), i.e., $x_l^* \in L(P)$. $S_1(x_l^*) = \{x|f(x) \geq f(x_l^*), g_i(x) \leq 0, i = 1, 2, \dots, m\}$ and $S_2(x_l^*) = \{x|f(x) < f(x_l^*), g_i(x) \leq 0, i = 1, 2, \dots, m\}$. A function $T(x, x_l^*)$ is called a modified filled function of $f(x)$ at x_l^*, if the following conditions hold.

- (i) $T(x, x_l^*)$ has no stationary points in the region $S_1(x_l^*)$ except the prefixed point x_l^*;
- (ii) If $x_l^* \in L(P)$ but $x_l^* \notin G(P)$, there is a point $\widehat{x} \in S_2(x_l^*)$ such that $\widehat{x}$ is a minimizer of $T(x, x_l^*)$.

Let x_l^* be a locally optimal point of problem (P), a class of augmented Lagrangian objective filled penalty functions can be expressed as

$$
\begin{aligned}
&L(x, x_l^*, M, u, \beta)\\
=&Q(f(x) - M) + \sum_{i=1}^{m} \max\{u_i g_i(x), 0\}^3 + \max\{u_{m+1}(f(x) - f(x_l^*) + \theta), 0\}^3\\
&+ \beta\gamma \sum_{i=1}^{m} P(\frac{g_i(x)}{\gamma}) + \beta\gamma P(\frac{h(x, x_l^*, \gamma)}{\gamma}),
\end{aligned}
$$

where

$$h(x, x_l^*, \gamma) = \frac{f(x) - f(x_l^*) + \theta}{\gamma + \frac{1}{\gamma}\|x - x_l^*\|},$$

$u_{m+1} > 1$, $0 < \gamma < \theta < \min\{|f(x_1^*) - f(x_2^*)|| : f(x_1^*) \neq f(x_2^*); x_1^*, x_2^* \in L(P)\}$, and the definitions of $Q(\cdot)$, $P(\cdot)$, $u_i, i \in I, \beta, \gamma$ are the same as in Section 2.

Under Assumption 2.1, there is a closed bounded domain Y such that $G(P) \subseteq Y$, so the filled penalty problem becomes

$$(GOP) \quad \min_{x \in Y} L(x, x_l^*, M, u, \beta).$$

We devote to using this filled penalty problem to find a better locally optimal point of (P) than the current locally optimal point x_l^*.

Definition 3.2. Let x_M^* be an optimal point of (GOP). If there is an $M' < 0$ such that $x_M^* \in S_2(x_l^*)$ for all $M \leq M'$, then $L(x, x_l^*, M, u, \beta)$ is an exact Lagrangian objective filled penalty function and M is an exact value of Lagrangian objective filled penalty parameter.

Theorem 3.1. *Let $x_g^* \in S_2(x_l^*)$ be an optimal point of (P) and x_M^* be an optimal point of (GOP). If $x_M^* \in S_2(x_l^*)$ and $M \leq f(x_g^*)$, then x_M^* is an optimal point of (P) which is better than x_l^*.*

The proof of this theorem is similar to that of Theorem 2.1 and is thus omitted.

Theorem 3.2. *Let x_M^* be a locally optimal point of (GOP) and $x^* \in G(P)$. The following statements hold*

(i) *If $L(x_M^*, x_l^*, M, u, \beta) = 0$, then $x_M^* \in S_2(x_l^*)$ and $f(x^*) < f(x_M^*) \leq M$.*

(ii) *If $L(x_M^*, x_l^*, M, u, \beta) > 0$ and $x_M^* \notin S_2(x_l^*)$, then $M < f(x)$, $f(x_M^*) < f(x)$ for all $x \in N(x_M^*) \cap X$, where $N(x_M^*)$ is a neighborhood of x_M^*.*

(iii) *If $L(x_M^*, x_l^*, M, u, \beta) > 0$ and $x_M^* \in S_2(x_l^*)$, then x_M^* is a locally optimal point of (P) which is better than x_l^*.*

The proof of this theorem is similar to that of Theorem 2.2 and is thus omitted.

Next, we will analyze filling properties of the augmented Lagrangian objective penalty function $L(x, x_l^*, M, u, \beta)$.

Theorem 3.3. *For sufficiently large $u, \beta > 1$ and sufficiently small $\gamma > 0$, where β and γ are two constants satisfies $\beta\gamma > 1$, then the penalty function $L(x, x_l^*, M, u, \beta)$ has no stationary points in the region $S_1(x_l^*)$ except the prefixed point x_l^*.*

Proof. For any $x \in S_1(x_l^*) \setminus x_l^*$, there is

$$\begin{aligned}
&\frac{(x - x_l*)^T}{\|x - x_l^*\|} \nabla L(x, x_l^*, M, u, \beta) \\
&= \frac{(x - x_l*)^T}{\|x - x_l^*\|} (Q'(f(x) - M)\nabla f(x) + 3\sum_{i=1}^{m} \max\{u_i g_i(x), 0\}^2 u_i \nabla g_i(x) \\
&\quad + 3u_{m+1} \max\{u_{m+1}(f(x) - f(x_l^*) + \theta), 0\}^2 \nabla f(x) \\
&\quad + \beta \sum_{i=1}^{m} P'(\frac{g_i(x)}{\gamma}) \nabla g_i(x) + \beta \sum_{i=1}^{m} P'(\frac{h(x, x_l^*, \gamma)}{\gamma}) h'(x, x_l^*, \gamma)) \\
&= \frac{(x - x_l*)^T}{\|x - x_l^*\|} (Q'(f(x) - M)\nabla f(x) + 3u_{m+1}^3 (f(x) - f(x_l^*) + \theta)^2 \frac{(x - x_l*)^T}{\|x - x_l^*\|} \nabla f(x) \\
&\quad + \beta \sum_{i=1}^{m} P'(\frac{h(x, x_l^*, \gamma)}{\gamma}) h'(x, x_l^*, \gamma))
\end{aligned}$$

$$\begin{aligned}
&= \frac{(x - x_l*)^T}{\|x - x_l^*\|} Q'(f(x) - M)\nabla f(x) + 3u_{m+1}^3 (f(x) - f(x_l^*) + \theta)^2 \frac{(x - x_l*)^T}{\|x - x_l^*\|} \nabla f(x) \\
&\quad + \frac{\beta\gamma}{\gamma^2 + \|x - x_l^*\|} \sum_{i=1}^{m} P'(\frac{1}{\gamma} h(x, x_l*, \gamma))[\frac{(x - x_l*)^T}{\|x - x_l^*\|} \nabla f(x) - \frac{1}{\gamma} h(x, x_l^*, \gamma)].
\end{aligned}$$

Since $Q(\cdot), f(\cdot)$ are continuously differentiable, $|\frac{(x-x_l*)^T}{\|x-x_l^*\|} Q'(f(x) - M)\nabla f(x)| \leq \|Q'(f(x) - M)\| \|\nabla f(x)\|$, and $\frac{1}{\gamma} h(x, x_l*, \gamma)$ is goes to a constant. At the same time, $\beta\gamma > 1$ is a constant. Both of the first term and the third term are constants, while the second term goes to infinity because of $u_{m+1} \to \infty$.

Then, $\frac{(x-x_l*)^T}{\|x-x_l^*\|} \nabla L(x, x_l^*, M, u, \beta) > 0$, when u_{m+1} is sufficiently large. This means $L(x, x_l^*, M, u, \beta)$ has no stationary points in the region $S_1(x_l^*) \setminus x_l^*$. □

Theorem 3.4. *If $x_l^* \in L(P)$, but $x_l^* \notin G(P)$, then filled penalty function $L(x, x_l^*, M, u, \beta)$ has a minimizer $\overline{x} \in S_2(x_l^*)$.*

Proof. Let $x^* \in G(P)$. Since $L(x, x_l^*, M, u, \beta)$ is continuously differentiable on X, there is a minimizer $\overline{x} \in X$ of $L(x, x_l^*, M, u, \beta)$ such that

$$\begin{cases} L(\overline{x}, x_l^*, M, u, \beta) \le L(x^*, x_l^*, M, u, \beta); \\ f(x^*) \le f(\overline{x}). \end{cases}$$

Recalling the definition of $L(\cdot)$,$Q(\cdot)$ and $P(\cdot)$, we obtain

$$\begin{aligned} & Q(f(\overline{x}) - M) + u_{m+1}^3 \max\{f(\overline{x}) - f(x_l^*) + \theta, 0\}^3 + \beta\gamma P(\frac{h(\overline{x}, x_l^*, \gamma)}{\gamma}) \\ \le\ & Q(f(x^*) - M) + u_{m+1}^3 \max\{f(x^*) - f(x_l^*) + \theta, 0\}^3 + \beta\gamma P(\frac{h(x^*, x_l^*, \gamma)}{\gamma}), \end{aligned}$$

and

$$Q(f(x^*) - M) \le Q(f(\overline{x}) - M).$$

then we get

$$\begin{aligned} & u_{m+1}^3 \max\{f(\overline{x}) - f(x_l^*) + \theta, 0\}^3 + \beta\gamma P(\frac{h(\overline{x}, x_l^*, \gamma)}{\gamma}) \\ \le\ & u_{m+1}^3 \max\{f(x^*) - f(x_l^*) + \theta, 0\}^3 + \beta\gamma P(\frac{h(x^*, x_l^*, \gamma)}{\gamma}) = 0, \end{aligned}$$

it implies

$$f(\overline{x}) - f(x_l^*) + \theta \le 0,$$

i.e., $f(\overline{x}) < f(x_l^*)$. Hence, the function $L(x, x_l^*, M, u, \beta)$ has a minimizer $\overline{x} \in S_2(x_l^*)$. □

Theorems 3.3 and 3.4 show that the penalty function $L(x, x_l^*, M, u, \beta)$ has those two filling properties. If x_l^* is not a globally optimal point, then the new function can be used to find an optimal point with smaller value of objective function $f(x)$.

Based on one of the locally optimal points of problem (P), a global optimization algorithm is given below. The algorithm is called the Augmented Lagrangian Objective Filled Penalty Functions (ALOFP) Algorithm.

Let the victor-valued function $\overline{\mathbf{g}}(x, \overline{x}_t) = [g_1(x), \cdots, g_m(x), f(x) - f(\overline{x_t}) + \theta]^T$,

$$\|\mathbf{g}(x, \overline{x}_t)\|_\infty = \max\{|g_1(x)|, \cdots, |g_m(x)|, |f(x) - f(\overline{x}_t) + \theta|\}.$$

ALOFP Algorithm.

Step 1: Given $N > 1$, $0 < \delta < 1$, $0 < eps < 1, 0 < p < 1$ and let $\overline{x}_1$ be the last or the limit iterate point of the ALOP Algorithm, $t := 1$.

Step 2: Choose $x_0 \in N(\overline{x}_t, \delta)$ randomly, given $u_{i,1} > 1, i \in I \cup \{m+1\}$, $\beta_1 > 1$, $0 < \gamma_1 < 1$, $a_1 < \min_{x \in X} f(x) < b_1$, $M_1 = \dfrac{a_1 + b_1}{2}$, $0 < \gamma_1 < \theta < 1$, $k := 1$.

Step 3: Solve $\min_{x \in Y} L(x, \overline{x}_t, M, u, \beta)$ starting at x_{k-1}, let x_k be a locally optimal point of (GOP).

Step 4: If $L(x, \overline{x}_t, M_k, u_{i,k}, \beta_k) = 0$, go to step 5; otherwise, go to Step 6.

Step 5: Let $a_{k+1} = a_k$, $b_{k+1} = M_k$, $M_{k+1} = \dfrac{a_{k+1} + b_{k+1}}{2}$, $k := k+1$, go to Step 3.

Step 6: If x_k is not γ_k-feasible to (P), go to Step 7; otherwise, let $\overline{x}_t(k) := x_k$, $\overline{x}_t(k)$ is a γ_k-locally optimal point of (P), go to Step 8.

Step 7: Let $a_{k+1} = M_k$, $b_{k+1} = b_k$, $M_{k+1} = \frac{a_{k+1} + b_{k+1}}{2}$. If $\|\mathbf{g}(x_k, \overline{x}_t)\|_\infty > \|\mathbf{g}(x_{k-1}, \overline{x}_t)\|_\infty$ or $\|\mathbf{g}(x_{k-1}, \overline{x}_t)\|_\infty \geq \|\mathbf{g}(x_k, \overline{x}_t)\|_\infty > \frac{1}{4}\|\mathbf{g}(x_{k-1}, \overline{x}_t)\|_\infty$, then $u_{i,k+1} = Nu_{i,k}$, $\beta_{k+1} = N\beta_k$, $\gamma_{k+1} = p\gamma_k$, $k := k+1$, and go to Step 3; if $\|\mathbf{g}(x_k, \overline{x}_t)\|_\infty \leq \frac{1}{4}\|\mathbf{g}(x_{k-1}, \overline{x}_t)\|_\infty$, then $u_{i,k+1} = Nu_{i,k}$, $\beta_{k+1} = \beta_k$, $\gamma_{k+1} = p\gamma_k$, $k := k+1$, and go to Step 3.

Step 8: If $|b_k - a_k| < eps$ or $\gamma_k < eps$, stop and x_k is an $eps-$globally optimal point of (P); otherwise, let $\overline{x}_t = \overline{x}_t(k), \theta := p\theta, t := t+1$, go to Step 2.

Remark 3.1. In Step 2, it is assumed that one can always obtain $a_1 < \min_{x \in X} f(x)$ and any efficient methods available can be used in Step 3. This algorithm has a nested algorithm, the inner loop is to find the locally optimal solutions of (P), meantime, the outer loop is to find a locally optimal point with a smaller value of objective functions of (P). That is, the ALOP Algorithm is a subroutine of the ALOFP Algorithm.

We need to find an $eps-$globally optimal point to (P) with these two algorithms, thus some notation follows:

(i) x_0 is the initial point to find locally optimal point of problem (P);

(ii) x_k is the $k-$th iterate point to solve $\min_{x \in Y} L(x, \overline{x}_t, M, u, \beta)$;

(iii) $\overline{x}_t(k)$ is the $k-$th point in internal loops and becomes the $t-$th locally optimal point in external loops;

(v) $\overline{x}_t$ is the $t-$th locally optimal point of (P).

Now we analyze the properties of the iterate sequence obtained by the ALOFP Algorithm.

Theorem 3.5. *Let $\{\overline{x}_t\}$ be a sequence generated by the ALOFP Algorithm and let $\{x_k\}$ be the sequence generated by solving $\min_{x \in Y} L(x, \overline{x}_{t_0}, M, u, \beta)$, for all $\overline{x}_{t_0} \in \{\overline{x}_t\}$. Suppose that the sequence $\{L(x_k, \overline{x}_{t_0}, M_k, u_k, \beta_k)\}$ is bounded.*

(i) In Step 6 of ALOFP Algorithm, there is a $k_0 \in N$, $k_0 > 0$, such that x_k is γ_k-feasible for problem (P) for any $k \geq k_0$ and x_k is a γ_k-locally optimal point for (P).

(ii) Suppose that ALOFP Algorithm stops at the t'th iteration, then the last point of $\{\overline{x}_t\}$ obtained by Step 7 is an eps$-$globally optimal point for problem (P) and there is

$$f(\overline{x}_{t'}) < \ldots < f(\overline{x}_{t+1}) < f(\overline{x}_t) < \ldots < f(\overline{x}_1).$$

Proof. (i) Suppose that for any $k > 0$, there is a $k_0 \geq k$ such that x_{k_0} is not γ_k-feasible for problem (P).

Let $g_{m+1}(x) = f(x) - f(x_l^*) + \theta$, $\overline{X} = \{x | g_i(x) \leq 0, i = 1, 2, \ldots, m+1\}$, there is

$$(\widehat{P}) \qquad \begin{aligned} &\min \ f(x) \\ &s.t. \ \ g_i(x) \leq 0, \quad i = 1, 2, \ldots, m+1. \end{aligned}$$

Obviously, x_{k_0} is also not γ_k-feasible for problem $(\widehat{P})$. Then, there is $i_0 \in \{1, 2, \ldots, m+1\}$ such that $g_{i_0}(x_{k_0}) \geq \alpha_0 > \gamma_{k_0} > 0$ where $\alpha_0 \in R^+$. So, we can obtain that

$$\begin{aligned}
&Q(f(x_{k_0}) - M_{k_0}) + u_{i_0,k_0}^3 \alpha_0^3 \\
&\leq Q(f(x_{k_0}) - M_{k_0}) + u_{i_0,k_0}^3 g_{i_0}^3(x_{k_0}) \\
&\leq L(x_{k_0}, \overline{x}_t, M_{k_0}, u_{i_0,k_0}, \beta_{k_0}) \\
&\leq L(x, \overline{x}_t, M_{k_0}, u_{i_0,k_0}, \beta_{k_0}) \\
&= Q(f(x) - M_{k_0}),
\end{aligned}$$

for any $x \in N(x_{k_0}) \cap \overline{X}$. There is $Q(f(x_{k_0}) - M_{k_0}) + u_{i_0,k_0}^3 \alpha_0^3 \leq Q(f(x) - M_{k_0})$. In Step 6 of this algorithm, u_{i_0,k_0} is sufficiently large when k is sufficiently large, then the left-hand side of this inequality tends to infinity, whereas the right-hand side of this inequality is finite. This creates a contradiction, so there is a $k_0 > 0$ such that x_k is γ_k-feasible for problem $(\widehat{P})$ for any $k \geq k_0$, and is also γ_k-feasible for problem (P).

In Step 6 of this algorithm, there is

$$0 < Q(f(x_k) - M_k) \leq L(x_k, \overline{x}_t, M_k, u_{i,k}, \beta_k) \leq L(x, \overline{x}_t, M_k, u_{i,k}, \beta_k) = Q(f(x) - M_k),$$

for any $x \in N(x_k) \cap \overline{X}$. It can be followed from the definition of $Q(\cdot)$ that $f(x_k) \leq f(x)$. In other words, x_k is a γ_k-locally optimal point for $(\widehat{P})$ and also a γ_k-locally optimal point for (P).

(ii) We suppose ALOFP Algorithm stops at the t'th iteration. Since $\{\overline{x}_t\}$ is γ_k-feasible to $(\widehat{P})$, there is $f(\overline{x}_{t+1}) - f(\overline{x}_t) + \theta \leq 0, t = 1, 2, \ldots, t'$. Hence,

$$f(\overline{x}_{t'}) < \ldots < f(\overline{x}_{t+1}) < f(\overline{x}_t) < \ldots < f(\overline{x}_1).$$

Let x^* be a globally optimal point for problem (P), $f^* = \min_{x \in X} f(x)$. From Theorem 3.6 in Tang etc [19], we obtain the sequence $\{a_k\}$ increases to a^* and $\{b_k\}$ decreases to b^*, $M_k \to M^*$ with

$$a_k < M_k = \frac{a_k + b_k}{2} < b_k, \quad b_{k+1} - a_{k+1} = \frac{b_k - a_k}{2}, \ k = 1, 2, \ldots.$$

Obviously, $a^* = b^* = M^* = f^*$. In Step 8 of this algorithm, there is

$$L(x_k, \overline{x}_t, M_k, u_{i,k}, \beta_k) \leq L(x^*, \overline{x}_t, M_k, u_{i,k}, \beta_k) = Q(f(x^*) - M_k) \to Q(f(x^*) - f^*) = 0,$$

when *eps* is sufficiently small. There is $k_0 > 0$ such that $L(x_k, \overline{x}_t, M_k, u_{i,k}, \beta_k) \leq 0$ for all $k \geq k_0$. So, we obtain that $Q(f(x_k) - M_k) = 0$, $f(x_k) \leq M_k \to f^*$. Because f^* is globally optimal value of problem (P), there is $f(x_k) \geq f^*$. Then, there is $f(x_k) \to f^*$. Let $\overline{x}_{t'} := \overline{x}_{t'}(k)$, $f(\overline{x}_{t'})$ can infinitely close to f^*, i.e., the last point $\overline{x}_{t'}$ is an *eps*-globally optimal solution to problem (P). □

4. Numerical experiments

In this section, two numerical experiments to explain the rationality of the ALOP Algorithm and ALOFP Algorithm are listed.

Example 4.1. Consider the following problem, which is taken from Di Pillo etc [16]:

$$(P_1)\qquad \begin{array}{ll} \min & f(x) = x_1^2 + x_2^2 + 2x_3^2 + x_4^2 - 5x_1 - 5x_2 - 21x_3 + 7x_4 \\ s.t. & g_1(x) = 2x_1^2 + x_2^2 + x_3^2 + 2x_1 + x_2 + x_4 - 5 \le 0, \\ & g_2(x) = x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_1 - x_2 + x_3 - x_4 - 8 \le 0, \\ & g_3(x) = x_1^2 + 2x_2^2 + x_3^2 + 2x_4^2 - x_1 - x_4 - 10 \le 0. \end{array}$$

Consider the augmented Lagrangian objective penalty function that is defined by $Q(t) = (t+|t|)^4$, $P(t) = \begin{cases} \sqrt{t^2+4} - 2, & t \ge 0 \\ 0, & t < 0. \end{cases}$ Let $eps = 10^{-5}$, $a_1 = -48$, $b_1 = -47$, $M_1 = -47.5$, $u = (25, 25, 25)^T$, $\gamma_1 = 0.1$, $\beta_1 = 9.045$ and $x_0 = (-7, 11, 10, 7)^T$. The authors ran the ALOP Algorithm to find local minimizers of (P_1) on Matlab and the results are listed in Table 1.

Table 1. Numerical Results of the ALOP Algorithm, $f^* = -44.2338$

k	M_k	γ_k	β_k	x_k	$f(x_k)$
1	-47.5	0.1	9.045	$(-7, 11, 10, 7)^T$	
2	-47.25	0.01	904.5	$(0.01029, -0.0087023, 2.4162, -0.84995)^T$	-44.2986

Example 4.2 (Greenwank Function)**.** Consider the following problem, which is taken from Wang etc [22]:

$$(P_2)\qquad \begin{array}{ll} \min & f(x) = \sum_{i=1}^{n} (x_i^2 - \frac{1}{10}\cos(5\pi x_i)) \\ s.t. & -1 \le x_i \le 1,\ i = 1, \ldots, n. \end{array}$$

The global optimal value of this problem is $f(x_1^*, x_2^*, \ldots, x_n^*) = -\frac{n}{10}$. Now, the authors consider $n = 10$ and $n = 15$, the global optimal value is -1 and -1.5. Consider the augmented Lagrangian objective filled penalty function that is defined by $Q(t) = (t+|t|)^3$, $P(t) = \begin{cases} \sqrt{t^2+4} - 2, & t \ge 0 \\ 0, & t < 0. \end{cases}$ Let $p = 0.1$, $eps = 10^{-2}$, $a_1 = -4$, $b_1 = 0$, $M_1 = -5$, $\gamma_1 = 0.5$, $\beta_1 = 1.1$.

Choose

$$x_0 = (0, 0, 0, 0, -2, -1, 1, -1, 0, 1)^T, \quad u = (3, 3, 3, 3, 3, 3, 3, 3, 3, 3)^T$$

while $n = 10$ and

$$x_0 = (0, 0, 0, 0, 0, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0)^T,$$
$$u = (3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3)^T$$

while $n = 15$. The authors ran the ALOP algorithm and ALOFP Algorithm on Matlab and produced the numerical results listed in Table 2.

Table 2. Locally and globally optimal point of Problem (P_2)

n	10	15
locally optimal point of ALOP	$(0.0335, 0.1286, -0.0018, -0.2581, 0.0063, 0.1645, 0.0825, 0.0160, 0.4175, -0.0049)^T$	$(0.0134, 0.3573, -0.0269, 0.0345, -0.3531, -0.3979, -0.2196, 0.1986, 0.1132, -0.0393, 0.0445, -0.0046, 0.4476, -0.0678, -0.1024)^T$
globally optimal point of ALOFP	$(-0.0924, 0.2182, -0.0006, 0.0410, 0.0185, -0.0651, 0.1276, -0.0197, 0.4331, 0.0394)^T$	$(0.0134, -0.1024, -0.0269, 0.0345, -0.3531, -0.3979, -0.2196, 0.3573, 0.1986, 0.1132, -0.0393, 0.0445, -0.0046, 0.4476, -0.0678)^T$
locally optimal value of ALOP	-0.9999	-1.5000
gobally optimal value of ALOFP	-1.000	-1.5000

Example 4.3. Consider the following constrained nonconvex optimization problem, which is taken from Wu [23]:

$$(P_3) \qquad \begin{aligned} \min \quad & f(x) = (x_1 - 1.125)^2 + \frac{x_2^2}{4} \\ s.t. \quad & x_1^2 - x_2^4 \le 0, \\ & x_1 \ge 0. \end{aligned}$$

Consider the augmented Lagrangian objective penalty function that is defined by $Q(t) = (t + |t|)^4$, $P(t) = \begin{cases} t + e^{-t} - 1, & t \ge 0 \\ 0, & t < 0. \end{cases}$

This problem has two global optimal solutions: $x^{1,*} = (1, 1)^T$ and $x^{2,*} = (1, -1)^T$, with the optimal value $f^* = 0.2656$. Let $eps = 10^{-5}$, $a_1 = -2$, $b_1 = 2$, $M_1 = 0$, $u = (5, 5, 5)^T$, $\gamma_1 = 0.1$, $\beta_1 = 2$ and $x_0 = (-0.58, 0)^T$. The authors ran the ALOFP Algorithm to solve (P_4) on Matlab and the results are listed in Table 3.

As shown in Table 3, the solution is obtained in the 1-st iteration by the Algorithm with approximate global optimal value 0.26511.

Table 3. Numerical Results of the ALOFP Algorithm (the external loop)

t	M	x	min
1	-0.5	$(0.6268, -0.26008)$	0.26511

Example 4.4. Consider the following problem, which is taken from Meng [25]:

$$(P_4)\quad \begin{aligned} \min \quad & f(x) = 1000 - x_1^2 - 2x_2^2 - x_3^2 - x_1x_2 - x_1x_3 \\ s.t. \quad & x_1^2 + x_2^2 + x_3^2 - 25 = 0, \\ & (x_1-5)^2 + x_2^2 + x_3^2 - 25 = 0, \\ & (x_1-5)^2 + (x_2-5)^2 + (x_3-5)^2 - 25 \le 0. \end{aligned}$$

Consider the augmented Lagrangian objective penalty function that is defined by $Q(t) = (t+|t|)^4$, $P(t) = \begin{cases} t + e^{-t} - 1, \ t \ge 0 \\ 0, \qquad\qquad t < 0. \end{cases}$ Let $eps = 10^{-5}$, $a_1 = 100$, $b_1 = 1000$, $M_1 = 550$, $u = (5,5,5)^T$, $\gamma_1 = 0.1$, $\beta_1 = 3$ and $x_0 = (2,2,2)^T$. The authors ran the ALOFP Algorithm to find the approximate global optimal minimisers of (P_3) on Matlab and the results are listed in Tables 4 and 5.

As shown in Tables 4 and 5, the solution is obtained in the 3-th iteration by the ALOFP Algorithm with approximate global optimal value 944.2015. while the solution is obtained in the 4-th iteration in with 944.215662.

Table 4. Numerical Results of the ALOFP Algorithm (the internal loop)

k	M_k	β_k	x_k	min
1	550	3	$(2,2,2)$	
2	775	25	$(7.9574, 6.4354, 2.5958)$	57.035
3	887.5	75	$(6.6516, 3.3819, 3.2613)$	33.4031
4	943.75	375	$(5.9865, 3.5088, 3.3836)$	26.7118

Table 5. Numerical Results of the ALOFP Algorithm (the external loop)

t	M	x	min
1	775	$(8.2751, 7.0082, 8.3031e-2)$	775.2985
2	887.5	$(5.9021, 4.918, 1.0299e-2)$	887.7668
3	943.75	$(2.9359, 4.1781, 1.2639e-2)$	944.2015

In fact, the ALOP Algorithm is implicit in the ALOFP algorithm. The authors used the ALOP algorithm to find the locally optimal points firstly, if the current point is one of the globally optimal points, then the ALOFP algorithm is the ALOP algorithm; otherwise, the ALOFP algorithm is used to obtain the approximate optimal solution next. It shows from these two numerical results that the two algorithms are respectively applicable to search locally optimal solution and globally optimal solution.

5. Conclusion

In this work, we introduce a class of augmented Lagrangian objective filled penalty functions to construct a global optimization methods. Until we find approximate global optimal points, we use the filled penalty function to find a better locally optimal points of optimization problems. Both of exactness of the filled penalty function and convergence of the two algorithms have been proved. Finally, numerical experiments have been showed to explain good applicability of the two algorithms.

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