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RANDOM ATTRACTORS FOR NON-AUTONOMOUS STOCHASTIC WAVE EQUATIONS WITH STRONG DAMPING AND ADDITIVE NOISE ON $\mathbb{R}^N$*

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Abstract This paper investigates the long-time behavior of a stochastic strongly damped wave equation with additive noise on $\mathbb{R}^N$. We establish that there exists a unique pullback random attractor for the equation in space $H^1(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$ with the nonlinearity $g(x,u)$ being of optimal subcritical growth p: $1 \leq p < p^* \equiv \frac{N+2}{(N-2)}(N \geq 3)$. In addition, we get the upper semicontinuity of the pullback random attractor as the intensity of noise goes to zero.

Keywords Random attractors, upper semicontinuity, Stochastic wave equation, additive noise, unbounded domains.

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1. Introduction

At the present paper, we consider the following initial-value problem for a stochastic strongly damped wave equation with additive noise defined on the whole space $\mathbb{R}^N (N \geq 3)$:

$$u_{tt} - \Delta u_t - \Delta u + u_t + g(x,u) = f(x,t) + \epsilon h(x)\frac{\mathrm{d}W}{\mathrm{d}t}, \quad t > \tau, \tag{1.1}$$

with initial conditions

$$u(x,\tau) = u_\tau(x),\ u_t(x,\tau) = u_{1,\tau}(x), \quad \tau \in \mathbb{R}, \tag{1.2}$$

where $u = u(x,t)$ is a real function of $x \in \mathbb{R}^N$ and $t \geq \tau$, $\epsilon \in (0,1]$, $g(x,u)$ is a nonlinear function satisfying some conditions, $f(x,t)$ and $h(x)$ are given functions in $L^2_{\mathrm{loc}}(\mathbb{R}, L^2(\mathbb{R}^N))$ and $H^2(\mathbb{R}^N)$, respectively, $W = W(t)$ is a two-sided real-valued Wiener process on a probability space $(\Omega, \mathcal{F}, \mathcal{P})$, which will be specified later.

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One of the most important things in studying evolution partial differential equations is to investigate the long-time behavior of solutions of the equations, and attractors are the ideal objects in this process. Up to now, there are abundant results related to attractors having been established for the deterministic infinite-dimensional dynamical systems (see e.g. monographs [2,7,20] and papers [9,17,30]). For the results on existence of attractors for the deterministic (when $\epsilon = 0$) wave equations corresponding to (1.1)-(1.2), one can refer to [4, 12, 19, 28] and the references therein. In this article, we are concerned with the existence of random attractor for the non-autonomous stochastic strongly damped wave equation (1.1)-(1.2) and the upper semicontinuity of the attractor as $\epsilon \to 0$.

When the external force f is time-independent, the equation (1.1) becomes an autonomous stochastic wave equation. The theories of random dynamical systems and random attractors for autonomous stochastic equations were first established by F. Flandoli et al. [5,6,13], in which the random attractor is a family of compact sets depending on random parameters and has some invariant property under the action of the random dynamical system. The concepts of pullback random attractors for non-autonomous random dynamical systems were introduced by Crauel et al. in [10] and then in details by Wang [22], where the pullback random attractor is a family of compact sets depending on both random parameters and time. For other results on the random attractors, one can refer to [3,21,23] for autonomous random dynamical system and [14, 29, 31, 32] for non-autonomous random dynamical system. Later, Cui and Langa in [11] introduced the concept of uniform random attractors for the non-autonomous random dynamical system. Recently, we establish the existence of uniform random attractors for the 2D stochastic Navier-Stokes equations in [15] and stochastic strongly damped wave equation in [16], respectively.

There are some results about pullback random attractors for non-autonomous stochastic wave equations on unbounded domains (see e.g. [24, 25, 27] and the references therein). Particularly, authors in [24] and [25] establish the existence of pullback random attractor for a stochastic strongly damped wave equation with additive noise and multiplicative noise, respectively, when the nonlinearity $g(x, u)$ is of subcritical growth $p : 1 \leq p < 3$ as $N = 3$. In the deterministic case, the existence of global attractor for strongly damped wave equation with growth exponent p: $1 \leq p \leq p^* \equiv \frac{N+2}{N-2}$ defined on $\mathbb{R}^N$ $(N \geq 3)$ has been established (see e.g. [8, 28]). Here, we establish the existence and upper semicontinuity of pullback random attractor for the stochastic strongly damped wave equation (1.1)-(1.2) defined on $\mathbb{R}^N$ with nonlinearity $g(x, u)$ satisfying $|g(x, u)| \leq C(1 + |u|^p)$, $1 \leq p < p^*$. The existence of pullback random attractor for system (1.1)-(1.2) in the critical case $(p = p^*)$ is still unsolved and we will study it in the near future.

This paper is organized as follows. In the next section, we recall some definitions and results on the pullback random attractors and non-autonomous random dynamical systems. In Section 3, we prove that there exists a unique pullback random attractor for problem (1.1)-(1.2). Section 4 is devoted to demonstrate the upper semicontinuity of the pullback random attractor as ϵ tends to zero. Throughout this article, we denote by C and c_i $(i = 1, 2, \cdots)$ the positive constants independent of ϵ.

2. Preliminaries

In this section, we present some basic concepts on non-autonomous random dynamical systems and pullback random attractors (see [22, 27] for details).

Let X be a complete separable metric space with Borel σ-algebra $\mathcal{B}(X)$. Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a probability space, where $\Omega = \{\omega \in \mathcal{C}(\mathbb{R}, \mathbb{R}) : \omega(0) = 0\}$, the Borel σ-algebra $\mathcal{F}$ on Ω is generated by the compact open topology, and $\mathcal{P}$ is the corresponding Wiener measure on $\mathcal{F}$. We identify "$\mathcal{P}$-a.e. $\omega \in \Omega$" with "$\omega \in \Omega$" hereinafter for simplicity. Consider the Wiener shift θ_t on the probability space $(\Omega, \mathcal{F}, \mathcal{P})$ defined by

$$\theta_t\omega(\cdot) = \omega(t + \cdot) - \omega(t), \text{ for all } \omega \in \Omega,\ t \in \mathbb{R},$$

and then $(\Omega, \mathcal{F}, \mathcal{P}, (\theta_t)_{t\in\mathbb{R}})$ is a metric dynamical system.

Definition 2.1. A mapping Ψ: $\mathbb{R}^+ \times \mathbb{R} \times \Omega \times X \mapsto X$ is called a continuous cocycle on X over $\mathbb{R}$ and $(\Omega, \mathcal{F}, \mathcal{P}, (\theta_t)_{t\in\mathbb{R}})$ if for all $\tau \in \mathbb{R}$, $t, s \in \mathbb{R}^+$ and $\omega \in \Omega$, the following conditions are fulfilled:

(i) $\Psi(\cdot, \tau, \cdot, \cdot) : \mathbb{R}^+ \times \Omega \times X \mapsto X$ is $(\mathcal{B}(\mathbb{R}^+) \times \mathcal{F} \times \mathcal{B}(X), \mathcal{B}(X))$-measurable;

(ii) $\Psi(0, \tau, \omega, \cdot) = \mathrm{id}_X$;

(iii) $\Psi(t + s, \tau, \omega, \cdot) = \Phi(t, \tau + s, \theta_s\omega, \cdot) \circ \Phi(s, \tau, \omega, \cdot)$;

(iv) $\Psi(t, \tau, \omega, \cdot) : X \mapsto X$ is continuous.

Let $D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$ be a family of nonempty bounded subsets of X and $\mathcal{D}$ be a collection of such families satisfying some conditions. A collection $\mathcal{D}_0$ is said to be inclusion-closed if $D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}_0$, then any family $O = \{O(\tau, \omega) \subseteq D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$ belongs to $\mathcal{D}_0$.

Definition 2.2. A family $K = \{K(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$ is said to be a $\mathcal{D}$-pullback absorbing set for Ψ if for all $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $D \in \mathcal{D}$, there exists some $T = T(\tau, \omega, D) > 0$ such that

$$\Psi(t, \tau - t, \theta_{-t}\omega, D(\tau - t, \theta_{-t}\omega)) \subseteq K(\tau, \omega) \quad \text{ for all } \quad t \geq T.$$

Moreover, if for every $\tau \in \mathbb{R}$ and $\omega \in \Omega, K(\tau, \omega)$ is a closed nonempty subset of X and K is measurable in ω with respect to $\mathcal{F}$, then K is said to be a closed measurable $\mathcal{D}$-pullback absorbing set for Ψ.

Definition 2.3. The cocycle Ψ is $\mathcal{D}$-pullback asymptotically compact in X if for every $\tau \in \mathbb{R}$, $\omega \in \Omega$, and $\{t_n\} \subseteq \mathbb{R}^+$ with $t_n \to +\infty$, the sequence

$$\{\Psi(t_n, \tau - t_n, \theta_{-t_n}\omega, x_n)\}_{n=1}^{\infty} \quad \text{possesses a convergent subsequence in } X,$$

where $x_n \in B(\tau - t_n, \theta_{-t_n}\omega)$ with $\{B(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$.

Definition 2.4. A family $\mathcal{A} = \{\mathcal{A}(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$ is said to be a $\mathcal{D}$-pullback random attractor for Ψ if the following properties hold for all $\tau \in \mathbb{R}$ and $\omega \in \Omega$:

(i) Measurability and Compactness: $\mathcal{A}$ is measurable in ω with respect to $\mathcal{F}$ and $\mathcal{A}(\tau, \omega)$ is compact in X.

(ii) Invariance: $\mathcal{A}$ is invariant in the sense that

$$\Psi(t,\tau,\omega,\mathcal{A}(\tau,\omega)) = \mathcal{A}(\tau+t,\theta_t\omega),\ \forall t \geq 0.$$

(iii) Attracting: $\mathcal{A}$ attracts each element of $\mathcal{D}$ in the sense of pullback, i.e., for each $\tau \in \mathbb{R}$ and $\omega \in \Omega$, $D = \{D(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$,

$$\lim_{t\to+\infty} \operatorname{dist}_X(\Psi(t,\tau-t,\theta_{-t}\omega, D(\tau-t,\theta_{-t}\omega)), \mathcal{A}(\tau,\omega)) = 0.$$

We end this section with following result, which can be found in [22, 27].

Lemma 2.1. *Let $\mathcal{D}$ be an inclusion-closed collection of some families of nonempty subsets of X and Ψ be a continuous cocycle on X over $\mathbb{R}$ and $(\Omega,\mathcal{F},\mathcal{P},(\theta_t)_{t\in\mathbb{R}})$. If Ψ possesses a closed measurable $\mathcal{D}$-pullback absorbing set $K = \{K(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$ and Ψ is $\mathcal{D}$-pullback asymptotically compact in X, then Ψ possesses a unique $\mathcal{D}$-pullback random attractor $\mathcal{A} = \{\mathcal{A}(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$, which is given by*

$$\mathcal{A}(\tau,\omega) = \Omega(K,\tau,\omega) = \bigcap_{r\geq 0} \overline{\bigcup_{t\geq r} \Phi(t,\tau-t,\theta_{-t}\omega, K(\tau-t,\theta_{-t}\omega))}.$$

3. Pullback random attractor

3.1. Basic settings

We first introduce some notations and function spaces:

- $L^p(\mathbb{R}^N)$-the usual Lebesgue space with norm $\|\cdot\|_p$. In particular, the norm in $L^2(\mathbb{R}^N)$ is denoted by $\|\cdot\|$.
- $H^m(\mathbb{R}^N)$-the usual Sobolev space with norm $\|\cdot\|_{H^m}$.
- $\mathcal{H}(\mathbb{R}^N) = H^1(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$-the Hilbert space endowed with norm $\|\cdot\|_{\mathcal{H}}$ and inner product $(\cdot,\cdot)_{\mathcal{H}}$, respectively, defined by

$$\|\varphi\|_{\mathcal{H}} = (\|\nabla u\|^2 + \|u\|^2 + \|v\|^2)^{\frac{1}{2}},\ \forall \varphi = (u,v) \in \mathcal{H}(\mathbb{R}^N),$$
$$(\varphi_1,\varphi_2)_{\mathcal{H}} = (\nabla u_1, \nabla u_2) + (u_1,u_2) + (v_1,v_2),\ \varphi_i = (u_i,v_i) \in \mathcal{H}(\mathbb{R}^N),\ i = 1,2.$$

- $(\cdot,\cdot)$ denotes the inner product in $L^2(\mathbb{R}^N)$ and also denotes the dual pairing between $H^1(\mathbb{R}^N)$ and its dual space $H^{-1}(\mathbb{R}^N)$.

Now, we show that the solution of problem (1.1)-(1.2) can generate a continuous cocycle Ψ_ϵ on $\mathcal{H}(\mathbb{R}^N)$.

Let $\varsigma = u_t + \delta u$, where δ is a positive number to be specified later, and then we can rewrite problem (1.1)-(1.2) as follows:

$$\frac{\mathrm{d}\varsigma}{\mathrm{d}t} - \Delta\varsigma - (1-\delta)\Delta u + (1-\delta)\varsigma + (\delta^2-\delta)u + g(x,u) = f(x,t) + \epsilon h(x)\frac{\mathrm{d}\omega}{\mathrm{d}t}, \tag{3.1}$$
$$\frac{\mathrm{d}u}{\mathrm{d}t} = \varsigma - \delta u, \tag{3.2}$$

with initial data

$$u(x,\tau) = u_\tau(x), \varsigma(x,\tau) = \varsigma_\tau(x), \tag{3.3}$$

where $\varsigma_\tau(x) = u_{1,\tau}(x) + \delta u_\tau(x)$, $x \in \mathbb{R}^N$.

In this article, we need the following assumptions:

(**i**) $g = g(x,u) \in \mathcal{C}(\mathbb{R}^N \times \mathbb{R})$ with $g(x,\cdot) \in \mathcal{C}^2(\mathbb{R})$ for almost all $x \in \mathbb{R}^N$, and

- (A_1) $g(\cdot,0) \in L^2(\mathbb{R}^N)$;
- (A_2) $|g'_u(x,0)| \le C$, $|g''_u(x,u)| \le c_1(1+|u|^{p-2})$, for $x \in \mathbb{R}^N$, $u \in \mathbb{R}$, $2 \le p < p^* \equiv \frac{N+2}{N-2}(N \le 6)$;
- (A_3) $\liminf\limits_{|u|\to\infty} \dfrac{g(x,u)}{u} \ge 0$ uniformly as $|x| \le r_0$, r_0 is a positive constant;
- (A_4) $\liminf\limits_{|u|\to\infty} \dfrac{g(x,u)u - c_2 G(x,u)}{u^2} \ge 0$ uniformly as $|x| \le r_0$, $G(x,u) = \int_0^u g(x,r)\mathrm{d}r$;
- (A_5) $(g(x,u) - g(x,0))u \ge c_3 u^2$, $g'_u(x,u) \ge -c_4$, for $|x| > r_0$ and $u \in \mathbb{R}$;
- (A_6) $g(x,u)u \ge c_5|u|^{p+1} - \phi_1(x)$, $\phi_1(x) \in L^1(\mathbb{R}^N)$, for $x \in \mathbb{R}^N$ and $u \in \mathbb{R}$.

(**ii**) $f \in L^2_{\mathrm{loc}}(\mathbb{R}, L^2(\mathbb{R}^N))$ and for any $\gamma > 0$, it holds

$$\int_{-\infty}^0 \mathrm{e}^{\gamma s}\|f(\cdot, s+\tau)\|^2 \mathrm{d}s < +\infty, \text{ for all } \tau \in \mathbb{R}. \tag{3.4}$$

Without loss of generality, we can assume $g(x,0) = 0$ (see [28] for details). In the following, we give an example of functions g and f satisfying assumptions (**i**) and (**ii**), respectively.

(1) Let
$$g(x,u) = a_0(x)|u|^{p-1}u + a_1(x)u, \ x \in \mathbb{R}^N, u \in \mathbb{R},$$
where $a_0(x), a_1(x) \in \mathcal{C}(\mathbb{R}^N)$ and $0 < c_5 \le a_0(x) \le C$, $0 < c_3 \le a_1(x) \le C$. It is obvious that $g(x,u)$ satisfies assumption (**i**).

(2) Let
$$f(x,t) = \frac{|t|^\alpha}{1+|x|^\beta}, \ x \in \mathbb{R}^N, t \in \mathbb{R},$$
where $0 < \alpha < +\infty$ and $\beta \ge \frac{N}{2} + 1$. Then $f(x,t)$ satisfies assumption (**ii**).

In order to define a random dynamical system, we transform problem (3.1)-(3.3) into pathwise deterministic one parameterized by ω. For any given $\omega \in \Omega$, let

$$y(\theta_t\omega) = -\int_{-\infty}^0 \mathrm{e}^s(\theta_t\omega)(s)\mathrm{d}s, \ t \in \mathbb{R}. \tag{3.5}$$

Then $y(\theta_t\omega)$ is a one-dimensional Ornstein-Uhlenbeck process and satisfies the Ornstein-Uhlenbeck equation

$$\mathrm{d}y(\theta_t\omega) + y(\theta_t\omega)\mathrm{d}t = \mathrm{d}\omega(t).$$

In addition, we can get from [1] that $y(\theta_t\omega)$ is continuous in t for $\omega \in \Omega$ and $|y(\omega)|$ is tempered. Let $z(\theta_t\omega) = z(x,\theta_t\omega) = h(x)y(\theta_t\omega)$, and then $z(\theta_t\omega)$ solves

$$\mathrm{d}z(\theta_t\omega) + z(\theta_t\omega)\mathrm{d}t = h(x)\mathrm{d}\omega(t).$$

Putting

$$v(t) = \varsigma(t) - \epsilon z(\theta_t\omega),$$

we can rewrite problem (3.1)-(3.3) as follows:

$$\frac{\mathrm{d}v}{\mathrm{d}t}-\Delta v-\epsilon\Delta z(\theta_t\omega)-(1-\delta)\Delta u+(\delta^2-\delta)u+(1-\delta)v-\epsilon\delta z(\theta_t\omega)+g(x,u)=f(x,t), \tag{3.6}$$

$$\frac{\mathrm{d}u}{\mathrm{d}t}=v+\epsilon z(\theta_t\omega)-\delta u, \tag{3.7}$$

with initial data

$$u(x,\tau)=u_\tau(x),\ v(x,\tau)=v_\tau(x), \tag{3.8}$$

where $v_\tau(x)=\varsigma_\tau(x)-\epsilon z(\theta_\tau\omega)=u_{1,\tau}(x)+\delta u_\tau(x)-\epsilon z(\theta_\tau\omega)$, $x\in\mathbb{R}^N$.

Notice that the initial-value problem (3.6)-(3.8) can be viewed as deterministic one with random parameter $\omega\in\Omega$. Under the conditions (A_1)-(A_3), (A_5) and (**ii**), the well-posedness for the deterministic wave equation with strong damping defined on $\mathbb{R}^N$ can be obtained by the method of [28]. By the similar proof as that of [27, 28], we have the following result.

Lemma 3.1. *Let assumptions* (**i**)-(**ii**) *hold. Then for each* $\omega\in\Omega$, *and initial data* $(u_\tau,v_\tau)\in\mathcal{H}(\mathbb{R}^N)$, *problem* (3.6)-(3.8) *possesses a unique solution* (u,v) *satisfying*

$$\big(u(\cdot,\tau,\omega,u_\tau),v(\cdot,\tau,\omega,v_\tau)\big)\in\mathcal{C}([\tau,+\infty),\mathcal{H}(\mathbb{R}^N)). \tag{3.9}$$

Moreover, for any $t\geq\tau$, $\big(u(t,\tau,\omega,u_\tau),v(t,\tau,\omega,v_\tau)\big)$ *is* $\big(\mathcal{F},\mathcal{B}(H^1(\mathbb{R}^N))\times\mathcal{B}(L^2(\mathbb{R}^N))\big)$-*measurable in* ω *and continuous with respect to initial data* (u_τ,v_τ) *in the norm of* $\mathcal{H}(\mathbb{R}^N)$.

Denote by $\varphi(\cdot,\tau,\omega,\varphi_\tau)$ the solution $\big(u(\cdot,\tau,\omega,u_\tau),v(\cdot,\tau,\omega,v_\tau)\big)$ of problem (3.6)-(3.8) with initial data $\varphi_\tau=(u_\tau,v_\tau)$. Then by Lemma 3.1, we can define a continuous cocycle Ψ_ϵ for system (3.6)-(3.8) by

$$\Psi_\epsilon(t,\tau,\omega,\varphi_\tau):=\varphi(t+\tau,\tau,\theta_{-\tau}\omega,\varphi_\tau),\ \epsilon\in(0,1]. \tag{3.10}$$

We will study the existence and upper semicontinuity of pullback random attractors for Ψ_ϵ. For a given bounded nonempty subset $B\subseteq\mathcal{H}(\mathbb{R}^N)$, we denote by $\|B\|=\sup\limits_{u\in B}\|u\|_{\mathcal{H}}$. A family $D=\{D(\tau,\omega):\tau\in\mathbb{R},\omega\in\Omega\}$ of nonempty bounded subsets of $\mathcal{H}(\mathbb{R}^N)$ is called tempered with respect to $(\theta_t)_{t\in\mathbb{R}}$ if for every $\gamma>0$, and $\omega\in\Omega$,

$$\lim_{t\to+\infty}\mathrm{e}^{-\gamma t}\|D(\tau-t,\theta_{-t}\omega)\|=0. \tag{3.11}$$

In the following, we denote by $\mathcal{D}$ the collection of all tempered families of nonempty bounded subsets of $\mathcal{H}(\mathbb{R}^N)$ and it is obvious that $\mathcal{D}$ is inclusion-closed.

We end this subsection with the following lemma that is useful for the proof of existence of $\mathcal{D}$-pullback random attractor for Ψ_ϵ.

Lemma 3.2. *Let assumptions* (A_1)-(A_5) *hold. Let* $\mathcal{G}(u)=\int_{\mathbb{R}^N}G(x,u)\mathrm{d}x$. *Then for any* $\nu>0$, *there exist* $\rho_i(\nu)\geq 0$, $i=1,2$ *such that*

$$\mathcal{G}(u)\geq-\nu\|u\|^2-\rho_1(\nu), \tag{3.12}$$

$$(g(x,u),u)-\eta\mathcal{G}(u)\geq-\nu\|u\|^2-\rho_2(\nu), \tag{3.13}$$

for $u \in H^1(\mathbb{R}^N)$ *and* $\eta : 0 < \eta < \min\{c_2, \frac{2c_3}{c_4+2c_3}\}$*. Furthermore, there exist* $\alpha > 0$ *and* $\beta \geq 0$ *such that*

$$(g(x,u),u) + \|\nabla u\|^2 \geq \alpha\|u\|^2 - \beta, \ \forall u \in H^1(\mathbb{R}^N). \tag{3.14}$$

Proof. Firstly, by assumption (A_5) and $g(x,0) = 0$, we can get that for any $\nu > 0$,

$$G(x,u) \geq \frac{c_3}{2}u^2, \text{ as } |x| > r_0. \tag{3.15}$$

When $|x| \leq r_0$, we get from assumption (A_3) that for any $\nu > 0$, there is a positive constant $M_0(\nu)$ such that for all $|u| > M_0(\nu)$,

$$g(x,u)u \geq -\nu u^2, \text{ and } G(x,u) \geq -\nu u^2.$$

Since $G(x,u) \in \mathcal{C}(\mathbb{R}^N \times \mathbb{R})$, we can get there exists a positive constant $C(\nu)$ such that

$$G(x,u) \geq -C(\nu) \text{ for } |x| \leq r_0, |u| \leq M_0(\nu).$$

Therefore, we get

$$G(x,u) \geq -\nu u^2 - C(\nu) \text{ for } |x| \leq r_0. \tag{3.16}$$

Integrating (3.16) with respect to x on $|x| \leq r_0$ and (3.15) on $|x| > r_0$, then we can get (3.12). Similar to the proof of Lemma 3.1 in [18], we can get (3.13) and (3.14), respectively, and we omit the details. □

3.2. Existence of $\mathcal{D}$-pullback random attractor for Ψ_ϵ

This subsection is devoted to show the existence of $\mathcal{D}$-pullback random attractor for Ψ_ϵ generated by the solution of problem (3.6)-(3.8). Now, we derive the uniform estimates of the solution of problem (3.6)-(3.8) to get a closed measurable $\mathcal{D}$-pullback absorbing set for Ψ_ϵ. Hereinafter, we denote by δ a fixed constant satisfying $\delta < \min\{\frac{1}{5}, \frac{\alpha}{2}, \frac{c_3}{4}\}$.

Lemma 3.3. *Let assumptions* (**i**)-(**ii**) *hold. Then for any* $\tau \in \mathbb{R}$*,* $\omega \in \Omega$*, and* $D = \{D(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$*, there exists* $T_1 = T_1(\tau,\omega,D) > 0$ *such that for all* $t \geq T_1$*, the solution* φ *of problem* (3.6)-(3.8) *satisfies*

$$\|\varphi(\tau,\tau-t,\theta_{-\tau}\omega,\varphi_{\tau-t})\|^2_{\mathcal{H}} \leq \mathcal{R}_{1,\epsilon}(\tau,\omega), \tag{3.17}$$

$$\int_{\tau-t}^{\tau} \mathrm{e}^{\frac{\delta\eta}{2}(s-\tau)}\|\varphi(s,\tau-t,\theta_{-\tau}\omega,\varphi_{\tau-t})\|^2_{\mathcal{H}}\mathrm{d}s \leq \mathcal{R}_{1,\epsilon}(\tau,\omega), \tag{3.18}$$

$$\int_{\tau-t}^{\tau} \mathrm{e}^{\frac{\delta\eta}{2}(s-\tau)}\|\nabla v(s,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})\|^2\mathrm{d}s \leq \mathcal{R}_{1,\epsilon}(\tau,\omega), \tag{3.19}$$

where $\varphi_{\tau-t} \in D(\tau-t,\theta_{-t}\omega)$ *and* $\mathcal{R}_{1,\epsilon}(\tau,\omega)$ *is given by*

$$\mathcal{R}_{1,\epsilon}(\tau,\omega) = M_1 + M_1\Big(\int_{-\infty}^{0}\mathrm{e}^{\frac{\delta\eta}{2}s}\|f(s+\tau)\|^2\mathrm{d}s + \epsilon\big(1 + \int_{-\infty}^{0}\mathrm{e}^{\frac{\delta\eta}{2}s}|y(\theta_s\omega)|^{p+1}\mathrm{d}s\big)\Big)$$

with M_1 *being a positive constant independent of* τ*,* ω*,* D *and* ϵ*.*

Proof. Using the multiplier v in equation (3.6) and then adding $\delta(u, \frac{\mathrm{d}u}{\mathrm{d}t})$ to the both sides of the equation, we get that

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v\|^2 + \delta(u, \frac{\mathrm{d}u}{\mathrm{d}t}) + \|\nabla v\|^2 - \epsilon(\Delta z(\theta_t\omega), v) - (1-\delta)(\Delta u, v)\\
&+ (\delta^2 - \delta)(u, v) + (1-\delta)\|v\|^2 - \epsilon\delta(z(\theta_t\omega), v)\\
&+ (g(x,u), v) = (f, v) + \delta(u, \frac{\mathrm{d}u}{\mathrm{d}t}). \qquad (3.20)
\end{aligned}$$

By (3.7), we obtain that

$$(u, v) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2 + \delta\|u\|^2 - \epsilon(u, z(\theta_t\omega)), \qquad (3.21)$$

$$-(\Delta u, v) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u\|^2 + \delta\|\nabla u\|^2 - \epsilon(\nabla u, \nabla z(\theta_t\omega)), \qquad (3.22)$$

$$(g(x,u), v) = \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{G}(u) + \delta(g(x,u), u) - \epsilon(g(x,u), z(\theta_t\omega)). \qquad (3.23)$$

From (3.20)-(3.23) and (3.7), we can get

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big(\|v\|^2 + (1-\delta)\|\nabla u\|^2 + \delta^2\|u\|^2 + 2\mathcal{G}(u)\Big) + \|\nabla v\|^2\\
&+ \delta^3\|u\|^2 - \epsilon(\delta^2 - \delta)(u, z(\theta_t\omega)) - \epsilon(\Delta z(\theta_t\omega), v)\\
&+ \delta(1-\delta)\|\nabla u\|^2 - \epsilon(1-\delta)(\nabla u, \nabla z(\theta_t\omega)) + (1-\delta)\|v\|^2\\
&- \epsilon\delta(z(\theta_t\omega), v) + \delta(g(x,u), u) - \epsilon(g(x,u), z(\theta_t\omega))\\
=&(f, v) + \delta(u, v + \epsilon z(\theta_t\omega)). \qquad (3.24)
\end{aligned}$$

By the Young inequality, we have

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big(\|v\|^2 + (1-\delta)\|\nabla u\|^2 + \delta^2\|u\|^2 + 2\mathcal{G}(u)\Big) + \|\nabla v\|^2\\
&+ \delta^3\|u\|^2 + \delta(1-\delta)\|\nabla u\|^2 + (1-\delta)\|v\|^2 + \delta(g(x,u), u)\\
\le&\frac{\delta^3}{8}\|u\|^2 + \frac{2\epsilon(\delta-\delta^2)^2}{\delta^3}\|z(\theta_t\omega)\|^2 + \frac{1}{2}\|\nabla v\|^2 + \frac{\epsilon}{2}\|\nabla z(\theta_t\omega)\|^2 + \frac{\delta(1-\delta)}{2}\|\nabla u\|^2\\
&+ \frac{\epsilon(1-\delta)}{2\delta}\|\nabla z(\theta_t\omega)\|^2 + \frac{\delta}{2}\|v\|^2 + \frac{\delta\epsilon}{2}\|z(\theta_t\omega)\|^2 + \epsilon(g(x,u), z(\theta_t\omega))\\
&+ \frac{\delta}{2}\|v\|^2 + \frac{1}{2\delta}\|f\|^2 + \frac{\delta^2}{2}\|u\|^2 + \frac{\|v\|^2}{2} + \frac{\delta^3}{8}\|u\|^2 + \frac{2\epsilon}{\delta}\|z(\theta_t\omega)\|^2. \qquad (3.25)
\end{aligned}$$

By assumption (A_2), the Cauchy inequality and the Hölder inequality, we have

$$\begin{aligned}
&\epsilon(g(x,u), z(\theta_t\omega))\\
=&\epsilon(g'_u(x, \theta u)u, z(\theta_t\omega))\\
\le&\epsilon c_6\int_{\mathbb{R}^N}|u||z|\mathrm{d}x + \epsilon c_6\int_{\mathbb{R}^N}|u|^p|z|\mathrm{d}x\\
\le&\frac{\delta^3}{4}\|u\|^2 + \frac{\epsilon c_6^2}{\delta^3}\|z(\theta_t\omega)\|^2 + \epsilon c_6\Big(\int_{\mathbb{R}^N}|u|^{p\cdot\frac{p+1}{p}}\mathrm{d}x\Big)^{\frac{p}{p+1}}\Big(\int_{\mathbb{R}^N}|z(\theta_t\omega)|^{p+1}\mathrm{d}x\Big)^{\frac{1}{p+1}}\\
\le&\frac{\delta^3}{4}\|u\|^2 + \frac{\epsilon c_6^2}{\delta^3}\|z(\theta_t\omega)\|^2 + \frac{\delta c_5}{4}\int_{\mathbb{R}^N}|u|^{p+1}\mathrm{d}x + \frac{\epsilon c_6^2}{\delta c_5}\|z(\theta_t\omega)\|_{H^1}^{p+1}, \qquad (3.26)
\end{aligned}$$

where $0 < \theta < 1$. Substituting (3.26) into (3.25), and using assumption (A_6), we can get that,

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big(\|v\|^2+(1-\delta)\|\nabla u\|^2+\delta^2\|u\|^2+2\mathcal{G}(u)\Big)+\frac{1}{2}\|\nabla v\|^2\\
&\quad+(\frac{\delta^3}{2}-\frac{\delta^2}{2})\|u\|^2+(\frac{\delta}{4}-\frac{\delta^2}{2})\|\nabla u\|^2+\frac{\delta}{4}\|\nabla u\|^2\\
&\quad+(\frac{1}{2}-2\delta)\|v\|^2+\frac{3\delta}{4}(g(x,u),u)\\
\le&\epsilon c_7\big(\|z(\theta_t\omega)\|^2_{H^1}+\|z(\theta_t\omega)\|^{p+1}_{H^1}\big)+\frac{1}{2\delta}\|f\|^2+\|\phi_1\|_1,
\end{aligned}$$

which along with (3.14) yields

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big(\|v\|^2+(1-\delta)\|\nabla u\|^2+\delta^2\|u\|^2+2\mathcal{G}(u)\Big)+\frac{1}{2}\|\nabla v\|^2\\
&\quad+(\frac{\delta^3}{2}-\frac{\delta^2}{2})\|u\|^2+(\frac{\delta}{4}-\frac{\delta^2}{2})\|\nabla u\|^2\\
&\quad+(\frac{1}{2}-2\delta)\|v\|^2+\frac{\delta}{2}(g(x,u),u)+\frac{\delta\alpha}{4}\|u\|^2\\
\le&\epsilon c_7\big(\|z(\theta_t\omega)\|^2_{H^1}+\|z(\theta_t\omega)\|^{p+1}_{H^1}\big)+\frac{1}{2\delta}\|f\|^2+\beta+\|\phi_1\|_1. \qquad (3.27)
\end{aligned}$$

Set

$$\begin{aligned}
H(t):=&\|v\|^2+(1-\delta)\|\nabla u\|^2+\delta^2\|u\|^2+2\mathcal{G}(u),\\
K(t):=&\frac{\delta}{2}(\delta^2-\delta+\frac{\alpha}{2})\|u\|^2+\frac{\delta}{4}(1-2\delta)\|\nabla u\|^2+(\frac{1}{2}-2\delta)\|v\|^2+\frac{\delta}{2}(g(x,u),u). \qquad (3.28)
\end{aligned}$$

By Lemma 3.2 we can get that, for any η: $0<\eta<\min\{c_2,\frac{2c_3}{c_4+2c_3},\delta\}$,

$$\begin{aligned}
&K(t)-\frac{\delta\eta}{4}H(t)\\
=&\frac{\delta}{2}(\delta^2-\delta+\frac{\alpha}{2}-\frac{\eta}{2}\delta^2)\|u\|^2+\frac{\delta}{4}(1-2\delta-\eta+\eta\delta)\|\nabla u\|^2\\
&+(\frac{1}{2}-2\delta-\frac{\delta\eta}{4})\|v\|^2+\frac{\delta}{2}(g(x,u),u)-\frac{\delta\eta}{2}\mathcal{G}(u)\\
\ge&\frac{\delta}{2}\big((g(x,u),u)-\eta\mathcal{G}(u)\big)+\frac{\delta}{2}(\frac{\delta^2}{2}+\frac{\alpha}{2}-\delta)\|u\|^2+\frac{1}{2}(1-5\delta)\|v\|^2+\frac{\delta(1-3\delta)}{4}\|\nabla u\|^2\\
\ge&\frac{\delta}{2}\big(-\nu\|u\|^2-\rho_2(\nu)\big)+\frac{\delta}{2}(\frac{\delta^2}{2}+\frac{\alpha}{2}-\delta)\|u\|^2+\frac{1}{2}(1-5\delta)\|v\|^2+\frac{\delta(1-3\delta)}{4}\|\nabla u\|^2\\
\ge&\frac{\delta}{2}(\frac{\alpha}{2}-\delta)\|u\|^2+\frac{1}{2}(1-5\delta)\|v\|^2+\frac{\delta(1-3\delta)}{4}\|\nabla u\|^2-\rho_2(\nu)\\
\ge&\sigma_0\|\varphi\|^2_{\mathcal{H}}-\rho_2(\nu), \qquad (3.29)
\end{aligned}$$

where we have chosen

$$\nu=\frac{\delta^2}{2},\ \sigma_0=\min\{\frac{\delta}{2}(\frac{\alpha}{2}-\delta),\frac{1}{2}(1-5\delta),\frac{\delta}{4}(1-3\delta)\}.$$

It follows from (3.27)-(3.29) that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}s}H(s)+\frac{\delta\eta}{2}H(s)+2\sigma_0\|\varphi\|_{\mathcal{H}}^2+\|\nabla v\|^2\\ \le&2\epsilon c_7\big(\|z(\theta_s\omega)\|_{H^1}^2+\|z(\theta_s\omega)\|_{H^1}^{p+1}\big)+\frac{1}{\delta}\|f\|^2+2\|\phi_1\|_1+2\rho_2(\nu)+2\beta.\end{aligned}\tag{3.30}$$

Multiplying (3.30) by $\mathrm{e}^{\frac{\delta\eta}{2}s}$ and then integrating over $[\tau-t,\tau]$ with $t>0$ with respect to s, we have

$$\begin{aligned}&\|v(\tau,\tau-t,\omega,v_{\tau-t})\|^2+2\sigma_0\int_{\tau-t}^{\tau}\mathrm{e}^{\frac{\delta\eta}{2}(s-\tau)}\|\varphi(s,\tau-t,\omega,\varphi_{\tau-t})\|_{\mathcal{H}}^2\mathrm{d}s\\ &+\delta^2\|u(\tau,\tau-t,\omega,u_{\tau-t})\|^2+(1-\delta)\|\nabla u(\tau,\tau-t,\omega,u_{\tau-t})\|^2\\ &+\int_{\tau-t}^{\tau}\mathrm{e}^{\frac{\delta\eta}{2}(s-\tau)}\|\nabla v(s,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})\|^2\mathrm{d}s+2\mathcal{G}(u(\tau,\tau-t,\omega,u_{\tau-t}))\\ \le&\Big(\|v_{\tau-t}\|^2+(1-\delta)\|\nabla u_{\tau-t}\|^2\mathrm{d}s+\delta^2\|u_{\tau-t}\|^2+2\mathcal{G}(u_{\tau-t})\Big)\mathrm{e}^{-\frac{\delta\eta}{2}t}\\ &+2\epsilon c_7\|h\|_{H^1}^2\int_{\tau-t}^{\tau}\mathrm{e}^{\frac{\delta\eta}{2}(s-\tau)}|y(\theta_s\omega)|^2\mathrm{d}s+\frac{1}{\delta}\int_{\tau-t}^{\tau}\mathrm{e}^{\frac{\delta\eta}{2}(s-\tau)}\|f(s)\|^2\mathrm{d}s\\ &+2\epsilon c_7\|h\|_{H^1}^{p+1}\int_{\tau-t}^{\tau}\mathrm{e}^{\frac{\delta\eta}{2}(s-\tau)}|y(\theta_s\omega)|^{p+1}\mathrm{d}s+\frac{4}{\delta\eta}\|\phi_1\|_1+\frac{4}{\delta\eta}\rho_2(\nu)+\frac{4\beta}{\delta\eta}.\end{aligned}\tag{3.31}$$

Substituting ω in (3.31) by $\theta_{-\tau}\omega$, we get that

$$\begin{aligned}&\|v(\tau,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})\|^2+2\sigma_0\int_{\tau-t}^{\tau}\mathrm{e}^{\frac{\delta\eta}{2}(s-\tau)}\|\varphi(s,\tau-t,\theta_{-\tau},\varphi_{\tau-t})\|_{\mathcal{H}}^2\mathrm{d}s\\ &+\delta^2\|u(\tau,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})\|^2+(1-\delta)\|\nabla u(\tau,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})\|^2\\ &+\int_{\tau-t}^{\tau}\mathrm{e}^{\frac{\delta\eta}{2}(s-\tau)}\|\nabla v(s,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})\|^2\mathrm{d}s+2\mathcal{G}(u(\tau,\tau-t,\theta_{-\tau}\omega,u_{\tau-t}))\\ \le&\Big(\|v_{\tau-t}\|^2+(1-\delta)\|\nabla u_{\tau-t}\|^2+\delta^2\|u_{\tau-t}\|^2+2\mathcal{G}(u_{\tau-t})\Big)\mathrm{e}^{-\frac{\delta\eta}{2}t}+\frac{4}{\delta\eta}\|\phi_1\|_1+\frac{4\beta}{\delta\eta}\\ &+\frac{4}{\delta\eta}\rho_2(\nu)+\epsilon c_8(1+\int_{\tau-t}^{\tau}\mathrm{e}^{\frac{\delta\eta}{2}(s-\tau)}|y(\theta_{s-\tau}\omega)|^{p+1}\mathrm{d}s)+\frac{1}{\delta}\int_{\tau-t}^{\tau}\mathrm{e}^{\frac{\delta\eta}{2}(s-\tau)}\|f(s)\|^2\mathrm{d}s\\ \le&\Big(\|v_{\tau-t}\|^2+(1-\delta)\|\nabla u_{\tau-t}\|^2+\delta^2\|u_{\tau-t}\|^2+2\mathcal{G}(u_{\tau-t})\Big)\mathrm{e}^{-\frac{\delta\eta}{2}t}+\frac{4}{\delta\eta}\|\phi_1\|_1+\frac{4\beta}{\delta\eta}\\ &+\frac{4}{\delta\eta}\rho_2(\nu)+\frac{1}{\delta}\int_{-\infty}^{0}\mathrm{e}^{\frac{\delta\eta}{2}s}\|f(s+\tau)\|^2\mathrm{d}s+\epsilon c_8(1+\int_{-\infty}^{0}\mathrm{e}^{\frac{\delta\eta}{2}s}|y(\theta_s\omega)|^{p+1}\mathrm{d}s).\end{aligned}\tag{3.32}$$

Notice that $|y(\theta_t\omega)|$ is tempered, and then we have

$$\int_{-\infty}^{0}\mathrm{e}^{\frac{\delta\eta}{2}s}|y(\theta_s\omega)|^{p+1}\mathrm{d}s<+\infty,\ \forall\omega\in\Omega.\tag{3.33}$$

Since $\varphi_{\tau-t}\in D(\tau-t,\theta_{-t}\omega)$, by assumptions (A_2), the embedding $H^1(\mathbb{R}^N)\hookrightarrow L^{p+1}(\mathbb{R}^N)$ and tempered property of D, we obtain that

$$\Big(\|v_{\tau-t}\|^2+(1-\delta)\|\nabla u_{\tau-t}\|^2+\delta^2\|u_{\tau-t}\|^2+2\mathcal{G}(u_{\tau-t})\Big)\mathrm{e}^{-\frac{\delta\eta}{2}t}$$

$$\leq C\Big(\|v_{\tau-t}\|^2+\|u_{\tau-t}\|^2+\|u_{\tau-t}\|_{H^1}^{p+1}+\|\nabla u_{\tau-t}\|^2\Big)\mathrm{e}^{-\frac{\delta\eta}{2}t}\to 0 \text{ as } t\to+\infty. \quad (3.34)$$

Moreover, by Lemma 3.2 we can get that

$$\begin{aligned}
&\|v(\tau,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})\|^2+(1-\delta)u(\tau,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})\|^2\\
&\quad+\delta^2\|u(\tau,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})\|^2+2\mathcal{G}(u(\tau,\tau-t,\theta_{-\tau}\omega,u_{\tau-t}))\\
\geq&\|v(\tau,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})\|^2+(1-\delta)\|\nabla u(\tau,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})\|^2\\
&\quad+\delta^2\|u(\tau,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})\|^2-2\nu\|u(\tau,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})\|^2-2\rho_1(\nu)\\
\geq&\sigma_1\|\varphi(\tau,\tau-t,\theta_{-\tau}\omega,\varphi_{\tau-t})\|_{\mathcal{H}}^2-2\rho_1(\nu),
\end{aligned} \quad (3.35)$$

where we have chosen $\nu=\dfrac{\delta^2}{4}$ and $\sigma_1=\dfrac{\delta^2}{2}$. The combination of (3.32)-(3.35) implies the result. □

Corollary 3.1. *Let assumptions* (**i**)-(**ii**) *hold. Then* Ψ_ϵ *generated by problem* (3.6)-(3.8) *possesses a closed measurable* $\mathcal{D}$*-pullback absorbing set* $K_\epsilon=\{K_\epsilon(\tau,\omega):\tau\in\mathbb{R},\omega\in\Omega\}\in\mathcal{D}$*, where*

$$K_\epsilon(\tau,\omega)=\{\varphi\in\mathcal{H}(\mathbb{R}^N):\|\varphi\|_{\mathcal{H}}^2\leq\mathcal{R}_{1,\epsilon}(\tau,\omega)\},$$

and $\mathcal{R}_{1,\epsilon}(\tau,\omega)$ *is given by Lemma 3.3.*

Proof. From Lemma 3.3 and (3.10), we immediately get that, for every $\tau\in\mathbb{R}$, $\omega\in\Omega$ and every $D\in\mathcal{D}$, K_ϵ satisfies that

$$\Psi_\epsilon(t,\tau-t,\theta_{-t}\omega,D(\tau-t,\theta_{-t}\omega))\subseteq K_\epsilon(\tau,\omega),\ \forall t\geq T_1,$$

where T_1 is given by Lemma 3.3. We next prove that K_ϵ is tempered, i.e. $K_\epsilon\in\mathcal{D}$. For any $\gamma>0$, $\epsilon\in(0,1]$, we have

$$\begin{aligned}
&\mathrm{e}^{-\gamma t}\|K_\epsilon(\tau-t,\theta_{-t}\omega)\|^2\leq\mathrm{e}^{-\gamma t}\mathcal{R}_{1,\epsilon}(\tau-t,\theta_{-t}\omega)\\
=&M_1\mathrm{e}^{-\gamma t}+M_1\mathrm{e}^{-\gamma t}\Big(\int_{-\infty}^0\mathrm{e}^{\frac{\delta\eta}{2}s}\|f(s+\tau-t)\|^2\mathrm{d}s+\epsilon\big(1+\int_{-\infty}^0\mathrm{e}^{\frac{\delta\eta}{2}s}|y(\theta_{s-t}\omega)|^{p+1}\mathrm{d}s\big)\Big)\\
\leq&2M_1\mathrm{e}^{-\gamma t}+M_1\mathrm{e}^{-\gamma t}\int_{-\infty}^0\mathrm{e}^{\frac{\delta\eta}{2}s}\big(\|f(s+\tau-t)\|^2+|y(\theta_{s-t}\omega)|^{p+1}\big)\mathrm{d}s.
\end{aligned} \quad (3.36)$$

Let $\tilde{\gamma}=\min\{\gamma,\frac{\delta\eta}{2}\}$, and we get that for all $t\geq0$,

$$\begin{aligned}
&M_1\mathrm{e}^{-\gamma t}\int_{-\infty}^0\mathrm{e}^{\frac{\delta\eta}{2}s}\big(\|f(s+\tau-t)\|^2+|y(\theta_{s-t}\omega)|^{p+1}\big)\mathrm{d}s\\
\leq&M_1\int_{-\infty}^0\mathrm{e}^{\tilde{\gamma}(s-t)}\big(\|f(s+\tau-t)\|^2+|y(\theta_{s-t}\omega)|^{p+1}\big)\mathrm{d}s\\
\leq&M_1\int_{-\infty}^{-t}\mathrm{e}^{\tilde{\gamma}s}\big(\|f(s+\tau)\|^2+|y(\theta_s\omega)|^{p+1}\big)\mathrm{d}s.
\end{aligned} \quad (3.37)$$

By assumption (**ii**) and the tempered property of $|y(\theta_s\omega)|$, we have

$$\int_{-\infty}^0\mathrm{e}^{\tilde{\gamma}s}\big(\|f(s+\tau)\|^2+|y(\theta_s\omega)|^{p+1}\big)\mathrm{d}s<+\infty,\ \text{for all } \tau\in\mathbb{R}. \quad (3.38)$$

It follows from (3.36)-(3.38) that

$$\lim_{t\to+\infty} \mathrm{e}^{-\gamma t}\|K_\epsilon(\tau-t,\theta_{-t}\omega)\|^2=0,$$

that is $K_\epsilon\in\mathcal{D}$. Furthermore, since for each $\tau\in\mathbb{R}$, $\mathcal{R}_{1,\epsilon}(\tau,\cdot):\Omega\mapsto\mathbb{R}$ is $(\mathcal{F},\mathcal{B}(\mathbb{R}))$-measurable, then $K_\epsilon(\tau,\cdot)$ is also measurable. Hence, $K_\epsilon\in\mathcal{D}$ is a closed measurable $\mathcal{D}$-pullback absorbing set for Ψ_ϵ. We complete the proof. □

To get the pullback asymptotic compactness for Ψ_ϵ, the following lemmas will be needed.

Lemma 3.4. *Let assumptions* (**i**)-(**ii**) *hold. Then for any* $\tau\in\mathbb{R}$, $\omega\in\Omega$, *and* $D=\{D(\tau,\omega):\tau\in\mathbb{R},\omega\in\Omega\}\in\mathcal{D}$, *there exists* $T_2=T_2(\tau,\omega,D)>0$ *such that for all* $t\ge T_2$, *and* $r\in[0,t]$, *the solution* φ *of problem* (3.6)-(3.8) *satisfies*

$$\|\varphi(\tau-r,\tau-t,\theta_{-\tau}\omega,\varphi_{\tau-t})\|^2_{\mathcal{H}(\mathbb{R}^N)}\le M_2+M_2\mathrm{e}^{\frac{\delta\eta}{2p}r}\mathcal{R}_{2,\epsilon}(\tau,\omega), \tag{3.39}$$

where $\varphi_{\tau-t}\in D(\tau-t,\theta_{-t}\omega)$ *and* $\mathcal{R}_{2,\epsilon}$ *is given by*

$$\mathcal{R}_{2,\epsilon}(\tau,\omega)=\epsilon(1+\int_{-\infty}^0\mathrm{e}^{\frac{\delta\eta}{2p}s}|y(\theta_s\omega)|^{2p}\mathrm{d}s)+\int_{-\infty}^0\mathrm{e}^{\frac{\delta\eta}{2p}s}\|f(s+\tau)\|^2\mathrm{d}s,$$

and M_2 *is a positive constant independent of* τ, ω, D *and* ϵ.

Proof. Similarly to (3.29), by Lemma 3.2 we can get that for any $p\ge1$ and $0<\eta<\min\{c_2,\frac{2c_3}{c_4+2c_3},\delta\}$,

$$K(t)-\frac{\delta\eta}{4p}H(t)\ge-\rho_2(\nu),$$

where $K(t)$ and $H(t)$ are given by (3.28). Then we can get

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}H(t)+\frac{\delta\eta}{2p}H(t)\\ \le&2\epsilon c_7\big(\|z(\theta_t\omega)\|^2_{H^1}+\|z(\theta_t\omega)\|^{p+1}_{H^1}\big)+\frac{1}{\delta}\|f\|^2+2\rho_2(\nu)+2\beta+2\|\phi_1\|_1.\end{aligned} \tag{3.40}$$

Applying the Gronwall inequality to (3.40) on interval $[\tau-t,\tau-r]$ with $r\in[0,t]$, we can get

$$\begin{aligned}&\|v(\tau-r,\tau-t,\omega,v_{\tau-t})\|^2+(1-\delta)\|\nabla u(\tau-r,\tau-t,\omega,u_{\tau-t})\|^2\\ &+\delta^2\|u(\tau-r,\tau-t,\omega,u_{\tau-t})\|^2+2\mathcal{G}(u(\tau-r,\tau-t,\omega,u_{\tau-t}))\\ \le&\Big(\|v_{\tau-t}\|^2+(1-\delta)\|\nabla u_{\tau-t}\|^2+\delta^2\|u_{\tau-t}\|^2+2\mathcal{G}(u_{\tau-t})\Big)\mathrm{e}^{\frac{\delta\eta}{2p}(r-t)}\\ &+2\epsilon c_7\|h\|^2_{H^1}\int_{\tau-t}^{\tau-r}\mathrm{e}^{\frac{\delta\eta}{2p}(s-\tau+r)}|y(\theta_s\omega)|^2\mathrm{d}s+\frac{1}{\delta}\int_{\tau-t}^{\tau-r}\mathrm{e}^{\frac{\delta\eta}{2p}(s-\tau+r)}\|f(s)\|^2\mathrm{d}s\\ &+2\epsilon c_7\|h\|^{p+1}_{H^1}\int_{\tau-t}^{\tau-r}\mathrm{e}^{\frac{\delta\eta}{2p}(s-\tau+r)}|y(\theta_s\omega)|^{p+1}\mathrm{d}s\\ &+\frac{4p}{\delta\eta}\rho_2(\nu)+\frac{4p}{\delta\eta}\beta+\frac{4p}{\delta\eta}\|\phi_1\|_1.\end{aligned} \tag{3.41}$$

Substituting ω by $\theta_{-\tau}\omega$ in above, we can obtain that

$$\|v(\tau-r,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})\|^2+(1-\delta)\|\nabla u(\tau-r,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})\|^2$$

$$
\begin{aligned}
&+\delta^2\|u(\tau-r,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})\|^2+2\mathcal{G}(u(\tau-r,\tau-t,\theta_{-\tau}\omega,u_{\tau-t}))\\
\le&\mathrm{e}^{\frac{\delta\eta}{2p}r}\Big(\big(\|v_{\tau-t}\|^2+(1-\delta)\|\nabla u_{\tau-t}\|^2+\delta^2\|u_{\tau-t}\|^2+2\mathcal{G}(u_{\tau-t})\big)\mathrm{e}^{-\frac{\delta\eta}{2p}t}\\
&+2\epsilon c_7\|h\|_{H^1}^2\int_{-t}^{-r}\mathrm{e}^{\frac{\delta\eta}{2p}s}|y(\theta_s\omega)|^2\mathrm{d}s+\frac{1}{\delta}\int_{-t}^{-r}\mathrm{e}^{\frac{\delta\eta}{2p}s}\|f(s+\tau)\|^2\mathrm{d}s\\
&+2\epsilon c_7\|h\|_{H^1}^{p+1}\int_{-t}^{-r}\mathrm{e}^{\frac{\delta\eta}{2p}s}|y(\theta_s\omega)|^{p+1}\mathrm{d}s\Big)+\frac{4p}{\delta\eta}\rho_2(\nu)+\frac{4p}{\delta\eta}\beta+\frac{4p}{\delta\eta}\|\phi_1\|_1\\
\le&\mathrm{e}^{\frac{\delta\eta}{2p}r}\Big(\big(\|v_{\tau-t}\|^2+(1-\delta)\|\nabla u_{\tau-t}\|^2+\delta^2\|u_{\tau-t}\|^2+2\mathcal{G}(u_{\tau-t})\big)\mathrm{e}^{-\frac{\delta\eta}{2p}t}\\
&+\epsilon c_8(1+\int_{-\infty}^{0}\mathrm{e}^{\frac{\delta\eta}{2p}s}|y(\theta_s\omega)|^{p+1}\mathrm{d}s)+\frac{1}{\delta}\int_{-\infty}^{0}\mathrm{e}^{\frac{\delta\eta}{2p}s}\|f(s+\tau)\|^2\mathrm{d}s\Big)\\
&+\frac{4p}{\delta\eta}\rho_2(\nu)+\frac{4p}{\delta\eta}\beta+\frac{4p}{\delta\eta}\|\phi_1\|_1.
\end{aligned}
$$

Setting

$$
\mathcal{R}_\epsilon(\tau,\omega)=\epsilon c_8(1+\int_{-\infty}^{0}\mathrm{e}^{\frac{\delta\eta}{2p}s}|y(\theta_s\omega)|^{p+1}\mathrm{d}s)+\frac{1}{\delta}\int_{-\infty}^{0}\mathrm{e}^{\frac{\delta\eta}{2p}s}\|f(s+\tau)\|^2\mathrm{d}s,
$$

then by the tempered property of $|y(\theta_s\omega)|$ and assumption (**ii**), we can get that $R_\epsilon(\tau,\omega)$ is well defined. Similarly to (3.34), we have

$$
\Big(\|v_{\tau-t}\|^2+(1-\delta)\|\nabla u_{\tau-t}\|^2+\delta^2\|u_{\tau-t}\|^2+2\mathcal{G}(u_{\tau-t})\Big)\mathrm{e}^{-\frac{\delta\eta}{2p}t}\to 0,\ \text{as } t\to+\infty.
$$

Then we obtain that there exists $T_2=T_2(\tau,\omega,D)$ such that for any $t\ge T_2$

$$
\begin{aligned}
&\|v(\tau-r,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})\|^2+(1-\delta)\|\nabla u(\tau-r,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})\|^2\\
&+\delta^2\|u(\tau-r,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})\|^2+2\mathcal{G}(u(\tau-r,\tau-t,\theta_{-\tau}\omega,u_{\tau-t}))\\
\le&2\mathrm{e}^{\frac{\delta\eta}{2p}r}\mathcal{R}_\epsilon(\tau,\omega)+\frac{4p}{\delta\eta}\rho_2(\nu)+\frac{4p}{\delta\eta}\beta+\frac{4p}{\delta\eta}\|\phi_1\|_1,
\end{aligned}
\tag{3.42}
$$

which along with (3.12) implies the result. □

Choose a smooth function ρ such that $0\le\rho(s)\le1$ for all $s\in\mathbb{R}$ and

$$
\rho(s)=\begin{cases}0, & |s|\le 1,\\ 1, & |s|\ge 2.\end{cases}
$$

Denote by $\psi(x)=\rho(\frac{|x|}{R})$ for all $x\in\mathbb{R}^N$. Then we have $0\le\psi(x)\le1$ for all $x\in\mathbb{R}^N$ and

$$
\psi(x)=\begin{cases}0, & |x|\le R,\\ 1, & |x|\ge 2R.\end{cases}\tag{3.43}
$$

In addition, we can get that there exist positive constants κ_1 and κ_2 such that

$$
|\nabla\psi(x)|^2\le\frac{\kappa_1}{R^2},\ \ |\Delta\psi(x)|\le\frac{\kappa_2}{R^2}.\tag{3.44}
$$

For any given $R>r_0$, let $B_R:=\{x\in\mathbb{R}^N:|x|\le R\}$ and $B_R^C:=\mathbb{R}^N\backslash B_R$.

Lemma 3.5. *Let assumptions* (**i**)-(**ii**) *hold. Then for any* $\varepsilon > 0$, $\tau \in \mathbb{R}$, $\omega \in \Omega$, *and* $D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$, *there exist* $T_3 = T_3(\tau, \varepsilon, \omega, D) > 0$ *and* $R_1 = R_1(\tau, \varepsilon, \omega) > r_0$, *such that for all* $t \geq T_3$, $R \geq R_1$, *the solution* φ *of problem* (3.6)-(3.8) *satisfies*

$$\|\varphi(\tau, \tau - t, \theta_{-\tau}\omega, \varphi_{\tau-t})\|^2_{\mathcal{H}(B^C_{2R})} \leq \varepsilon,$$

where $\varphi_{\tau-t} \in D(\tau - t, \theta_{-t}\omega)$.

Proof. Using the multiplier $\varphi^2 v$ in equation (3.6), we have

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\psi v\|^2 + (\nabla v, \nabla(\psi^2 v)) + \epsilon(\nabla z(\theta_t\omega), \nabla(\psi^2 v)) + (1-\delta)(\nabla u, \nabla(\psi^2 v)) \\
&\quad + (\delta^2 - \delta)(u, \psi^2 v) + (1-\delta)\|\psi v\|^2 - \delta\epsilon(z(\theta_t\omega), \psi^2 v) \\
=&(f(x), \psi^2 v) - (g(x,u), \psi^2 v).
\end{aligned} \tag{3.45}$$

By (3.7) and integration by parts, we can get

$$\begin{aligned}
(\nabla u, \nabla(\psi^2 v)) &= (\nabla u, 2\psi\nabla\psi v) + (\psi\nabla u, \psi\nabla v) \\
&= (\nabla u, 2\psi\nabla\psi v) + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\psi\nabla u\|^2 + \delta\|\psi\nabla u\|^2 - \epsilon(\psi\nabla u, \psi\nabla z(\theta_t\omega)),
\end{aligned} \tag{3.46}$$

$$(u, \psi^2 v) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\psi u\|^2 + \delta\|\psi u\|^2 - \epsilon(\psi u, \psi z(\theta_t\omega)), \tag{3.47}$$

$$(g(x,u), \psi^2 v) = \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^N}\psi^2 G(x,u)\mathrm{d}x + \delta(g(x,u), \psi^2 u) - \epsilon(g(x,u), \psi^2 z(\theta_t\omega)), \tag{3.48}$$

$$(\nabla v, \nabla(\psi^2 v)) = (\nabla v, 2\psi\nabla\psi v) + (\psi\nabla v, \psi\nabla v), \tag{3.49}$$

$$(\nabla z, \nabla(\psi^2 v)) = (\psi\nabla z(\theta_t\omega), \psi\nabla v) + (\nabla z(\theta_t\omega), 2\psi\nabla\psi v). \tag{3.50}$$

Then we can get from (3.45)-(3.50) that

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big(\|\psi v\|^2 + (1-\delta)\|\psi\nabla u\|^2 + 2\int_{\mathbb{R}^N}\psi^2 G(x,u)\mathrm{d}x + (\delta^2-\delta)\|\psi u\|^2\Big) \\
&\quad + (\nabla v, 2\psi\nabla\psi v) + \|\psi\nabla v\|^2 + \epsilon(\psi\nabla z(\theta_t\omega), \psi\nabla v) + \epsilon(\nabla z(\theta_t\omega), 2\psi\nabla\psi v) \\
&\quad + (1-\delta)(\nabla u, 2\psi\nabla\psi v) + (1-\delta)\|\psi v\|^2 + \delta(g(x,u), \psi^2 u) \\
&\quad + (1-\delta)\delta\|\psi\nabla u\|^2 + \delta(\delta^2-\delta)\|\psi u\|^2 \\
=&\epsilon(1-\delta)(\psi\nabla u, \psi\nabla z(\theta_t\omega)) + \epsilon(\delta^2-\delta)(\psi u, \psi z(\theta_t\omega)) + \delta\epsilon(\psi z(\theta_t\omega), \psi v) \\
&\quad + \epsilon(g(x,u), \psi^2 z(\theta_t\omega)) + (f(x), \psi^2 v).
\end{aligned} \tag{3.51}$$

Set

$$H_1(t) := \|\psi v\|^2 + (1-\delta)\|\psi\nabla u\|^2 + 2\int_{\mathbb{R}^N}\psi^2 G(x,u)\mathrm{d}x + (\delta^2-\delta)\|\psi u\|^2.$$

Then by the Young inequality, assumption (A_5) and some calculations, we can get from (3.51) that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}H_1(t) + \frac{\delta c_3}{4}\|\psi u\|^2 + \frac{3\delta}{4}(g(x,u), \psi^2 u) + \|\psi\nabla v\|^2 + \delta(1-\delta)\|\psi\nabla u\|^2$$

$$
\begin{aligned}
&+\delta(\delta^2-\delta)\|\psi u\|^2+(1-\delta)\|\psi v\|^2\\
\leq&\frac{1}{2}\|\psi\nabla v\|^2+C\|\nabla\psi v\|^2+\epsilon C\|\psi\nabla z(\theta_t\omega)\|^2+\epsilon C\|\psi z(\theta_t\omega)\|^2\\
&+\delta\|\psi v\|^2+\frac{1}{2\delta}\|\psi f\|^2+\epsilon(g(x,u),\psi^2 z(\theta_t\omega))\\
&+\frac{\delta}{2}(1-\delta)\|\psi\nabla u\|^2+\frac{\delta}{4}(\frac{c_3}{4}+\delta^2-\delta)\|\psi u\|^2. \qquad (3.52)
\end{aligned}
$$

Using assumption (A_2), we have

$$
\begin{aligned}
\epsilon(g(x,u),\psi^2 z(\theta_t\omega)) =& \epsilon C\int_{\mathbb{R}^N}(|u|+|u|^p)\psi^2 z(\theta_t\omega)\mathrm{d}x\\
\leq& \epsilon C\int_{\mathbb{R}^N}\psi^2|u||z(\theta_t\omega)|\mathrm{d}x+\epsilon C\int_{\mathbb{R}^N}\psi^2|z(\theta_t\omega)||u|^p\mathrm{d}x\\
\leq& \frac{\delta}{4}(\frac{c_3}{4}+\delta^2-\delta)\|\psi u\|^2+\epsilon C\|\psi z(\theta_t\omega)\|^2\\
&+\epsilon C(\int_{\mathbb{R}^N}\psi^2|u|^{p+1}\mathrm{d}x)^{\frac{p}{p+1}}(\int_{\mathbb{R}^N}\psi^2|z(\theta_t\omega)|^{p+1}\mathrm{d}x)^{\frac{1}{p+1}}\\
\leq& \frac{\delta}{4}(\frac{c_3}{4}+\delta^2-\delta)\|\psi u\|^2+\epsilon C\|\psi z(\theta_t\omega)\|^2\\
&+\frac{\delta c_5}{4}\int_{\mathbb{R}^N}\psi^2|u|^{p+1}\mathrm{d}x+\epsilon C\int_{\mathbb{R}^N}\psi^2|z(\theta_t\omega)|^{p+1}\mathrm{d}x. \qquad (3.53)
\end{aligned}
$$

Therefore, by assumption (A_6), we have

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}H_1(t)+\frac{\delta}{2}(g(x,u),\psi^2 u)+\frac{1}{2}\|\psi\nabla v\|^2+\frac{\delta}{2}(1-\delta)\|\psi\nabla u\|^2\\
&+\frac{\delta}{2}(\frac{c_3}{4}+\delta^2-\delta)\|\psi u\|^2+(1-2\delta)\|\psi v\|^2\\
\leq&\frac{C}{R^2}\|v\|^2+\epsilon C(\|\psi\nabla z(\theta_t\omega)\|^2+\|\psi z(\theta_t\omega)\|^2)\\
&+\frac{1}{2\delta}\|\psi f\|^2+\epsilon C\int_{\mathbb{R}^N}\psi^2|z(\theta_t\omega)|^{p+1}\mathrm{d}x+\|\psi\phi_1\|_1. \qquad (3.54)
\end{aligned}
$$

Denote by

$$
K_1(t)=\frac{\delta}{2}(1-\delta)\|\psi\nabla u\|^2+\frac{\delta}{2}(\frac{c_3}{4}+\delta^2-\delta)\|\psi u\|^2+(1-2\delta)\|\psi v\|^2+\frac{\delta}{2}(g(x,u),\psi^2 u).
$$

Then, for any $0<\eta<\frac{2c_3}{c_4+2c_3}$, we have

$$
\begin{aligned}
K_1(t)-\frac{\eta\delta}{4}H_1(t)\geq&(1-2\delta-\frac{\eta\delta}{4})\|\psi v\|^2+\frac{\delta}{2}(1-\delta)(1-\frac{\eta}{2})\|\psi\nabla u\|^2\\
&+\frac{\delta}{2}\int_{\mathbb{R}^N}\psi^2 g(x,u)u-\eta\psi^2 G(x,u)\mathrm{d}x\\
\geq&\frac{\delta}{2}\int_{\mathbb{R}^N}\psi^2 g(x,u)u-\eta\psi^2 G(x,u)\mathrm{d}x. \qquad (3.55)
\end{aligned}
$$

Actually, we can obtain that

$$
\int_{\mathbb{R}^N}\psi^2 g(x,u)u-\eta\psi^2 G(x,u)\mathrm{d}x\geq 0, \ \text{ for all } \eta\in(0,\frac{2c_3}{c_4+2c_3}).
$$

Notice that $R > r_0$. Then by assumption (A_5) and $\eta < \frac{2c_3}{c_4+2c_3}$, we can get that

$$\begin{aligned}g(x,u)u - \eta G(x,u) &= \eta(g(x,u)u - G(x,u)) + (1-\eta)g(x,u)u \\ &= \eta \int_0^u g(x,u) - g(x,s)\mathrm{d}s + (1-\eta)g(x,u)u \\ &\geq -\frac{c_4}{2}\eta u^2 + (1-\eta)c_3u^2 \geq 0.\end{aligned}$$

Hence, we have

$$K_1(t) - \frac{\eta\delta}{4}H_1(t) \geq 0,$$

which along with (3.54) implies that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}H_1(t) + \frac{\delta\eta}{2}H_1(t) \leq& \frac{C}{R^2}\|v\|^2 + \epsilon C(\|\psi\nabla z(\theta_t\omega)\|^2 + \|\psi z(\theta_t\omega)\|^2) \\ &+ \frac{1}{\delta}\|\psi f\|^2 + \epsilon C\int_{\mathbb{R}^N}\psi^2|z(\theta_t\omega)|^{p+1}\mathrm{d}x + 2\|\psi\phi_1\|_1 \\ \leq& \frac{C}{R^2}\|v\|^2 + \epsilon C\|\psi h\|_{H^1}^2|y(\theta_t\omega)|^2 + 2\|\psi\phi_1\|_1 \\ &+ \frac{1}{\delta}\|\psi f\|^2 + \epsilon C|y(\theta_t\omega)|^{p+1}\int_{\mathbb{R}^N}\psi^2|h|^{p+1}\mathrm{d}x. \qquad (3.56)\end{aligned}$$

By the similar way as the derivation of (3.32), we can obtain that

$$\begin{aligned}&\|\psi v(\tau,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})\|^2 + (1-\delta)\|\psi\nabla u(\tau,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})\|^2 \\ &+2\int_{\mathbb{R}^N}\psi^2G(x,u(\tau,\tau-t,\theta_{-\tau}\omega,u_{\tau-t}))\mathrm{d}x + (\delta^2-\delta)\|\psi u(\tau,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})\|^2 \\ \leq& \Big(\|v_{\tau-t}\|^2 + (1-\delta)\|\nabla u_{\tau-t}\|^2 + 2\int_{\mathbb{R}^N}\psi^2G(x,u_{\tau-t})\mathrm{d}x \\ &+ (\delta^2-\delta)\|u_{\tau-t}\|^2\Big)\mathrm{e}^{-\frac{\eta\delta}{2}t} + \frac{4}{\delta\eta}\|\psi\phi_1\|_1 \\ &+ \epsilon C\int_{\mathbb{R}^N}\psi^2|h|^{p+1}\mathrm{d}x\int_{-t}^0\mathrm{e}^{\frac{\eta\delta}{2}s}|y(\theta_s\omega)|^{p+1}\mathrm{d}s + \epsilon C\|\psi h\|_{H^1}^2\int_{-t}^0\mathrm{e}^{\frac{\eta\delta}{2}s}|y(\theta_s\omega)|^2\mathrm{d}s \\ &+ \frac{C}{R^2}\int_{\tau-t}^{\tau}\mathrm{e}^{\frac{\eta\delta}{2}s}\|v(s,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})\|^2\mathrm{d}s + \frac{1}{\delta}\int_{-t}^0\mathrm{e}^{\frac{\eta\delta}{2}s}\|\psi f(s+\tau)\|^2\mathrm{d}s. \quad (3.57)\end{aligned}$$

Since $\varphi_{\tau-t} \in D(\tau-t,\theta_{-t}\omega)$, by assumptions (A_2), the embedding $H^1(\mathbb{R}^N) \hookrightarrow L^{p+1}(\mathbb{R}^N)$, and the tempered property of D, we can get that

$$\begin{aligned}&\Big(\|v_{\tau-t}\|^2 + (1-\delta)\|\nabla u_{\tau-t}\|^2 + 2\int_{\mathbb{R}^N}\psi^2G(x,u_{\tau-t})\mathrm{d}x \\ &+ (\delta^2-\delta)\|u_{\tau-t}\|^2\Big)\mathrm{e}^{-\frac{\eta\delta}{2}t} \to 0, \text{ as } t \to +\infty. \qquad (3.58)\end{aligned}$$

On the other hand, note that $h \in H^2(\mathbb{R}^N)$, $H^2(\mathbb{R}^N) \hookrightarrow L^{p+1}(\mathbb{R}^N)$, $\phi_1 \in L^1(\mathbb{R}^N)$ and $|y(\theta_t\omega)|$ is tempered. Therefore, by (3.18) and assumption (**ii**), we can get that for any $\varepsilon > 0$, there exist $T_3 = T_3(\tau,\varepsilon,\omega,D) > 0$ and $R_1 = R_1(\tau,\varepsilon,\omega) > r_0$ such that for any $t \geq T_3$ and $R \geq R_1$,

$$\|\psi v(\tau,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})\|^2 + (1-\delta)\|\psi\nabla u(\tau,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})\|^2$$

$$+ 2\int_{\mathbb{R}^N} \psi^2 G(x, u(\tau, \tau - t, \theta_{-\tau}\omega, u_{\tau-t}))\mathrm{d}x + (\delta^2 - \delta)\|\psi u(\tau, \tau - t, \theta_{-\tau}\omega, u_{\tau-t})\|^2$$
$$\leq \varepsilon. \tag{3.59}$$

Let $\alpha_2 = \delta^2$. Then by (3.15), we get

$$\begin{aligned}
&\|\psi v(\tau, \tau - t, \theta_{-\tau}\omega, v_{\tau-t})\|^2 + (1-\delta)\|\psi \nabla u(\tau, \tau - t, \theta_{-\tau}\omega, u_{\tau-t})\|^2 \\
&\quad + 2\int_{\mathbb{R}^N} \psi^2 G(x, u(\tau, \tau - t, \theta_{-\tau}\omega, u_{\tau-t}))\mathrm{d}x + (\delta^2 - \delta)\|\psi u(\tau, \tau - t, \theta_{-\tau}\omega, u_{\tau-t})\|^2 \\
\geq &\|\psi v(\tau, \tau - t, \theta_{-\tau}\omega, v_{\tau-t})\|^2 + (1-\delta)\|\psi \nabla u(\tau, \tau - t, \theta_{-\tau}\omega, u_{\tau-t})\|^2 \\
&\quad + (c_3 + \delta^2 - \delta)\|\psi u(\tau, \tau - t, \theta_{-\tau}\omega, u_{\tau-t})\|^2 \\
\geq &\alpha_2 \|\psi \varphi(\tau, \tau - t, \theta_{-\tau}\omega, \varphi_{\tau-t})\|^2_{\mathcal{H}},
\end{aligned}$$

which along with (3.59) implies the result. □

Denote by $\tilde{\rho}(s) = 1 - \rho(s)$ for $s \in \mathbb{R}$ and $\tilde{\psi} = \tilde{\rho}(\frac{|x|}{R})$ for all $x \in \mathbb{R}^N$. Set

$$\begin{cases} \tilde{u}(t, \tau, \omega, \tilde{u}_0) = \tilde{\psi} u(t, \tau, \omega, u_0), \\ \tilde{v}(t, \tau, \omega, \tilde{v}_0) = \tilde{\psi} v(t, \tau, \omega, v_0), \end{cases} \tag{3.60}$$

where (u, v) is the solution of problem (3.6)-(3.8). Notice that $\tilde{\varphi} = (\tilde{u}, \tilde{v})$ satisfies the following equations in bounded domain B_{2R}:

$$\begin{aligned}
&\frac{\mathrm{d}\tilde{v}}{\mathrm{d}t} - \Delta\tilde{v} - \epsilon\tilde{\psi}\Delta z(\theta_t\omega) - (1-\delta)\Delta\tilde{u} + (\delta^2 - \delta)\tilde{u} + (1-\delta)\tilde{v} \\
&= -\tilde{\psi} g + \epsilon\delta\tilde{\psi} z(\theta_t\omega) + \tilde{\psi} f - \Delta\tilde{\psi} v - 2\nabla\tilde{\psi}\nabla v - (1-\delta)(\Delta\tilde{\psi} u + 2\nabla\tilde{\psi}\nabla u), \qquad (3.61)\\
&\frac{\mathrm{d}\tilde{u}}{\mathrm{d}t} = \tilde{v} - \delta\tilde{u} + \epsilon\tilde{\psi} z(\theta_t\omega), \qquad (3.62)
\end{aligned}$$

with boundary condition

$$\tilde{u} = \tilde{v} = 0 \text{ for } |x| = 2R.$$

Let $\{\mathrm{e}_k\}_{k=1}^{\infty}$ be an orthonormal basis of $L^2(B_{2R})$ such that

$$-\Delta \mathrm{e}_k = \lambda_k \mathrm{e}_k, \tag{3.63}$$

with zero boundary condition in B_{2R}, where

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots, \ \lambda_k \to +\infty, \text{ as } k \to +\infty.$$

Let $P_n : L^2(B_{2R}) \mapsto X_n$ be the orthogonal projection operator from $L^2(B_{2R})$ onto the space $X_n = \mathrm{span}\{\mathrm{e}_1, \mathrm{e}_2, \cdots, \mathrm{e}_n\}$.

Lemma 3.6. *Let assumptions* (**i**)-(**ii**) *hold. Then for any* $\varepsilon > 0$, $\tau \in \mathbb{R}$, $\omega \in \Omega$ *and* $D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$, *there exist* $R_2 = R_2(\tau, \varepsilon, \omega) \geq 1$, $T_4 = T_4(\tau, \varepsilon, \omega, D) > 0$ *and* $N_1 = N_1(\tau, \varepsilon) \geq 1$ *such that for any* $R \geq R_2$, $t \geq T_4$ *and* $n \geq N_1$,

$$\|(I - P_n)\tilde{\varphi}(\tau, \tau - t, \theta_{-\tau}\omega, \tilde{\varphi}_{\tau-t})\|^2_{\mathcal{H}(B_{2R})} \leq \varepsilon,$$

with $\tilde{\varphi}_{\tau-t} = \tilde{\psi}\varphi_{\tau-t} \in D(\tau - t, \theta_{-t}\omega)$.

Proof. Set $\tilde{u}_{n,1} = P_n\tilde{u}$, $\tilde{u}_{n,2} = (I - P_n)\tilde{u}$, $\tilde{v}_{n,1} = P_n\tilde{v}$, $\tilde{v}_{n,2} = (I - P_n)\tilde{v}$. Applying operator $(I - P_n)$ to (3.61)-(3.62), we can obtain

$$\begin{aligned}
&\frac{\mathrm{d}\tilde{v}_{n,2}}{\mathrm{d}t} - \Delta\tilde{v}_{n,2} - \epsilon(I-P_n)\tilde{\psi}\Delta z(\theta_t\omega) - (1-\delta)\Delta\tilde{u}_{n,2} + (\delta^2-\delta)\tilde{u}_{n,2}\\
&\quad + (1-\delta)\tilde{v}_{n,2} - \epsilon(I-P_n)\delta\tilde{\psi}z(\theta_t\omega) + (I-P_n)\tilde{\psi}g\\
&= (I-P_n)\tilde{\psi}f - (I-P_n)\Delta\tilde{\psi}v - 2(I-P_n)\nabla\tilde{\psi}\nabla v\\
&\quad - (I-P_n)(1-\delta)(\Delta\tilde{\psi}u + 2\nabla\tilde{\psi}\nabla u), \qquad (3.64)\\
&\frac{\mathrm{d}\tilde{u}_{n,2}}{\mathrm{d}t} = \tilde{v}_{n,2} - \delta\tilde{u}_{n,2} + \epsilon(I-P_n)\tilde{\psi}z(\theta_t\omega). \qquad (3.65)
\end{aligned}$$

Using the multiplier $\tilde{v}_{n,2}$ in equation (3.64) and then adding $\delta(\tilde{u}_{n,2}, \frac{\mathrm{d}}{\mathrm{d}t}\tilde{u}_{n,2})$ to the both sides of the equation, we have

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big(\|\tilde{v}_{n,2}\|^2 + (1-\delta)\|\nabla\tilde{u}_{n,2}\|^2 + \delta^2\|\tilde{u}_{n,2}\|^2\Big) + \|\nabla\tilde{v}_{n,2}\|^2\\
&\quad + \frac{3\delta(1-\delta)}{4}\|\nabla\tilde{u}_{n,2}\|^2 + \frac{\delta(1-\delta)\lambda_n}{4}\|\tilde{u}_{n,2}\|^2 + \delta^3\|\tilde{u}_{n,2}\|^2\\
&\quad + (1-\delta)\|\tilde{v}_{n,2}\|^2 - (1-\delta)\epsilon(\nabla\tilde{u}_{n,2}, \nabla(\tilde{\psi}z(\theta_t\omega)))\\
&\quad - (\delta^2-\delta)\epsilon(\tilde{u}_{n,2}, \tilde{\psi}z(\theta_t\omega)) - \epsilon(\delta\tilde{\psi}z(\theta_t\omega), \tilde{v}_{n,2}) + (\tilde{\psi}g, \tilde{v}_{n,2})\\
=&(\tilde{\psi}f, \tilde{v}_{n,2}) - (\Delta\tilde{\psi}v, \tilde{v}_{n,2}) - 2(\nabla\tilde{\psi}\nabla v, \tilde{v}_{n,2})\\
&\quad + \epsilon(\tilde{\psi}\Delta z(\theta_t\omega), \tilde{v}_{n,2}) - (1-\delta)(\Delta\tilde{\psi}u + 2\nabla\tilde{\psi}\nabla u, \tilde{v}_{n,2})\\
&\quad + \delta(\tilde{u}_{n,2}, \tilde{v}_{n,2}) + \delta\epsilon(\tilde{u}_{n,2}, \tilde{\psi}z(\theta_t\omega)). \qquad (3.66)
\end{aligned}$$

Since $\lambda_n \to +\infty$, we can choose $\tilde{N}_1 \geq 1$ such that for any $n \geq \tilde{N}_1$, $\lambda_n \geq 4$. Thus it follows from (3.66) that

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big(\|\tilde{v}_{n,2}\|^2 + (1-\delta)\|\nabla\tilde{u}_{n,2}\|^2 + \delta^2\|\tilde{u}_{n,2}\|^2\Big) + \|\nabla\tilde{v}_{n,2}\|^2\\
&\quad + \frac{3\delta(1-\delta)}{4}\|\nabla\tilde{u}_{n,2}\|^2 + \delta(1+\delta^2-\delta)\|\tilde{u}_{n,2}\|^2\\
&\quad + (1-\delta)\|\tilde{v}_{n,2}\|^2 - (1-\delta)\epsilon(\nabla\tilde{u}_{n,2}, \nabla(\tilde{\psi}z(\theta_t\omega)))\\
&\quad - (\delta^2-\delta)\epsilon(\tilde{u}_{n,2}, \tilde{\psi}z(\theta_t\omega)) - \epsilon(\delta\tilde{\psi}z(\theta_t\omega), \tilde{v}_{n,2}) + (\tilde{\psi}g, \tilde{v}_{n,2})\\
=&(\tilde{\psi}f, \tilde{v}_{n,2}) - (\Delta\tilde{\psi}v, \tilde{v}_{n,2}) - 2(\nabla\tilde{\psi}\nabla v, \tilde{v}_{n,2})\\
&\quad + \epsilon(\tilde{\psi}\Delta z(\theta_t\omega), \tilde{v}_{n,2}) - (1-\delta)(\Delta\tilde{\psi}u + 2\nabla\tilde{\psi}\nabla u, \tilde{v}_{n,2})\\
&\quad + \delta(\tilde{u}_{n,2}, \tilde{v}_{n,2}) + \delta\epsilon(\tilde{u}_{n,2}, \tilde{\psi}z(\theta_t\omega)). \qquad (3.67)
\end{aligned}$$

By the Gagliardo-Nirenberg inequality, the Cauchy inequality and assumption (A_2), we can obtain that

$$\begin{aligned}
(\tilde{\psi}g, \tilde{v}_{n,2}) \leq& C\int_{\mathbb{R}^N} |g'_u(x,\theta u)u||\tilde{v}_{n,2}|\mathrm{d}x\\
\leq& C\int_{\mathbb{R}^N} (|u| + |u|^p)|\tilde{v}_{n,2}|\mathrm{d}x\\
\leq& C\int_{\mathbb{R}^N} |u||\tilde{v}_{n,2}|\mathrm{d}x + C\big(\int_{\mathbb{R}^N} |u|^{p+1}\mathrm{d}x\big)^{\frac{p}{p+1}}\big(\int_{\mathbb{R}^N} |\tilde{v}_{n,2}|^{p+1}\mathrm{d}x\big)^{\frac{1}{p+1}}
\end{aligned}$$

$$
\begin{aligned}
&\le C\|u\|\|\tilde{v}_{n,2}\| + C\|u\|_{p+1}^{p}\|\tilde{v}_{n,2}\|_{p+1}\\
&\le C\lambda_n^{-\frac{1}{2}}\|u\|\|\nabla\tilde{v}_{n,2}\| + C\|u\|_{H^1}^{p}\|\nabla\tilde{v}_{n,2}\|^{\frac{N(p+1)-2N}{2(p+1)}}\|\tilde{v}_{n,2}\|^{\frac{2N+(2-N)(p+1)}{2(p+1)}}\\
&\le C\lambda_n^{-1}\|u\|^2 + \frac{1}{4}\|\nabla\tilde{v}_{n,2}\|^2 + C\lambda_n^{-\frac{2N+(2-N)(p+1)}{(p+1)}}\|u\|_{H^1}^{2p} + \frac{1}{4}\|\nabla\tilde{v}_{n,2}\|^2,
\end{aligned}
\tag{3.68}
$$

where $0 < \theta < 1$. By using the Cauchy inequality, (3.68), (3.44), we can obtain from (3.67) that

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big(\|\tilde{v}_{n,2}\|^2 + (1-\delta)\|\nabla\tilde{u}_{n,2}\|^2 + \delta^2\|\tilde{u}_{n,2}\|^2\Big)\\
&\quad + \frac{\delta}{2}(1-\delta)\|\nabla\tilde{u}_{n,2}\|^2 + (\frac{1}{2}-2\delta)\|\tilde{v}_{n,2}\|^2 + \frac{\delta}{2}(1+\delta^2-2\delta)\|\tilde{u}_{n,2}\|^2\\
&\le\epsilon C\|(I-P_n)\nabla(\tilde{\psi}z(\theta_t\omega))\|^2 + \epsilon C\|(I-P_n)\tilde{\psi}z(\theta_t\omega)\|^2 + \epsilon C\|(I-P_n)\tilde{\psi}\Delta z(\theta_t\omega)\|^2\\
&\quad + C\|(I-P_n)\tilde{\psi}f\|^2 + C\lambda_n^{-1}\|u\|^2 + \frac{C}{R^4}\|v\|^2 + \frac{C}{R^2}\|\nabla v\|^2\\
&\quad + C\lambda_n^{-\frac{2N+(2-N)(p+1)}{(p+1)}}\|u\|_{H^1}^{2p} + \frac{C}{R^4}\|u\|^2 + \frac{C}{R^2}\|\nabla u\|^2.
\end{aligned}
\tag{3.69}
$$

Setting

$$
\begin{aligned}
&H_2(t) = \|\tilde{v}_{n,2}\|^2 + (1-\delta)\|\nabla\tilde{u}_{n,2}\|^2 + \delta^2\|\tilde{u}_{n,2}\|^2,\\
&K_2(t) = \frac{\delta}{2}(1-\delta)\|\nabla\tilde{u}_{n,2}\|^2 + (\frac{1}{2}-2\delta)\|\tilde{v}_{n,2}\|^2 + \frac{\delta}{2}(1+\delta^2-2\delta)\|\tilde{u}_{n,2}\|^2,
\end{aligned}
$$

we have

$$
K_2(t) - \frac{\delta}{2}H_2(t) \ge 0.
$$

Then, (3.69) yields

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}H_2(t) + \delta H_2(t)\\
&\le\epsilon C\|(I-P_n)\nabla\tilde{\psi}z(\theta_t\omega)\|^2 + \epsilon C\|(I-P_n)\tilde{\psi}\nabla z(\theta_t\omega)\|^2 + \epsilon C\|(I-P_n)\tilde{\psi}z(\theta_t\omega)\|^2\\
&\quad + \epsilon C\|(I-P_n)\tilde{\psi}\Delta z(\theta_t\omega)\|^2 + C\|(I-P_n)\tilde{\psi}f\|^2\\
&\quad + C\lambda_n^{-1}\|u\|^2 + C\lambda_n^{-\frac{2N+(2-N)(p+1)}{(p+1)}}\|u\|_{H^1}^{2p}\\
&\quad + \frac{C}{R^2}(\|u\|^2 + \|\nabla u\|^2 + \|v\|^2) + \frac{C}{R^2}\|\nabla v\|^2.
\end{aligned}
\tag{3.70}
$$

Similar to the proof of (3.32), we can get

$$
\begin{aligned}
&\|\tilde{v}_{n,2}(\tau,\tau-t,\theta_{-\tau}\omega,\tilde{v}_{\tau-t})\|^2 + (1-\delta)\|\nabla\tilde{u}_{n,2}(\tau,\tau-t,\theta_{-\tau}\omega,\tilde{u}_{\tau-t})\|^2\\
&\quad + \delta^2\|\tilde{u}_{n,2}(\tau,\tau-t,\theta_{-\tau}\omega,\tilde{u}_{\tau-t})\|^2\\
&\le\Big(\|v_{\tau-t}\|^2 + (1-\delta)\|\nabla u_{\tau-t}\|^2 + \delta^2\|u_{\tau-t}\|^2\Big)\mathrm{e}^{-\delta t}\\
&\quad + C\|(I-P_n)h(x)\|_{H^2(B_{2R})}^2\int_{-t}^{0}\mathrm{e}^{\delta s}|y(\theta_s\omega)|^2\mathrm{d}s + \frac{C}{R^2}\|h(x)\|^2\int_{-t}^{0}\mathrm{e}^{\delta s}|y(\theta_s\omega)|^2\mathrm{d}s\\
&\quad + C\int_{-t}^{0}\mathrm{e}^{\delta s}\|(I-P_n)\tilde{\psi}f(s+\tau)\|^2\mathrm{d}s
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{C}{R^2}\int_{\tau-t}^{\tau}\mathrm{e}^{\delta(s-\tau)}\|\nabla v(s,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})\|^2\mathrm{d}s\\
&+\frac{C}{R^2}\int_{\tau-t}^{\tau}\mathrm{e}^{\delta(s-\tau)}\|\varphi(s,\tau-t,\theta_{-\tau}\omega,\varphi_{\tau-t})\|^2_{\mathcal{H}}\mathrm{d}s\\
&+C\lambda_n^{-\frac{2N+(2-N)(p+1)}{(p+1)}}\int_{\tau-t}^{\tau}\mathrm{e}^{\delta(s-\tau)}\|u(s,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})\|^{2p}_{H^1}\mathrm{d}s\\
&+C\lambda_n^{-1}\int_{\tau-t}^{\tau}\mathrm{e}^{\delta(s-\tau)}\|u(s,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})\|^2\mathrm{d}s. \qquad (3.71)
\end{aligned}
$$

Since $h\in H^2(\mathbb{R}^N)$ and $\lambda_n\to+\infty$ as $n\to\infty$, we can get that for any $\varepsilon>0$, there exist $\tilde R=\tilde R(\varepsilon)\ge 1$ and $\tilde N_2=\tilde N_2(\varepsilon)\ge 1$ such that for any $R\ge\tilde R$ and $n\ge\tilde N_2$

$$
\begin{aligned}
&\|\tilde v_{n,2}(\tau,\tau-t,\theta_{-\tau}\omega,\tilde v_{\tau-t})\|^2+(1-\delta)\|\nabla\tilde u_{n,2}(\tau,\tau-t,\theta_{-\tau}\omega,\tilde u_{\tau-t})\|^2\\
&+\delta^2\|\tilde u_{n,2}(\tau,\tau-t,\theta_{-\tau}\omega,\tilde u_{\tau-t})\|^2\\
\le&\Big(\|v_{\tau-t}\|^2+(1-\delta)\|\nabla u_{\tau-t}\|^2+\delta^2\|u_{\tau-t}\|^2\Big)\mathrm{e}^{-\delta t}\\
&+\varepsilon\int_{\tau-t}^{\tau}\mathrm{e}^{\delta(s-\tau)}\|u(s,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})\|^{2p}_{H^1}\mathrm{d}s\\
&+\varepsilon\int_{\tau-t}^{\tau}\mathrm{e}^{\delta(s-\tau)}\|u(s,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})\|^2\mathrm{d}s\\
&+\varepsilon\int_{\tau-t}^{\tau}\mathrm{e}^{\delta(s-\tau)}\|\nabla v(s,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})\|^2\mathrm{d}s\\
&+\varepsilon\int_{\tau-t}^{\tau}\mathrm{e}^{\delta(s-\tau)}\|\varphi(s,\tau-t,\theta_{-\tau}\omega,\varphi_{\tau-t})\|^2_{\mathcal{H}}\mathrm{d}s\\
&+C\int_{-\infty}^{0}\mathrm{e}^{\delta s}\|(I-P_n)\tilde\psi f(s+\tau)\|^2\mathrm{d}s+\varepsilon\int_{-\infty}^{0}\mathrm{e}^{\delta s}|y(\theta_s\omega)|^2\mathrm{d}s. \qquad (3.72)
\end{aligned}
$$

We now estimate every term on the right-hand side of (3.72). For the first term, similarly to (3.34), we can obtain that there exists $\tilde T=\tilde T(\tau,\omega,D,\varepsilon)>0$ such that for any $t\ge\tilde T$

$$
\Big(\|v_{\tau-t}\|^2+(1-\delta)\|\nabla u_{\tau-t}\|^2+\delta^2\|u_{\tau-t}\|^2\Big)\mathrm{e}^{-\delta t}<\varepsilon. \qquad (3.73)
$$

For the second term on the right-hand side of (3.72), applying Lemma 3.4, we can get that there exists $\tilde T_1=\tilde T_1(\tau,\omega,D)>0$ such that for all $t\ge\tilde T_1$

$$
\begin{aligned}
&\varepsilon\int_{\tau-t}^{\tau}\mathrm{e}^{\delta(s-\tau)}\|u(s,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})\|^{2p}_{H^1}\mathrm{d}s\\
\le&\varepsilon\int_{0}^{t}\mathrm{e}^{-\delta s}\|u(\tau-s,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})\|^{2p}_{H^1}\mathrm{d}s\\
\le&\varepsilon C\bigg(\int_0^t\mathrm{e}^{-\delta s}M_2^p\mathrm{d}s+M_2^p\mathcal{R}^p_{2,\epsilon}(\tau,\omega)\int_0^t\mathrm{e}^{-\delta s}\mathrm{e}^{\frac{\delta\eta}{2}s}\mathrm{d}s\bigg)\\
\le&\varepsilon C\big(1+\mathcal{R}^p_{2,\epsilon}(\tau,\omega)\big). \qquad (3.74)
\end{aligned}
$$

Also, by Lemma 3.3 we can get that there exists $\tilde T_2=\tilde T_2(\tau,\omega,D)>0$ such that for all $t\ge\tilde T_2$

$$
\varepsilon\int_{\tau-t}^{\tau}\mathrm{e}^{\delta(s-\tau)}\|\nabla v(s,\theta_s\omega,v_{\tau-t})\|^2\mathrm{d}s+\varepsilon\int_{\tau-t}^{\tau}\mathrm{e}^{\delta(s-\tau)}\|\varphi(s,\tau-t,\theta_{-\tau}\omega,\varphi_{\tau-t})\|^2_{\mathcal{H}}\mathrm{d}s
$$

$$
\begin{aligned}
&+\varepsilon\int_{\tau-t}^{\tau}\mathrm{e}^{\delta(s-\tau)}\|u(s,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})\|^2\mathrm{d}s\\
\le&3\varepsilon\mathcal{R}_{1,\epsilon}(\tau,\omega).
\end{aligned}
\tag{3.75}
$$

For the second to the last term on the right-hand side of (3.72), by assumption (**ii**) we can get that

$$
\int_{-\infty}^{0}\mathrm{e}^{\delta s}\|\tilde{\psi}f(s+\tau)\|^2\mathrm{d}s<+\infty.
$$

Therefore, by the Lebesgue dominated convergence theorem one can get that

$$
\int_{-\infty}^{0}\mathrm{e}^{\delta s}\|(I-P_n)\tilde{\psi}f(s+\tau)\|^2\mathrm{d}s\to 0 \text{ as } n\to+\infty,
$$

which implies that there exists $\tilde{N}_3=\tilde{N}_3(\tau,\varepsilon)\ge 1$ such that for all $n\ge\tilde{N}_3$,

$$
\int_{-\infty}^{0}\mathrm{e}^{\delta s}\|(I-P_n)\tilde{\psi}f(s+\tau)\|^2\mathrm{d}s<\varepsilon.
\tag{3.76}
$$

Substituting the estimates (3.73)-(3.76) to (3.72), we obtain the result. □

We are now in the position to prove the pullback asymptotic compactness for Ψ_ϵ generated by problem (3.6)-(3.8).

Lemma 3.7. *Let assumptions* (**i**)-(**ii**) *hold. Then* Ψ_ϵ *for problem* (3.6)-(3.8) *is* $\mathcal{D}$*-pullback asymptotically compact.*

Proof. Let $\{t_k\}\subseteq\mathbb{R}^+$ such that $t_k\to+\infty$ as $k\to+\infty$, and $D=\{D(\tau,\omega):\tau\in\mathbb{R},\omega\in\Omega\}\in\mathcal{D}$, $\varphi_{\tau-t_k}\in D(\tau-t_k,\theta_{-t_k}\omega)$. By Lemma 3.3 we can get that for each $\omega\in\Omega$ and $\tau\in\mathbb{R}$, there exists $K_1=K_1(\tau,\omega,D)\ge 1$ such that the sequence of solutions $\{\varphi(\tau,\tau-t_k,\theta_{-\tau}\omega,\varphi_{\tau-t_k})\}$ for problem (3.6)-(3.8) satisfies

$$
\|\varphi(\tau,\tau-t_k,\theta_{-\tau}\omega,\varphi_{\tau-t_k})\|^2_{\mathcal{H}(\mathbb{R}^N)}\le\mathcal{R}_{1,\epsilon}(\tau,\omega),\ \forall k\ge K_1.
\tag{3.77}
$$

Similarly, we can deduce from Lemma 3.5 that there exist $\hat{R}_1=\hat{R}_1(\tau,\varepsilon,\omega)>\max\{1,r_0\}$ and $K_2=K_2(\tau,\omega,D,\varepsilon)\ge K_1$ such that for all $k\ge K_2$,

$$
\|\varphi(\tau,\tau-t_k,\theta_{-\tau}\omega,\varphi_{\tau-t_k})\|^2_{\mathcal{H}(\mathbb{R}^N\setminus B_{\hat{R}_1})}\le\varepsilon.
\tag{3.78}
$$

Then by Lemma 3.6, we get there exist $\hat{N}_1=\hat{N}_1(\tau,\omega,\varepsilon)\ge 1$, $\hat{R}_2=\hat{R}_2(\tau,\omega,\varepsilon)\ge\hat{R}_1$ and $K_3=K_3(\tau,\omega,D,\varepsilon)\ge K_2$ such that for all $k\ge K_3$,

$$
\|(I-P_{\hat{N}_1})\tilde{\varphi}(\tau,\tau-t_k,\theta_{-\tau}\omega,\tilde{\varphi}_{\tau-t_k})\|^2_{\mathcal{H}(B_{2\hat{R}_2})}\le\varepsilon,
\tag{3.79}
$$

where $\tilde{\varphi}(\tau,\tau-t_k,\theta_{-\tau}\omega,\tilde{\varphi}_{\tau-t_k})=\tilde{\rho}(\frac{|x|}{\hat{R}_2})\varphi(\tau,\tau-t_k,\theta_{-\tau}\omega,\varphi_{\tau-t_k})$. By (3.77), we have

$$
\{\tilde{\varphi}(\tau,\tau-t_k,\theta_{-\tau}\omega,\tilde{\varphi}_{\tau-t_k})\} \text{ is bounded in } E(B_{2\hat{R}_2}).
$$

With this fact and (3.79), we can get that $\{\tilde{\varphi}(\tau,\tau-t_k,\theta_{-\tau}\omega,\tilde{\varphi}_{\tau-t_k})\}$ is precompact in $\mathcal{H}(B_{2\hat{R}_2})$. Notice that $\tilde{\rho}(\frac{|x|}{\hat{R}_2})=1$ for $|x|\le\hat{R}_2$, and then we can get that $\{\varphi(\tau,\tau-t_k,\theta_{-\tau}\omega,\varphi_{\tau-t_k})\}$ is also precompact in $\mathcal{H}(B_{\hat{R}_2})$. Finally, by (3.78), we can obtain that $\{\varphi(\tau,\tau-t_k,\theta_{-\tau}\omega,\varphi_{\tau-t_k})\}$ is precompact in $\mathcal{H}(\mathbb{R}^N)$, which along with (3.10) implies that the result. □

By Corollary 3.1, Lemma 3.7 and Lemma 2.1, we get the main result of this section.

Theorem 3.1. *Let assumptions* (**i**)-(**ii**) *hold. Then the continuous cocycle* Ψ_ϵ *corresponding to problem* (3.6)-(3.8) *possesses a unique* $\mathcal{D}$*-pullback random attractor* $\mathcal{A}_\epsilon = \{\mathcal{A}_\epsilon(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$ *in* $\mathcal{H}(\mathbb{R}^N)$.

4. Upper semicontinuity of the $\mathcal{D}$-pullback random attractor

In this section, we devote to show the upper semicontinuity of the $\mathcal{D}$-pullback random attractor $\mathcal{A}_\epsilon$ for Ψ_ϵ as ϵ goes to zero. In the following, to express the dependence of solution on ϵ, we denote by $\varphi^\epsilon = (u^\epsilon, v^\epsilon)$ the solution of problem (3.6)-(3.8) with initial data $\varphi^\epsilon_\tau = (u^\epsilon_\tau, v^\epsilon_\tau)$.

When $\epsilon = 0$, the random equations (3.6)-(3.8) degenerate into the following deterministic equations:

$$\frac{\mathrm{d}v}{\mathrm{d}t} - \Delta v - (1-\delta)\Delta u + (\delta^2-\delta)u + (1-\delta)v + g(x,u) = f(x,t), \tag{4.1}$$

$$\frac{du}{dt} = v - \delta u, \tag{4.2}$$

with initial data

$$u(x,\tau) = u_\tau(x),\ v(x,\tau) = v_\tau(x) = u_{1,\tau}(x) + \delta u_\tau(x). \tag{4.3}$$

Let Ψ_0 be the continuous deterministic cocycle generated by problem (4.1)-(4.3). Let $\mathcal{D}_0$ be the collection of all tempered families of deterministic nonempty bounded subsets of $\mathcal{H}(\mathbb{R}^N)$. Similarly, we can also get Ψ_0 possesses a unique $\mathcal{D}_0$-pullback attractor $\mathcal{A}_0 = \{\mathcal{A}(\tau) : \tau \in \mathbb{R}\}$ under the assumptions (**i**)-(**ii**).

We first introduce a result about upper semicontinuity of pullback random attractor with respect to some parameter, which has been proved in [26].

Lemma 4.1. *Let* Ψ_0 *be a continuous deterministic cocycle on* X *over* $\mathbb{R}$*, which has a unique* $\mathcal{D}_0$*-pullback attractor* $\mathcal{A}_0 = \{\mathcal{A}_0(\tau) : \tau \in \mathbb{R}\} \in \mathcal{D}_0$*. Suppose that* Ψ_ϵ *is a continuous cocycle on* X *over* $\mathbb{R}$ *and* $(\Omega, \mathcal{F}, \mathcal{P}, (\theta_t)_{t\in\mathbb{R}})$ *such that:*

(a) Ψ_ϵ *possesses a closed measurable* $\mathcal{D}$*-pullback absorbing set* $K_\epsilon = \{K_\epsilon(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$ *and a unique* $\mathcal{D}$*-pullback random attractor* $\mathcal{A}_\epsilon = \{\mathcal{A}_\epsilon(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$*.*

(b) *There is a mapping* $\zeta : \mathbb{R} \mapsto \mathbb{R}$ *such that for every* $\tau \in \mathbb{R}$*,* $\omega \in \Omega$*,*

$$\limsup_{\epsilon\to 0} \|K_\epsilon(\tau,\omega)\|_X \le \zeta(\tau),$$

and

$$K_0 = \{K_0(\tau) = \{u \in X : \|u\|_X \le \zeta(\tau)\} : \tau \in \mathbb{R}\} \in \mathcal{D}_0.$$

(c) *For any* $t > 0$*,* $\tau \in \mathbb{R}$*,* $\omega \in \Omega$*,* $\epsilon \to 0$ *and* x_n*,* $x \in X$ *with* $x_n \to x$ *as* $n \to +\infty$*, it holds*

$$\lim_{\epsilon\to 0} \Psi_\epsilon(t,\tau,\omega,x_n) = \Psi_0(t,\tau,x).$$

(d) *There exists an* $\epsilon_0 > 0$ *such that for each* $\tau \in \mathbb{R}$ *and* $\omega \in \Omega$*,*

$$\bigcup_{0<\epsilon\le\epsilon_0} \mathcal{A}_\epsilon(\tau,\omega) \text{ is precompact in } X.$$

Then for any $\tau \in \mathbb{R}$ and $\omega \in \Omega$,

$$\operatorname{dist}_X(\mathcal{A}_\epsilon(\tau,\omega), \mathcal{A}_0(\tau)) \to 0, \ \ \textit{as } \epsilon \to 0.$$

For every $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $T > 0$, let $\varphi^\epsilon(t,\tau,\omega,\varphi_\tau^\epsilon)$ and $\varphi(t,\tau,\varphi_\tau)$, respectively, be the solution of problem (3.6)-(3.8) and problem (4.1)-(4.3) with initial conditions φ_τ^ϵ and φ_τ. The following result implies the convergence of solution $\varphi^\epsilon(t,\tau,\omega,\varphi_\tau^\epsilon) \to \varphi(t,\tau,\varphi_\tau)$ when $\varphi_\tau^\epsilon \to \varphi_\tau$ $(\epsilon \to 0)$, which plays an important role in the proof of upper semicontinuity of the $\mathcal{D}$-pullback random attractor $\mathcal{A}_\epsilon$.

Lemma 4.2. *Let assumptions* (**i**)-(**ii**) *hold. Assume that there exists a constant R_0 such that $\|\varphi_\tau^\epsilon\|_{\mathcal{H}}^2 + \|\varphi_\tau\|_{\mathcal{H}}^2 \le R_0$. Then for all $t \in [\tau, \tau+T]$, it holds*

$$\|\varphi^\epsilon(t,\tau,\omega,\varphi_\tau^\epsilon) - \varphi(t,\tau,\varphi_\tau)\|_{\mathcal{H}}^2 \le C\mathrm{e}^{C(t-\tau)}\|\varphi_\tau^\epsilon - \varphi_\tau\|_{\mathcal{H}}^2 + C\epsilon\int_\tau^t \mathrm{e}^{C(t-s)}|y(\theta_s\omega)|^2\mathrm{d}s.$$

Proof. Let $\bar{u}^\epsilon = u^\epsilon - u$, $\bar{v}^\epsilon = v^\epsilon - v$, and then $\bar{\varphi}^\epsilon = \varphi^\epsilon - \varphi = (\bar{u}^\epsilon, \bar{v}^\epsilon)$. By equations (3.6)-(3.7) and equations (4.1)-(4.2), we can obtain that

$$\begin{aligned} &\frac{\mathrm{d}\bar{v}^\epsilon}{\mathrm{d}t} - \Delta\bar{v}^\epsilon - \epsilon\Delta z(\theta_t\omega) - (1-\delta)\Delta\bar{u}^\epsilon + (\delta^2-\delta)\bar{u}^\epsilon + (1-\delta)\bar{v}^\epsilon \\ &- \epsilon\delta z(\theta_t\omega) + g(x,u^\epsilon) - g(x,u) = 0, \qquad (4.4)\\ &\frac{d\bar{u}^\epsilon}{\mathrm{d}t} = \bar{v}^\epsilon + \epsilon z(\theta_t\omega) - \delta\bar{u}^\epsilon. \qquad (4.5)\end{aligned}$$

Using the multiplier $\bar{v}^\epsilon$ in equation (4.4) and then adding $\delta(\bar{u}^\epsilon, \frac{\mathrm{d}\bar{u}^\epsilon}{\mathrm{d}t})$ to the both sides of the equation, we get that

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big(\|\bar{v}^\epsilon\|^2 + (1-\delta)\|\nabla\bar{u}^\epsilon\|^2 + \delta^2\|\bar{u}^\epsilon\|^2\Big) + \|\nabla\bar{v}^\epsilon\|^2 \\ &+ \delta(\delta^2-\delta)\|\bar{u}^\epsilon\|^2 - \epsilon(\delta^2-\delta)(\bar{u}^\epsilon, z(\theta_t\omega)) - \epsilon(\Delta z(\theta_t\omega), \bar{v}^\epsilon) \\ &+ \delta(1-\delta)\|\nabla\bar{u}^\epsilon\|^2 - \epsilon(1-\delta)(\nabla\bar{u}^\epsilon, \nabla z(\theta_t\omega)) + (1-\delta)\|\bar{v}^\epsilon\|^2 \\ &- \epsilon\delta(z(\theta_t\omega), \bar{v}^\epsilon) + (g(x,u^\epsilon) - g(x,u), \bar{v}^\epsilon) \\ =&\delta(\bar{u}^\epsilon, \bar{v}^\epsilon + \epsilon z(\theta_t\omega) - \delta\bar{u}^\epsilon). \qquad (4.6)\end{aligned}$$

Notice that $H^1(\mathbb{R}^N) \hookrightarrow L^{p+1}(\mathbb{R}^N)$. Then by assumption (A_2), we get

$$\begin{aligned} &|(g(x,u^\epsilon) - g(x,u), \bar{v}^\epsilon)| \\ \le& C\int_{\mathbb{R}^N}(|\theta u + (1-\theta)u^\epsilon|^{p-1} + 1)|\bar{u}^\epsilon||\bar{v}^\epsilon|\mathrm{d}x \\ \le& C\int_{\mathbb{R}^N}(|u|^{p-1} + |u^\epsilon|^{p-1})|\bar{u}^\epsilon||\bar{v}^\epsilon|\mathrm{d}x + C\int_{\mathbb{R}^N}|\bar{u}^\epsilon||\bar{v}^\epsilon|\mathrm{d}x \\ \le& \frac{C}{2}\|\bar{v}^\epsilon\|^2 + \frac{C}{2}\|\bar{u}^\epsilon\|^2 + C(\|u\|_{H^1}^{p+1} + \|u^\epsilon\|_{H^1}^{p+1})\|\bar{u}^\epsilon\|_{p+1}\|\bar{v}^\epsilon\|_{p+1} \\ \le& \frac{C}{2}\|\bar{v}^\epsilon\|^2 + \frac{C}{2}\|\bar{u}^\epsilon\|^2 + C(\|u\|_{H^1}^{p+1} + \|u^\epsilon\|_{H^1}^{p+1})\|\bar{u}^\epsilon\|_{H^1}\|\bar{v}^\epsilon\|_{H^1}. \qquad (4.7)\end{aligned}$$

Using (3.27) for (u^ϵ, v^ϵ) and (u,v), respectively, and then integrating on $[\tau, t]$ with $t \in [\tau, \tau+T]$, we can obtain that there exists $C(\tau,\omega,R_0,T)$ such that

$$\|u(t,\tau,u_\tau)\|_{H^1}^2 + \|u^\epsilon(t,\tau,\omega,u_\tau^\epsilon)\|_{H^1}^2 \le C(\tau,\omega,R_0,T). \qquad (4.8)$$

It follows from (4.7)-(4.8) and the Cauchy inequality that

$$\begin{aligned}&|(g(x,u_\epsilon)-g(x,u),\bar{v}^\epsilon)|\\ \le&\frac{C}{2}\|\bar{v}^\epsilon\|^2+\frac{C}{2}\|\bar{u}^\epsilon\|^2+C\|\bar{u}^\epsilon\|_{H^1}\|\bar{v}^\epsilon\|_{H^1}\\ \le&\frac{C}{2}\|\bar{v}^\epsilon\|^2+\frac{C}{2}\|\bar{u}^\epsilon\|^2+C\|\bar{u}^\epsilon\|_{H^1}^2+\frac{1-\delta}{2}\|\bar{v}^\epsilon\|_{H^1}^2.\end{aligned}\tag{4.9}$$

Combining (4.6) and (4.9), and using the Cauchy inequality, we can get that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big(\|\bar{v}^\epsilon\|^2+(1-\delta)\|\nabla\bar{u}^\epsilon\|^2+\delta^2\|\bar{u}^\epsilon\|^2\big)\\ \le&C(\|\bar{v}^\epsilon\|^2+(1-\delta)\|\nabla\bar{u}^\epsilon\|^2+\delta^2\|\bar{u}^\epsilon\|^2)+C\epsilon|y(\theta_t\omega)|^2.\end{aligned}\tag{4.10}$$

Applying the Gronwall inequality to (4.10) on $[\tau,t]$, we obtain the desired result. □

We now prove the upper semicontinuity of $\mathcal{D}$-pullback random attractor $\mathcal{A}_\epsilon$ when $\epsilon\to 0$.

Theorem 4.1. *Let assumptions* (**i**)-(**ii**) *hold. Then for every $\tau\in\mathbb{R}$ and $\omega\in\Omega$,*

$$\lim_{\epsilon\to0}\mathrm{dist}_{\mathcal{H}(\mathbb{R}^N)}(\mathcal{A}_\epsilon(\tau,\omega),\mathcal{A}_0(\tau))=0.$$

Proof. We only need to show that Ψ_ϵ satisfies conditions (a)-(d) of Lemma 4.1.

From Lemma 3.3, Corollary 3.1 and Theorem 3.1, we can get that Ψ_ϵ possesses a closed measurable $\mathcal{D}$-pullback absorbing set $K_\epsilon=\{K_\epsilon(\tau,\omega):\tau\in\mathbb{R},\omega\in\Omega\}\in\mathcal{D}$ given by

$$K_\epsilon(\tau,\omega)=\{\varphi\in\mathcal{H}(\mathbb{R}^N):\|\varphi\|_{\mathcal{H}}^2\le\mathcal{R}_{1,\epsilon}(\tau,\omega)\},\tag{4.11}$$

where

$$\mathcal{R}_{1,\epsilon}(\tau,\omega)=M_1+M_1\Big(\int_{-\infty}^0\mathrm{e}^{\frac{\delta\eta}{2}s}\|f(s+\tau)\|^2\mathrm{d}s+\epsilon(1+\int_{-\infty}^0\mathrm{e}^{\frac{\delta\eta}{2}s}|y(\theta_s\omega)|^{p+1}\mathrm{d}s)\Big)\tag{4.12}$$

and Ψ_ϵ possesses a unique $\mathcal{D}$-pullback random attractor $\mathcal{A}_\epsilon=\{\mathcal{A}_\epsilon(\tau,\omega):\tau\in\mathbb{R},\omega\in\Omega\}\in\mathcal{D}$. Moreover, for every $\tau\in\mathbb{R}$ and $\omega\in\Omega$, we have $\mathcal{A}_\epsilon(\tau,\omega)\subseteq K_\epsilon(\tau,\omega)$. Thus, condition (a) is hold.

Let

$$\mathcal{R}(\tau)=M_1+M_1\int_{-\infty}^0\mathrm{e}^{\frac{\delta\eta}{2}s}\|f(s+\tau)\|^2\mathrm{d}s.$$

By (4.11) and (4.12), we can get that for each $\tau\in\mathbb{R}$ and $\omega\in\Omega$,

$$\limsup_{\epsilon\to0}\|K_\epsilon(\tau,\omega)\|\le\mathcal{R}(\tau).$$

Moreover, similar to the derivation of Lemma 3.3 and Corollary 3.1, we can get

$$K_0=\{K_0(\tau)=\{\varphi\in\mathcal{H}(\mathbb{R}^N):\|\varphi\|_{\mathcal{H}}^2\le\mathcal{R}(\tau)\}:\tau\in\mathbb{R}\}\in\mathcal{D}_0$$

is a closed $\mathcal{D}_0$-pullback absorbing set for Ψ_0. The condition (b) is obtained.

By Lemma 4.2, we know that Ψ_ϵ and Ψ_0 associated to problem (3.6)-(3.8) and problem (4.1)-(4.3), respectively, satisfy condition (c) of Lemma 4.1. Now, it remains to prove condition (d). Denote by $K=\{K(\tau,\omega):\tau\in\mathbb{R},\omega\in\Omega\}$, and

$$K(\tau,\omega)=\{\varphi\in\mathcal{H}(\mathbb{R}^N):\|\varphi\|_{\mathcal{H}}^2\le\mathcal{R}(\tau,\omega)\},\tag{4.13}$$

where

$$\mathcal{R}(\tau,\omega)=M_1+M_1\Big(\int_{-\infty}^{0}\mathrm{e}^{\frac{\delta\eta}{2}s}\|f(s+\tau)\|^2\mathrm{d}s+(1+\int_{-\infty}^{0}\mathrm{e}^{\frac{\delta\eta}{2}s}|y(\theta_s\omega)|^{p+1})\mathrm{d}s\Big).$$

Since $\epsilon\in(0,1]$, we can get from (4.11) and (4.13) that, for any $\tau\in\mathbb{R}$ and $\omega\in\Omega$,

$$\bigcup_{0<\epsilon\leq 1}\mathcal{A}_\epsilon(\tau,\omega)\subseteq\bigcup_{0<\epsilon\leq 1}K_\epsilon(\tau,\omega)\subseteq K(\tau,\omega). \tag{4.14}$$

It follows from (4.13)-(4.14), Lemma 3.5, and the invariance of $\mathcal{A}_\epsilon$ that for any $\varepsilon>0$, $\tau\in\mathbb{R}$ and $\omega\in\Omega$, there exists $\tilde{R}_1=\tilde{R}_1(\varepsilon,\tau,\omega)>\max\{1,r_0\}$ such that

$$\|\varphi\|^2_{\mathcal{H}(B^C_{\tilde{R}_1})}\leq\varepsilon,\quad\text{for every }\varphi\in\bigcup_{0<\epsilon\leq 1}\mathcal{A}_\epsilon(\tau,\omega). \tag{4.15}$$

In addition, we can get from Lemma 3.6, (4.13)-(4.14) and the invariance of $\mathcal{A}_\epsilon$ that there exists $\tilde{R}_2\geq\tilde{R}_1$ such that, for all $\tau\in\mathbb{R}$, $\omega\in\Omega$,

$$\bigcup_{0<\epsilon\leq 1}\mathcal{A}_\epsilon(\tau,\omega)\text{ is precompact in }\mathcal{H}(B_{\tilde{R}_2}),$$

which together with (4.15) implies that, for all $\tau\in\mathbb{R}$, $\omega\in\Omega$,

$$\bigcup_{0<\epsilon\leq 1}\mathcal{A}_\epsilon(\tau,\omega)\text{ is precompact in }\mathcal{H}(\mathbb{R}^N).$$

We complete the proof. □

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References

[1] L. Arnold, *Random Dynamical Systems*, Springer-Verlag, Berlin, 1998.

[2] A. V. Babin and M. I. Vishik, *Attractors of Evolutionary Equations*, North-Holland, Amsterdam, 1992.

[3] P. W. Bates, K. Lu and B. Wang, *Random attractors for stochastic reaction-diffusion equations on unbounded domains*, J. Diff. Eqs., 2009, 246(2), 845–869.

[4] V. Belleri and V. Pata, *Attractors for Semilinear Strongly Damped Wave Equations on* $\mathbb{R}^3$, Discrete Contin. Dyn. Syst., 2001, 7(4), 719–735.

[5] H. Crauel and F. Flandoli, *Attractors for random dynamical systems*, Probab. Theory Relat. Fields, 1994, 100(3), 365–393.

[6] H. Crauel, A. Debussche and F. Flandoli, *Random attractors*, J. Dyn. Diff. Eqs., 1997, 9(2), 307–341.

[7] V. V. Chepyzhov and M. I. Vishik, *Attractors for Equations of Mathematical Physics*, Amer. Math. Soc. Colloq. Publ., vol. 49, Amer. Math. Soc., Providence, RI, 2002.

[8] M. Conti, V. Pata and M. Squassina, *Strongly damped wave equations on* $\mathbb{R}^3$ *with critical nonlinearities*, Commun. Appl. Anal., 2005, 9(2), 161–176.

[9] A. Cheskidov, *Global attractors of evolutionary systems*, J. Dyn. Diff. Eqs., 2009, 21(2), 249–268.

[10] H. Crauel, P. E. Kloeden and M. Yang, *Random attractors of stochastic reaction-diffusion equations on variable domains*, Stoch. Dyn., 2011, 11(2–3), 301–314.

[11] H. Cui and J. A. Langa, *Uniform attractors for non-autonomous random dynamical systems*, J. Diff. Eqs., 2017, 263(2), 1225–1268.

[12] P. Ding and Z. Yang, *Well-posedness and attractor for a strongly damped wave equation with supercritical nonlinearity on* $\mathbb{R}^N$, Commun. Pure. Appl. Anal., 2021, 20(3), 1059–1076.

[13] F. Flandoli and B. Schmalfuß, *Random attractors for the 3D stochastic Navier-Stokes equation with multiplicative white noise*, Stoch. Stoch. Rep., 1996, 59, 21–45.

[14] Z. Guo and L. Yang, *Stochastic semi-linear degenerate parabolic model with multiplicative noise and deterministic non-autonomous forcing*, Stoch. Anal. Appl., 2019, 37(1), 90–140.

[15] X. Li, *Uniform random attractors for 2D non-autonomous stochastic Navier-Stokes equations*, J. Diff. Eqs., 2021, 276, 1–42.

[16] Y. Li, B. Li and X. Li, *Uniform random attractors for a non-autonomous stochastic strongly damped wave equation on* $\mathbb{R}^N$, Z. Angew. Math. Phys., 2022, 73(3), 106, 30 pp.

[17] N. Lei and S. Zhou, *Upper semicontinuity of pullback attractors for non-autonomous lattice systems under singular perturbations*, Discrete Contin. Dyn. Syst., 2022, 42(1), 73–108.

[18] V. Pata, *Attractors for a Damped Wave Equation on* $\mathbb{R}^3$ *with Linear Memory*, Math. Meth. Appl. Sci., 2000, 23(7), 633–653.

[19] V. Pata and M. Squassina, *On the Strongly Damped Wave Equation*, Comm. Math. Phys., 2005, 253(3), 511–533.

[20] R. Temam, *Infinite-Dimensional Dynamical Systems in Mechanics and Physics*, Springer-Verlag, New York, 1997.

[21] B. Wang, *Asymptotic behavior of stochastic wave equations with critical exponents on* $\mathbb{R}^3$, Trans. Amer. Math. Soc., 2011, 363(7), 3639–3663.

[22] B. Wang, *Sufficient and necessary criteria for existence of pullback attractors for non-compact random dynamical systems*, J. Diff. Eqs., 2012, 253(5), 1544–1583.

[23] Z. Wang, S. Zhou and A. Gu, *Random attractor for a stochastic damped wave equation with multiplicative noise on unbounded domains*, Nonlinear Anal. RWA, 2011, 12(6), 3468–3482.

[24] Z. Wang and S. Zhou, *Random attractor for non-autonomous stochastic strongly damped wave equation on unbounded domains*, J. Appl. Anal. Comput., 2015, 5(3), 363–387.

[25] Z. Wang and S. Zhou, *Existence and upper semicontinuity of random attractor attractors for non-autonomous stochastic strongly damped wave equation with multiplicative noise*, Discrete Contin. Dyn. Syst., 2017, 37(5), 2787–2812.

[26] B. Wang, *Existence and upper semicontinuity of attractor for stochastic equations with deterministic non-autonomous terms*, Stoch. Dyn., 2014, 14(4), 31.

[27] B. Wang, *Random attractors for non-autonomous stochastic wave equations with multiplicative noise*, Discrete Contin. Dyn. Syst., 2014, 34(1), 269–300.

[28] Z. Yang and P. Ding, *Longtime dynamics of Kirchhoff equation with strong damping and critical nonlinearity on* $\mathbb{R}^N$, J. Math. Anal. Appl., 2016, 434(2), 1826–1851.

[29] W. Zhao, *Random dynamics of non-autonomous semi-linear degenerate parabolic equations on* $\mathbb{R}^N$ *driven by an unbounded additive noise*, Discrete Contin. Dyn. Syst. Ser. B, 2018, 23(6), 2499–2526.

[30] C. Zhao and T. Caraballo, *Asymptotic regularity of trajectory attractor and trajectory statistical solution for the 3D globally modified Navier-Stokes equations*, J. Diff. Eqs., 2019, 266(11), 7205–7229.

[31] S. Zhou and M. Hua, *Random attractor and exponentially stability for the second order nonautonomous retarded lattice systems with multiplicative white noise*, J. Math. Anal. Appl., 2022, 507(2), 26.

[32] W. Zhao and Y. Zhang, *Upper semi-continuity of random attractors for a non-autonomous dynamical system with a weak convergence condition*, Acta Math. Sci. Ser. B (Engl. Ed.), 2020, 40(4), 921–933.