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DIFFERENCE QUOTIENT ESTIMATES AND ACCURACY ENHANCEMENT OF DISCONTINUOUS GALERKIN METHODS FOR NONLINEAR CONVECTION-DIFFUSION EQUATIONS*

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Abstract In this paper, we apply the post-processing technique to the improvement of the superconvergence of the discontinuous Galerkin method for the nonlinear convection-diffusion equations. We firstly analyze the error estimate and convergence accuracy under L^2-norm, and then demonstrate that the α-order difference quotient of DG error is of order $k + 3/2 - \alpha/2$ when the upwind fluxes are used. By the duality argument, we construct an appropriate dual equation, and futher obtain superconvergence results of order in the negative-order norm, namely $2k + 3/2 - \alpha/2$ order superconvergence accuracy. Finally, we choose an appropriate kernel function and apply the SIAC filter to the nonlinear convection-diffusion equation to obtain at least $3k/2 + 1$ order superconvergence for post-processed solutions. All theoretical results are proved by numerical experiments.

Keywords Discontinuous Galerkin method, nonlinear convection-diffusion equation, SIAC filter, negative-order norm, post-processing.

MSC(2010) 65M12, 65M15, 65M60.

1. Introduction

In the past twenty years, the superconvergence of the discontinuous Galerkin (DG) methods has yielded many research results. For details, please refer to [19]. Smoothness-increasing accuracy-conserving (SIAC) filters are used to enhance the accuracy of DG solutions by means of the post-processing technique. The basic idea of post-processing based on the theory of superconvergence negative-order norm estimates is to use an appropriate kernel function to convolve the DG solution in order to obtain a higher convergence order for the exact solutions in L^2-norm and lower errors.

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The following gives the superconvergence results of the negative-order norm estimates of the hyperbolic equations and the post-processing of DG methods. Inspired by Bramble and Schatz [2] for the continuous Galerkin method of elliptic equations, Cockburn et al. [3] established the post-processing technique theory of DG method for solving hyperbolic equations by means of negative-order norm estimates and obtained the $(2k+1)$-th order superconvergence results when the exact solution is globally smooth. This post-processing technique was later applied by Ryan et al. to different aspects of the problem. Ryan and Shu [20] proposed an unilateral post-processing technique that fully considers the vicinity of the regional boundary, the discontinuous solution, and the cell interface. The result of the $(2k+1)$-th order superconvergence covered the entire computational region. Mirzaee et al. [15] extended the linear hyperbolic equations on the uniform grid in the literature to the variable coefficient hyperbolic equations on the triangular structure. For the study of negative-order norm error estimate and superconvergence for general nonuniform mesh and structural tetrahedral mesh cases, see [4, 16]. Since the kernel function in the post-processing technique is related to the grid size, the encryption grid will greatly affect the computational efficiency of post-processing. King et al. [7] analyzed intrinsic connection between the error level and superconvergence accuracy caused by the encrypted grid, and gave a detailed comparison of the calculation effects of different grids. These post-processing methods that improve the smoothness of numerical solutions and maintain negative-order norm accuracy are collectively referred to as SIAC filtering methods [17, 21]. Li and Ryan et al. [8, 9] studied the connection between the error of the filtered solution and the nonuniform mesh and developed a filter scaling that approximates the optimal error reduction. The filtered solution has demonstrated the improvement in accuracy order as well as reducing error compared to the original DG solution. Recently, Ryan [18] and Decampo showed the application of one-dimensional line SIAC filtering in streamline visualization.

For the one-dimensional linear hyperbolic equation with δ-singularities in the initial value condition or the source term, Yang and Shu [23] gave the DG error estimates of the negative $(k+1)$-th order norm and the negative $(k+2)$-th order norm over the entire calculation region Ω. Ji et al. [5] generalized the SIAC filter to the multidimensional linear convection-diffusion equations, and obtained the $(2k+m)$-th order superconvergence result in the negative-order norm by means of the dual argumentation technique in [3], and m depends on the selection of numerical fluxes. From the post-processing theory [2, 3], it is known that the negative-order norm estimates of the difference quotient of DG error are an important tool for obtaining the superconvergence error estimates of the post-processed solution in the L^2-norm. Once the negative-order norm estimates of the DG error are obtained, the negative-order norm estimates of the difference quotient of the DG error are readily available. Ji et al. [6] gave the negative-order norm error estimates of the DG method for multidimensional evolutionary hyperbolic conservation laws. The numerical example is used to verify that the local post-processing SIAC filtering method based on convolution kernel function can make the convergence order of the linear problem DG method increase to $2k+1$. For the nonlinear hyperbolic conservation law, Meng and Ryan [12] studied the problems of difference quotient estimates and accuracy improvement of the nonlinear scalar hyperbolic conservation law equation. They proved that the difference quotient of the DG error can reach the $k+3/2-\alpha/2$ order convergence precision in the L^2-norm when the upwind fluxes are used and by using the dual argument, the superconvergence accuracy achieved $2k+3/2-\alpha/2$ in the negative-order norm, and obtained at least $(3k/2+1)$-th order superconvergence to post-processed solutions. Furthermore, Meng and Ryan [13] extended the above method to the difference quotient estimates and accuracy enhancement of nonlinear symmetric systems, and obtained the same results.

In recent years, for the study of nonlinear convection-diffusion equations, Xu and Shu [22] studied error estimates of the semi-discrete local discontinuous Galerkin method for nonlinear convection-diffusion equations and achieved $(k+1)$-th order superconvergence result. For nonlinear convection-diffusion equations, although more research can be found in [1, 10, 11, 14], the post-processing technique applied to the improvement of the superconvergence discontinuous Galerkin method is not provided. In the past two decades, people have studied various superconvergence properties of DG methods, which not only deepens the understanding of DG solutions, but also is very useful for practitioners. The superconvergence of the subsequent understanding is achieved by establishing a negative order norm error estimate, which enables us to obtain a higher order approximation by simply post-processing the specially designed check DG solution at the end of calculation. For the nonlinear convection diffusion equation considered in this paper, it is important and interesting to solve the above problems by establishing L^2 norm and negative order norm error estimates of the difference. The key technical issue is how to construct a suitable dual problem for the difference of the nonlinear convection diffusion equation. However, for two-dimensional expansion, especially for establishing the relationship between the spatial derivative and the time derivative of the error, this does not seem trivial. The main tool used to derive L^2 norm error estimates for the divided differences is energy analysis. In order to deal with the nonlinearity of the flux function, Taylor expansion is used in the error estimation of nonlinear problems, following standard techniques. What we want to point out is that the superconvergence analysis in this paper shows the possible relationship between the supercloseness and negative-order norm estimates.

In this paper, we study accuracy-enhancement semi-discrete LDG methods for solving the nonlinear convection-diffusion equation.

$$u_t + f(u)_x = \varepsilon u_{xx}, \qquad (x,t) \in (a,b) \times (0,T], \tag{1.1a}$$
$$u(x,0) = u_0(x), \qquad x \in (a,b). \tag{1.1b}$$

The boundary conditions of the equation were chosen as periodic boundary conditions and $u_0(x)$ is a smooth function. We assume the nonlinear flux function $f(u)$ and the exact solution u are sufficiently smooth. We demonstrate that the α-th order $(1 \le \alpha \le k+1)$ difference quotient of the LDG error achieves $(k+3/2-\alpha/2)$-th order in the L^2-norm when the upwind fluxes are used. By the duality argument, we obtain superconvergence results of order $2k+3/2-\alpha/2$ in the negative-order norm. Then we extend the SIAC filter to the nonlinear convection-diffusion equation to obtain at least $(3k/2+1)$-th order superconvergence results.

The main structure is as follows: In Section 2, we mainly give some properties and definitions about the discontinuous finite element space. In Section 3, we give main conclusions and proofs of nonlinear convection-diffusion equation in the L^2-norm. In Section 4, we give the accuracy-enhancement superconvergence analysis based on L^2-norm error estimates of difference quotients. In Section 5, we demonstrates the theoretical results through numerical experiments.

2. Preliminaries

2.1. Meshing and function spaces

First, we divide $\Omega = (a,b)$ into N cells as follows $a = x_{\frac{1}{2}} < x_{\frac{3}{2}} < \cdots < x_{N+\frac{1}{2}} = b$. We denote $x_j = (x_{j-\frac{1}{2}} + x_{j+\frac{1}{2}})/2$ as cell center. We introduce two overlapping uniform meshes for Ω, namely $I_j = (x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}})$ and $I_{j+\frac{1}{2}} = (x_j, x_{j+1})$ with $h = x_{j+\frac{1}{2}} - x_{j-\frac{1}{2}}$. Combining with the above meshes, we define the discontinuous finite element space as

$$V_h^{\alpha} = \{v : v|_{I_{j'}} \in P^k(I_{j'}), \forall j' = j + \frac{l}{2}, l = \alpha \bmod 2, j = 1, \cdots, N\},$$

where $P^k(I_{j'})$ denotes the set of polynomial of degree up to k defined on the cell $I_{j'} = (x_{j'-\frac{1}{2}}, x_{j'+\frac{1}{2}})$, α represents the order of difference quotient for a known function, whose definition is given in preliminaries.

Obviously, V_h^{α} is a piecewise polynomial space on mesh $I_{j'} = I_j$ for even α and a piecewise polynomial space on mesh $I_{j'} = I_{j+\frac{1}{2}}$ for odd α. For the sake of simplicity, we use V_h to denote standard finite element space of degree k defined on the cell I_j , if there is no confusion. We use w_i^- and w_i^+ to denote the values of $w(x)$ at the discontinuity point x_i from the left cell and right cell, respectively. We use $[[w]] = w^+ - w^-$ and $\{\{w\}\} = (w^+ + w^-)/2$ to denote the jump and average of $w(x)$ at each element boundary point.

For any integer $s > 0$, we denote by $W^{s,p}(D)$ the Sobolev space on sub-domain $D \subset \Omega$ equipped with the norm $\|\cdot\|_{s,p,D}$. Especially, if $p = 2$, then $W^{s,p}(D) = H^s(D)$, and $\|\cdot\|_{s,p,D} = \|\cdot\|_{s,D}$. If $s = 0$, then $\|\cdot\|_{s,D} = \|\cdot\|_D$. When $D = \Omega$, we will omit the index D. Furthermore, the broken Sobolev space can be defined as

$$W^{s,p}(\Omega_h) = \{u \in L^2(\Omega) : u|_{I_j} \in W^{s,p}(I_j), j = 1, 2, \cdots, N\}$$

with Ω_h being the union of all cells. Additionally, we denote by $\|\cdot\|_{\partial I_{j'}}$ the standard L^2-norm on cell interfaces of the mesh $I_{j'}$, and $\|v\|_{\Gamma_h}^2 = \sum_{j=1}^N \|v\|_{\partial I_{j'}}^2$ with $\|v\|_{\partial I_{j'}} = ((v_{j'-\frac{1}{2}}^+)^2 + (v_{j'+\frac{1}{2}}^-)^2)^{\frac{1}{2}}$. for the sake of simplicity, referring to [3], we introduce the so-called jump semi-norm $|[v]| = (\sum_{j=1}^N [[v]]_{j'-\frac{1}{2}}^2)^{\frac{1}{2}}$ for $v \in H^1(\Omega_h)$.

In post-processing, we need to consider the definition of negative-order norm: given $l > 0$ and domain Ω, we have

$$\|v\|_{-l,\Omega} = \sup_{\Phi \in C_0^{\infty}(\Omega)} \frac{(v, \Phi)}{\|\Phi\|_l}. \tag{2.1}$$

2.2. Difference quotient, projection and DG spatial discretization properties

For different constants, we denote by C a positive constant independent of h, but dependent on the exact solution of the equation, which could have different values in different situations. To emphasize non-linearity of the flux function $f(u)$, we denote by C_* a non-negative constant about $f(x)$, $u(x,t)$ and their derivatives maximum, likewise $\max |f'(u) u_x|$.

2.2.1. Properties for the difference quotient

For any function $w(x)$ and integer α, the central difference quotient is defined as

$$\partial_h^\alpha w(x) = \frac{1}{h^\alpha}\sum_{i=1}^{\alpha}(-1)^\alpha \begin{pmatrix}\alpha\\ i\end{pmatrix} w\Big(x+\Big(\tfrac{\alpha}{2}-i\Big)h\Big). \tag{2.2a}$$

Note that the above definition still hold even if w is a piecewise function with possible discontinuities at the cell boundaries. The difference quotient has the following properties: for any functions w and v

$$\partial_h^\alpha(w(x)v(x)) = \sum_{i=1}^{\alpha}\begin{pmatrix}\alpha\\ i\end{pmatrix}\partial_h^i w\Big(x+\tfrac{\alpha-i}{2}h\Big)\partial_h^{\alpha-i}v\Big(x-\tfrac{i}{2}h\Big), \tag{2.2b}$$

$$\partial_h^\alpha(w(x),v(x)) = (-1)^\alpha(w(x),\partial_h^\alpha v(x)). \tag{2.2c}$$

Here (2.2b) and (2.2c) are the so-called Leibniz rule and summation by parts for the difference quotient.

2.2.2. The inverse and projection properties

For any $u \in V_h^\alpha$, there exists a positive constant C independent of u and h such that

$$(i)\|\partial_x p\| \le Ch^{-1}\|p\|; \qquad (ii)\|p\|_{\Gamma_h} \le Ch^{-1/2}\|p\|; \qquad (iii)\|p\|_\infty \le Ch^{-1/2}\|p\|.$$

Introduce the standard L^2 projection of function $u \in L^2(\Omega)$ into the finite element space V_h^k , recorded as $P_h u$, which is a unique function in V_h^k, and satisfies

$$(u - P_h u, v_h) = 0, \quad \forall v_h \in V_h^k. \tag{2.3}$$

For the proof below, we also need to introduce two special projections $P_h^\pm$ into V_h, which appear in [22-23]. For any known function $u \in H^1(\Omega_h)$ and an arbitrary element $I_{j'} = (x_{j'-\frac{1}{2}}, x_{j'+\frac{1}{2}})$, the special Gauss-Radau projection of u , denoted by $P_h^\pm u$, is the unique functions in V_h^k , for each $j^{'}$ satisfing

$$(u - P_h^+ u, v_h)_{j'} = 0, \ \ \forall v_h \in P^{k-1}(I_{j'}), \ \ (u - P_h^+ u)^+_{j'-\frac{1}{2}} = 0, \tag{2.4a}$$

$$(u - P_h^- u, v_h)_{j'} = 0, \ \ \forall v_h \in P^{k-1}(I_{j'}), \ \ (u - P_h^- u)^-_{j'+\frac{1}{2}} = 0. \tag{2.4b}$$

We denote by $\eta = u(x) - Q_h u(x)(Q_h = P_h \ or \ P_h^\pm)$ the projection error for sufficiently smooth $u(x)$ such that

$$\|\eta_u\| + h\|(\eta_u)_x\| + h^{\frac{1}{2}}\|\eta_u\|_{\Gamma_h} \le Ch^{k+1}\|u\|_{k+1}. \tag{2.5a}$$

Furthermore, we obtain

$$\|\eta_u\|_\infty \le Ch^{k+1}\|u\|_{k+1,\infty}. \tag{2.5b}$$

2.2.3. The inverse and projection properties

The DG spatial discretization operators $H_{j'}(\cdot,\cdot)$ are defined on each cell $I_{j'} = (x_{j'-\frac{1}{2}}, x_{j'+\frac{1}{2}})$, namely

$$H_{j'}(w,v) = (w, v_x)_{j'} - \hat{w}v^-|_{j'+\frac{1}{2}} + \hat{w}v^+|_{j'-\frac{1}{2}}. \tag{2.6}$$

We use H to denote the summation of $H_{j'}$ with respect to cell $I_{j'}$, that is

$$H^-(w,v) = (w, v_x) + \sum_{j=1}^{N}(w^-[[v]])|_{j'+\frac{1}{2}} = -(w_x, v) - \sum_{j=1}^{N}([[w]]v^+)|_{j'-\frac{1}{2}}, \tag{2.7a}$$

$$H^+(w,v) = (w, v_x) + \sum_{j=1}^{N}(w^+[[v]])|_{j'+\frac{1}{2}} = -(w_x, v) - \sum_{j=1}^{N}([[w]]v^-)|_{j'-\frac{1}{2}}. \tag{2.7b}$$

And the DG spatial discretization operator has the following property

$$H^+(w,v) + H^-(v,w) = 0. \tag{2.8}$$

In order to perform the L^2 error estimate of difference quotients, we need the following properties of DG operator H.

Property 2.1. Suppose that $r(u)(r = f^{'}(u))$ is sufficiently smooth with respect to each variable. For any $w, v \in V_h^\alpha$, there holds the following inequality

$$H(rw, v) \leq C_*(\|w\| + \|w_x\| + h^{-\frac{1}{2}}|[w]|)\|v\|, \tag{2.9a}$$

and especially if $r = f^{'}(u) \geq \delta > 0$, there holds

$$H(rw, w) \leq C_*\|w\|^2 - \frac{\delta}{2}|[w]|^2. \tag{2.9b}$$

Property 2.2. Under the conditions of Property 2.1, for sufficiently small h, there holds

$$H((\partial_h^\alpha r)w, v) \leq C_*(\|w\| + \|w_x\| + h^{-\frac{1}{2}}|[w]|)\|v\|, \quad \forall \alpha \geq 0. \tag{2.10}$$

Property 2.3. Suppose that $r(u)$ is sufficiently smooth with respect to each variable. For any $w \in H^{k+1}(\Omega_h)$ and $v \in V_h^\alpha$, there holds the following inequality

$$H(r(w - P_h^- w), v) \leq C_* h^{k+1}\|v\|. \tag{2.11}$$

Property 2.4. Suppose that $r(u)$ is sufficiently smooth with respect to each variable. For any $w \in H^{k+1}(\Omega_h)$ and $v \in V_h^\alpha$, there holds the following inequality

$$H(\partial_h^\alpha(r(w - P_h^- w)), v) \leq C_* h^{k+1}\|v\|, \quad \forall \alpha \geq 0. \tag{2.12}$$

For details of the proof of the above properties, please refer to [12].

2.3. Smoothness-increasing accuracy-conserving (SIAC) filter

The SIAC filters are used to extract the hidden accuracy of the DG methods by post-processing technique, which uses a specially selected kernel function to convolve with the DG solutions to improve the accuracy and reduce the oscillation of the error. namely

$$u_h^* = K_h^{2k+1,k+1} * u_h,$$

where u_h^* is the post-processed solution and u_h are DG solutions at final time.

As for the kernel function $K_h^{2k+1,k+1}$, it is a linear combination of B-splines of order $k+1$ obtained by convolving the feature function $\psi^{(1)} = \chi$ of the interval $(-1/2, 1/2)$ with itself k times. We give the definition as follows

$$K_h^{2k+1,k+1}(x) = \frac{1}{h}\sum_{\gamma \in Z} c_\gamma^{2k+1,k+1}\psi^{(k+1)}\left(\frac{x}{h} - \gamma\right),$$

where $c_\gamma^{2k+1,k+1}$ can be obtained by the property of the kernel function, namely $K_h^{2k+1,k+1} * p = p$, p is a polynomial of degree $2k$. The convolution kernel has the following important property.

Theorem 2.1 ([3]). *For $0 < T < T^*$, where T^* is the maximum time for the existence of a smooth solution, let $u \in L^\infty([0,T]; H^{2k+1}(\Omega))$ is the exact solution of the equation (1.1), $\Omega_0 + 2supp(K_h^{2k+1,k+1}(x)) \subset\subset \Omega$ and u_h is the approximation of u, then*

$$\|u - u_h^*\| \leq \frac{h^{2k+1}}{(2k+1)!}C_1|u|_{2k+1} + C_1C_2\sum_{\alpha \leq k+1}\|\partial_h^\alpha(u-u_h)\|_{-(k+1),\Omega},$$

where C_1 and C_2 depend on Ω_0, k, but is independent of h.

3. The Convergence results for difference quotient in L^2-norm

3.1. The convergence results in L^2-norm

In this section, we give the LDG scheme of the difference quotients of equations (1.1a) and (1.1b) according to the marks in [12, 24].

The difference quotients of nonlinear convection-diffusion equations (1.1a) and (1.1b) are

$$\partial_h^\alpha u_t + \partial_h^\alpha f(u)_x = \partial_h^\alpha \varepsilon u_{xx}, \quad (x,t) \in \Omega \times (0,T], \tag{3.1a}$$
$$\partial_h^\alpha u(x,0) = \partial_h^\alpha u_0(x), \qquad x \in \Omega^\alpha, \tag{3.1b}$$

where $\Omega^\alpha = (a + l\frac{h}{2}, b + l\frac{h}{2})$ with $l = \alpha \ mod\ 2$. When $\alpha = 0$, equations (3.1a) and (3.1b) reduce to (1.1a) and (1.1b).

For partial differential equations with higher-order spatial derivatives, we use the semi-discrete LDG schemes constructed in [1] to rewrite equations (1.1a) and (1.1b) as

$$\partial_h^\alpha u_t + \partial_h^\alpha f(u)_x = \partial_h^\alpha \sqrt{\varepsilon} q_x, \quad (x,t) \in \Omega \times (0,T], \tag{3.2a}$$
$$\partial_h^\alpha q = \sqrt{\varepsilon}\partial_h^\alpha u_x. \tag{3.2b}$$

The semi-discrete LDG method for solving equations (3.2a) and (3.2b) can be approximated as: find the unique functions $u_h, q_h \in V_h$ such that the weak forms

$$((\partial_h^\alpha u_h)_t, v_h)_{j'} = H_{j'}(\partial_h^\alpha f(u_h), v_h) - H_{j'}(\sqrt{\varepsilon}\partial_h^\alpha q_h, v_h), \tag{3.3a}$$

$$(\partial_h^\alpha q_h, w_h)_{j'} = -H_{j'}(\sqrt{\varepsilon}\partial_h^\alpha u_h, w_h), \tag{3.3b}$$

hold for all $v_h, w_h \in V_h^\alpha$.

Without loss of generality, we only consider the case $f^{'}(u) > \delta > 0$ and $\varepsilon = 1$. The numerical fluxes are selected as the upwind and alternating fluxes

$$\hat{f}(u_h^-, u_h^+) = f(u_h^-), \hat{u_h} = u_h^-, \hat{q_h} = q_h^+.$$

Error between exact solution and numerical solution is denoted by $e_u = u - u_h$. Inserting the Gauss-Radau projection, we have

$$e_u = u - P_h^- u + P_h^- u - u_h = \eta_u + \xi_u,$$
$$e_q = q - P_h^+ q + P_h^+ q - q_h = \eta_q + \xi_q,$$

where η is the projection error and ξ is the projection of the error.

Theorem 3.1. *For any $0 \le \alpha \le k+1$, let $\partial_h^\alpha u, \partial_h^\alpha q$ be the exact solutions of equations (3.2a) and (3.2b), which are assumed to be sufficiently smooth with bounded derivative, and assume that $|f'(u)|$ is uniformly lower bounded by a positive constant. Let $\partial_h^\alpha u_h, \partial_h^\alpha q_h$ be the numerical solutions of the LDG schemes (3.3a) and (3.3b) with initial conditions $\partial_h^\alpha u_h(0) = P_h^-(\partial_h^\alpha u_0)$, $\partial_h^\alpha q_h(0) = P_h(\partial_h^\alpha q_0)$ when the upwind flux is used. For a uniform mesh of $\Omega = (a, b)$, if the finite element space $V_h^\alpha (k \ge 1)$ is taken as the k order piecewise polynomial space, then for small enough h and any $T > 0$ there holds the following error estimate*

$$\|\partial_h^\alpha \xi_u(T)\|^2 + \int_0^T |[\partial_h^\alpha \xi_u]|^2 dt \le C_* h^{2k+3-\alpha}, \tag{3.4}$$

where the positive integer C_* depends on u, $\|u\|_{k+1}$, $\|u_t\|_{k+1}$, $\|u_{tt}\|_{k+1}$, δ, T, but is independent of h.

Corollary 3.1. *Under the conditions of Theorem 2.1, if $\alpha \ge 1$, we have the following error estimate:*

$$\|\partial_h^\alpha (u - u_h)(T)\| \le C_* h^{k+\frac{3}{2}-\frac{\alpha}{2}}. \tag{3.5}$$

More details of proof are given in [12].

3.2. Proof of the main results in the L^2-norm

Equations (3.3a) and (3.3b) become

$$((\partial_h^\alpha u_h)_t, v_h)_{j'} = H_{j'}(\partial_h^\alpha f(u_h), v_h) - H_{j'}^+(\partial_h^\alpha q_h, v_h), \tag{3.6a}$$

$$(\partial_h^\alpha q_h, w_h)_{j'} = -H_{j'}^-(\partial_h^\alpha u_h, w_h). \tag{3.6b}$$

For the selection of initial conditions, we denote by $P_h^- \partial_h^\alpha u_0$ the Gauss-Radau projection of $\partial_h^\alpha u_0(x)$ as initial value of equation (3.2a), and the variable $\partial_h^\alpha q(x,0) = \partial_h^\alpha \partial_x u_0(x)$. $\partial_h^\alpha q_h$ satisfies

$$(\partial_h^\alpha (q - q_h), w_h)_{j'}$$

$$=(\partial_h^\alpha u_0 - P_h^- \partial_h^\alpha u_0) w_h^- |_{j'+\frac{1}{2}} - (\partial_h^\alpha u_0 - P_h^- \partial_h^\alpha u_0) w_h^+ |_{j'-\frac{1}{2}}$$
$$- ((\partial_h^\alpha u_0 - P_h^- \partial_h^\alpha u_0), (w_h)_x) = 0.$$

Namely $\partial_h^\alpha q_h$ is L^2 projection of $\partial_h^\alpha q(x,0)$. Initial conditions satisfy

$$\|\partial_h^\alpha \xi_u(\cdot,0)\| = \|P_h^- \partial_h^\alpha u - P_h^- \partial_h^\alpha u_h(\cdot,0)\| = 0,$$
$$\|\xi_q(\cdot,0)\| = \|P_h^+ \partial_h^\alpha q - \partial_h^\alpha q_h(\cdot,0)\| = \|P_h^+ \partial_h^\alpha q - P_h \partial_h^\alpha q\| \le Ch^{k+1}.$$

As for the proof of Theorem 3.1, for the case $\alpha = 0$ has been proved in [1], the following conclusions are obtained.

$$\|\xi_u\| \le C_* h^{k+\frac{3}{2}}, \tag{3.7a}$$
$$\|(\xi_u)_x\| \le C_*(\|(\xi_u)_t\| + h^{k+1}), \tag{3.7b}$$
$$\|(\xi_u)_t\| \le C_* h^{k+1}. \tag{3.7c}$$

We only need to consider $1 \le \alpha \le k+1$. In order to clearly display the main idea of how to perform the L^2-norm error estimates for difference quotients, in the following two parts we mainly prove Theorem 3.1 with $\alpha = 1$ and $\alpha = 2$. Then $3 \le \alpha \le k+1$ can be proved by induction, which are omitted to save space.

3.2.1. Proof of first order difference quotient

when $\alpha = 1$, the LDG scheme (3.6) becomes

$$((\partial_h u_h)_t, v_h)_{j'} = H_{j'}(\partial_h f(u_h), v_h) - H_{j'}^+(\partial_h q_h, v_h), \tag{3.8}$$
$$(\partial_h q_h, w_h)_{j'} = -H_{j'}^-(\partial_h u_h, w_h),$$

with $j' = j + \frac{1}{2}$, which holds for any $v_h, w_h \in V_h^\alpha$, $j = 1, \cdots, N$. By Galerkin orthogonality, we have the following error equations

$$((\partial_h e_u)_t, v_h)_{j'} = H_{j'}(\partial_h(f(u) - f(u_h)), v_h) - H_{j'}^+(\partial_h e_q, v_h), \tag{3.9}$$
$$(\partial_h e_q, w_h)_{j'} = -H_{j'}^-(\partial_h e_u, w_h),$$

for all $v_h, w_h \in V_h^\alpha$. For the sake of simplicity, we denote

$$\partial_h e_u = \bar{e}_u = \bar{\eta}_u + \bar{\xi}_u, \bar{\eta}_u = \partial_h \eta_u, \bar{\xi}_u = \partial_h \xi_u,$$
$$\partial_h e_q = \bar{e}_q = \bar{\eta}_q + \bar{\xi}_q, \bar{\eta}_q = \partial_h \eta_q, \bar{\xi}_q = \partial_h \xi_q.$$

Due to $\bar{\eta}_u^-|_{j'+\frac{1}{2}} = \bar{\eta}_q^+|_{j'+\frac{1}{2}} = 0$ and projection orthogonality, summing over all j', we have

$$((\partial_h e_u)_t, v_h) = H(\partial_h(f(u) - f(u_h)), v_h) - H^+(\bar{\xi}_q, v_h), \tag{3.10a}$$
$$(\partial_h e_q, w_h) = -H^-(\bar{\xi}_u, w_h). \tag{3.10b}$$

When taking $v_h = \bar{\xi}_u, w_h = \bar{\xi}_q$ in equations (3.10), we get the following equations

$$\frac{1}{2}\frac{d}{dt}\|\bar{\xi}_u\|^2 + ((\bar{\eta}_u)_t, \bar{\xi}_u) = H(\partial_h(f(u) - f(u_h)), \bar{\xi}_u) - H^+(\bar{\xi}_q, \bar{\xi}_u), \tag{3.11a}$$
$$\|\bar{\xi}_q\|^2 + (\bar{\eta}_q, \bar{\xi}_q) = -H^-(\bar{\xi}_u, \bar{\eta}_q). \tag{3.11b}$$

In order to estimate the nonlinear term on the right side of equation (3.11a), we give the following lemmas.

Lemma 3.1. *Assuming that the conditions of Theorem 3.1 hold, we have*

$$H(\partial_h(f(u)-f(u_h)),\bar{\xi}_u) \le C_*\|\bar{\xi}_u\|^2 - \frac{\delta}{2}|[\bar{\xi}_u]|^2 + h^{-1}|[\xi_u]|^2 + Ch^{2k+2}, \tag{3.12}$$

where the positive constants C and C_* are independent of h and u_h.

More details of proof are given in [12].

We are now ready to obtain the estimate of $\bar{\xi}_u$ in L^2-norm. Adding equations (3.11a) and (3.11b), we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\bar{\xi}_u\|^2 + \|\bar{\xi}_q\|^2\\ &= H(\partial_h(f(u)-f(u_h)),\bar{\xi}_u) - ((\bar{\eta}_u)_t,\bar{\xi}_u) - (\bar{\eta}_q,\bar{\xi}_q) - H^+(\bar{\xi}_q,\bar{\xi}_u) - H^-(\bar{\xi}_u,\bar{\xi}_q).\end{aligned}$$

By the property of DG discrete operator (2.8), we have

$$-H^+(\bar{\xi}_q,\bar{\xi}_u) - H^-(\bar{\xi}_u,\bar{\xi}_q) = 0.$$

Then, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\bar{\xi}_u\|^2 + \|\bar{\xi}_q\|^2 \le |H(\partial_h(f(u)-f(u_h)),\bar{\xi}_u)| + |((\bar{\eta}_u)_t,\bar{\xi}_u)| + |(\bar{\eta}_q,\bar{\xi}_q)|. \tag{3.13}$$

For the integral terms in (3.13), we have

$$|((\bar{\eta}_u)_t,\bar{\xi}_u)| + |(\bar{\eta}_q,\bar{\xi}_q)| \le Ch^{2k+2} + C_*\|\bar{\xi}_u\|^2 + C_*\|\bar{\xi}_q\|^2. \tag{3.14}$$

Substituting (3.12) and (3.14) into (3.13), taking into account the bound for $\bar{\eta}_u$ and $(\bar{\eta}_u)_t$ and applying the Cauchy-Schwarz inequality and the Young's inequality, we have

$$\frac{1}{2}\frac{d}{dt}\|\bar{\xi}_u\|^2 + \|\bar{\xi}_q\|^2 + \frac{\delta}{2}|[\bar{\xi}_u]|^2 \le C_*\|\bar{\xi}_u\|^2 + C_*\|\bar{\xi}_q\|^2 + h^{-1}|[\xi_u]|^2 + Ch^{2k+2}.$$

We integrate the above inequality with respect to time t from 0 to T, notice that $\bar{\xi}_u(0)=0$ due to $\xi_u(0)=0$, and convergence result (3.8a), we obtain

$$\frac{1}{2}\|\bar{\xi}_u\|^2 + \frac{\delta}{2}\int_0^T |[\bar{\xi}_u]|^2 dt \le C_*\int_0^T \|\bar{\xi}_u\|^2 dt + Ch^{2k+2}.$$

Finally, by Gronwall's inequality we obtain

$$\|\bar{\xi}_u\|^2 + \int_0^T |[\bar{\xi}_u]|^2 dt \le C_* h^{2k+2}. \tag{3.15}$$

Thus, we finish the proof of Theoerm 3.1 when $\alpha = 1$.

3.2.2. Proof of second order difference quotient

When $\alpha = 2$, the LDG schemes (3.6a) and (3.6b) become

$$\begin{aligned}((\partial_h^2 u_h)_t, v_h)_{j'} &= H_{j'}(\partial_h^2 f(u_h), v_h) - H_{j'}^+(\partial_h^2 q_h, v_h),\\ (\partial_h^2 q_h, w_h)_{j'} &= -H_{j'}^-(\partial_h^2 u_h, w_h),\end{aligned} \tag{3.16}$$

with $j^{'} = j$, which hold for any $v_h, w_h \in V_h^\alpha$, $j = 1, \cdots, N$. By Galerkin orthogonality, there is the following error equations

$$((\partial_h^2 e_u)_t, v_h)_{j'} = H_{j'}(\partial_h^2(f(u) - f(u_h)), v_h) - H_{j'}^+(\partial_h^2 e_q, v_h), \quad (3.17)$$
$$(\partial_h^2 e_q, w_h)_{j'} = -H_{j'}^-(\partial_h^2 e_u, w_h),$$

for all $v_h, w_h \in V_h^\alpha$. For the sake of simplicity, we take

$$\partial_h^2 e_u = \tilde{e}_u = \tilde{\eta}_u + \tilde{\xi}_u, \tilde{\eta}_u = \partial_h^2 \eta_u, \tilde{\xi}_u = \partial_h^2 \xi_u,$$
$$\partial_h^2 e_q = \tilde{e}_q = \tilde{\eta}_q + \tilde{\xi}_q, \tilde{\eta}_q = \partial_h^2 \eta_q, \tilde{\xi}_q = \partial_h^2 \xi_q.$$

Noting that $\tilde{\eta}_u^-|_{j'+\frac{1}{2}} = \tilde{\eta}_q^+|_{j'+\frac{1}{2}} = 0$ and summing over all $j^{'}$, we have

$$((\partial_h^2 e_u)_t, v_h) = H(\partial_h^2(f(u) - f(u_h)), v_h) - H^+(\tilde{\xi}_q, v_h), \quad (3.18)$$
$$(\partial_h^2 e_q, w_h) = -H^-(\tilde{\xi}_u, w_h).$$

If we take $v_h = \tilde{\xi}_u, w_h = \tilde{\xi}_q$ in equation (3.18), we get the following equations

$$\frac{1}{2}\frac{d}{dt}\|\tilde{\xi}_u\|^2 + ((\tilde{\eta}_u)_t, \tilde{\xi}_u) = H(\partial_h^2(f(u) - f(u_h)), \tilde{\xi}_u) - H^+(\tilde{\xi}_q, \tilde{\xi}_u), \quad (3.19a)$$
$$\|\bar{\xi}_q\|^2 + (\tilde{\eta}_q, \tilde{\xi}_q) = -H^-(\tilde{\xi}_u, \tilde{\xi}_q). \quad (3.19b)$$

Since the estimation of the nonlinear term on the right side of equation (3.19a) is complicated, we give the following lemma.

Lemma 3.2. *Suppose that the conditions of Theorem 3.1 hold, we have*

$$H(\partial_h^2(f(u) - f(u_h)), \tilde{\xi}_u) \le C_*\|\tilde{\xi}_u\|^2 - \frac{\delta}{2}|[\tilde{\xi}_u]|^2 + h^{-1}|[\xi_u]|^2 + Ch^{2k+2}, \quad (3.20)$$

where the positive constant C and C_* are independent of h and u_h.

Proof By the second order Taylor expansion, we have

$$\begin{aligned} & H(\partial_h^2(f(u) - f(u_h)), \tilde{\xi}_u) \\ = & H(\partial_h^2(f^{'}(u)\xi_u), \tilde{\xi}_u) + H(\partial_h^2(f^{'}(u)\eta_u), \tilde{\xi}_u) - H(\partial_h^2(R_1 e_u^2), \tilde{\xi}_u) \quad (3.21) \\ = & P + Q - S. \end{aligned}$$

The estimates are made separately below.

For P, we use the Leibniz rule (2.2b) to rewrite $\partial_h^2(f^{'}(u)\xi_u)$ as

$$\begin{aligned} \partial_h^2(f^{'}(u)\xi_u) = & f^{'}(u(x+h))\tilde{\xi}_u(x) + 2\partial_h f^{'}(u(x+h/2))\bar{\xi}_u(x-h/2) \\ & + \partial_h^2(f^{'}(u(x))\xi_u(x-h). \end{aligned}$$

Therefore, we know

$$\begin{aligned} P = & H(f^{'}(u)\tilde{\xi}_u(x), \tilde{\xi}_u) + 2H(\partial_h f^{'}(u)\bar{\xi}_u, \tilde{\xi}_u) + H((\partial_h^2(f^{'}(u))\xi_u, \tilde{\xi}_u) \\ = & P_1 + P_2 + P_3, \end{aligned}$$

where we have omitted the dependence of x for convenience if there is no confusion. Directly applying (2.9a) and (2.9b) in Property 2.1 together with the assumption $f^{'}(u) \geq \delta > 0$, we get the estimation of P_1 as follows

$$P_1 \leq C_* \|\tilde{\xi}_u\|^2 - \frac{\delta}{2} |[\tilde{\xi}_u]|^2. \tag{3.22a}$$

By Property 2.2, we obtain the estimates of P_2 and P_3 as follows

$$P_2 \leq C_*(\|\bar{\xi}_u\| + \|(\bar{\xi}_u)_x\| + h^{-\frac{1}{2}} |[\bar{\xi}_u]|) \|\tilde{\xi}_u\|, \tag{3.22b}$$

$$P_3 \leq C_*(\|\xi_u\| + \|(\xi_u)_x\| + h^{-\frac{1}{2}} |[\xi_u]|) \|\tilde{\xi}_u\|. \tag{3.22c}$$

Substituting (3.7a) - (3.7c) into (3.22b) - (3.22c), and combining (3.22a) with (3.15), after directly applying Young's inequality, we obtain

$$P \leq C_* \|\tilde{\xi}_u\|^2 - \frac{\delta}{2} |[\tilde{\xi}_u]|^2 + h^{-1}(|[\xi_u]|^2 + |[\bar{\xi}_u]|^2) + \|(\bar{\xi}_u)_x\|^2 + Ch^{2k+2}. \tag{3.23}$$

For integral terms in (3.23), we also need to estimate $(\bar{\xi}_u)_x$, which is given in the following lemma.

Lemma 3.3. *Suppose that the conditions in Theorem 3.1 hold, we have*

$$\|(\bar{\xi}_u)_x\| \leq C_*(\|(\bar{\xi}_u)_t\| + h^{k+1}), \tag{3.24}$$

where the positive constant C and C_* are independent of h and u_h.

Proof Firstly, we use the Taylor expansion (3.14a) and the Leibniz rule (2.2b) to rewrite $\partial_h(f(u) - f(u_h))$ as

$$\begin{aligned}
\partial_h(f(u) - f(u_h)) &= \partial_h(f^{'}(u)\xi_u) + \partial_h(f^{'}(u)\eta_u) - \partial_h(R_1 e_u^2) \\
&= f^{'}(u(x + h/2))\bar{\xi}_u + (\partial_h(f^{'}(u))\xi_u(x - h/2) + \partial_h(f^{'}(u)\eta_u) \\
&\quad - R_1(u(x + h/2))(\partial_h e_u^2) - (\partial_h R_1) e_u^2(x - h/2) \\
&= \pi_1 + \cdots + \pi_5.
\end{aligned}$$

Equation (3.10) can be written as

$$\begin{aligned}
((\bar{e}_u)_t, v_h) &= H(\partial_h(f(u) - f(u_h)), v_h) - H^+(\bar{\xi}_q, v_h) \\
&= \Pi_1 + \cdots + \Pi_5 + \Theta,
\end{aligned} \tag{3.25}$$

with $\Pi_i = H(\pi_i, v_h)(i = 1, \cdots, 5)$ and $\Theta = H^+(\bar{\xi}_q, v_h)$. Next, we separatly estimate each term.

Firstly consider Π_1. By the definition of the DG discrete operator, we obtain

$$\Pi_1 = H(f^{'}(u)\bar{\xi}_u, \xi_u) = -(f^{'}(\bar{\xi}_u)_x, v_h) - \sum_{j=1}^{N} (f^{'}[[\bar{\xi}_u]]v_h^+)_j.$$

We take $v_h = (\bar{\xi}_q)_x - rL_k(d)$ and note that $r = (-1)^k((\bar{\xi}_q)_x)^-_{j+1}$ is a constant and $d = 2(x - x_{j+\frac{1}{2}})/h \in [-1, 1]$, where L_k is the standard Legendre polynomial of degree k in $[-1,1]$. So $L_k(-1)=(-1)^k$ and L_k is orthogonal to any polynomial of degree at most $k-1$. Due to $v_j^+=v_{j+1}^-=0$, we have

$$\Pi_1=-(\partial_x f^{'}(u)\xi_u)-(f^{'}(\bar{\xi}_u)_x,(\bar{\xi}_q)_x-bL_k(s)=-A-B. \qquad (3.26)$$

On each element $I_{j'}=I_{j+\frac{1}{2}}=(x_j,x_{j+1})$, by the linearization $f^{'}(u)=f^{'}(u_{j+\frac{1}{2}})+(f^{'}(u)-f^{'}(u_{j+\frac{1}{2}}))$ with $u_{j+\frac{1}{2}}=u(x_{j+\frac{1}{2}},t)$ and $((\bar{\xi}_u)_x,L_k)_{j+\frac{1}{2}}=0$, we obtain an equation about B as follows

$$B=B_1+B_2, \qquad (3.27)$$

where

$$\begin{aligned}B_1&=\sum_{j=1}^{N}f^{'}(u_{j+\frac{1}{2}})(\|(\bar{\xi}_u)_x\|\|(\bar{\xi}_q)_x\|)_{I_{j+\frac{1}{2}}},\\ B_2&=((f^{'}(u)-f^{'}(u_{j+\frac{1}{2}}))(\bar{\xi}_u)_x,(\bar{\xi}_q)_x-rL_k).\end{aligned}$$

By inverse property (ii), for $v_h=(\bar{\xi}_q)_x-rL_k$, we have

$$\|v_h\|\le C\|(\bar{\xi}_q)_x\|.$$

Substituting the above results into (3.25) and noting that $f^{'}(u)\ge\delta>0$, we obtain

$$\delta\|(\bar{\xi}_u)_x\|\|(\bar{\xi}_q)_x\|\le Y_1=\sum_{i=2}^{5}\Pi_i-A-B_2+\Theta-((\bar{e}_u)_t,(\bar{\xi}_q)_x-rL_k). \qquad (3.28)$$

Next, we estimate each term on the right side of (3.28).

For Π_2, by using the definition of the DG discrete operator and $(v_h)_j^+=0$, we obtain

$$\Pi_2=-((\partial_h(f^{'}(u)\xi_u),v_h).$$

Furthermore, by Cauchy-Schwarz inequality, we obtain a bound for Π_2. That is

$$\|\Pi_2\|\le C_*(\|\xi_u\|+\|(\xi_u)_x\|)\|(\bar{\xi}_q)_x\|. \qquad (3.29a)$$

Directly applying Property 2.4 to estimate Π_3, we obtain

$$\|\Pi_3\|\le C_*h^{k+1}\|(\bar{\xi}_q)_x\|. \qquad (3.29b)$$

Using the proof method similar to lemma3.1, we have

$$\|\Pi_4\|\le C_*h^{-1}\|e_u\|_\infty(\|\bar{\xi}_u\|+h^{k+1})\|(\bar{\xi}_q)_x\|, \qquad (3.29c)$$
$$\|\Pi_5\|\le C_*h^{-1}\|e_u\|_\infty(\|\xi_u\|+h^{k+1})\|(\bar{\xi}_q)_x\|. \qquad (3.29d)$$

By the Cauchy-Schwarz inequality, we have

$$\|A\|\le C_*\|\bar{\xi}_u\|\|(\bar{\xi}_q)_x\|, \qquad (3.29e)$$
$$\|B_2\|\le C_*\|\bar{\xi}_u\|\|(\bar{\xi}_q)_x\|. \qquad (3.29f)$$

Using the triangular inequality, we have

$$|((\bar{e}_u)_t, (\bar{\xi}_q)_x - rL_k)| \leq C(\|(\bar{\xi}_u)_t\| + h^{k+1})\|(\bar{\xi}_q)_x\|. \tag{3.29g}$$

Noting that $(v_h)^-_{j+1} = 0$, we obtain

$$\Theta = H(\bar{\xi}_q, v_h) = ((\bar{\xi}_q)_x, v_h).$$

Finally, it follows from Cauchy-Schwarz inequality that

$$|\Theta| \leq C_* \|(\bar{\xi}_q)_x\|^2. \tag{3.29h}$$

If we want to get the conclusion of Lemma 3.3, we need to estimate $\|(\bar{\xi}_q)_x\|$. Since $H^+(\bar{e}_q, v_h) = H^+(\bar{\xi}_q, v_h)$ and $(v_h)^+_j = 0$, we obtain

$$\begin{aligned}((\bar{\xi}_q)_x, v_h) + \sum_{j=1}^{N}([[\bar{\xi}_q]](v_h)^+)_j &= H(\partial_h(f(u) - f(u_h)), v_h) - ((\bar{e}_u)_t, v_h) \\ &\leq |H(\partial_h(f(u) - f(u_h)), v_h)| + |((\bar{e}_u)_t, v_h)|.\end{aligned}$$

According to Lemma 3.1 and Cauchy-Schwarz inequality, we have

$$|((\bar{\xi}_q)_x, v_h)| \leq C_* \|\bar{\xi}_u\|^2 - \frac{\delta}{2}|[\bar{\xi}_u]|^2 + h^{-1}|[\xi_u]|^2 + \|(\bar{e}_u)_t\| + Ch^{2k+2}.$$

When $v_h = (\bar{\xi}_q)_x - rL_k(d)$, we have

$$\|(\bar{\xi}_q)_x\|^2 \leq (C_* \|\bar{\xi}_u\|^2 - \frac{\delta}{2}|[\bar{\xi}_u]|^2 + h^{-1}|[\xi_u]|^2 + \|(\bar{e}_u)_t\| + Ch^{2k+2})\|(\bar{\xi}_q)_x\|.$$

It is easy to see that

$$\|(\bar{\xi}_q)_x\| \leq C_* \|\bar{\xi}_u\|^2 - \frac{\delta}{2}|[\bar{\xi}_u]|^2 + h^{-1}|[\xi_u]|^2 + \|(\bar{\xi}_u)_t\| + Ch^{2k+2}. \tag{3.30}$$

Substituting (3.29a)-(3.29h) into (3.28) and using estimates (3.7a)-(3.7c), (3.15) and (3.30), we complete the proof of Lemma 3.3.

In order to estimate $\|(\bar{\xi}_u)_t\|$, we also need to estimate the terms $\|((\xi_u)_t)_x\|$ and $\|(\xi_u)_{tt}\|$, whose results are shown in Lemma 3.4 and 3.5.

Lemma 3.4. *Suppose that the conditions in Theorem 3.1 hold, we have*

$$\|((\xi_u)_t)_x\| \leq C_*(\|(\xi_u)_{tt}\| + h^{k+1}). \tag{3.31}$$

The proof of Lemma 3.4 is similar to Lemma 3.3.

Lemma 3.5. *Suppose that the conditions in Theorem 3.1 hold, we have*

$$\|(\xi_u)_{tt}\|^2 + \int_0^T |[(\xi_u)_{tt}]|^2 dt \leq C_* h^{2k+1}. \tag{3.32}$$

Proof Considering $\xi_u(0) = 0$ and $(\xi_u)_t(0) \le Ch^{k+1}$ has been proved in [1], we find that the first-order time derivative of the original error equations

$$((e_u)_{tt}, v_h) = H(\partial_t(f(u) - f(u_h)), v_h) - H^+((\xi_q)_t, v_h), \tag{3.33a}$$
$$((e_q)_t, w_h) = -H^-((\xi_u)_t, w_h), \tag{3.33b}$$

still hold at $t = 0$ for any $v_h, w_h \in V_h^\alpha$. Taking $v_h = (\xi_u)_{tt}(0) = 0$ in (3.41a) and using similar proof method to $\|(\xi_u)_t(0)\|$ in [12], we get a bound for $\|(\xi_u)_{tt}(0)\|$ as follows

$$\|(\xi_u)_{tt}(0)\| \le Ch^{k+1}. \tag{3.34}$$

For $(\xi_u)_{tt}(T)$, $T > 0$, taking into account the second-order time derivative of the original error equation and letting $v_h = (\xi_u)_{tt}$, $w_h = (\xi_q)_{tt}$, we have

$$((e_u)_{ttt}, (\xi_u)_{tt}) = H(\partial_{tt}(f(u) - f(u_h)), (\xi_u)_{tt}) - H^+((\xi_q)_{tt}, (\xi_u)_{tt}),$$
$$((e_q)_{tt}, (\xi_q)_{tt}) = -H^-((\xi_u)_{tt}, (\xi_q)_{tt}).$$

Adding the above two equations together, according to the property of the DG discrete operator, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|(\xi_u)_{tt}\|^2 + \|(\xi_q)_{tt}\|^2 \\
&\le |H(\partial_{tt}(f(u) - f(u_h)), (\xi_u)_{tt}| + |((\eta_u)_{ttt}, (\xi_u)_{tt})| + |((\eta_q)_{tt}, (\xi_q)_{tt})|.
\end{aligned} \tag{3.35}$$

Next, we use Taylor expansion and the Leibniz rule (2.2b) for spatial derivatives to estimate the right nonlinear term of inequality (3.35), respectively. Rewrite $\partial_{tt}(f(u) - f(u_h))$ as

$$\begin{aligned}
\partial_{tt}(f(u) - f(u_h) =& \partial_{tt}(f^{'}(u)\xi_u) + \partial_{tt}(f^{'}(u)\eta_u) - \partial_{tt}(R_1 e_u^2) \\
=& (\partial_{tt} f^{'}(u))\xi_u + 2(\partial_t f^{'}(u))(\xi_u)_t + f^{'}(u)(\xi_u)_{tt} + (\partial_{tt} f^{'}(u))\eta_u \\
&+ 2(\partial_t f^{'}(u))(\eta_u)_t + f^{'}(u)(\eta_u)_{tt} - (\partial_{tt} R_1)e_u^2 \\
&- 2(\partial_t R_1)\partial_t e_u^2 - R_1(\partial_{tt} e_u^2) \\
=& \psi_1 + \cdots + \psi_9.
\end{aligned}$$

Then, the estimation of the nonlinear term

$$H(\partial_{tt}(f(u) - f(u_h)), (\xi_u)_{tt}) = \Psi_1 + \cdots + \Psi_9, \tag{3.36}$$

with $\Psi_i = H(\psi_i, (\xi_u)_{tt})(i = 1, \cdots, 9)$.

By (2.9a) in Property 2.1, we obtain the estimation of Ψ_1 as follows

$$\begin{aligned}
|\Psi_1| &\le C_*(\|\xi_u\| + \|(\xi_u)_x\| + h^{-\frac{1}{2}}|[\xi_u]|)\|(\xi_u)_{tt}\| \\
&\le C_*(h^{k+1} + h^{-\frac{1}{2}}|[\xi_u]|)\|(\xi_u)_{tt}\| \\
&\le C_*(\|(\xi_u)_{tt}\|^2 + h^{-1}|[\xi_u]|^2 + h^{2k+2}),
\end{aligned} \tag{3.37a}$$

where error estimates (3.7a) - (3.7c) and Young's inequality are used. Analogously, we obtain

$$|\Psi_2| \le C_*(\|(\xi_u)_t\| + \|((\xi_u)_t)_x\| + h^{-\frac{1}{2}}|[(\xi_u)_t]|)\|(\xi_u)_{tt}\|$$

$$\le C_*(h^{k+1} + \|(\xi_u)_{tt}\| + h^{-\frac{1}{2}}|[(\xi_u)_t]|)\|(\xi_u)_{tt}\|$$
$$\le C_*(\|(\xi_u)_{tt}\|^2 + h^{-1}|[(\xi_u)_{tt}]|^2 + h^{2k+2}). \tag{3.37b}$$

Directly applying (2.9b) in Property 2.1, we get

$$|\Psi_3| \le C_*\|(\xi_u)_{tt}\|^2 - \frac{\delta}{2}|[(\xi_u)_{tt}]|^2. \tag{3.37c}$$

Noting that $(\eta_u)_t = u_t - P_h^- u_t$ and $(\eta_u)_{tt} = u_{tt} - P_h^- u_{tt}$, by Property 2.3, we have

$$|\Psi_4| + |\Psi_5| + |\Psi_6| \le C_* h^{k+1}\|(\xi_u)_{tt}\|. \tag{3.37d}$$

Using similar proof method as lemma3.1, we obtain

$$|\Psi_7| \le C_* h^{-1}\|e_u\|_\infty(\|\xi_u\| + h^{k+1})\|(\xi_u)_{tt}\|,$$
$$|\Psi_8| \le C_* h^{-1}\|e_u\|_\infty(\|(\xi_u)_t\| + h^{k+1})\|(\xi_u)_{tt}\|,$$
$$|\Psi_9| \le C_* h^{-1}(\|e_u\|_\infty + \|(e_u)_t\|_\infty)(\|(\xi_u)_t\| + \|(\xi_u)_{tt}\| + h^{k+1})\|(\xi_u)_{tt}\|.$$

By using the inverse property (iii), the superconvergence result (3.7a), (3.7c) and the approximate error estimate (2.5b), for small enough h, we have

$$C_* h^{-1}\|e_u\|_\infty \le C_* h^{-1}(\|\xi_u\|_\infty + \|\eta_u\|_\infty) \le C_* h^k \le C,$$
$$C_* h^{-1}\|(e_u)_t\|_\infty \le C_* h^{-1}(\|(\xi_u)_t\|_\infty + \|(\eta_u)_t\|_\infty) \le C_* h^{k-\frac{1}{2}} \le C,$$

where C is a positive constant independent of h. Therefore, we have

$$|\Psi_7| \le C(\|\xi_u\| + h^{k+1})\|(\xi_u)_{tt}\|, \tag{3.37e}$$
$$|\Psi_8| \le C(\|(\xi_u)_t\| + h^{k+1})\|(\xi_u)_{tt}\|, \tag{3.37f}$$
$$|\Psi_9| \le C(\|(\xi_u)_t\| + \|(\xi_u)_{tt}\| + h^{k+1})\|(\xi_u)_{tt}\|. \tag{3.37g}$$

Substituting (3.37a)-(3.37g) into (3.36), combining estimates (3.7a)-(3.7c) and Cauchy-Schwarz inequality, we obtain

$$H(\partial_{tt}(f(u) - f(u_h), (\xi_u)_{tt})$$
$$\le C_*(\|(\xi_u)_{tt}\|^2 + h^{-1}(|[\xi_u]|^2 + |[(\xi_u)_t]|^2) + h^{2k+2}). \tag{3.38}$$

For the the integral terms in (3.35), we have

$$|((\eta_u)_{ttt}, (\xi_u)_{tt})| + |((\eta_q)_{tt}, (\xi_q)_{tt})| \le Ch^{2k+2} + C_*\|(\xi_u)_{tt}\|^2 + C_*\|(\xi_q)_{tt}\|^2. \tag{3.39}$$

Substituting (3.44) and (3.39) into (3.35), directly applying Cauchy-Schwarz inequality and Young's inequality, we obtain

$$\frac{1}{2}\frac{d}{dt}\|(\xi_u)_{tt}\|^2 + \frac{\delta}{2}|[(\xi_u)_{tt}]|^2 + \|(\xi_q)_{tt}\|^2$$
$$\le C_*\|(\xi_u)_{tt}\|^2 + C_* h^{-1}(|[\xi_u]|^2 + |[(\xi_u)_t]|) + Ch^{2k+2} + C_*\|(\xi_q)_{tt}\|.$$

Integrating the above inequality with respect to time t from 0 to T and combining the initial error estimate (3.33) with the superconvergence results (3.7a) and (3.7c), we obtain

$$\frac{1}{2}\|(\xi_u)_{tt}\|^2 + \frac{\delta}{2}\int_0^T |[(\xi_u)_{tt}]|^2 dt \le C_*\int_0^T \|(\xi_u)_{tt}\|^2 dt + Ch^{2k+1}.$$

By using the Gronwall's inequality, we obtain

$$\|(\xi_u)_{tt}\|^2 + \int_0^T |[(\xi_u)_{tt}]|^2 dt \le C_* h^{2k+1}. \tag{3.40}$$

This finishes the proof of Lemma 3.5.

By Lemmas 3.4 and 3.5, we can give the bound for $(\bar{\xi}_u)_t$.

Lemma 3.6. *Suppose that the conditions in Theorem 3.1 hold, we have*

$$\|(\bar{\xi}_u)_t\|^2 + \int_0^T |[(\bar{\xi}_u)_t]|^2 dt \le C_* h^{2k+1}, \tag{3.41}$$

where the positive constants C and C_* are independent of h and u_h.

Proof Since $\xi_u(0) = 0$, it is easy to show that $\bar{\xi}_u(0) = 0$. Original error equation still hold for any $v_h \in V_h^\alpha$ at $t = 0$. In equation (3.7a), let $v_h = (\bar{\xi}_u)_t(0)$. Using similar proof method of $\|(\xi_u)_t(0)\|$ in [1], we get the estimate of $(\bar{\xi}_u)_t(0)$ as follows

$$\|(\bar{\xi}_u)_t(0)\| \le C_* h^{k+1}. \tag{3.42}$$

We are going to obtain the estimate of $\|(\bar{\xi}_u)_t(T)\|$ for $T > 0$. Taking the time derivative of the original error equation and letting $v_h = (\bar{\xi}_u)_t$, $w_h = (\bar{\xi}_q)_t$, we have

$$((e_u)_{tt}, (\bar{\xi}_u)_t) = H(\partial_t\partial_h(f(u) - f(u_h)), (\bar{\xi}_u)_t) - H^+((\bar{\xi}_q)_t, (\bar{\xi}_u)_t), \tag{3.43a}$$

$$((e_q)_t, (\bar{\xi}_q)_t) = -H^-((\bar{\xi}_u)_t, (\bar{\xi}_q)_t). \tag{3.43b}$$

Adding (3.51a) to (3.43b), we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|(\bar{\xi}_u)_t\|^2 + \|(\bar{\xi}_q)_t\|^2 \\ &\le |H(\partial_t\partial_h(f(u) - f(u_h)), (\bar{\xi}_u)_t)| + |(\|(\bar{\eta}_u)_{tt}\|, \|(\bar{\xi}_u)_t\|)| + |(\|(\bar{\eta}_q)_t\|, \|(\bar{\xi}_q)_t\|)|.\end{aligned} \tag{3.44}$$

In order to estimate the right side of (3.44), we use Taylor expansion (3.14a) and Leibniz rule (2.2b) for spatial derivatives to rewrite $\partial_t\partial_h(f(u) - f(u_h))$ as

$$\begin{aligned}&\partial_t\partial_h(f(u) - f(u_h)) \\ =\ &\partial_h\partial_t(f^{'}(u)\xi_u) + \partial_h\partial_t(f^{'}(u)\eta_u) - \partial_h\partial_t(f^{'}(u)R_1e_u^2) \\ =\ &\partial_h(\partial_t f^{'}(u)\xi_u) + \partial_h(f^{'}(u)(\xi_u)_t) + \partial_h(\partial_t f^{'}(u)\eta_u) + \partial_h(f^{'}(u)(\eta_u)_t) \\ &- \partial_h(R_1\partial_t e_u^2) - \partial_h(\partial_t R_1 e_u^2) \\ =\ &\partial_t f^{'}(u(x+\frac{h}{2}))\xi_u(x) + \partial_h(\partial_t f^{'}(u))\xi_u(x-\frac{h}{2}) + f^{'}(u(x+\frac{h}{2}))(\bar{\xi}_u)_t(x) \\ &+ \partial_t f^{'}(u)(\bar{\xi}_u)_t(x-\frac{h}{2}) + \partial_h(\partial_t f^{'}(u)\eta_u) + \partial_h(f^{'}(u)(\eta_u)_t) - R_1(u(x+\frac{h}{2}))\partial_h(\partial_t e_u^2) \\ &- \partial_h R_1\partial_t e_u^2(x-\frac{h}{2}) - \partial_t R_1(u(x+\frac{h}{2}))\partial_h e_u^2 - \partial_h(\partial_t R_1)e_u^2(x-\frac{h}{2}) \\ =\ &\theta_1 + \cdots + \theta_{10}.\end{aligned}$$

The right side term of inequality (3.52) can be written as

$$H(\partial_t\partial_h(f(u) - f(u_h)), (\bar{\xi}_u)_t) = \Theta_1 + \cdots + \Theta_{10}, \tag{3.45}$$

with $\Theta_i = H(\theta_i, (\bar{\xi}_u)_t)$ for $i = 1, \cdots, 10$. Next, we will estimate these terms separately.

By (2.9a) in Property 2.1, (3.23), (3.32) and Young's inequality, we obtain the estimation of Θ_1 as follows

$$\begin{aligned}|\Theta_1| &\le C_*(\|\bar{\xi}_u\| + \|(\bar{\xi}_u)_x\| + h^{-\frac{1}{2}}|[\bar{\xi}_u]|)\|(\bar{\xi}_u)_t\| \\ &\le C_*(h^{k+1} + \|(\bar{\xi}_u)_t\| + h^{-\frac{1}{2}}|[\bar{\xi}_u]|)\|(\bar{\xi}_u)_t\| \\ &\le C_*(h^{2k+2} + \|(\bar{\xi}_u)_t\|^2 + h^{-1}|[\bar{\xi}_u]|^2). \end{aligned} \tag{3.46a}$$

Analogously, for Θ_2 and Θ_4, we apply Property 2.2, (3.7a)-(3.7c) and (3.31) to get

$$|\Theta_2| \le C_*(\|(\bar{\xi}_u)_t\|^2 + h^{-1}|[\xi_u]|^2 + h^{2k+2}), \tag{3.46b}$$

$$|\Theta_4| \le C_*(\|(\bar{\xi}_u)_t\|^2 + \|(\bar{\xi}_u)_{tt}\|^2 + h^{-1}|[(\xi_u)_t]|^2 + h^{2k+2}). \tag{3.46c}$$

By using (2.9b) in Property 2.1 together with the assumption that $f^{'}(u) \ge \delta > 0$, we obtain the estimate of Θ_3:

$$|\Theta_3| \le C_*(\|(\bar{\xi}_u)_t\|^2 - \frac{\delta}{2}|[(\bar{\xi}_u)_t]|^2). \tag{3.46d}$$

Noting that $(\eta_u)_t = u_t - P_h^-(u_t)$, by Property 2.4, we have

$$|\Theta_5| + |\Theta_6| \le C_* h^{k+1}\|(\bar{\xi}_u)_t\|. \tag{3.46e}$$

Using the similar proof method as lemma3.1, we obtain

$$|\Theta_7| \le C_*(\|(\xi_u)_t\| + \|(\bar{\xi}_u)_t\| + h^{k+1})\|(\bar{\xi}_u)_t\|, \tag{3.46f}$$

$$|\Theta_8| \le C_*(\|(\xi_u)_t\| + h^{k+1})\|(\bar{\xi}_u)_t\|, \tag{3.46g}$$

$$|\Theta_9| \le C_*(\|\bar{\xi}_u\| + h^{k+1})\|(\bar{\xi}_u)_t\|, \tag{3.46h}$$

$$|\Theta_{10}| \le C_*(\|\xi_u\| + h^{k+1})\|(\bar{\xi}_u)_t\|. \tag{3.46i}$$

Substituting (3.46a)-(3.46i) into (3.45) and applying the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}&H(\partial_t\partial_h(f(u) - f(u_h)), (\bar{\xi}_u)_t) \\ &\le C_*(\|(\bar{\xi}_u)_t\|^2 + h^{-1}(|[\xi_u]|^2 + |[(\xi_u)_t]|^2 + |[\bar{\xi}_u]|^2) + h^{2k+1}),\end{aligned} \tag{3.48}$$

where the estimates (3.7a)-(3.7c) and (3.31) are used.

For the integral terms in (3.44), we have

$$|(\|(\bar{\eta}_u)_{tt}\|, \|(\bar{\xi}_u)_t\|)| + |(\|(\bar{\eta}_q)_t\|, \|(\bar{\xi}_q)_t\|)| \le C_*(h^{2k+2} + \|(\bar{\xi}_u)_t\|^2 + \|(\bar{\xi}_q)_t\|^2). \tag{3.49}$$

Substituting (3.48) and (3.49) into (3.44), we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|(\bar{\xi}_u)_t\|^2 + \frac{\delta}{2}|[(\bar{\xi}_u)_t]|^2 + \|(\bar{\xi}_q)_t\|^2 \\ &\le C_*(\|(\bar{\xi}_u)_t\|^2 + \|(\bar{\xi}_q)_t\|^2 + h^{2k+1}) + C_* h^{-1}(|[\xi_u]|^2 + |[(\xi_u)_t]|^2 + |[\bar{\xi}_u]|).\end{aligned}$$

We integrate the above inequality with respect to time t from 0 to T. Combining the initial estimate (3.42) with (3.7a), (3.7c), (3.15) and (3.48), we obtain

$$\frac{1}{2}\|(\bar{\xi}_u)_t\|^2 + \frac{\delta}{2}\int_0^T |[(\bar{\xi}_u)_t]|^2 dt \le C_*\int_0^T \|(\bar{\xi}_u)_t\|^2 dt + Ch^{2k+1}.$$

Finally, according to Gronwall's inequality, we have

$$\|(\bar{\xi}_u)_t\|^2 + \int_0^T |[(\bar{\xi}_u)_t]|^2 dt \le C_* h^{2k+1}. \tag{3.50}$$

This finishes the proof of Lemma 3.6.

We take the estimates in Lemmas 3.3 and 3.6 into (3.31) to obtain

$$P \le C_*\|\tilde{\xi}_u\|^2 - \frac{\delta}{2}|[\tilde{\xi}_u]|^2 + h^{-1}(|[\xi_u]|^2 + |[\bar{\xi}_u]|^2) + Ch^{2k+1}. \tag{3.51}$$

For the estimate of Q, direct applying Property 2.4, we have

$$Q \le C_* h^{k+1}\|\tilde{\xi}_u\|. \tag{3.52}$$

For the estimate of S, we use the Leibniz rule (2.2b) to rewrite $\partial_h^2(R_1 e_u^2)$ as

$$\begin{aligned}
&\partial_h^2(R_1 e_u^2)\\
&= R_1(u(x+h))\partial_h^2 e_u^2 + 2\partial_h R_1(u(x+\frac{h}{2}))\partial_h e_u^2(x-\frac{h}{2}) + \partial_h^2 R_1(u(x))e_u^2(x-h)\\
&= E_1 + E_2 + E_3,
\end{aligned}$$

where

$$\begin{aligned}
&E_1 = R_1(u(x+h))(e_u(x+h)\tilde{e}_u(x) + 2\bar{e}_u(x+\frac{h}{2})\bar{e}_u(x-\frac{h}{2}) + \tilde{e}_u(x)e_u(x-h)),\\
&E_2 = 2\partial_h R_1(u(x+\frac{h}{2}))\bar{e}_u(x-\frac{h}{2})(e_u(x) + e_u(x-h)),\\
&E_3 = \partial_h^2 R_1(u(x))e_u^2(x-h).
\end{aligned}$$

Thus, we obtain

$$S = H(E_1, \tilde{\xi}_u) + H(E_2, \tilde{\xi}_u) + H(E_3, \tilde{\xi}_u) = S_1 + S_2 + S_3.$$

By using the similar proof method of lemma3.1, we have

$$\begin{aligned}
S_1 &\le C_* h^{-1}(\|e_u\|_\infty + \|\bar{e}_u\|_\infty)(\|\tilde{\xi}_u\| + \|\bar{\xi}_u\| + h^{k+1})\|\tilde{\xi}_u\|\\
&\le C(\|\tilde{\xi}_u\| + \|\|\bar{\xi}_u + h^{k+1})\|\tilde{\xi}_u\|,\\
S_2 &\le C_* h^{-1}\|e_u\|_\infty(\|\bar{\xi}_u\| + h^{k+1})\|\tilde{\xi}_u\| \le C(\|\|\bar{\xi}_u + h^{k+1})\|\tilde{\xi}_u\|,\\
S_3 &\le C_* h^{-1}\|e_u\|_\infty(\|\xi_u\| + h^{k+1})\|\tilde{\xi}_u\| \le C(\|\xi_u\| + h^{k+1})\|\tilde{\xi}_u\|.
\end{aligned}$$

Note that h is small enough. We assume $C_* h^{-1}(\|e_u\|_\infty + \|\bar{e}_u\|_\infty) \le C$ when $k > 1$. Thus, we have

$$S \le C(\|\tilde{\xi}_u\| + \|\|\bar{\xi}_u + \|\xi_u\|h^{k+1})\|\tilde{\xi}_u\|. \tag{3.53}$$

Substituting (3.51)-(3.53) into (3.21) and considering (3.7a) and (3.15), we obtain the result of Lemma 3.2.

Next, we will go on estimating the L^2-norm of $\tilde{\xi}_u$. Adding (3.19a) and (3.19b), we obtain

$$\frac{1}{2}\frac{d}{dt}\|\tilde{\xi}_u\|^2 + \|\tilde{\xi}_q\|^2$$

$$= H(\partial_h(f(u)-f(u_h)),\tilde{\xi}_u) - ((\tilde{\eta}_u)_t,\tilde{\xi}_u) - (\tilde{\eta}_q,\tilde{\xi}_q) - H^+(\tilde{\xi}_q,\tilde{\xi}_u) - H^-(\tilde{\xi}_u,\tilde{\eta}_q).$$

According to the property of DG discrete operator (2.8), we know

$$-H^+(\tilde{\xi}_q,\tilde{\xi}_u) - H^-(\tilde{\xi}_u,\tilde{\eta}_q) = 0.$$

Then, we have

$$\frac{1}{2}\frac{d}{dt}\|\tilde{\xi}_u\|^2 + \|\tilde{\xi}_q\|^2 = H(\partial_h(f(u)-f(u_h)),\tilde{\xi}_u) - ((\tilde{\eta}_u)_t,\tilde{\xi}_u) - (\tilde{\eta}_q,\tilde{\xi}_q). \tag{3.54}$$

For the integral terms in (3.54), we have

$$|((\tilde{\eta}_u)_t,\tilde{\xi}_u)| + |(\tilde{\eta}_q,\tilde{\xi}_q)| \le Ch^{2k+2} + C_*\|\tilde{\xi}_u\|^2 + C_*\|\tilde{\xi}_q\|^2. \tag{3.55}$$

Substituting (3.20) and (3.55) into (3.54), applying Cauchy-Schwarz inequality and Young's inequality, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\tilde{\xi}_u\|^2 + \frac{\delta}{2}|[\tilde{\xi}_u]|^2 + \|\tilde{\xi}_q\|^2 \le C_*\|\tilde{\xi}_u\|^2 + C_*\|\tilde{\xi}_q\|^2 + h^{-1}(|[\xi_u]|^2 + |[\bar{\xi}_u]|^2) + Ch^{2k+1}.$$

Integrating the above inequality with respect to time t between 0 and T. Thus $\tilde{\xi}_u(0) = \partial_h^2\xi(0) = 0$ due to $\xi(0) = 0$. Combining with estimates (3.7a) and (3.15), we have

$$\frac{1}{2}\|\tilde{\xi}_u\|^2 + \frac{\delta}{2}\int_0^T |[\tilde{\xi}_u]|^2 dt \le C_*\int_0^T \|\tilde{\xi}_u\|^2 dt + Ch^{2k+1}.$$

Finally, it is easy to show by Gronwall's inequality that

$$\|\tilde{\xi}_u\|^2 + \int_0^T |[\tilde{\xi}_u]|^2 dt \le C_* h^{2k+1}, \tag{3.56}$$

which finishes the proof of Theorem 3.1 when $\alpha = 2$.

By the proof of Theorem 3.1 when $\alpha = 1$ and $\alpha = 2$, we find that it is necessary to estimate the relevant low order difference quotient and the corresponding spatial and time derivatives if we want to obtain the L^2-norm estimates of high order difference quotients. Therefore, when $\alpha = 3$, we need to estimate $(\tilde{\xi}_u)_x$, $(\tilde{\xi}_u)_t$, $((\bar{\xi}_u)_t)_x$, $(\bar{\xi}_u)_{tt}$, $((\xi_u)_{tt})_x$ and $(\xi_u)_{ttt}$. Thus, Theorem 3.1 can be proved in the same way for $\alpha \le k+1$.

4. Superconvergent error estimates in the negative-order norm

For nonlinear convection-diffusion equation, by post-processing theory[3,13], in order to obtain the superconvergence error estimate of the post-processed solution, it is necessary to obtain the negative-order norm estimates of the difference quotients of LDG error. Using dual argumentation and combining with the previously obtained L^2-norm estimate, we have the following results.

Theorem 4.1. *For any $0 \le \alpha \le k+1$, let $\partial_h^\alpha u$, $\partial_h^\alpha q$ be the exact solutions of equations (3.2a) and (3.2b), which are assumed to be sufficiently smooth with bounded* *derivatives. Let $\partial_h^\alpha u_h$, $\partial_h^\alpha q_h$ be the numerical solutions of the LDG schemes (3.3a) and (3.3b) with initial conditions $\partial_h^\alpha u_h(0) = P_h^-(\partial_h^\alpha u_0)$, $\partial_h^\alpha q_h(0) = P_h(\partial_h^\alpha q_0)$ when the upwind flux is used. For a uniform mesh of $\Omega = (a, b)$, if the finite element space $V_h^\alpha (k \geq 1)$ is taken as the k order piecewise polynomial space, then for small enough h and any $T > 0$ there holds the following error estimate*

$$\|\partial_h^\alpha (u - u_h)\|_{-(k+1),\Omega} \leq C_* h^{2k+\frac{3}{2}-\frac{\alpha}{2}}, \tag{4.1}$$

where the positive integer C_* depends on u, $\|u\|_{k+1}$, $\|u_t\|_{k+1}$, $\|u_{tt}\|_{k+1}$, δ and T, but is independent of h.

Combining Theorems 4.1 and 2.1, we obtain the following corollary.

Corollary 4.1. *Assume that the conditions in Theorem 4.1 hold. If $K_h^{2k+1,k+1}$ is a convolution kernel consisting of $2k+1$ B-splines of order $k+1$ such that it reproduces polynomials of degree $2k$, then we have*

$$\|u - u_h^*\| \leq C_* h^{\frac{3k}{2}+1}, \tag{4.2}$$

where $u_h^* = K_h^{2k+1,k+1} * u_h$.

For $k+1$ order B-splines, using similar argument for proof of the negative $k+1$ order norm estimates, we can obtain the following superconvergent error result

$$\|\partial_h^\alpha (u - u_h)(T)\|_{-(k+1),\Omega} \leq Ch^{k-\frac{3}{2}-\frac{\alpha}{2}+k+1-1} \leq Ch^{\frac{3k}{2}+1}.$$

4.1. Proof of the main results in the negative-order norm

Similar to the proof of the L^2-norm estimates of the difference quotient in Section 3.2, we only consider the case $f'(u) \geq \delta > 0$. According to the definition of negative-order norm, we first use dual argument in [3,12] to estimate $(\partial_h^\alpha (u-u_h)(T), \Phi)$ with $\Phi \in C_0^\infty(\Omega)$. For the nonlinear convection-diffusion equations (1.1a) and (1.1b) when $\varepsilon = 1$, we choose the dual equation as follows.

Find a function φ such that $\varphi(\cdot, t)$ is periodic for all $t \in [0, T]$ and satisfies

$$\partial_h^\alpha \varphi_t + f'(u)\partial_h^\alpha \varphi_x + \partial_h^\alpha \varphi_{xx} = 0, (x, t) \in \Omega \times [0, T), \tag{4.3a}$$

$$\varphi(x, T) = \Phi(x), x \in \Omega. \tag{4.3b}$$

Furthermore, if we multiply equation (3.1a) by φ and (4.3a) by $(-1)^\alpha u$ and integrate on Ω, we have

$$\frac{d}{dt}(\partial_h^\alpha u, \varphi) + \Gamma(u; \varphi) = 0, \tag{4.4}$$

$$\Gamma(u; \varphi) = (-1)^\alpha ((f'(u)u - f(u)), \partial_h^\alpha \varphi_x),$$

where we use integration by parts and summation by parts (2.2c).

We integrate (4.4) with respect to time t between 0 and T to obtain

$$(\partial_h^\alpha u, \varphi)(T) = (\partial_h^\alpha u, \varphi)(0) - \int_0^T \Gamma(u; \varphi) dt. \tag{4.5}$$

To estimate $(\partial_h^\alpha (u-u_h)(T), \Phi)$, for any $\lambda \in V_h^\alpha$, we use (4.5) to get an equivalent form:

$$(\partial_h^\alpha (u - u_h)(T), \Phi)$$

$$
\begin{aligned}
&= (\partial_h^\alpha (u-u_h)(T), \varphi(T)) \\
&= (\partial_h^\alpha u, \varphi)(0) - \int_0^T \Gamma(u;\varphi)dt - (\partial_h^\alpha u_h, \varphi)(0) - \int_0^T \frac{d}{dt}(\partial_h^\alpha u_h, \varphi)dt \\
&= (\partial_h^\alpha (u-u_h), \varphi)(0) - \int_0^T (((\partial_h^\alpha u_h)_t, \varphi) + (\partial_h^\alpha u_h, \varphi_t))dt - \int_0^T \Gamma(u;\varphi)dt \\
&= \Sigma_1 + \Sigma_2 + \Sigma_3,
\end{aligned}
$$

where

$$
\begin{aligned}
&\Sigma_1 = (\partial_h^\alpha (u-u_h), \varphi)(0), \\
&\Sigma_2 = -\int_0^T (((\partial_h^\alpha u_h)_t, \varphi-\lambda) - B_1(\partial_h^\alpha f(u_h), \partial_h^\alpha q_h, \varphi-\lambda))dt, \\
&\Sigma_3 = -\int_0^T ((\partial_h^\alpha u_h, \varphi_t) + B_1(\partial_h^\alpha f(u_h), \partial_h^\alpha q_h, \varphi) + \Gamma(u;\varphi))dt, \\
&B_1(\partial_h^\alpha f(u_h), \partial_h^\alpha q_h, v_h) = H(\partial_h^\alpha f(u_h), v_h) - H(\partial_h^\alpha q_h, v_h).
\end{aligned}
$$

Next we will estimate Σ_1, Σ_2 and Σ_3 respectively.

Lemma 4.1. *There exists a positive constant C independent of h, such that*

$$|\Sigma_1| \le Ch^{2k+1}\|\partial_h^\alpha u_0\|_{k+1}\|\varphi(0)\|_{k+1}. \tag{4.6}$$

The proof of Lemma 4.1 can refer to [12].

Lemma 4.2. *There exists a positive constant C independent of h, such that*

$$|\Sigma_2| \le Ch^{2k+\frac{3}{2}-\frac{\alpha}{2}}\|\varphi\|_{k+1}. \tag{4.7}$$

Proof According to the definition of Ξ_2, taking $\lambda = P_h\varphi$, we have

$$((\partial_h^\alpha u_h)_t, \varphi - P_h\varphi) = 0.$$

And the integral term inside Σ_2 is

$$
\begin{aligned}
\sigma_2 &= -B_1(\partial_h^\alpha f(u_h), \partial_h^\alpha q_h, \varphi-\lambda) \\
&= -H(\partial_h^\alpha f(u_h), \varphi-\lambda) + H(\partial_h^\alpha q_h, \varphi-\lambda) \\
&= -H(\partial_h^\alpha f(u_h), \varphi-P_h\varphi) + H(\partial_h^\alpha q_h, \varphi-P_h\varphi) \\
&= M + N.
\end{aligned}
$$

First, we give the estimate of M, namely

$$
\begin{aligned}
M &= -H(\partial_h^\alpha f(u_h), \varphi-P_h\varphi) \\
&= -(\partial_h^\alpha (f(u_h)-f(u)), (\varphi-P_h\varphi)_x) + (\partial_h^\alpha f(u)_x, \varphi-P_h\varphi) \\
&\quad + \sum_{j=1}^N (\partial_h^\alpha (f(u)-f(u_h^-))[[\varphi-P_h\varphi]])_{j'-\frac{1}{2}} \\
&= M_1 + M_2 + M_3.
\end{aligned}
$$

Next, we consider the estimates of M_1, M_2 and M_3. For M_1, by using the second order Taylor expansion, we obtain

$$\begin{aligned} M_1 &= (\partial_h^\alpha(f'(u)e_u - R_1 e_u^2), (\varphi - P_h\varphi)_x) \\ &= (\partial_h^\alpha(f'(u)e_u), (\varphi - P_h\varphi)_x) - (\partial_h^\alpha(R_1 e_u^2), (\varphi - P_h\varphi)_x) \\ &= M_1^l - M_1^n, \end{aligned}$$

where M_1^l and M_1^n are the linear and nonlinear part of M_1 , respectively. By using the Leibniz rule (2.2b), Cauchy-Schwarz inequality, (3.5) and (2.5a), we have

$$|M_1^l| \le C \sum_{l=0}^{\alpha} \|\partial_h^{\alpha-l} e_u\| \|(\varphi - P_h\varphi)_x\| \le C_* h^{2k+\frac{3}{2}-\frac{\alpha}{2}} \|\varphi\|_{k+1}. \tag{4.8a}$$

Analogously, for nonlinear part M_1^n, by using the Leibniz rule (2.2b) twice for $\partial_h^\alpha(R_1 e_u^2)$, Cauchy-Schwarz inequality and (2.5a), we obtain

$$\begin{aligned} |M_1^n| &\le C \sum_{l=0}^{\alpha} \|\partial_h^{\alpha-l} e_u^2\| \|(\varphi - P_h\varphi)_x\| \\ &\le \sum_{m=0}^{\alpha} \|\partial_h^m e_u\|_\infty \|\partial_h^{\alpha-m} e_u\| \|(\varphi - P_h\varphi)_x\| \\ &\le \|C_* h^{3k+\frac{5}{2}-\frac{\alpha}{2}} \|\varphi\|_{k+1}. \end{aligned} \tag{4.8b}$$

Combining estimates (4.8a) and (4.8b), we obtain

$$|M_1| \le C_* h^{2k+\frac{3}{2}-\frac{\alpha}{2}} \|\varphi\|_{k+1}. \tag{4.9}$$

Then, we give the estimate of M_2 by making use of the orthogonal property of the L^2 projection P_h.

$$M_2 = (\partial_h^\alpha f(u)_x - P_h(\partial_h^\alpha f(u)_x), \varphi - P_h\varphi),$$

namely

$$|M_2| \le C h^{2k+2} \|\partial_h^\alpha f(u)_x\|_{k+1} \|\varphi\|_{k+1}, \tag{4.10}$$

where we have used error estimate (2.5a).

Finally, we give the estimate for M_3. Applying the Taylor expansion, the Cauchy-Schwarz inequality, inverse properties (ii)-(iii) and estimates (2.5a) and (3.5), we obtain

$$\begin{aligned} |M_3| \le\, & C \sum_{l=0}^{\alpha} \|\partial_h^l e_u\|_{\Gamma_h} \|\varphi - P_h\varphi\|_{\Gamma_h} \\ & + C_* \sum_{m=0}^{\alpha} \|\partial_h^m e_u\|_\infty \|\partial_h^{\alpha-m} e_u\|_{\Gamma_h} \|(\varphi - P_h\varphi)_x\|_{\Gamma_h} \\ \le\, & C h^{2k+\frac{3}{2}-\frac{\alpha}{2}} \|\varphi\|_{k+1} + C_* h^{3k+\frac{5}{2}-\frac{\alpha}{2}} \|\varphi\|_{k+1} \\ \le\, & C h^{2k+\frac{3}{2}-\frac{\alpha}{2}} \|\varphi\|_{k+1}. \end{aligned} \tag{4.11}$$

According to the estimates (4.9)-(4.11), we obtain

$$|M| \leq Ch^{2k+\frac{3}{2}-\frac{\alpha}{2}} \|\varphi\|_{k+1}. \tag{4.12}$$

Using a similar method to M, we have

$$N = H(\partial_h^\alpha q_h, \varphi - P_h\varphi) = H(\partial_h^\alpha f(u_h), \varphi - P_h\varphi) - ((\partial_h^\alpha u_h)_t, \varphi - P_h\varphi).$$

Since $((\partial_h^\alpha u_h)_t, \varphi - P_h\varphi) = 0$, we have

$$|N| \leq Ch^{2k+\frac{3}{2}-\frac{\alpha}{2}} \|\varphi\|_{k+1}. \tag{4.13}$$

Collecting the estimates (4.12) and (4.13), we obtain

$$|\Sigma_2| \leq Ch^{2k+\frac{3}{2}-\frac{\alpha}{2}} \|\varphi\|_{k+1}. \tag{4.14}$$

The estimate of Σ_3 is given in the following lemma.

Lemma 4.3. *There exists a positive constant C independent of h, such that*

$$|\Sigma_3| \leq Ch^{2k+3-\frac{\alpha}{2}} \|\varphi\|_{k+1}. \tag{4.15}$$

More details of proof are given in [12]. We have completed the proof of Theorem 4.1.

5. Numerical experiments

In this section, we present some numerical results to confirm that we can indeed improve the convergence rate of the LDG solution from $k+1$ to $2k+1$ for the nonlinear convection-diffusion equation. We consider the LDG method combined with the third order implicit-explicit Runge-Kutta method in time. We take a small enough time step such that the spatial errors dominate and present the results for P^1 and P^2 polynomials only to save space. The time step is $\tau = 0.5h$ when the linear piecewise polynomial finite element space is used. For quadratic piecewise polynomial finite element space, the time step is set as $\tau = 0.1h$. For numerical initial conditions, we take the standard L^2 projection of the initial condition. Uniform meshes are used in all experiments. Only one-dimensional scalar equations are tested, whose theoretical results are covered in our theorems.

Example 5.1. We consider the nonlinear convection-diffusion equation on the domain $\Omega = [0, 2\pi]$, where boundary conditions are periodic. The exact solution is $u(x,t) = e^{-\varepsilon t} \sin x$.

$$\begin{aligned} &u_t + \left(\frac{u^3}{3}\right)_x = \varepsilon u_{xx} + e^{-3\varepsilon t} \sin^2 x \cos x, \\ &u(x,0) = \sin x. \end{aligned} \tag{5.1}$$

When $\varepsilon = 0.5$, 0.01 and 5, we give the L^2 errors at the final time $T = 1$ in Tables 1, 2 and 3, respectively. From Tables 1 to 3, we observe that the errors of after post-processed are lower than before post-processed, and the orders of convergence can be improved from $k+1$ to at least $2k+1$ when $\varepsilon = 0.01$, but the orders of

Table 1. L^2 errors when $\varepsilon = 0.5$ before and after post-processed for Example 5.1

	Before post-processed		After post-processed	
N	L^2 error	order	L^2 error	order
		P^1		
10	0.0263	-	0.0040	-
20	0.0065	2.0200	3.8153e-04	3.3901
40	0.0016	2.0051	4.0062e-05	3.2515
80	4.0369e-04	2.0013	4.4759e-06	3.1620
160	1.0056e-04	2.0002	4.8841e-07	3.1960
		P^2		
10	0.0013	-	4.2728e-04	-
20	1.6118e-04	3.0335	1.2214e-05	5.1305
40	1.9993e-05	3.0111	3.5850e-07	5.0905
80	3.4936e-06	3.0033	1.1302e-08	4.9873
160	3.9637e-07	3.0341	3.5302e-10	5.0007

Table 2. L^2 errors when $\varepsilon = 0.01$ before and after post-processed for Example 5.1

	Before post-processed		After post-processed	
N	L^2 error	order	L^2 error	order
		P^1		
10	0.0425	-	0.0112	-
20	0.0105	2.0058	0.0015	2.8706
40	0.0026	2.0040	1.9655e-04	2.9596
80	6.5914e-04	2.0019	2.4707e-05	2.9919
160	1.6471e-04	2.0006	3.0873e-06	3.0005
		P^2		
10	0.0023	-	4.8397e-04	-
20	5.7108e-04	2.0258	1.5138e-05	4.9986
40	9.1448e-05	2.6427	5.3028e-07	4.8353
80	1.1721e-05	2.9638	1.6125e-08	5.0393
160	1.1777e-06	3.3150	4.8117e-10	5.0667

convergence don't achieve the desired $2k+1$ order accuracy when $\varepsilon = 5$. The post-processing superconvergence is very remarkale for smaller ε. Meanwhile we give the L^2 errors at the final time $T = 10$ and $T = 100$ when $\varepsilon = 0.5$ in Tables 4 and 5. We find that the orders of convergence can also be improved from $k+1$ to $2k+1$, which shows that the superconvergence can be maintained for a long time.

Example 5.2. We consider the equation (5.2) on the domain $\Omega = [0, 2\pi]$ with strongly nonlinear flux function, where boundary conditions are periodic. The exact

Table 3. L^2 errors when $\varepsilon = 5$ before and after post-processed for Example 5.1

N	Before post-processed		After post-processed	
	L^2 error	order	L^2 error	order
		P^1		
10	0.0018	-	0.0018	-
20	4.1176e-04	2.1605	4.0711e-04	2.1774
40	7.4536e-05	2.4658	7.2452e-05	2.4903
80	1.0880e-05	2.7762	9.9203e-06	2.8685
160	1.7788e-06	2.6127	1.3817e-06	2.8439
		P^2		
10	0.0018	-	5.7736e-04	-
20	4.0640e-04	2.1760	6.6371e-05	3.1208
40	7.2378e-05	2.4892	5.2269e-06	3.6665
80	9.9126e-06	2.8682	2.5354e-07	4.3656
160	1.3592e-06	2.8664	1.1456e-08	4.4680

Table 4. L^2 errors at final time $T = 10$ before and after post-processed for Example 5.1

N	Before post-processed		After post-processed	
	L^2 error	order	L^2 error	order
		P^1		
10	2.8930e-04	-	4.4355e-05	-
20	7.1923e-05	2.0081	5.1747e-06	3.0996
40	1.7947e-05	2.0027	6.1676e-07	3.0687
80	4.4814e-06	2.0017	7.5515e-08	3.0299
		P^2		
10	1.3941e-05	-	4.4589e-06	-
20	1.6844e-06	3.0490	1.3812e-07	5.0127
40	2.0623e-07	3.0299	4.5538e-09	4.9227
80	2.5403e-08	3.0212	1.4134e-10	5.0098

solution is $u(x,t) = e^{-\varepsilon t} \sin x$.

$$\begin{aligned} &u_t + (e^u)_x = \varepsilon u_{xx} + e^{e^{-\varepsilon t} \sin x} e^{-\varepsilon t} \cos x, \\ &u(x,0) = \sin x. \end{aligned} \tag{5.2}$$

Taking $\varepsilon = 0.5$, 0.01 and 5, we give the L^2 errors at the final time $T = 1$ in Tables 6, 7 and 8, respectively. We observe that the errors of after post-processed are lower than before post-processed, and the orders of convergence can be improved from $k+1$ to at least $2k+1$. Meanwhile we give the L^2 errors at the final time $T = 10$ and $T = 100$ as $\varepsilon = 0.5$ in Tables 9 and 10. We find that the orders of convergence can also be improved from $k+1$ to $2k+1$, which shows that the superconvergence can be maintained for a long time.

Table 5. L^2errors at final time $T = 100$ before and after post-processed for Example 5.1

N	Before post-processed		After post-processed	
	L^2 error	order	L^2 error	order
		P^1		
10	1.4564e-08	-	9.7895e-09	-
20	5.5464e-09	1.3928	1.6326e-09	2.5841
40	1.6566e-09	1.7433	2.3521e-10	2.7952
80	4.6565e-10	1.8309	3.1685e-11	2.8921
		P^2		
10	3.5406e-16	-	4.1386e-17	-
20	6.1914e-17	2.5157	2.1156e-18	4.2900
40	9.4914e-18	2.7056	7.8450e-20	4.7531
80	1.3494e-18	2.8143	2.8438e-21	4.7859

Table 6. L^2 errors at $\varepsilon = 0.5$ before and after post-processed for Example 5.2

N	Before post-processed		After post-processed	
	L^2 error	order	L^2 error	order
		P^1		
10	0.0261	-	0.0059	-
20	0.0065	2.0104	6.9001e-04	3.0960
40	0.0016	2.0050	8.2820e-05	3.0960
80	4.0373e-04	2.0013	9.9892e-06	3.0515
160	1.0090e-04	2.0004	1.2016e-06	3.0554
		P^2		
10	0.0013	-	4.2865e-04	-
20	1.6146e-04	3.0384	1.3010e-05	5.0421
40	2.0004e-05	3.0128	4.0510e-07	5.0052
80	2.4938e-06	3.0038	1.2730e-08	4.9807
160	3.1866e-07	2.9683	4.0010e-10	4.9917

Table 7. L^2 errors when $\varepsilon = 0.01$ before and after post-processed for Example 5.2

N	Before post-processed		After post-processed	
	L^2 error	order	L^2 error	order
		P^1		
10	0.0427	-	0.0112	-
20	0.0106	2.0058	0.0014	2.9549
40	0.0026	2.0040	1.8216e-04	2.9959
80	6.5908e-04	2.0019	2.2635e-05	3.0086
160	1.5824e-04	2.0583	2.8124e-06	3.0087
		P^2		
10	0.0018	-	5.0099e-04	-
20	2.0676e-04	3.1143	1.5258e-05	5.0371
40	2.5484e-05	3.0203	4.7432e-07	5.0076
80	3.1772e-06	3.0037	1.4633e-08	5.0185
160	3.9492e-07	3.0081	4.5517e-10	5.0067

Table 8. L^2 errors when $\varepsilon = 5$ before and after post-processed for Example 5.2

N	Before post-processed		After post-processed	
	L^2 error	order	L^2 error	order
		P^1		
10	0.0035	-	7.2737e-04	-
20	8.7726e-04	2.0128	8.1203e-05	3.1631
40	2.1870e-04	2.0040	9.5017e-06	3.0953
80	5.4633e-05	2.0011	1.1433e-06	3.0550
160	1.3652e-05	2.0007	1.3766e-07	3.0539
		P^2		
10	3.2047e-04	-	5.8198e-05	-
20	4.0543e-05	2.9827	1.8227e-06	4.9968
40	5.1050e-06	2.9895	4.5481e-08	5.3247
80	6.3766e-07	3.0010	1.2633e-09	5.1700
160	7.9690e-08	3.0003	3.5717e-11	5.1445

Table 9. L^2 errors at final time $T = 10$ before and after post-processed for Example 5.2

N	Before post-processed		After post-processed	
	L^2 error	order	L^2 error	order
		P^1		
10	2.9571e-04	-	6.0897e-05	-
20	7.2362e-05	2.0309	8.2300e-06	2.8874
40	1.7974e-05	2.0093	1.0701e-06	2.9431
80	4.4857e-06	2.0025	1.2611e-07	3.0850
		P^2		
10	1.2916e-05	-	4.5149e-06	-
20	1.5713e-06	3.0391	1.5070e-07	4.9049
40	1.9444e-07	3.0146	4.9745e-09	4.9210
80	2.4134e-08	3.0102	1.5414e-10	5.0122

Table 10. L^2 errors at final time $T = 100$ before and after post-processed for Example 5.2

N	Before post-processed		After post-processed	
	L^2 error	order	L^2 error	order
		P^1		
10	1.2530e-08	-	2.0494e-09	-
20	4.2592e-09	1.5567	3.5507e-10	2.5290
40	1.2102e-09	1.8153	5.7485e-11	2.6268
80	3.3485e-10	1.8537	8.6766e-12	2.7280
		P^2		
10	9.2557e-09	-	5.0271e-10	-
20	2.1736e-09	2.0903	4.1586e-11	3.5956
40	3.2297e-10	2.7506	1.7955e-12	4.5336
80	4.5485e-11	2.8279	6.5194e-13	4.7835

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