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NEW APPROACH BASED ON COLLOCATION AND SHIFTED CHEBYSHEV POLYNOMIALS FOR A CLASS OF THREE-POINT SINGULAR BVPS

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Abstract In the recent decades, variety of real-life problems arises in astrophysics have been mimic using the class of three-point singular boundary value problems (BVPs). Finding an effective and accurate approach for a class of three-point BVPs is still a difficult problem, though. The goal of this paper is to design a numerical strategy for approximating a class of three-point singular boundary value problems using the collocation technique and shifted Chebyshev polynomials. Utilizing shifted Chebyshev polynomials, the problem is reduced to a matrix form, which is then converted into a system of nonlinear algebraic equations by employing the collocation points. The key advantages of the new approach are (a) it is a straightforward mathematical formulation, which makes it effortless to code, and (b) it is easily adaptable to solve various classes of three-point singular boundary value problems. The convergence analysis is carried out to ensure the viability of the proposed scheme. Various examples are considered and tested in order to illustrate its applicability and efficiency. The results show excellent accuracy and efficiency compared to the other existing methods.

Keywords Shifted Chebyshev polynomials, collocation method, three-point singular BVPs, convergence analysis.

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1. Introduction

Singular Boundary Value Problems (SBVPs) have been a topic of interest for the researchers working in the area of physics, chemistry and engineering due to their

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wide applications. Several complex physical phenomena such as the thermal explosion in the cylindrical reactor [32], the theory of stellar structure, the polytropic and isothermal gas sphere using the Lane-Emden equation of index n [22,41] can be formulated mathematically using SBVPs. Recently, real-life applications including large bridges with multi-point support, the vibration of a guy wire of uniform cross section and composed of N parts, or the elasticity of an equally loaded three layered sandwich beam, have been modeled using multipoint BVPS (for detail see [8, 47] and references therein).

Different approaches have been developed in the literature to find analytical solutions of SBVPs [6, 7, 15, 20, 21, 24]. Pandey [21] presented a study about the existence of a unique solution of a class of SBVPs with the boundary conditions $\lim_{x\to 0} y'(x) = 0,\ y(b) = B$. Later, Pandey and Verma [20] presented an analytical result on the existence and uniqueness of the solution of two point SBVPs with boundary conditions $y(0) = 0,\ \alpha_1 y(b) + \beta_1 y'(b) = \gamma_1$. Russell and Shampine [24] proved the existence of a unique solution of a class of two point SBVPs. They have also developed a method based on patch bases, traditional finite differences and collocation to get an approximate solution. Kelevedjiev [15] has studied the existence of positive solutions of the boundary value problem $x^{''} = f(t, x, x^{'}),\ x(0) = a,\ x(1) = b,\ a, b > 0$. Chawla and Shivakumar [6] have examined the existence of a unique solution of the class of SBVP with the boundary conditions $y'(0^+) = 0,\ y(1) = A$, when $\sup \frac{\partial f}{\partial y} < k_1$, where k_1 is first positive zero of $J_{\frac{\alpha-1}{2}}\sqrt{k}$, being $J_\nu(z)$ the Bessel's function of first kind and order ν.

In the literature, several successful attempts have been made for obtaining the analytical solutions of a three-point SBVP [28, 43, 44]. Wang and Tsai [43] have used the upper and lower solutions method with Schauder's fixed point theorem to ensure the existence of a solution of a three-point BVP. Xie [44] has shown an existence result on three point singular boundary value problems using upper and lower solutions. Singh et al. [28] have shown some existence results for the solution of a class of nonlinear SBVPs $-y^{''} - \frac{1}{x}y^{'} = f(x, y),\ 0 < x < 1,\ y'(0) = 0$ and $y(1) = \delta y(\eta)$ using the monotone iterative technique in the presence of upper and lower solutions. In order to prove these results, they have proved the maximum and anti-maximum principles for the differential inequalities $-(xy^{'}(x))' - \lambda x y(x) \geq 0,\ 0 < x < 1,\ y^{'}(0) = 0$ and $y(1) - \delta y(\eta) \geq 0$. Singh et al. [29] have used the variational iteration method for the numerical solution of three-point singular boundary value problems.

There are several numerical techniques available for the numerical solution of boundary value problems [25, 26, 40]. Other methods to find approximate solutions of SBVPs are cubic spline method [13, 23], modified Homotopy analysis method [30, 32], mixed decomposition analysis method [14], Hermite functional collocation method [19], Adomian decomposition method [11, 33], advanced Adomian decomposition method [39], artificial neural network [35], hybrid functions approximation [38], optimal decomposition method [31] and combination of iterative method and homotopy perturbation method [27], variational iteration method [37, 45]. Ahmad et al. [3] have presented a bio-inspired numerical technique for solving a boundary value problem arising in the modelling of corneal shape. Although these methods have various advantages, the implementation is not easy and time consuming.

In the present work, a class of three-point SBVPs [2, 29, 42] is considered

$$-\left(p(x)y'(x)\right)' = p(x)f(x, y(x)), \quad 0 \ < \ x \leq \ 1, \tag{1.1}$$

subject to the boundary conditions

$$y(0) = 0,\ y(1) = \alpha y(\eta), \tag{1.2}$$

or else to

$$y^{'}(0) = 0,\ y(1) = \alpha y(\eta). \tag{1.3}$$

Here, we assume that

$$\begin{cases} 0 < \alpha < \dfrac{h(1)}{h(\eta)},\ h(x) = \displaystyle\int_0^x \frac{1}{p(t)}dt, & \text{in the case of boundary conditions (1.2)}, \\ \alpha > 0, & \text{in the case of boundary conditions (1.3)}, \end{cases}$$

$0 < \eta < 1$ and $f \in C([0,1] \times [0,\infty), [0,\infty))$.

For the SBVP (1.1)-(1.2) the function $p \in C[0,1] \bigcap C^1(0,1]$ with $p(x) > 0$ on $(0,1]$ and $\frac{1}{p} \in L^1(0,1)$. Moreover, in case of SBVP (1.1) and (1.3), the function $p \in C[0,1] \bigcap C^1(0,1]$ with $p(x) > 0$ on $(0,1]$ and $\frac{1}{p}$ is locally integrable on $(0,1]$, but not integrable in any neighborhood of 0. When $p(0) = 0$ the problems (1.1), (1.2) or (1.1), (1.3) are singular, which constitutes a challenge to find analytical or numerical solutions.

The collocation method is a powerful mathematical tool that has been used to obtain the numerical solution of BVPs. The Chebyshev collocation method has been used intensively to solve delay differential equations, integro-differential equations and integro-differential-difference equations (see [9, 10, 46] and references therein). Öztürk and Gülsu [18] have solved the Lane-Emden equations arising in astrophysics using truncated shifted Chebyshev series together with the operational matrix. Öztürk [17] has applied the collocation method together with Chebyshev polynomials to solve systems of Lane-Emden type equations [32, 34] with initial boundary conditions.

In this article, the collocation method in the presence of shifted Chebyshev polynomials is employed to obtain numerical solutions of the three-point SBVPs (1.1)-(1.2) or (1.1)-(1.3). The method is based on the representation of the unknown solution as a truncated shifted Chebyshev series with unknown coefficients. This leads to the transformation of the SBVP into a matrix form. The next step uses the collocation points to convert the matrix form of the SBVP into a system of nonlinear algebraic equations. Consequently, the solution of the system of algebraic equations yields the approximate solution of the SBVP.

The rest of the article is structured as follows: Section 2 introduces the Chebyshev polynomials and the approximation of the unknown solution using shifted Chebyshev polynomials. In Section 3, the methodology to deal with the SBVPs (1.1)-(1.3) is developed and the convergence analysis of the method is discussed in detail in Section 4. The performance of the proposed method is tested against existing numerical methods by considering several examples of three-point SBVP in Section 5. Finally, some remarks and conclusions are made in Section 6.

2. Chebyshev Polynomials

The Chebyshev polynomial of first kind of degree n, $T_n(x)$, is given by

$$T_n(x) = \cos(n\theta),\ \text{where } x = \cos\theta. \tag{2.1}$$

Since the independent variable lies in the half interval [0, 1] rather than $[-1,1]$, then we use the shifted Chebyshev polynomials which are defined as

$$T_n^*(x) = T_n(2x-1),\ x \in [0,1], \quad n = 0,1,2,\dots \tag{2.2}$$

The first three shifted Chebyshev polynomials are given by

$$\begin{aligned} T_0^*(x) &= 1, \\ T_1^*(x) &= 2x-1, \\ T_2^*(x) &= 8x^2-8x+1. \end{aligned}$$

The Chebyshev polynomials $T_n(x)$ verify the following recursive relation

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x), n = 1,2,\dots. \tag{2.3}$$

The shifted Chebyshev polynomials have many properties [5,16], the most important are as follows

1. All the roots of the polynomial $T_n^*(x)$ are real and lie in the interval $[0,1]$. These roots are given as

$$x_{i-1} = \frac{1}{2}\left(1+\cos\left(\frac{\left(n-i+\frac{3}{2}\right)\pi}{n+1}\right)\right), \quad i = 1,2,3,...,n. \tag{2.4}$$

   These points are used as collocation points in the present study to obtain a numerical solution of the considered BVPs. These points are called Chebyshev-Gauss points.

2. The relation between $T_n^*(x)$ and x^n has been discussed in detail by Öztürk and Gülsu [18], being

$$x^n = 2^{-2n+1}\sum_{k=0}^{n}{}' \begin{pmatrix} 2n \\ n-k \end{pmatrix} T_k^*(x), \qquad 0 \le x \le 1, \tag{2.5}$$

   where the symbol $\sum'$ denotes that the first term is halved.

Now, we can approximate any function $y(x) \in L^2[0,1]$ using shifted Chebyshev polynomials, by

$$y(x) = \sum_{n=0}^{\infty}{}' c_n T_n^*(x), \tag{2.6}$$

where

$$c_n = \langle y(x), T_n^*(x)\rangle = \frac{2}{\pi}\int_0^1 \frac{1}{\sqrt{x(1-x)}} y(x)T_n^*(x)dx, \quad n = 0,1,\dots . \tag{2.7}$$

Let us now consider the approximate solution of the SBVPs (1.1)-(1.2) or (1.1)-(1.3) in terms of a truncated shifted Chebyshev series which can be written as follows:

$$y_N(x) = \sum_{n=0}^{N}{}' c_n T_n^*(x), \tag{2.8}$$

where N is any arbitrary positive integer. The matrix representation of the summation in (2.8) and the derivatives of the function $y_N(x)$ are given by

$$y_N(x) = T^*(x)C, \quad y_N^{(k)}(x) = T^{*(k)}(x)C, \quad k = 1, 2, \ldots, N, \tag{2.9}$$

where

$$T^*(x) = [T_0^*(x), T_1^*(x), T_2^*(x), ..., T_N^*(x)], \quad C = \left[\frac{1}{2}c_0, c_1, ..., c_N\right]^T \tag{2.10}$$

and $T^{*(k)}(x)$ denotes the derivatives of order k of each component of $T^*(x)$. Using the formula in (2.5) we can write

$$(X(x))^T = D(T^*(x))^T \text{ or } X(x) = (T^*(x))D^T, \tag{2.11}$$

where $X(x) = [1,\ x,\ x^2,\ \ldots\ , x^N]$, and

$$D = \begin{bmatrix} 2^0\binom{0}{0} & 0 & 0 & \ldots & 0 \\ 2^{-2}\binom{2}{1} & 2^{-1}\binom{2}{0} & 0 & \ldots & 0 \\ 2^{-4}\binom{4}{2} & 2^{-3}\binom{4}{1} & 2^{-3}\binom{4}{0} & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 2^{-2N}\binom{2N}{N} & 2^{-2N+1}\binom{2N}{N-1} & 2^{-2N+1}\binom{2N}{N-2} & \ldots & 2^{-2N+1}\binom{2N}{0} \end{bmatrix}.$$

Now, using equation (2.11), the following expression is obtained

$$T^*(x) = X(x)(D^{-1})^T \tag{2.12}$$

and

$$T^{*(k)}(x) = X^{(k)}(x)(D^{-1})^T. \tag{2.13}$$

Further, the relation between $X(x)$ and the vector of derivatives of order k of each component of $X(x)$, denoted by $X^k(x)$, is given by

$$X^k(x) = X(x)\left(B^T\right)^k, \tag{2.14}$$

where

$$B = \begin{bmatrix} 0 & 0 & 0 \ldots 0 \\ 1 & 0 & 0 \ldots 0 \\ 0 & 2 & 0 \ldots 0 \\ \vdots & \vdots & \vdots \ddots \vdots \\ 0 \ldots & 0 & N \;\; 0 \end{bmatrix}. \tag{2.15}$$

Using equations (2.12)-(2.14) in (2.9) gives

$$y_N^{(k)}(x) = X(x)(B^T)^k (D^{-1})^T C, \quad k = 0, 1, 2, \ldots, N. \tag{2.16}$$

Note that using (2.9) and (2.13) for $k = 0$, the m-th power of $y_N(x)$ can be expressed as

$$(y_N(x))^m = (T^*(x)C)^{m-1} X(x)(D^{-1})^T C. \tag{2.17}$$

3. Methodology

In this section, a numerical technique for the solution of the SBVP (1.1)-(1.2) or (1.1)-(1.3) is developed. This method is based on the shifted Chebyshev polynomials and the collocation approach. Using the approximations of $y(x)$ and its derivatives given in (2.16) the differential equation in (1.1) can be expressed in a matrix form given by

$$\begin{aligned} &p(x)X(x)\left(B^T\right)^2 \left(D^T\right)^{-1} C + p^{'}(x)X(x)B^T \left(D^T\right)^{-1} C \\ &+ p(x)f(x, X(x)\left(D^T\right)^{-1} C) = 0. \end{aligned} \tag{3.1}$$

Now, we evaluate the equation in (3.1) at each of the roots of $T_{N+1}^*(x)$, as given in (2.4), and obtain the matrix equation

$$P\bar{\bar{X}}\left(B^T\right)^2 \left(D^T\right)^{-1} C + P_1\bar{\bar{X}}B^T \left(D^T\right)^{-1} C + F = 0, \tag{3.2}$$

where F is given by

$$F = [p(x_0)f(x_0, X(x_0)\left(D^T\right)^{-1} C), \ldots, p(x_N)f(x_N, X(x_N)\left(D^T\right)^{-1} C)]^T$$

and

$$P = \begin{bmatrix} p(x_0) & 0 & 0 & \cdots & 0 \\ 0 & p(x_1) & 0 & \cdots & 0 \\ 0 & 0 & p(x_2) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & p(x_N) \end{bmatrix}, \; P_1 = \begin{bmatrix} p^{'}(x_0) & 0 & 0 & \cdots & 0 \\ 0 & p^{'}(x_1) & 0 & \cdots & 0 \\ 0 & 0 & p^{'}(x_2) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & p^{'}(x_N) \end{bmatrix},$$

$$\bar{\bar{X}} = \begin{bmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^N \\ 1 & x_1 & x_1^2 & \cdots & x_1^N \\ 1 & x_2 & x_2^2 & \cdots & x_2^N \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_N & x_N^2 & \cdots & x_N^N \end{bmatrix}.$$

Furthermore, the matrix representation of the boundary conditions in (1.2) is given by

$$X(0)\left(D^{-1}\right)^T C = 0, X(1)\left(D^{-1}\right)^T C - \alpha X(\eta)\left(D^{-1}\right)^T C = 0, \tag{3.3}$$

and the boundary conditions in (1.3) are given by

$$X(0)B^T\left(D^{-1}\right)^T C=0,\quad X(1)\left(D^{-1}\right)^T C-\alpha X(\eta)\left(D^{-1}\right)^T C=0. \tag{3.4}$$

The aforementioned procedure leads to a system of $(N+1)$ equations in (3.2), which is a discretization of the considered BVP. Furthermore, two pair of algebraic equations given in (3.3) and (3.4) corresponding to the conditions (1.2) and (1.3), respectively, are obtained. In order to get the solution of a SBVP, replace two of the equations given in (3.2) by (3.3) or (3.4) according to the corresponding boundary conditions. The solution of the system of algebraic equations is obtained using the "Maple 18" software. The desired numerical solution is obtained by replacing the values of the coefficients in equation (2.8).

It is worth mentioning that most of the numerical examples involve an integer power of y. Therefore, it is necessary to show how an integer power of y can be approximate using truncated shifted Chebyshev polynomials and the collocation method. The approximation of y^m in terms of shifted Chebyshev polynomials was given in Section 2. Now, using the collocation points in (2.17), we obtain the matrix representation given by

$$\begin{aligned}&\left[\left(y_N(x_0)\right)^m,\left(y_N(x_1)\right)^m,\ldots,\left(y_N(x_N)\right)^m\right]^T\\=&\left[\left(T^*(x_0)C\right)^m,\left(T^*(x_1)C\right)^m,\ldots,\left(T^*(x_N)C\right)^m\right]^T=\left(T_1\right)^{m-1}\bar{\bar{X}}\left(D^{-1}\right)^T C,\end{aligned} \tag{3.5}$$

where

$$T_1=\begin{bmatrix} T^*(x_0)C & 0 & 0 & \cdots & 0\\ 0 & T^*(x_1)C & 0 & \cdots & 0\\ 0 & 0 & T^*(x_2)C & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & T^*(x_N)C\end{bmatrix}. \tag{3.6}$$

The flowchart of the methodology is depicted in Figure 1.

4. Convergence Analysis

In order to support the reliability of the proposed method, a convergence analysis is addressed.

Theorem 4.1. *Suppose that $y''(x)\in L_2[0,1]$ and that it is bounded on $[0,1]$, that is, $|y''(x)|\le l$ for some $l>0$. Then the infinite series $y(x)=\sum'^{\infty}_{n=0}c_nT_n^*(x)$ converges uniformly, with*

$$|c_n|<\frac{l}{2n(n-1)}.$$

Proof. The coefficients c_n in the infinite series $y(x)=\sum'^{\infty}_{n=0}c_nT_n^*(x)$ are given by

$$c_n=\frac{2}{\pi}\int_0^1\frac{1}{\sqrt{x(1-x)}}y(x)T_n^*(x)dx=\frac{2}{\pi}\int_0^1\frac{1}{\sqrt{x(1-x)}}y(x)T_n(2x-1)dx. \tag{4.1}$$

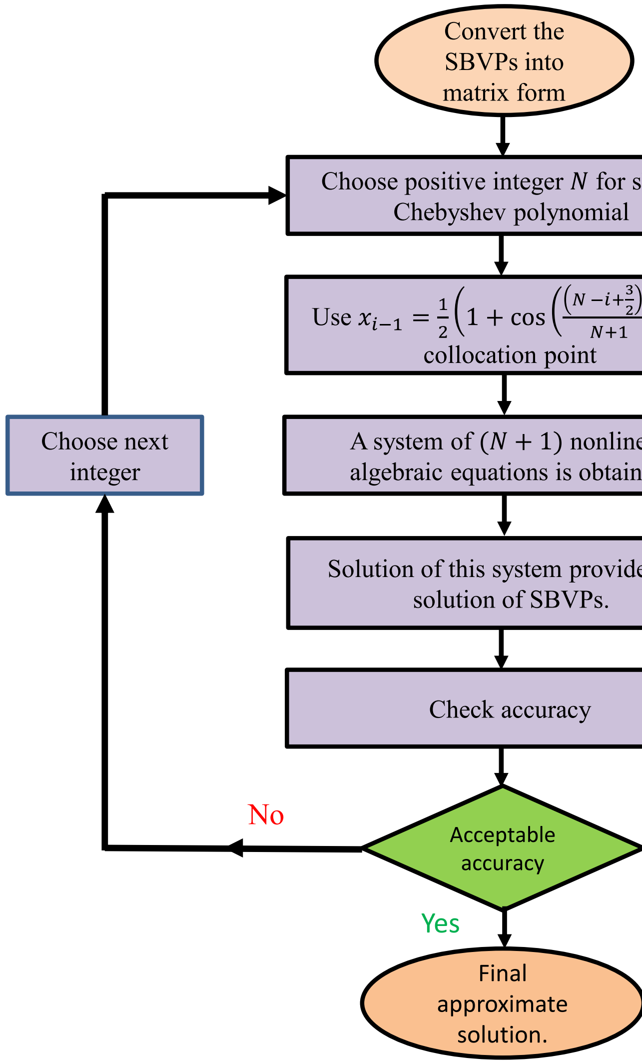


Figure 1. Schematic representation of the new approach.

Using the change of variable $\cos^{-1}(2x-1) = t$, the integral (4.1) can be written as

$$\begin{aligned} c_n &= -\frac{2}{\pi}\int_{\pi}^{0} y\left(\frac{1+\cos(t)}{2}\right)\cos(nt)dt \\ &= \frac{2}{\pi}\int_{0}^{\pi} y\left(\frac{1+\cos(t)}{2}\right)\cos(nt)dt. \end{aligned} \tag{4.2}$$

Using integration by parts on (4.2), we get

$$c_n = \frac{2}{\pi}\left\{y\left(\frac{1+\cos(t)}{2}\right)\left(\frac{\sin(nt)}{n}\right)\right\}_0^{\pi} + \frac{2}{n\pi}\int_0^{\pi} y'\left(\frac{1+\cos(t)}{2}\right)\left(\frac{\sin(t)}{2}\right)\sin(nt)dt. \tag{4.3}$$

From equation (4.3), we have

$$c_n = \frac{2}{n\pi}\int_0^{\pi} y^{'}\left(\frac{1+\cos(t)}{2}\right)\left(\frac{\sin(t)}{2}\right)\sin(nt)dt,$$
$$= \frac{1}{2n\pi}\int_0^{\pi} y^{'}\left(\frac{1+\cos(t)}{2}\right)\{\cos(n-1)t-\cos(n+1)t\}\,dt. \tag{4.4}$$

Similarly, on performing integration by parts on (4.4), we get

$$c_n = \frac{1}{4n(n-1)\pi}\int_0^{\pi} y^{''}\left(\frac{1+\cos(t)}{2}\right)\sin(t)\sin(n-1)tdt$$
$$-\frac{1}{4n(n+1)\pi}\int_0^{\pi} y^{''}\left(\frac{1+\cos(t)}{2}\right)\sin(t)\sin(n+1)tdt, \tag{4.5}$$

and therefore

$$|c_n| < \frac{l}{4n}\left(\frac{1}{n-1}+\frac{1}{n+1}\right) < \frac{l}{2n(n-1)}. \tag{4.6}$$

Hence, the series converges to $y(x)$ uniformly. □

Theorem 4.2. *Assume that the $y(x) \in L_2[0,1]$ with a bounded second derivative such that $|y^{''}(x)| < l$. Then, if $e_1 = y(x) - y_N(x)$, it hold that $\|e_1\|$ tend to zero when $N \to \infty$, where $\|.\|$ denotes the canonical form.*

Proof. We have that

$$e_1 = y(x) - y_N(x) = \sum_{n=N+1}^{\infty} c_n T_n^*(x), \tag{4.7}$$

and thus,

$$\|e_1\|^2 = \langle e_1, e_1\rangle = \frac{2}{\pi}\int_0^1 \left(\sum_{n=N+1}^{\infty} c_n T_n^*(x)\right)\left(\sum_{m=N+1}^{\infty} c_m T_m^*(x)\right)dx$$
$$= \sum_{n=N+1}^{\infty} c_n \sum_{m=N+1}^{\infty} c_m \frac{2}{\pi}\int_0^1 \frac{1}{\sqrt{x(1-x)}} T_n^*(x) T_m^*(x)dx,$$
$$= \sum_{n=N+1}^{\infty} |c_n|^2.$$

From Theorem 4.1, we have

$$\|e_1\|^2 < \sum_{n=N+1}^{\infty} \frac{l^2}{4(n-1)^4},$$
$$= \frac{l^2}{4}\sum_{n=N+1}^{\infty}\frac{1}{(n-1)^4},$$
$$= \frac{l^2}{4}\sum_{n=N}^{\infty}\frac{1}{n^4},$$
$$= \frac{l^2}{4}\left(\frac{\pi^4}{90} - Q_N\right), \tag{4.8}$$

where

$$Q_N = \sum_{n=1}^{N-1} \frac{1}{n^4}.$$

Thus from (4.8), we have

$$-\frac{l}{2}\left(\frac{\pi^4}{90} - Q_N\right)^{\frac{1}{2}} < \|e_1\| < \frac{l}{2}\left(\frac{\pi^4}{90} - Q_N\right)^{\frac{1}{2}}. \tag{4.9}$$

An inequality [1] is given by

$$\frac{1}{3N^3} < \left(\frac{\pi^4}{90} - Q_N\right) < \frac{1}{3(N-1)^3}. \tag{4.10}$$

Using the following inequality in (4.9), we have

$$I_N < \|e_1\| < U_N, \tag{4.11}$$

where

$$I_N = -\frac{l}{2\sqrt{3}(N-1)^{\frac{3}{2}}} \quad \text{and} \quad U_N = \frac{l}{2\sqrt{3}(N-1)^{\frac{3}{2}}}. \tag{4.12}$$

Since $I_N \to 0$ and $U_N \to 0$ as $N \to \infty$, we have that $\|e_1\|$ tends to zero as $N \to \infty$, which proves the convergence of the method. This completes the proof. □

5. Numerical Testing and Discussion

In this section, the accuracy of the proposed method is contrasted with exact and numerical values provided by other methods, considering different examples.

Example 5.1.

$$-y'' - \frac{2}{x}y' = \frac{3}{4}e^y, \quad 0 < x < 1, \tag{5.1}$$

subject to

$$y'(0) = 0, \quad y(1) = \frac{2}{5}y\left(\frac{1}{2}\right). \tag{5.2}$$

In this problem $f(x,y) = \frac{3}{4}e^y$. Expand the function $f(x,y)$ in Taylor series, we have

$$\frac{3}{4}e^y = \frac{3}{4}\left(1 + y + \frac{y^2}{2} + \frac{y^3}{6} + \frac{y^4}{24} + \frac{y^5}{120} + ...\right).$$

Using the methodology developed in Section 3, we have the following matrix form of the differential equation (5.1)

$$-X(x)\left(B^T\right)^2\left(D^T\right)^{-1}C - \left(\frac{2}{x}\right)X(x)B^T\left(D^T\right)^{-1}C - \frac{3}{4}X(x)\left(D^T\right)^{-1}C$$

$$-\frac{3}{8}(T^*C)X(x)\left(D^T\right)^{-1}C - \frac{1}{8} * (T^*C)^2X(x)\left(D^T\right)^{-1}C \tag{5.3}$$

$$-\frac{1}{32} * (T^*C)^3X(x)\left(D^T\right)^{-1}C - \frac{1}{160} * (T^*C)^4X(x)\left(D^T\right)^{-1}C = \frac{3}{4}. \tag{5.4}$$

Also, the matrix representation of the boundary conditions (5.2) is given by

$$X(0)B^T\left(D^T\right)^{-1}C=0, \tag{5.5}$$

$$X(1)\left(D^T\right)^{-1}C-\frac{2}{5}X\left(\frac{1}{2}\right)\left(D^T\right)^{-1}C=0. \tag{5.6}$$

Now, using the collocation points (2.4), the equation in (5.4) results in

$$\begin{aligned}&-\bar{\bar{X}}\left(B^T\right)^2\left(D^T\right)^{-1}C-H\bar{\bar{X}}B^T\left(D^T\right)^{-1}C-\frac{3}{4}\bar{\bar{X}}\left(D^T\right)^{-1}C-\frac{3}{8}T_1\bar{\bar{X}}\left(D^T\right)^{-1}C\\&-\frac{1}{8}{T_1}^2\bar{\bar{X}}\left(D^T\right)^{-1}C-\frac{1}{32}{T_1}^3\bar{\bar{X}}\left(D^T\right)^{-1}C-\frac{1}{160}{T_1}^4\bar{\bar{X}}\left(D^T\right)^{-1}C-g=0.\end{aligned} \tag{5.7}$$

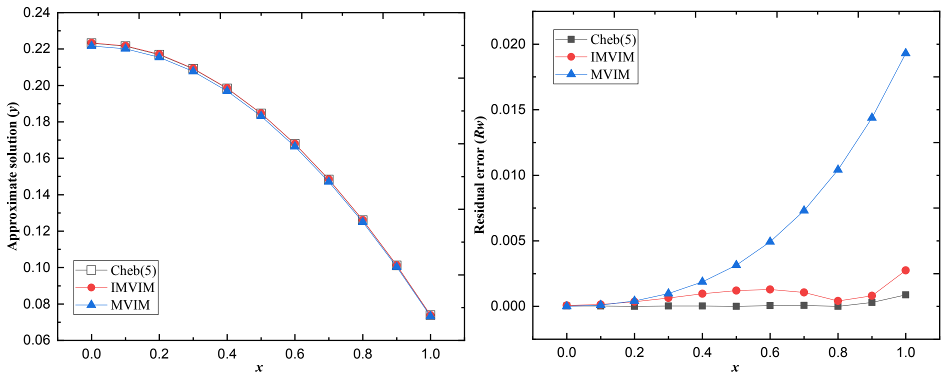
Here, the matrices $B^T,\left(D^T\right)^{-1}$ and C are defined in section 2 and $\bar{\bar{X}}$ and T_1 are defined in Section 3. The matrices H and g are given by

$$H=\begin{bmatrix}\frac{2}{x_0} & 0 & 0 & \cdots & 0\\ 0 & \frac{2}{x_1} & 0 & \cdots & 0\\ 0 & 0 & \frac{2}{x_2} & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & & \cdots & \frac{2}{x_N}\end{bmatrix},\ g=\begin{bmatrix}\frac{3}{4}\\ \frac{3}{4}\\ \frac{3}{4}\\ \vdots\\ \frac{3}{4}\end{bmatrix}.$$

Table 1. Comparison of Cheb(N) with IMVIM and MVIM at $N=5$ of Example 5.1.

x	Cheb(5)	Rw	IMVIM	Rw	MVIM	Rw
0.0	0.223148	3.0×10^{-05}	0.223377	7.8×10^{-05}	0.221653	0.000000
0.1	0.221587	3.3×10^{-05}	0.221814	1.5×10^{-04}	0.220093	1.0×10^{-04}
0.2	0.216910	2.7×10^{-06}	0.217135	3.5×10^{-04}	0.215425	4.1×10^{-04}
0.3	0.209145	3.6×10^{-05}	0.209366	6.4×10^{-04}	0.207674	9.8×10^{-04}
0.4	0.198334	3.9×10^{-05}	0.198547	9.6×10^{-04}	0.196889	1.8×10^{-03}
0.5	0.184537	5.5×10^{-11}	0.184736	1.2×10^{-03}	0.183133	3.1×10^{-03}
0.6	0.167828	5.6×10^{-11}	0.168007	1.2×10^{-03}	0.166489	4.9×10^{-03}
0.7	0.148295	7.8×10^{-05}	0.148447	1.0×10^{-03}	0.147056	7.3×10^{-03}
0.8	0.126038	1.0×10^{-05}	0.126161	4.1×10^{-04}	0.124949	1.0×10^{-02}
0.9	0.101170	2.9×10^{-04}	0.101267	8.1×10^{-04}	0.100299	1.4×10^{-02}
1.0	0.073815	8.8×10^{-04}	0.073894	2.7×10^{-03}	0.073253	1.9×10^{-02}

In order to get the solution, we replace two equations of (5.7) by (5.5) and (5.6). Now, solving this system for unknown coefficients $c_i,\ i=0,1,2,...,N$ and substituting them into equation (2.8), the required numerical solution is obtained. The approximated results are presented against the modified variational iteration method (MVIM) [29] and the improved modified variational iteration method (IMVIM) [42]. Numerical results are listed and presented in comparison with MVIM and IMVIM in Table 1 and Figure 2 for $N = 5$. To check the accuracy the residual errors $Rw = |-y^{''} - \frac{2}{x}y^{'} - \frac{3}{4}e^{y}|$ are presented in Table 1. The values of the Chebyshev coefficients are given in Table 2 for $N = 5$. From Figure 2 and Tables 1 and 2, one can easily conclude that the proposed approach is very promising for solving a three point SBVPs.



(a) Numerical vs VIM

(b) Residual error (Rw)

Figure 2. Comparison of results for Example 5.1.

Table 2. Chebyshev coefficients at $N = 5$ for Example 5.1.

	c_0	c_1	c_2	c_3	c_4	c_5
$N = 5$	0.33292864	-0.07507989	-0.01802781	0.00041474	0.00004520	-0.00000162

Example 5.2.

$$-y^{''} - \frac{2}{x}y^{'} = 1 - 2y^{3}, \quad 0 < x < 1, \tag{5.8}$$

subject to

$$y'(0) = 0, \quad y(1) = \frac{1}{3}y\left(\frac{1}{4}\right). \tag{5.9}$$

Applying the methodology developed in Section 3, we have the following matrix form of the differential equation (5.8)

$$-X(x)\left(B^{T}\right)^{2}\left(D^{T}\right)^{-1}C - \left(\frac{2}{x}\right)X(x)B^{T}\left(D^{T}\right)^{-1}C + 2*(T^{*}C)^{2}X(x)\left(D^{T}\right)^{-1}C = 1. \tag{5.10}$$

Also, the matrix representation of the boundary conditions (5.9) is given by

$$X(0)B^{T}\left(D^{T}\right)^{-1}C = 0, \tag{5.11}$$

$$X(1)\left(D^{T}\right)^{-1}C - \frac{1}{3}X\left(\frac{1}{4}\right)\left(D^{T}\right)^{-1}C = 0. \tag{5.12}$$

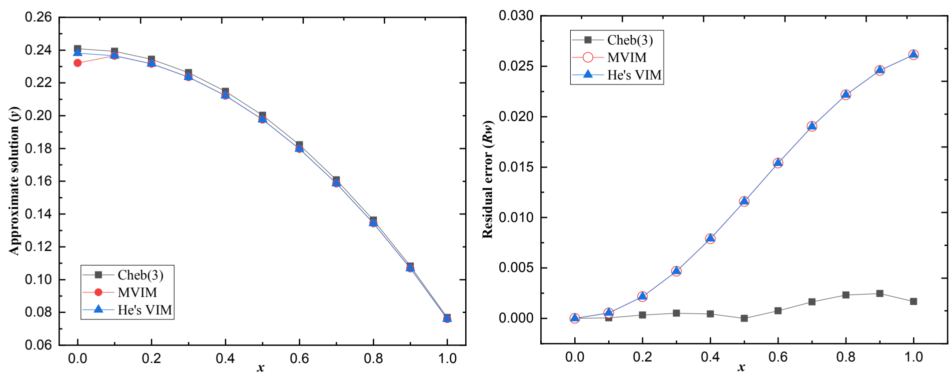
Now, using the collocation points (2.4), we get the following system

$$-\bar{\bar{X}}\left(B^{T}\right)^{2}\left(D^{T}\right)^{-1}C - H\bar{\bar{X}}B^{T}\left(D^{T}\right)^{-1}C - {T_1}^{2}\bar{\bar{X}}\left(D^{T}\right)^{-1}C - g = 0. \tag{5.13}$$

Here, the matrices $B^T, \left(D^T\right)^{-1}$ and C are defined in section 2 and $\bar{\bar{X}}$ and T_1 are defined in Section 3. The matrices H and g are given by

$$H = \begin{bmatrix} \frac{2}{x_0} & 0 & 0 & \cdots & 0 \\ 0 & \frac{2}{x_1} & 0 & \cdots & 0 \\ 0 & 0 & \frac{2}{x_2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & & \cdots & \frac{2}{x_N} \end{bmatrix}, g = \begin{bmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}.$$

In order to get the solution, replace two equations of (5.13) by (5.11) and (5.12).



(a) Numerical vs VIM (b) Residual error (Rw)

Figure 3. Comparison of results for Example 5.2.

Now, solving these equations for the unknown coefficients and substituting them into equation (2.8), the required numerical solution is obtained. The approximate results obtained using the proposed method (Cheb(N)) are compared against the MVIM [29] and He's VIM [12]. Numerical results are also listed and presented in comparison with MVIM and He's VIM in Table 3 and Figure 3 for $N = 3$. The approximate results using MVIM are presented for the parameter $\omega = 0$ considering the best result in [29]. Also, the values of approximate solution using He's VIM are given at first iteration. The methods are also compared in terms of the residual error $Rw = | - y^{''} - \frac{2}{x}y^{'} - 1 + 2y^3|$, which are provided in Table 3. The values of Chebyshev coefficients have presented in Table 4 for $N = 3$.

From Figure 3, one can easily conclude that the proposed approach is very promising for solving a three point SBVPs(see Figure 3(a) and 3(b)). Moreover, in terms of residual errors, the proposed method shows better precision than the existing method and the similar trends to the previous case are obtained for errors. From the above results and discussion, it can be concluded easily that the new approach not only approximate the three point SBVPs with higher precision, but also consumes lesser computations to obtained these results.

Example 5.3.

$$-(xy')' = x(-92 + 198x - 23x^2 + 22x^3 + y) \tag{5.14}$$

Table 3. Comparison of Cheb(N) with VIM at $N = 3$ and 4 of Example 5.2.

x	Cheb(3)	Rw	MVIM	Rw	He's VIM	Rw
0.0	0.240916	1.3×10^{-11}	0.238177	4.5×10^{-17}	0.238177	2.8×10^{-08}
0.1	0.239296	5.8×10^{-05}	0.236555	5.4×10^{-04}	0.236555	5.4×10^{-04}
0.2	0.234428	3.2×10^{-04}	0.231690	2.1×10^{-03}	0.231690	2.1×10^{-03}
0.3	0.226298	5.2×10^{-04}	0.223582	4.6×10^{-03}	0.223582	4.6×10^{-03}
0.4	0.214893	4.4×10^{-04}	0.212231	7.9×10^{-03}	0.212231	7.9×10^{-03}
0.5	0.200199	6.6×10^{-12}	0.197636	1.1×10^{-02}	0.197636	1.1×10^{-02}
0.6	0.182202	7.5×10^{-04}	0.179798	1.5×10^{-02}	0.179798	1.5×10^{-02}
0.7	0.160890	1.6×10^{-03}	0.158717	1.9×10^{-02}	0.158717	1.9×10^{-02}
0.8	0.136248	2.3×10^{-03}	0.134392	2.2×10^{-02}	0.134392	2.2×10^{-02}
0.9	0.108264	2.3×10^{-03}	0.106825	2.4×10^{-02}	0.106825	2.4×10^{-02}
1.0	0.076924	1.6×10^{-03}	0.076013	2.6×10^{-02}	0.076014	2.6×10^{-02}

Table 4. Chebyshev coefficients $N = 3$ for Example 5.2.

	c_0	c_1	c_2	c_3
$N = 3$	0.35911848	-0.08192586	-0.02063932	-0.00007015

subject to

$$y^{'}(0) = 0, \quad y(1) = \frac{1}{3} y\left(\frac{1}{2}\right). \tag{5.15}$$

The exact solution of the SBVPs (5.14)-(5.15) is $-22x^3 + 23x^2$. The qualitative and quantitative comparison of the approximate solution of the present method (Cheb(N)) at $N = 3$ against the exact solution (Exact) and the He's variational iteration method (He's VIM) [12] is provided in Table 5 and Figure 4(a), respectively. Table 5 and Figure 4(b) provides the absolute error $e = |$exact solution $-$ approximate solution$|$. The values of unknown Chebyshev coefficient have presented in Table 6. We have provided the numerical results of He's VIM at third iteration. From Figures (4(a) and 4(b)) and Table 5, it can be observed that the proposed method is computing highly accurate results and matching well with the exact results.

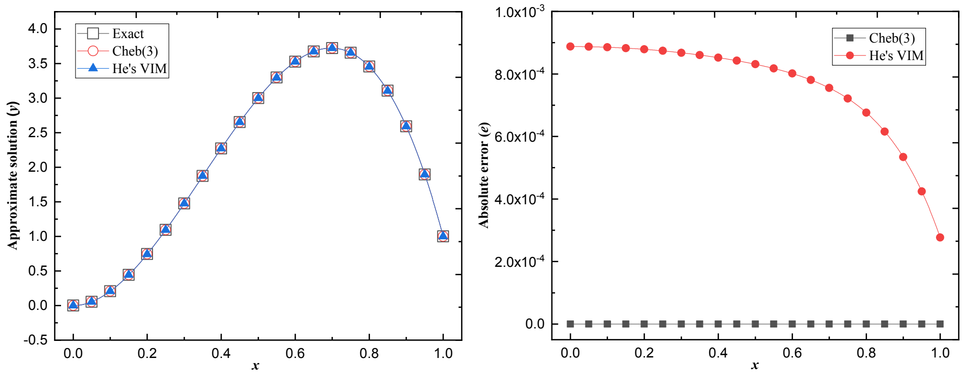
Example 5.4.

$$-\left(x^2 y^{'}\right)^{'} = x^2 \left(-1 + \frac{324}{53} x + \frac{54}{53} x^3 - \frac{729}{2809} x^6 + y^2\right), \quad 0 < x < 1, \tag{5.16}$$

subject to

$$y'(0) = 0, \quad y(1) = \frac{1}{2} y\left(\frac{1}{3}\right). \tag{5.17}$$

The exact solution of the SBVPs (5.16)-(5.17) is $-\frac{27}{53}x^3 + 1$. For a problem 5.4, the approximate solution using present method (Cheb(N)) is compared with the exact solution and He's VIM [12] graphically in Figure 5 along with the quantitative values of the solutions at different values of x in Table 7. The absolute errors e are also calculated and provided in Table 7 and Figure 5. The values of unknown



(a) Numerical vs Exact (b) Absolute error (e)

Figure 4. Comparison of results for Example 5.3.

Table 5. Comparison of Cheb(N) with exact solution and He's VIM at $N = 3$ of Example 5.3.

x	Exact	Cheb(3)	e	He's VIM	e
0.0	0.000000	-6.8×10^{-09}	6.8×10^{-09}	-0.000887	0.000887
0.1	0.208000	0.207999	6.8×10^{-09}	0.207114	0.000885
0.2	0.744000	0.743999	6.7×10^{-09}	0.743121	0.000878
0.3	1.476000	1.475999	6.6×10^{-09}	1.475132	0.000867
0.4	2.272000	2.271999	6.4×10^{-09}	2.271147	0.000852
0.5	3.000000	2.999999	6.1×10^{-09}	2.999169	0.000830
0.6	3.528000	3.527999	5.8×10^{-09}	3.527198	0.000801
0.7	3.724000	3.723999	5.5×10^{-09}	3.723244	0.000755
0.8	3.456000	3.455999	5.1×10^{-09}	3.455323	0.000676
0.9	2.592000	2.591999	4.7×10^{-09}	2.591465	0.000534
1.0	1.000000	0.999999	4.2×10^{-09}	0.999722	0.000277

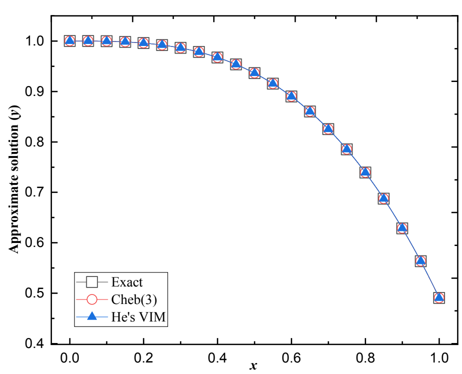
Table 6. Chebyshev coefficients at $N = 3$ for Example 5.3.

	c_0	c_1	c_2	c_3
$N = 3$	3.4999999882	1.1875000013	-1.2499999968	-0.6875000000

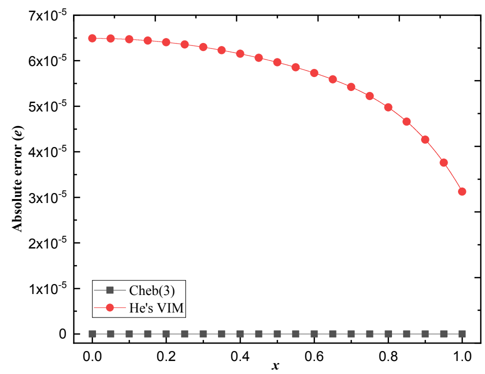
Chebyshev coefficients are listed in Table 8 for $N = 3$. The numerical results of He's VIM at third iteration are presented in Table 6 and Figure 5. Once again the plots show that the proposed approach estimates the results well and overlaps with the exact solutions (refer to Figures 5(a) and 5(b)).

Example 5.5.

$$-\left(x^{0.7}y'\right)' = x^{0.7}\left(\frac{567}{85}x - \frac{17}{5}\right), \tag{5.18}$$



(a) Numerical vs Exact



(b) Absolute error (e)

Figure 5. Comparison of results for Example 5.4.

Table 7. Comparison of Cheb(N) with exact solution and He's VIM at $N = 3$ of Example 5.4.

x	Exact	Cheb(3)	e	He's VIM	e
0.0	1.000000	1.000000	8.5×10^{-10}	1.000065	6.4×10^{-05}
0.1	0.999491	0.999490	8.4×10^{-10}	0.999555	6.4×10^{-05}
0.2	0.995925	0.995924	8.3×10^{-10}	0.995989	6.4×10^{-05}
0.3	0.986245	0.986245	8.0×10^{-10}	0.986308	6.2×10^{-05}
0.4	0.967396	0.967396	7.7×10^{-10}	0.967458	6.1×10^{-05}
0.5	0.936321	0.936320	7.4×10^{-10}	0.936380	5.9×10^{-05}
0.6	0.889962	0.889962	7.0×10^{-10}	0.890020	5.7×10^{-05}
0.7	0.825264	0.825264	6.5×10^{-10}	0.825318	5.4×10^{-05}
0.8	0.739170	0.739169	6.0×10^{-10}	0.739220	4.9×10^{-05}
0.9	0.628623	0.628622	5.4×10^{-10}	0.628665	4.2×10^{-05}
1.0	0.490566	0.490566	4.8×10^{-10}	0.490597	3.1×10^{-05}

Table 8. Chebyshev coefficients at $N = 3$ for Example 5.4.

	c_0	c_1	c_2	c_3
$N = 3$	1.681603	-.238797	-0.095518	-0.015919

subject to

$$y(0) = 0, \quad y(1) = 1.2y\left(\frac{1}{2}\right). \tag{5.19}$$

The exact solution of the boundary value problem (5.18)-(5.19) is

$$y(x) = -\frac{14}{17}x^3 + x^2. \tag{5.20}$$

Applying the methodology developed in Section 3, the following matrix form of the differential equation (5.14) is obtained

$$-X(x)\left(B^T\right)^2\left(D^T\right)^{-1}C - \left(\frac{7}{10x}\right)X(x)B^T\left(D^T\right)^{-1}C = \left(\frac{567}{85}x - \frac{17}{5}\right), \tag{5.21}$$

and the matrix representation of boundary conditions (5.19) is given by

$$X(0)\left(D^T\right)^{-1}C=0, \tag{5.22}$$

$$X(1)\left(D^T\right)^{-1}C-1.2X\left(\frac{1}{2}\right)\left(D^T\right)^{-1}C=0. \tag{5.23}$$

Now, using the collocation points (2.4), the matrix form (5.21) of the differential equation (5.18) has a new matrix representation which is a system of $(N+1)$ algebraic equations given by

$$-\bar{\bar{X}}\left(B^T\right)^2\left(D^T\right)^{-1}C-H\bar{\bar{X}}B^T\left(D^T\right)^{-1}C-g=0. \tag{5.24}$$

Here, the matrices H and g are given by

$$H=\begin{bmatrix}\frac{7}{10x_0} & 0 & 0 & \cdots & 0\\ 0 & \frac{7}{10x_1} & 0 & \cdots & 0\\ 0 & 0 & \frac{7}{10x_2} & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & & \cdots & \frac{7}{10x_N}\end{bmatrix},\quad g=\begin{bmatrix}\left(\frac{567}{85}x_0-\frac{17}{5}\right)\\ \left(\frac{567}{85}x_1-\frac{17}{5}\right)\\ \left(\frac{567}{85}x_2-\frac{17}{5}\right)\\ \vdots\\ \left(\frac{567}{85}x_N-\frac{17}{5}\right)\end{bmatrix}.$$

In order to get the solution of problem 5.5, following from the proposed methodology replace the boundary conditions given in equations (5.22) and (5.23) into two out of $(N+1)$ algebraic equations (5.24). Now, on solving these equations for unknown coefficient $c_i,\ i=0,1,2,...,N$ at $N=3$, the approximate solution given as follows is obtained

$$\begin{aligned}y(x)&=T^*(x)C,\\ &=-0.8235294115*x^3+0.9999999996*x^2+2.99\times10^{-10}*x+1.04\times10^{-17},\\ &\approx-\frac{14}{17}x^3+x^2.\end{aligned} \tag{5.25}$$

Here,

$$C=\left[0.11764705\ 0.11397058\ -0.02941176\ -0.02573529\right]^T.$$

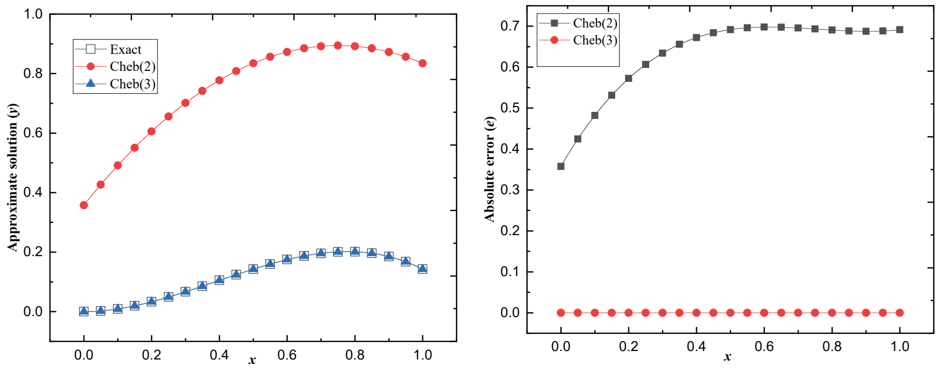
From equation (5.25), it can be seen that the numerical solution is approximately equal to the exact solution (5.20). Hence, the proposed method has the ability to find the solution with higher precision at a less computational cost.

Example 5.6.

$$-\left(x^{0.5}y'\right)'=x^{0.5}\left(-3+\frac{45}{7}x-x^4+\frac{12}{7}x^5-\frac{36}{49}x^6+y^2\right) \tag{5.26}$$

subject to

$$y(0)=0,\quad y(1)=y\left(\frac{1}{2}\right). \tag{5.27}$$



(a) Numerical vs Exact (b) Absolute error (e)

Figure 6. Comparison of results for Example 5.6.

Table 9. Comparison of Cheb(N) with exact solution at $N = 2$ and $N = 3$ of Example 5.6.

x	Exact	Cheb(2)	e	Cheb(3)	e
0.0	0.000000	0.357649	0.357649	1.0×10^{-11}	1.0×10^{-11}
0.1	0.009143	0.491171	0.482029	0.009142	1.3×10^{-11}
0.2	0.033143	0.605619	0.572476	0.033142	1.3×10^{-11}
0.3	0.066857	0.700992	0.634135	0.066857	1.1×10^{-11}
0.4	0.105143	0.777291	0.672148	0.105142	7.3×10^{-12}
0.5	0.142857	0.834515	0.691657	0.142857	2.8×10^{-12}
0.6	0.174857	0.872664	0.697807	0.174857	9.0×10^{-13}
0.7	0.196000	0.891738	0.695738	0.196000	2.2×10^{-12}
0.8	0.201143	0.891738	0.690596	0.201142	1.9×10^{-12}
0.9	0.185143	0.872664	0.687521	0.185142	2.9×10^{-12}
1.0	0.142857	0.834515	0.691657	0.142857	1.2×10^{-11}

Table 10. Chebyshev coefficients at different values of N for Example 5.6.

	c_0	c_1	c_2	c_3
$N = 2$	0.71529815	0.23843271	-0.11921635	-
$N = 3$	0.21428571	0.09821428	-0.03571428	-0.02678571

The exact solution of the boundary value problem (5.26)-(5.27) is $y(x) = -\frac{6}{7}x^3 + x^2$.

We have presented the approximate solution (Cheb(N)) in comparison with the exact solution (Exact) for $N = 2$ and 3 quantitatively and qualitatively in Table 9 and Figure 6, respectively. To ensure the accuracy and reliability of the method, the absolute error $e = |\text{Exact} - \text{Cheb}(N)|$ have also been presented in Table 9 and Figure 6. The values of Chebyshev coefficient have presented in Table 10 for $N = 2$ and 3.

One can observe that the numerical solution matches well with the exact solution for $N = 3$ (refer to Figure 6(a)). Moreover, Table 10 demonstrates that as the value of N increased from 2 to 3, the error values (e) decreased significantly.

6. Conclusions and Remarks

This article provides an efficient and accurate approach for finding the approximate solution of a class of three-point SBVPs based on collocation method in the presence of shifted Chebyshev polynomials. The mathematical formulations of the proposed method is very simple in terms of (a) easy to code and, (b) reduces the mathematical complexities due to utilization of lower order Chebyshev polynomials (in particular $N = 3$ and $N = 4$ for our case). The qualitative and quantitative behavior of the solution suggest that the proposed method is highly adoptable to solve various kind of three point SBVPs. The residual error calculated using the proposed methodology shows better results than the existing variational iterative methods [12, 29, 42]. The proposed method takes fewer Chebychev polynomials to determine the solution of a class of three-point SBVPs which makes it highly efficient. Detailed convergence of the method has also been discussed. Finally, it can be concluded that the proposed method is one of the most smartest approaches to solve a class of three-point SBVPs.

Moreover, there is vast future scope of the study made in this article. The methodology developed in this article can be modify and verify over system of nonlinear singular four point boundary value problems [4]. Multi-pantograph [36] delay can be incorporated to create a novel mathematical model over SBVPs (1.1)-(1.3), and the current methods can be improved and validated.

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