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HYERS-ULAM-RASSIAS STABILITY OF A NONLINEAR STOCHASTIC FRACTIONAL VOLTERRA INTEGRO-DIFFERENTIAL EQUATION

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Abstract In this paper, we apply the fixed point technique to study the Hyers-Ulam and the Hyers-Ulam-Rassias stability of the stochastic fractional Volterra integro-differential equation with uncertainty for a kind of ϕ-Hilfer stochastic fractional differential equations. The findings represent an extension of some results found in the current literature.

Keywords ϕ-Hilfer stochastic fractional derivative, Hyers-Ulam-Rassias stability, fixed point technique, stochastic fractional Volterra.

MSC(2010) 45Dxx, 58Cxx, 33Exx.

1. Introduction

Fractional order equations have been proven better than the integer order ones to describe the dynamics of many real-world problems. Such kind of equations involve fractional derivatives, and we can find them in the fields of electromagnetics, aerodynamics, electrodynamics, control, materials, biologic systems, and many others [1, 6, 11, 21]. Besides, since most real problems are stochastic, stochastic integro-differential equations naturally arise in the scope of many areas of mathematics, sciences, and engineering. In addition, the proposal of general results involving stochastic fractional integro-differential equations has been gaining prominence, being that the Hyers-Ulam (HU) and Hyers-Ulam-Rassias (HUR) stability of linear differential equations can be investigated by the Laplace transform [13, 14, 16–20].

The main purpose of this paper is to study the existence of solutions for the stochastic fractional nonlinear Volterra integro-differential equation ($VIDE$) using the Banach's contraction principle:

$$\begin{cases} {}^H\mathbb{D}_{0+}^{\iota,\kappa;\phi}\boldsymbol{z}(p,r) = \Theta(p,r,\boldsymbol{z}(p,r)) + \displaystyle\int_0^r \boldsymbol{k}(p,r,\vartheta,\boldsymbol{z}(p,r))d\vartheta, \\ \mathcal{I}_{0+}^{1-\gamma}\boldsymbol{z}(p,0) = \sigma, \end{cases} \tag{1.1}$$

where $r \in [0,T]$, $\Theta(p,r,\boldsymbol{z})$ is a continuous random operator (RO) with respect to r and $\boldsymbol{z}$ on $\Upsilon \times [0,T] \times \mathbb{R}$, $\boldsymbol{k}(p,r,\vartheta,\boldsymbol{z})$ stands for a continuous random function

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with respect to r, ϑ and $\boldsymbol{z}$ on $\Upsilon \times [0,T] \times \mathbb{R} \times \mathbb{R}$, σ is a given constant, ${}^{H}\mathbb{D}_{0+}^{\iota,\kappa;\phi}\boldsymbol{z}(.)$ denotes the right-sided ϕ-Hilfer stochastic fractional derivative (Eq. (2.1)) where $p \in \Upsilon$, $0 < \iota < 1$, $0 \leq \kappa \leq 1$, and $\mathcal{I}_{0+}^{1-\gamma}(.)$ is ϕ-Riemann-Liouville stochastic fractional integral with $0 \leq \gamma < 1$.

The paper is organized as follows. In Section 2 we give some preliminary definitions, namely the ϕ-Hilfer stochastic fractional derivative, and the HU and HUR stability. In Section 3 we present the main results of the paper, specifically the HU and HUR stability conditions. Finally, in Section 4, we draw the main conclusions.

2. Brief Mathematical Background

In this section, we introduce the ϕ-Hilfer stochastic fractional derivative. Afterwards, we present the stability definitions of HU and HUR by means of the ϕ-Hilfer stochastic fractional derivative, and important results for the study of the stochastic fractional nonlinear differential equation (1.1) and its stability.
Let us assume that $\boldsymbol{A} = [0,T]$, with $T > 0$, $\boldsymbol{B} = (0,\infty)$, $\boldsymbol{C} = (0,1]$, $\boldsymbol{D} = [0,\infty]$ and $\boldsymbol{E} = [0,1]$. Consider that $(\Upsilon, \boldsymbol{B}, \xi)$ is a probability measure space, and $(S, \boldsymbol{B}_S)$ and $(T, \boldsymbol{B}_T)$ are Borel measurable spaces, in which S and T are complete fuzzy normed (FN) spaces. If $\{p : \mathcal{F}(p,\xi) \in B\} \in \boldsymbol{B}$ for each $\xi \in T$ and $B \in \boldsymbol{B}_S$, then $\mathcal{F} : \Upsilon \times T \to S$ is a RO. A RO $\mathcal{F} : \Upsilon \times T \to S$ is linear if $\mathcal{F}(p, \mathbf{a}\xi_1 + \mathbf{b}\xi_2) = \mathbf{a}\mathcal{F}(p,\xi_1) + \mathbf{b}\mathcal{F}(p,\xi_2)$ almost everywhere for every ξ_1, υ_2 in T and $\mathbf{a}, \mathbf{b} \in \mathbb{R}$, and is bounded if there exists a nonnegative real-valued random variable $M(p)$ such that

$$\delta\left(\mathcal{F}(p,\xi_1) - \mathcal{F}(p,\xi_2), M(p)s\right) \geq \delta\left(\xi_1 - \xi_2, s\right),$$

almost everywhere for every ξ_1, ξ_2 in T, $s \in \boldsymbol{B}$ and $p \in \Upsilon$.

Some stability results for fractional differential equations and stochastic integrals can be found in references [2–5, 7–10, 14].

Theorem 2.1 ([5], The alternative of fixed point)**.** *Consider that (U,δ) is a generalized metric space and that $\hbar : U \to U$ is a strictly contractive mapping where $\iota < 1$ is a Lipschitz constant. Assume that $\xi \in U$, thus, in this case, either*

$$\delta\left(\hbar^m \xi, \hbar^{m+1}\xi\right) = \infty,$$

for all $m \in \mathbb{N}$, or there exists $m_0 \in \mathbb{N}$ in such a way that

(a) $\delta\left(\hbar^m \xi, \hbar^{m+1}\xi\right) < \infty$, *for every $m \geq m_0$;*

(b) $\{\hbar^m \xi\}$ *converges to a FP ζ^* from $\hbar$;*

(c) ζ^* *is the unique FP of $\hbar$ in $V = \{\zeta \in U \mid \delta(\hbar^{m_0}\xi, \zeta) < \infty\}$;*

(d) $(1-\iota)\delta\left(\zeta,\zeta^*\right) \leq \delta(\zeta, \hbar\zeta)$ *for every $\zeta \in V$.*

Consider $\iota \in \mathring{\boldsymbol{E}}$, the integrable RO Θ on $\boldsymbol{A}$, and the nondecreasing RO $\phi \in C^1(\Upsilon \times \boldsymbol{A})$ with $\phi'(p,r) \neq 0$, for each $r \in \boldsymbol{A}$. The right-sided ϕ-Hilfer stochastic fractional derivative is [12, 15]:

$${}^{H}\mathbb{D}_{0+}^{\iota,\kappa;\phi}\Theta(p,r) = \mathcal{I}_{0+}^{\kappa(1-\iota);\phi}\left(\frac{1}{\phi'(p,r)}\frac{d}{dr}\right)\mathcal{I}_{0+}^{(1-\kappa)(1-\iota);\phi}\Theta(p,r). \tag{2.1}$$

Definition 2.1 ([12]). If for each continuously differentiable RO $\boldsymbol{z}(p,r)$ and continuous fuzzy set $\varpi(r,s)$ satisfying

$$\delta\left({}^{H}\mathbb{D}_{0+}^{\iota,\kappa;\phi}\boldsymbol{z}(p,r)-\Theta(p,r,\boldsymbol{z}(p,r))-\int_0^r \boldsymbol{k}(p,r,\vartheta,\boldsymbol{z}(p,r))d\vartheta,s\right)\geq\varpi(r,s),$$

for each $r\in\boldsymbol{A}$, $s\in\boldsymbol{B}$ and $p\in\Upsilon$, there exists a solution $\boldsymbol{z}_0(p,r)$ of the $VIDE$ (1.1) and a fixed number $\mathbf{C}>0$ with

$$\delta\left(\boldsymbol{z}(p,r)-\boldsymbol{z}_0(p,r),s\right)\geq\varpi\left(r,\frac{s}{\mathbf{C}}\right),$$

for each $r\in\boldsymbol{A}$, $s\in\boldsymbol{B}$ and $p\in\Upsilon$, where $\mathbf{C}$ is independent of $\boldsymbol{z}(p,r)$ and $\boldsymbol{z}_0(p,r)$, then (1.1) has HUR stability. If $\varpi(r,s)$ is a constant continuous fuzzy set in the above inequalities, then (1.1) has HU stability.

3. Mains Results

In this section, we introduce the Lipschitz condition, in order to present and discuss the main result of this paper, that is, the study of the stability of HU and HUR.

First, we introduce the following hypothesis.
$(H0)$ Consider the fixed numbers $M,L_\Theta,L_{\boldsymbol{k}}>0$ with $M\left(L_\Theta+L_{\boldsymbol{k}}\right)\in\mathring{\boldsymbol{E}}$. Consider the continuous ROs $\Theta:\Upsilon\times\boldsymbol{A}\times\mathbb{R}\to\mathbb{R}$ and $\boldsymbol{k}:\Upsilon\times\boldsymbol{A}\times\boldsymbol{A}\times\mathbb{R}\to\mathbb{R}$ satisfying

$$\delta\left(\Theta(p,r,\boldsymbol{z}_1)-\Theta(p,r,\boldsymbol{z}_2),s\right)\geq\delta\left(\boldsymbol{z}_1-\boldsymbol{z}_2,\frac{s}{L_\Theta}\right), \tag{3.1}$$

for each $r\in\boldsymbol{A}$, $\boldsymbol{z}_1,\boldsymbol{z}_2\in\mathbb{R}$, $s\in\boldsymbol{B}$ and $p\in\Upsilon$,

$$\delta\left(\boldsymbol{k}(p,r,\vartheta,\boldsymbol{z}_1)-\boldsymbol{k}(p,r,\vartheta,\boldsymbol{z}_2),s\right)\geq\delta\left(\boldsymbol{z}_1-\boldsymbol{z}_2,\frac{s}{L_{\boldsymbol{k}}}\right), \tag{3.2}$$

for each $r,\vartheta\in\boldsymbol{A}$, $\boldsymbol{z}_1,\boldsymbol{z}_2\in\mathbb{R}$, $s\in\boldsymbol{B}$ and $p\in\Upsilon$.

Theorem 3.1. *Suppose $(H0)$, and consider the nondecreasing RO $\phi\in C(\Upsilon\times\boldsymbol{A})$ with $\phi'(p,r)\neq0$ and the continuously differentiable RO $\boldsymbol{z}:\Upsilon\times\boldsymbol{A}\to\mathbb{R}$ satisfying*

$$\delta\left({}^{H}\mathbb{D}_{0+}^{\iota,\kappa;\phi}\boldsymbol{z}(p,r)-\Theta(p,r,\boldsymbol{z}(p,r))-\int_0^r \boldsymbol{k}(p,r,\vartheta,\boldsymbol{z}(p,\vartheta))d\vartheta,s\right)\geq\varpi(r,s), \tag{3.3}$$

for each $r,\vartheta\in\boldsymbol{A}$, $\boldsymbol{z}\in\mathbb{R}$, $s\in\boldsymbol{B}$ and $p\in\Upsilon$, where $\varpi:\boldsymbol{A}\times\boldsymbol{B}\to\boldsymbol{C}$ is continuous with

$$\delta\left(\frac{1}{\Gamma(\iota)}\int_0^r\phi'(p,\xi)(\phi(p,r)-\phi(p,\xi))^{\iota-1}\varpi(\xi,s)d\xi,s\right)\geq\varpi\left(r,\frac{s}{M}\right), \tag{3.4}$$

for each $r\in\boldsymbol{A}$, $s\in\boldsymbol{B}$ and $p\in\Upsilon$. Then, there is a unique continuous RO $\boldsymbol{z}_0:\Upsilon\times\boldsymbol{A}\to\mathbb{R}$, in such a way that

$$\begin{aligned}\boldsymbol{z}_0(p,r)=&\frac{(\phi(p,r)-\phi(p,0))^{\gamma-1}}{\Gamma(\gamma)}\sigma\\&+\mathcal{I}_{0+}^{\iota;\phi}\Theta(p,r,\boldsymbol{z}_0(p,r))\end{aligned} \tag{3.5}$$

$$+\mathcal{I}_{0+}^{\iota;\phi}\left[\int_0^{\xi}\boldsymbol{k}(p,r,\vartheta,\boldsymbol{z}_0(p,\vartheta))d\vartheta\right],$$

with $\mathcal{I}_{0+}^{1-\gamma;\phi}\boldsymbol{z}(p,0)=\sigma$, $0<\iota<1$, $\kappa\in\boldsymbol{E}$ *and*

$$\delta\left(\boldsymbol{z}(p,r)-\boldsymbol{z}_0(p,r),s\right)\geq\varpi\left(r,\frac{Ms}{1-M\left(L_\Theta+L_{\boldsymbol{k}}\right)}\right), \tag{3.6}$$

for each $r\in\boldsymbol{A}$, $s\in\boldsymbol{B}$ *and* $p\in\Upsilon$.

Proof. In fact, for each $\boldsymbol{x},\boldsymbol{y}\in U$, we set

$$q(\boldsymbol{x},\boldsymbol{y})=\inf\left\{\mathbf{C}\in\boldsymbol{D}:\delta\left(\boldsymbol{x}(p,r)-\boldsymbol{y}(p,r),s\right)\geq\varpi\left(r,\frac{s}{\mathbf{C}}\right)\right\}, \tag{3.7}$$

for all $r\in\boldsymbol{A}$, $s\in\boldsymbol{B}$ and $p\in\Upsilon$,

$$U=\{\boldsymbol{x}:\Upsilon\times\boldsymbol{A}\to\mathbb{R}\text{ is continuous RO}\}.$$

Consider $\Lambda:U\to U$, as

$$\begin{aligned}\Lambda\boldsymbol{x}(p,r)=&\frac{(\phi(p,r)-\phi(p,0))^{\gamma-1}}{\Gamma(\gamma)}\sigma\\&+\mathcal{I}_{0+}^{\iota;\phi}\Theta(p,r,\boldsymbol{x}(p,r))\\&+\mathcal{I}_{0+}^{\iota;\phi}\left[\int_0^{\xi}\boldsymbol{k}(p,r,\vartheta,\boldsymbol{x}(p,\vartheta))d\vartheta\right],\end{aligned} \tag{3.8}$$

for each $\boldsymbol{x}\in\boldsymbol{A}$, $r\in\boldsymbol{A}$ and $p\in\Upsilon$.

We have Λ strictly contractive on U.

Consider $\mathbf{C}_{\boldsymbol{xy}}\in\boldsymbol{D}$ with $q(\boldsymbol{x},\boldsymbol{y})\leq\mathbf{C}_{\boldsymbol{xy}}$ for each $\boldsymbol{x},\boldsymbol{y}\in U$. By (3.7), we get

$$\delta\left(\boldsymbol{x}(p,r)-\boldsymbol{y}(p,r),s\right)\geq\varpi\left(r,\frac{s}{\mathbf{C}_{\boldsymbol{xy}}}\right), \tag{3.9}$$

for all $r\in\boldsymbol{A}$, $s\in\boldsymbol{B}$ and $p\in\Upsilon$. So, by (3.1), (3.2), (3.4), (3.8) and (3.9), we can write

$$\begin{aligned}&\delta\left(\Lambda\boldsymbol{x}(p,r)-\Lambda\boldsymbol{y}(p,r),s\right)\\&=\delta\Bigg(\frac{1}{\Gamma(\iota)}\int_0^r\phi'(p,\xi)(\phi(p,r)-\phi(p,\xi))^{\iota-1}\delta\Bigg(\Theta(p,\xi,\boldsymbol{x}(p,\xi))-\Theta(p,\xi,\boldsymbol{y}(p,\xi))\\&\quad+\int_0^{\xi}\boldsymbol{k}(p,r,\vartheta,\boldsymbol{x}(p,\vartheta))-\boldsymbol{k}(p,r,\vartheta,\boldsymbol{y}(p,\vartheta))d\vartheta,s\Bigg)d\xi,s\Bigg)\\&\geq\delta\Bigg(\frac{1}{\Gamma(\iota)}\int_0^r\phi'(p,\xi)(\phi(p,r)-\phi(p,\xi))^{\iota-1}\min\Bigg\{\delta(\Theta(p,\xi,\boldsymbol{x}(p,\xi))\\&\quad-\Theta(p,\xi,\boldsymbol{y}(p,\xi)),s),\delta\left(\int_0^{\xi}\boldsymbol{k}(p,r,\vartheta,\boldsymbol{x}(p,\vartheta))-\boldsymbol{k}(p,r,\vartheta,\boldsymbol{y}(p,\vartheta))d\vartheta,s\right)\Bigg\}d\xi,s\Bigg)\\&\geq\delta\Bigg(\frac{1}{\Gamma(\iota)}\int_0^r\phi'(p,\xi)(\phi(p,r)\\&\quad-\phi(p,\xi))^{\iota-1}\min\left\{\delta\left(\boldsymbol{x}(p,\xi)-\boldsymbol{y}(p,\xi),\frac{s}{L_\Theta}\right),\delta\left(\boldsymbol{x}(p,\xi)-\boldsymbol{y}(p,\xi),\frac{s}{L_{\boldsymbol{k}}}\right)\right\}d\xi,s\Bigg)\end{aligned}$$

$$
\begin{aligned}
&\geq \delta\left(\frac{1}{\Gamma(\iota)}\int_0^r \phi'(p,\xi)(\phi(p,r)-\phi(p,\xi))^{\iota-1}\delta\left(\boldsymbol{x}(p,\xi)-\boldsymbol{y}(p,\xi),\frac{s}{L_\Theta+L_{\boldsymbol{k}}}\right)d\xi, s\right)\\
&\geq \delta\left(\frac{1}{\Gamma(\iota)}\int_0^r \phi'(p,\xi)(\phi(p,r)-\phi(p,\xi))^{\iota-1}\varpi\left(\xi,\frac{s}{\mathbf{C}_{\boldsymbol{xy}}(L_\Theta+L_{\boldsymbol{k}})}\right)d\xi, s\right)\\
&\geq \varpi\left(r,\frac{s}{M\mathbf{C}_{\boldsymbol{xy}}(L_\Theta+L_{\boldsymbol{k}})}\right).
\end{aligned}
\tag{3.10}
$$

By applying (3.10), we conclude that

$$
q(\Lambda\boldsymbol{x},\Lambda\boldsymbol{y}) \leq \varpi(r,\frac{s}{M\mathbf{C}_{\boldsymbol{xy}}(L_\Theta+L_{\boldsymbol{k}})}),
$$

for each $r\in\boldsymbol{A}$ and $s\in\boldsymbol{B}$. Hence, we can deduce that $q(\Lambda\boldsymbol{x},\Lambda\boldsymbol{y})\leq[M\,(L_\Theta+L_{\boldsymbol{k}})]q(\boldsymbol{x},\boldsymbol{y})$ for each $\boldsymbol{x},\boldsymbol{y}\in U$, where $0<M\,(L_\Theta+L_{\boldsymbol{k}})<1$.

By (3.8), we can find a fixed number $\mathbf{C}\in\boldsymbol{B}$, in such a way that

$$
\begin{aligned}
&\delta\left(\Lambda\boldsymbol{y}_0(p,r)-\boldsymbol{y}_0(p,r),s\right)\\
=&\delta\Bigg(\frac{(\phi(p,r)-\phi(p,0))^{\gamma-1}}{\Gamma(\gamma)}\sigma+\mathcal{I}_{0+}^{\iota;\phi}\Theta(p,r,\boldsymbol{y}_0(p,r))\\
&+\mathcal{I}_{0+}^{\iota;\phi}\left[\int_0^\xi\Theta(p,r,\vartheta,\boldsymbol{y}_0(p,\vartheta))d\vartheta\right]-\boldsymbol{y}_0(p,r),s\Bigg)\\
\geq&\varpi(r,\frac{s}{\mathbf{C}}),
\end{aligned}
$$

for arbitrary $\boldsymbol{y}_0\in U$, for each $r\in\boldsymbol{A}$, $s\in\boldsymbol{B}$ and $p\in\Upsilon$. With the boundedness of $\Theta(p,\xi,\boldsymbol{y}_0(p,\xi))$, $\boldsymbol{k}(p,r,\vartheta,\boldsymbol{y}_0(p,\vartheta))$ and $\boldsymbol{y}_0(p,r)$, then, from (3.7) we get $q(\Lambda\boldsymbol{y}_0,\boldsymbol{y}_0)<\infty$. By Theorem 2.1, we can find a continuous *RO* $\boldsymbol{z}_0:\Upsilon\times\boldsymbol{A}\to\mathbb{R}$ in such a way that $\Lambda^n\boldsymbol{z}_0\to\boldsymbol{z}_0$ in (U,q) and $\Lambda\boldsymbol{z}_0=\boldsymbol{z}_0$.

Since $\boldsymbol{z}_0$ is bounded, for all $\boldsymbol{y}\in U$ and $\min_{r\in\boldsymbol{A}}\varpi(r,s)>0$, then we get a fixed number $\mathbf{C}_{\boldsymbol{xy}}\in\boldsymbol{D}$ with

$$
\delta\left(\boldsymbol{y}_0(p,r)-\boldsymbol{y}(p,r),s\right)\geq\varpi\left(r,\frac{s}{\mathbf{C}_{\boldsymbol{y}}}\right),
$$

for each $r\in\boldsymbol{A}$, $s\in\boldsymbol{B}$ and $p\in\Upsilon$. We have $q(\boldsymbol{y}_0,\boldsymbol{y})<\infty$ for all $\boldsymbol{y}\in U$.

So, we get that $U=\{\boldsymbol{y}\in U:q(\boldsymbol{y}_0,\boldsymbol{y})<\infty\}$. Also Theorem 2.1 and (3.5), imply the uniqueness of $\boldsymbol{z}_0$.

Using the (3.3) and [15, Theorem 5], implies that

$$
\begin{aligned}
&\delta\Bigg(\boldsymbol{z}(p,r)-\frac{(\phi(p,r)-\phi(p,0))^{\gamma-1}}{\Gamma(\gamma)}\sigma\\
&-\mathcal{I}_{0+}^{\iota;\phi}\Theta(p,r,\boldsymbol{z}(p,r))-\mathcal{I}_{0+}^{\iota;\phi}\left[\int_0^\xi\boldsymbol{k}(p,r,\vartheta,\boldsymbol{z}(p,\vartheta))d\vartheta\right],s\Bigg)\\
\geq&\frac{1}{\Gamma(\iota)}\int_0^r\phi'(p,\xi)(\phi(p,r)-\phi(p,\xi))^{\iota-1}\varpi(\xi,s)d\xi.
\end{aligned}
$$

Then, by (3.4) and (3.8), we get

$$
\delta\left(\boldsymbol{z}(p,r)-\Lambda\boldsymbol{z}(p,r),s\right)
$$

$$\geq \frac{1}{\Gamma(\iota)} \int_0^r \phi'(p,\xi)(\phi(p,r)-\phi(p,\xi))^{\iota-1}\varpi(\xi,s)d\xi$$
$$\geq \varpi\left(r, \frac{s}{M}\right),$$

for all $r \in \boldsymbol{A}$, $s \in \boldsymbol{B}$ and $p \in \Upsilon$, which implies

$$q(\boldsymbol{z}, \Lambda \boldsymbol{z}) \leq M. \tag{3.11}$$

Again by Theorem 2.1 and (3.11), we deduce that

$$q(\boldsymbol{z}, \boldsymbol{z}_0) \leq \frac{1}{1 - M(L_\Theta + L_{\boldsymbol{k}})} q(\Lambda \boldsymbol{z}, \boldsymbol{z}) \leq \frac{M}{1 - M(L_\Theta + L_{\boldsymbol{k}})},$$

which implies (3.6). □

It is important to note that using the same hypotheses H_0, one can presume HUR stability, always as a theorem, assuming a limited and closed interval.

In what follows we present and prove a theorem associated with the stability involving (3.5).

Theorem 3.2. *Consider $\iota, \kappa \in \mathring{\boldsymbol{E}}$, the nondecreasing RO $\phi \in C^1(\Upsilon \times \boldsymbol{A})$ with $\phi'(p,r) \neq 0$ for all $r \in \boldsymbol{A}$ and the fixed number $L_\Theta, L_{\boldsymbol{k}} > 0$ in such a way that $\frac{(L_\Theta + L_{\boldsymbol{k}})}{\Gamma(\iota+1)} \in \mathring{\boldsymbol{E}}$. Consider the continuous ROs $\Theta : \Upsilon \times \boldsymbol{A} \times \mathbb{R} \to \mathbb{R}$ and $\boldsymbol{k} : \Upsilon \times \boldsymbol{A} \times \boldsymbol{A} \times \mathbb{R} \to \mathbb{R}$ satisfying (3.1) and (3.2), respectively. Consider $\varepsilon \in \mathring{\boldsymbol{E}}$ and the continuously differentiable RO $\boldsymbol{z} : \Upsilon \times \boldsymbol{A} \to \mathbb{R}$ in such a way that*

$$\delta\left({}^H\mathbb{D}_{0+}^{\iota,\kappa;\phi} \boldsymbol{z}(p,r) - \Theta(p,r,\boldsymbol{z}(p,r)) - \int_0^r \boldsymbol{k}(p,r,\vartheta,\boldsymbol{z}(p,\vartheta))d\vartheta, s\right) \geq \varepsilon,$$

for all $r, \vartheta \in \boldsymbol{A}$, $\boldsymbol{z} \in \mathbb{R}$, $s \in \boldsymbol{B}$ and $p \in \Upsilon$, and

$$\delta\Big((\phi(p,r)-\phi(p,0))^\iota, s\Big) \geq \delta\left(r, s\right).$$

Then, we can find unique continuous RO $\boldsymbol{z}_0 : \Upsilon \times \boldsymbol{A} \to \mathbb{R}$ satisfying (3.5) and

$$\delta\left(\boldsymbol{z}(p,r) - \boldsymbol{z}_0(p,r), s\right) \geq \frac{(\phi(p,T)-\phi(p,0))^\iota \varepsilon}{\Gamma(\iota+1) - (\phi(p,T)-\phi(p,0))^\iota [T\left(L_\Theta + L_{\boldsymbol{k}}\right)]}, \tag{3.12}$$

for all $r \in \boldsymbol{A}$, $\boldsymbol{z} \in \mathbb{R}$, $s \in \boldsymbol{B}$ and $p \in \Upsilon$.

Proof. Consider $U = \{\boldsymbol{x} : \Upsilon \times \boldsymbol{A} \to \mathbb{R} \text{ is continuous RO}\}$ and

$$q(\boldsymbol{x}, \boldsymbol{y}) = \inf\left\{\mathbf{C} \in \boldsymbol{D} : \delta\left(\boldsymbol{x}(p,r) - \boldsymbol{y}(p,r), s\right) \geq \left(\frac{s}{s+\mathbf{C}}\right)\right\}, \tag{3.13}$$

for each $r \in \boldsymbol{A}$, $s \in \boldsymbol{B}$ and $p \in \Upsilon$. Sevgin and Sevli [12] proved the completeness of (U, q).

Now, consider $\Lambda : U \to U$ as

$$\Lambda \boldsymbol{x}(p,r) = \frac{(\phi(p,r)-\phi(p,0))^{\gamma-1}}{\Gamma(\gamma)} \sigma + \mathcal{I}_{0+}^{\iota;\phi} \Theta(p,r,\boldsymbol{x}(p,r)) \tag{3.14}$$

$$+\mathcal{I}_{0+}^{\iota;\phi}\bigg[\int_0^{\xi}\boldsymbol{k}(p,r,\vartheta,\boldsymbol{x}(p,\vartheta))d\vartheta\bigg],$$

for every $r\in\boldsymbol{A}$ and $p\in\Upsilon$.

Consider $\boldsymbol{x},\boldsymbol{y}\in U$ and $\mathbf{C}_{\boldsymbol{xy}}\in\boldsymbol{D}$ in such a way that $q(\boldsymbol{x},\boldsymbol{y})\leq\mathbf{C}_{\boldsymbol{xy}}$ and

$$\delta\left(\boldsymbol{x}(p,r)-\boldsymbol{y}(p,r),s\right)\geq\left(\frac{s}{s+\mathbf{C}_{\boldsymbol{xy}}}\right), \tag{3.15}$$

for each $r\in\boldsymbol{A}$, $s\in\boldsymbol{B}$ and $p\in\Upsilon$. Using the (3.1), (3.2), (3.14) and (3.15), we deduce

$$\begin{aligned}
&\delta\left(\Lambda\boldsymbol{x}(p,r)-\Lambda\boldsymbol{y}(p,r),s\right)\\
&=\delta\bigg(\frac{1}{\Gamma(\iota)}\int_0^r\phi'(p,\xi)(\phi(p,r)-\phi(p,\xi))^{\iota-1}\delta\bigg(\Theta(p,\xi,\boldsymbol{x}(p,\xi))-\Theta(p,\xi,\boldsymbol{y}(p,\xi))\\
&\quad+\int_0^{\xi}\boldsymbol{k}(p,r,\vartheta,\boldsymbol{x}(p,\vartheta))-\boldsymbol{k}(p,r,\vartheta,\boldsymbol{y}(p,\vartheta))d\vartheta,s\bigg)d\xi,s\bigg)\\
&\geq\delta\bigg(\frac{1}{\Gamma(\iota)}\int_0^r\phi'(p,\xi)(\phi(p,r)-\phi(p,\xi))^{\iota-1}\min\bigg\{\delta(\Theta(p,\xi,\boldsymbol{x}(p,\xi))\\
&\quad-\Theta(p,\xi,\boldsymbol{y}(p,\xi)),s),\delta\left(\int_0^{\xi}\boldsymbol{k}(p,r,\vartheta,\boldsymbol{x}(p,\vartheta))-\boldsymbol{k}(p,r,\vartheta,\boldsymbol{y}(p,\vartheta))d\vartheta,s\right)\bigg\}d\xi,s\bigg)\\
&\geq\delta\bigg(\frac{1}{\Gamma(\iota)}\int_0^r\phi'(p,\xi)(\phi(p,r)-\phi(p,\xi))^{\iota-1}\\
&\quad\times\min\left\{\delta\left(\boldsymbol{x}(p,\xi)-\boldsymbol{y}(p,\xi),\frac{s}{L_\Theta}\right),\delta\left(\boldsymbol{x}(p,\xi)-\boldsymbol{y}(p,\xi),\frac{s}{L_{\boldsymbol{k}}}\right)\right\}d\xi,s\bigg)\\
&\geq\delta\left(\frac{1}{\Gamma(\iota)}\int_0^r\phi'(p,\xi)(\phi(p,r)-\phi(p,\xi))^{\iota-1}\delta\left(\boldsymbol{x}(p,\xi)-\boldsymbol{y}(p,\xi),\frac{s}{L_\Theta+L_{\boldsymbol{k}}}\right)d\xi,s\right)\\
&\geq\delta\left(\frac{1}{\Gamma(\iota)}\int_0^r\phi'(p,\xi)(\phi(p,r)-\phi(p,\xi))^{\iota-1}\left(\frac{s\left(L_\Theta+L_{\boldsymbol{k}}\right)}{s+\mathbf{C}_{\boldsymbol{xy}}}\right)d\xi,s\right)\\
&\geq\delta\left((\phi(p,r)-\phi(p,0))^{\iota}\left(\frac{s\Gamma(\iota+1)\left(L_\Theta+L_{\boldsymbol{k}}\right)}{\Gamma(\iota+1)\left(s+\mathbf{C}_{\boldsymbol{xy}}\right)}\right),s\right)\\
&\geq\delta\left((\phi(p,r)-\phi(p,0))^{\iota},\frac{s}{\left(\frac{s(L_\Theta+L_{\boldsymbol{k}})}{\Gamma(\iota+1)(s+\mathbf{C}_{\boldsymbol{xy}})}\right)}\right)\\
&\geq\delta\left(r,\frac{s}{\left(\frac{s(L_\Theta+L_{\boldsymbol{k}})}{\Gamma(\iota+1)(s+\mathbf{C}_{\boldsymbol{xy}})}\right)}\right),
\end{aligned}$$

for every $r\in\boldsymbol{A}$, $s\in\boldsymbol{B}$ and $p\in\Upsilon$. We also have that

$$q(\Lambda\boldsymbol{x},\Lambda\boldsymbol{y})\leq\left(\frac{s\left(L_\Theta+L_{\boldsymbol{k}}\right)}{\Gamma(\iota+1)\left(s+\mathbf{C}_{\boldsymbol{xy}}\right)}\right)q(\boldsymbol{x},\boldsymbol{y}),$$

for each $\boldsymbol{x},\boldsymbol{y}\in U$ and $p\in\Upsilon$. Let $\boldsymbol{y}_0\in U$. So, there is $\mathbf{C}\in\boldsymbol{B}$

$$\delta\left(\Lambda\boldsymbol{y}_0(p,r)-\boldsymbol{y}_0(p,r),s\right)$$

$$
\begin{aligned}
&=\delta\Bigg(\frac{(\phi(p,r)-\phi(p,0))^{\gamma-1}}{\Gamma(\gamma)}\sigma+\mathcal{I}_{0+}^{\iota;\phi}\Theta(p,r,\boldsymbol{y}_0(p,r))\\
&\quad+\mathcal{I}_{0+}^{\iota;\phi}\left[\int_0^{\xi}\boldsymbol{k}(p,r,\vartheta,\boldsymbol{y}_0(p,\vartheta))d\vartheta\right]-\boldsymbol{y}_0(p,r),s\Bigg)\\
&\geq\frac{s}{s+\mathbf{C}},
\end{aligned}
$$

for all $r\in\boldsymbol{A}$ and $p\in\Upsilon$.

The boundedness of $\Theta(p,\xi,\boldsymbol{y}_0(p,\xi))$, $\boldsymbol{k}(p,r,\vartheta,\boldsymbol{y}_0(p,\vartheta))$, $\boldsymbol{y}_0(p,r)$ and (3.13), imply that $q(\Lambda\boldsymbol{y}_0,\boldsymbol{y}_0)<\infty$.

By Theorem 2.1, we can find a continuous RO $\boldsymbol{z}_0:\Upsilon\times\boldsymbol{A}\to\mathbb{R}$ with $\Lambda^n\boldsymbol{y}_0\to\boldsymbol{z}_0$ in (U,q) and $\Lambda\boldsymbol{z}_0=\boldsymbol{z}_0$. Therefore, $\boldsymbol{z}_0$ satisfies (3.5). Using a method similar to that in Theorem 3.1, we get $\{\boldsymbol{y}\in U:q(\boldsymbol{y}_0,\boldsymbol{y})<\infty\}=U$. Also Theorem 2.1 and (3.5) imply the uniqueness of $\boldsymbol{z}_0$.

By (3.3) and [15, Theorem 5], we get

$$
\begin{aligned}
&\delta\Bigg(\boldsymbol{z}(p,r)-\frac{(\phi(p,r)-\phi(p,0))^{\gamma-1}}{\Gamma(\gamma)}\sigma-\mathcal{I}_{0+}^{\iota;\phi}\Theta(p,r,\boldsymbol{z}_0(p,r))\\
&\quad-\mathcal{I}_{0+}^{\iota;\phi}\left[\int_0^{\xi}\boldsymbol{k}(p,r,\vartheta,\boldsymbol{z}_0(p,\vartheta))d\vartheta\right],\frac{s\Gamma(\iota+1)}{(\phi(p,T)-\phi(p,0))^{\iota}}\Bigg)\\
&\geq\varepsilon,
\end{aligned}
$$

for every $r\in\boldsymbol{A}$ and $p\in\Upsilon$, which implies

$$
q(\boldsymbol{z},\Lambda\boldsymbol{z})\leq\varepsilon\frac{(\phi(p,T)-\phi(p,0))^{\iota}}{\Gamma(\iota+1)}.
$$

Again by Theorem 2.1 and (3.7), we deduce that

$$
\delta\left(\boldsymbol{z}(p,r)-\boldsymbol{z}_0(p,r),\frac{s\left(\Gamma(\iota+1)-(\phi(p,T)-\phi(p,0))^{\iota}[L_\Theta+\frac{T}{2}L_{\boldsymbol{k}}]\right)}{(\phi(p,T)-\phi(p,0))^{\iota}}\right)\geq\varepsilon,
$$

which implies (3.12) for all $r\in\boldsymbol{A}$. □

4. Conclusions

In this research article we derived necessary conditions for the existence, uniqueness, and HU and HUR stability of systems (1.1) and (2.1). The results were established by using the Diaz-Margolis's fixed-point theorem. Further, we derived some proper conditions for various kinds of HU and HUR stability of the solution of the those systems.

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