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SOME RESULTS ON VALUE DISTRIBUTION OF MEROMORPHIC FUNCTIONS CONCERNING DIFFERENCE POLYNOMIALS*

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Abstract In this paper, we study value distribution of meromorphic functions concerning difference polynomials and solve an open problem posed by Zheng and Chen [J. Math. Anal. Appl. 397 (2013)]. By using different methods, we improve and extend some results due to Zheng and Chen [J. Math. Anal. Appl. 397 (2013)], Zhang and Huang [Chinese Ann. Math. Ser. A 40 (2019)].

Keywords Meromorphic function, Borel exceptional value, hyper-order, difference.

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1. Introduction and main results

In this paper, we assume that the reader is familiar with the basic notions of Nevanlinna's value distribution theory, see [14,16,22,23]. In the following, a meromorphic function always means meromorphic in the whole complex plane. By $S(r, f)$, we denote any quantity satisfying $S(r, f) = o(T(r, f))$ as $r \to \infty$ possible outside of an exceptional set E with finite logarithmic measure $\int_E dr/r < \infty$. A meromorphic function α is said to be a small function of f if it satisfies $T(r, \alpha) = S(r, f)$.

Let f be a nonconstant meromorphic function. The order of f is defined by

$$\rho(f) = \overline{\lim_{r\to\infty}} \frac{\log^+ T(r, f)}{\log r}.$$

Let f be a nonconstant meromorphic function, and let α be a small function of f. The exponent of convergence of zeros of $f - \alpha$ is defined by

$$\lambda(f-\alpha) = \overline{\lim_{r\to\infty}} \frac{\log^+ N\left(r, \frac{1}{f-\alpha}\right)}{\log r}.$$

If

$$\lambda(f-\alpha) < \rho(f)$$

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for $\rho(f) > 0$; and $N\left(r, \frac{1}{f-\alpha}\right) = O(\log r)$ for $\rho(f) = 0$, then α is called a Borel exceptional function of f. If α is a constant, then α is called a Borel exceptional value of f.

In 1959, Hayman [13] proved the following theorem.

Theorem 1.1. *Let f be a transcendental entire (meromorphic) function, let $a(\neq 0), c$ be two finite complex numbers, and let n be a positive integer. If $n \geq 3$ $(n \geq 5)$, then $f' - af^n - c$ has infinitely many zeros.*

Recently, value distribution in difference analogue of meromorphic functions has become a subject of some interests, see [1, 3, 5, 6, 9–12, 15, 17, 18, 21, 24, 26, 27].

In 2013, Zheng and Chen [27] proved:

Theorem 1.2. *Let f be a transcendental entire function of finite order, let m, n be two distinct positive integers, let a, c be two nonzero complex numbers, let $c_1, c_2, \cdots, c_m$ be complex numbers such that at least one of them is nonzero, and let $\varphi(z) = f(z + c_1)f(z + c_2) \cdots f(z + c_m) - af^n(z)$. If $N\left(r, \frac{1}{f}\right) = S(r, f)$, then*

(i) for $|n - m| = 1$, φ has infinitely many zeros;

(ii) for $\min\{n, m\} = d \geq 2$, $\varphi - c$ has infinitely many zeros.

Theorem 1.3. *Let f be a transcendental entire function of finite order with a Borel exceptional value b, let m, n be two distinct positive integers, let $a(\neq 0), c(\neq b^m - ab^n)$ be two complex numbers, let $c_1, c_2, \cdots, c_m$ be complex numbers such that at least one of them is nonzero, and let $\varphi(z) = f(z + c_1)f(z + c_2) \cdots f(z + c_m) - af^n(z)$. If $n > m \geq 1$, then $\varphi - c$ has infinitely many zeros and $\lambda(\varphi - c) = \rho(f)$.*

In [27], Zheng and Chen posed the following problem.

Problem 1.1. *Whether Theorem 1.2 is valid or not for $n = m$ and whether Theorem 1.3 is valid or not for $n \leq m$?*

In this paper, we give a positive answer to Problem 1.1 and prove:

Theorem 1.4. *Let f be a transcendental meromorphic function of finite order, let m, n be two positive integers, let b be a complex number, let $\alpha(\not\equiv 0), c(\not\equiv b^m - \alpha b^n)$ be two small functions of f, let $c_j (j = 1, 2, \cdots, m)$ be complex numbers such that at lest one of them is nonzero, and let $\varphi(z) = f(z+c_1)f(z+c_2) \cdots f(z+c_m) - \alpha f^n(z)$. If $N(r, f) + N\left(r, \frac{1}{f-b}\right) = S(r, f)$ and $\varphi \not\equiv b^m - \alpha b^n$, then*

(i) for $n \neq m$, φ has infinitely many zeros and $\lambda(\varphi) = \rho(f)$;

(ii) $\varphi - c$ has infinitely many zeros and $\lambda(\varphi - c) = \rho(f)$.

The following examples show that $\varphi \not\equiv b^m - \alpha b^n$ is necessary in Theorem 1.4.

Example 1.1. Let $b = 0$, let $f = e^z$, and let $m = n = 2$, $\alpha = 1$, $c_1 = \frac{5\pi}{4}i$, $c_2 = \frac{3\pi}{4}i$, then $\varphi(z) \equiv 0 \neq c$. Hence $\varphi - c$ does not have zeros.

Example 1.2. Let $b = 2$, let $f = e^{2z} + 2$, and let $m = n = 2$, $\alpha = 1$, $c_1 = \pi i$, $c_2 = 2\pi i$, then $\varphi(z) \equiv 0 \neq c$. Hence $\varphi - c$ does not have zeros.

Example 1.3. Let $b = 2$, let $f = e^z + 2$, and let $m = n = 1$, $\alpha = -1$, $c = 1$, $c_1 = \pi i$, then $\varphi(z) \equiv 4 \neq 1$. Hence $\varphi - 1$ does not have zeros.

Corollary 1.1. *Let f be a transcendental entire function of finite order, let m, n be two distinct positive integers, let $a(\neq 0), c$ be two complex numbers, let $c_1, c_2, \cdots, c_m$ be complex numbers such that at least one of them is nonzero, and let $\varphi(z) = f(z+c_1)f(z+c_2)\cdots f(z+c_m) - af^n(z)$. If $N\left(r, \frac{1}{f}\right) = S(r, f)$, then $\varphi - c$ has infinitely many zeros and $\lambda(\varphi - c) = \rho(f)$.*

Corollary 1.2. *Let f be a transcendental entire function of finite order with a Borel exceptional value b, let m, n be two distinct positive integers, let $a(\neq 0), c(\neq b^m - ab^n)$ be two complex numbers, let $c_1, c_2, \cdots, c_m$ be complex numbers such that at least one of them is nonzero, and let $\varphi(z) = f(z+c_1)f(z+c_2)\cdots f(z+c_m) - af^n(z)$. Then $\varphi - c$ has infinitely many zeros and $\lambda(\varphi - c) = \rho(f)$.*

Remark 1.1. Corollary 1.1 improves Theorem 1.2, Corollary 1.2 improves Theorem 1.3.

In 1959, Hayman [13] proved the following theorem.

Theorem 1.5. *Let f be a transcendental entire (meromorphic) function, let a be a nonzero finite complex number, and let n be a positive integer. If $n \geq 2$ $(n \geq 3)$, then $f^n f' - a$ has infinitely many zeros.*

Clunie [7, 8], Mues [20], Bergweiler and Eremenko [2], Chen and Fang [4], Zalcman [25] proved:

Theorem 1.6 ([2, 4]). *Let f be a transcendental meromorphic function, let a be a nonzero finite complex number. Then $f^n f' - a$ has infinitely many zeros.*

In 2007, Laine and Yang [17] obtained the difference analogue to Theorem 1.6 and proved:

Theorem 1.7. *Let f be a transcendental entire function of finite order, let a, c be two nonzero finite complex numbers, and let n be a positive integer. If $n \geq 2$, then $f^n(z)f(z+c) - a$ has infinitely many zeros.*

In 2011, Liu et al. [18] considered the case of meromorphic function and proved the following result.

Theorem 1.8. *Let f be a transcendental meromorphic function of finite order, let $\alpha(\not\equiv 0)$ be a small function of f, let c be a nonzero finite complex number, and let n be a positive integer. If $n \geq 2$, then $f^n(z)f(z+c) - \alpha(z)$ has infinitely many zeros.*

The following example shows that Theorem 1.7 and Theorem 1.8 do not valid if $n = 1$.

Example 1.4. Let $f = e^z + 1$, and let $n = 1$, $\alpha = 1$, $c = \pi i$, then $f(z)f(z+\pi i) - 1 = -e^{2z}$. Hence $f^n(z)f(z+c) - 1$ does not have zeros.

In 2019, Zhang and Huang [26] proved:

Theorem 1.9. *Let f be a transcendental meromorphic function of finite order, let c be a nonzero complex number, let n be a positive integer, let a, b be two distinct Borel exceptional values of f on extend complex plane, and let $\alpha(\not\equiv 0)$ be a small function of f. If $n \geq 2$ and one of the following conditions is satisfied:*

(i) $a, b \in \mathbb{C}, a^{n+1} - \alpha \not\equiv 0$ *and* $b^{n+1} - \alpha \not\equiv 0$;

(ii) $a \in \mathbb{C}, b = \infty, a^{n+1} - \alpha \not\equiv 0$,

then $f^n(z)f(z+c)-\alpha(z)$ *has infinitely many zeros.*

According to the above theorems and Example 1.4, we naturally pose the following problem.

Problem 1.2. *Whether Theorem 1.9 is valid or not for* $n=1$*?*

In this paper, we give a positive answer to Problem 1.2 and prove the following result.

Theorem 1.10. *Let* f *be a transcendental meromorphic function of finite order, let* c *be a nonzero complex number, let* a, b *be two distinct Borel exceptional values of* f *on extend complex plane, let* n *be a positive integer, let* $\alpha(\not\equiv 0)$ *be a small function of* f*, and let* $\varphi_1(z)=f^n(z)f(z+c)$*. If one of the following conditions is satisfied:*

(i) $a, b\in\mathbb{C}, a^{n+1}-\alpha\not\equiv 0$ *and* $b^{n+1}-\alpha\not\equiv 0$;

(ii) $a\in\mathbb{C}, b=\infty, a^{n+1}-\alpha\not\equiv 0$,

then $\varphi_1-\alpha$ *has infinitely many zeros and* $\lambda(\varphi_1-\alpha)=\rho(f)$*.*

2. Some Lemmas

Lemma 2.1 ([5, 9])**.** *Let* f *be a trancendental meromorphic function of finite order, and let* c *be a nonzero complex number. Then*

$$m\left(r,\frac{f(z+c)}{f(z)}\right)=S(r,f).$$

Lemma 2.2 ([5, 11])**.** *Let* f *be a trancendental meromorphic function of finite order, and let* c *be a nonzero complex number. Then*

$$N(r,f(z+c))=N(r,f(z))+S(r,f),$$
$$N\left(r,\frac{1}{f(z+c)}\right)=N\left(r,\frac{1}{f(z)}\right)+S(r,f).$$

Lemma 2.3 ([14])**.** *Let* f *be a trancendental meromorphic function, and let* α,β *be two distinct small functions of* f*. Then*

$$T(r,f)\le\overline{N}(r,f)+\overline{N}\left(r,\frac{1}{f-\alpha}\right)+\overline{N}\left(r,\frac{1}{f-\beta}\right)+S(r,f).$$

Lemma 2.4 ([19])**.** *Let* f *be a nonconstant meromorphic function and* $R(f)=\frac{P(f)}{Q(f)}$*, where* $P(f)=\sum\limits_{i=0}^{p}\alpha_i f^i$ *and* $Q(f)=\sum\limits_{j=0}^{q}\beta_j f^j$ *are two mutually prime polynomials in* f*. If the coefficients* $\{\alpha_i(z)\}$*,* $\{\beta_j(z)\}$ *are small functions of* f *and* $\alpha_p(z)\not\equiv 0$*,* $\beta_q(z)\not\equiv 0$*, then*

$$T(r,R(f))=\max\{p,q\}\cdot T(r,f)+S(r,f).$$

Lemma 2.5 ([22])**.** *Let* f *be a trancendental meromorphic function with* $\rho(f)>0$*, and let* a, b *be two distinct Borel exceptional values of* f*. Then*

$$N\left(r,\frac{1}{f-a}\right)=S(r,f),\ N\left(r,\frac{1}{f-b}\right)=S(r,f).$$

Remark 2.1. For $\rho(f)=0$, Lemma 2.5 is still valid.

3. Proof of Theorem 1.4

We consider two cases.

Case 1. $b = 0$. Then we obtain

$$N(r, f) + N\left(r, \frac{1}{f}\right) = S(r, f). \tag{3.1}$$

Now, we consider three subcases.

Case 1.1. $m = n$. Then we have

$$\varphi(z) = \left[\frac{f(z+c_1)}{f(z)}\frac{f(z+c_2)}{f(z)}\cdots\frac{f(z+c_n)}{f(z)} - \alpha(z)\right] f^n(z) = A(z)f^n(z), \tag{3.2}$$

where $A(z) = \frac{f(z+c_1)}{f(z)}\frac{f(z+c_2)}{f(z)}\cdots\frac{f(z+c_n)}{f(z)} - \alpha(z)$.

Since α is a small function of f, then by (3.1), Lemma 2.1 and Lemma 2.2, we obtain

$$\begin{aligned} T(r, A) \leq & T\left(r, \frac{f(z+c_1)}{f(z)}\cdots\frac{f(z+c_n)}{f(z)}\right) + T(r, \alpha) + S(r, f) \\ \leq & N\left(r, \frac{f(z+c_1)}{f(z)}\frac{f(z+c_2)}{f(z)}\cdots\frac{f(z+c_n)}{f(z)}\right) + S(r, f) \\ \leq & \sum_{i=1}^{n} N(r, f(z+c_i)) + nN\left(r, \frac{1}{f(z)}\right) + S(r, f) \\ \leq & nN(r, f) + nN\left(r, \frac{1}{f}\right) + S(r, f) \leq S(r, f). \end{aligned} \tag{3.3}$$

Hence, A is a small function of f.

It follows from (3.2), (3.3) and Lemma 2.4 that

$$T(r, \varphi) = nT(r, f) + S(r, f). \tag{3.4}$$

Let $c(\not\equiv 0)$ be a small function of f, then by (3.4), we know that c is a small function of φ. Hence, by (3.1), (3.2), (3.3) and Lemma 2.3, we have

$$\begin{aligned} T(r, \varphi) \leq & \overline{N}(r, \varphi) + \overline{N}\left(r, \frac{1}{\varphi}\right) + \overline{N}\left(r, \frac{1}{\varphi - c}\right) + S(r, \varphi) \\ \leq & \overline{N}\left(r, \frac{1}{\varphi - c}\right) + S(r, \varphi). \end{aligned} \tag{3.5}$$

It follows that $\varphi - c$ has infinitely many zeros.

By (3.4) and (3.5), we obtain

$$T(r, f) \leq \frac{1}{n}\overline{N}\left(r, \frac{1}{\varphi - c}\right) + S(r, f).$$

Hence, we obtain $\lambda(\varphi - c) = \rho(f)$.

Case 1.2. $m > n$. Then, we have

$$\varphi(z) = \left[\frac{f(z+c_1)}{f(z)}\frac{f(z+c_2)}{f(z)}\cdots\frac{f(z+c_m)}{f(z)}\right] f^m(z) - \alpha(z)f^n(z)$$

$$=B(z)f^m(z)-\alpha(z)f^n(z), \tag{3.6}$$

where $B(z)=\frac{f(z+c_1)}{f(z)}\frac{f(z+c_2)}{f(z)}\cdots\frac{f(z+c_m)}{f(z)}$.

Obviously, $B\not\equiv 0$. Since α is a small function of f, then by (3.1), Lemma 2.1 and Lemma 2.2, we obtain

$$\begin{aligned}
T(r,B)&=m(r,B)+N(r,B)\\
&\le N\left(r,\frac{f(z+c_1)}{f(z)}\frac{f(z+c_2)}{f(z)}\cdots\frac{f(z+c_m)}{f(z)}\right)+S(r,f)\\
&\le\sum_{i=1}^{m}N(r,f(z+c_i))+mN\left(r,\frac{1}{f(z)}\right)+S(r,f)\\
&\le mN(r,f)+mN\left(r,\frac{1}{f}\right)+S(r,f)\le S(r,f).
\end{aligned} \tag{3.7}$$

Thus, B is a small function of f.

By (3.6), (3.7) and Lemma 2.4, we obtain

$$T(r,\varphi)=mT(r,f)+S(r,f). \tag{3.8}$$

Now, we prove conclusion (i). By (3.6), we have

$$\varphi(z)=f^n(z)(B(z)f^{m-n}(z)-\alpha(z)). \tag{3.9}$$

It follows from (3.1), (3.9), Lemma 2.3 and Lemma 2.4 that

$$\begin{aligned}
(m-n)T(r,f)&=T(r,f^{m-n})+S(r,f)\\
&\le\overline{N}(r,f^{m-n})+\overline{N}\left(r,\frac{1}{f^{m-n}}\right)+\overline{N}\left(r,\frac{1}{f^{m-n}-\frac{\alpha}{B}}\right)+S(r,f)\\
&\le\overline{N}\left(r,\frac{1}{f^{m-n}-\frac{\alpha}{B}}\right)+S(r,f)\le\overline{N}\left(r,\frac{1}{\varphi}\right)+S(r,f).
\end{aligned} \tag{3.10}$$

By (3.8) and (3.10), we obtain

$$T(r,\varphi)=mT(r,f)\le\frac{m}{m-n}\overline{N}\left(r,\frac{1}{\varphi}\right)+S(r,f).$$

It follows that φ has infinitely many zeros and $\lambda(\varphi)=\rho(f)$.

Thus, conclusion (i) is proved for Case 1.2. Next, we prove conclusion (ii).

Let $c(\not\equiv 0)$ be a small function of f, then by (3.8), we know that c is a small function of φ. Hence, by (3.1), (3.6) and Lemma 2.3, we have

$$\begin{aligned}
T(r,\varphi)\le&\overline{N}(r,\varphi)+\overline{N}\left(r,\frac{1}{\varphi}\right)+\overline{N}\left(r,\frac{1}{\varphi-c}\right)+S(r,\varphi)\\
\le&\overline{N}\left(r,\frac{1}{Bf^m-\alpha f^n}\right)+\overline{N}\left(r,\frac{1}{\varphi-c}\right)+S(r,\varphi)\\
\le&\overline{N}\left(r,\frac{1}{f^n(Bf^{m-n}-\alpha)}\right)+\overline{N}\left(r,\frac{1}{\varphi-c}\right)+S(r,\varphi)\\
\le&(m-n)T(r,f)+\overline{N}\left(r,\frac{1}{\varphi-c}\right)+S(r,\varphi).
\end{aligned} \tag{3.11}$$

By (3.8) and (3.11), we have

$$T(r,\varphi) \le \frac{m}{n}\overline{N}\left(r,\frac{1}{\varphi - c}\right) + S(r,\varphi). \tag{3.12}$$

It follows that $\varphi - c$ has infinitely many zeros.

By (3.8) and (3.12), we have

$$T(r,f) \le \frac{1}{n}\overline{N}\left(r,\frac{1}{\varphi - c}\right) + S(r,f).$$

Hence, we obtain $\lambda(\varphi - c) = \rho(f)$.

Case 1.3. $m < n$. By using the same argument as used in Case 1.2, we prove that Theorem 1.4 is valid for this case.

Case 2. $b \ne 0$. Then we have

$$\begin{aligned}\varphi(z) =& f(z+c_1)f(z+c_2)\cdots f(z+c_m) - \alpha(z)f^n(z)\\ =& [(f(z+c_1)-b)+b]\cdots[(f(z+c_m)-b)+b] - \alpha(z)[(f(z)-b)+b]^n\\ =& [f(z+c_1)-b]\cdots[f(z+c_m)-b] + \cdots + \sum_{i=1}^{m} b^{m-1}\left(f(z+c_i)-b\right) + b^m\\ & - \alpha(z)[(f(z)-b)^n + nb(f(z)-b)^{n-1} + \cdots + nb^{n-1}(f(z)-b) + b^n].\end{aligned} \tag{3.13}$$

Set

$$g(z) = f(z) - b. \tag{3.14}$$

Thus, we have

$$T(r,f) = T(r,g) + S(r,g). \tag{3.15}$$

It follows from $N(r,f) + N\left(r,\frac{1}{f-b}\right) = S(r,f)$ that

$$N(r,g) + N\left(r,\frac{1}{g}\right) = S(r,g). \tag{3.16}$$

By (3.13) and (3.14), we have

$$\begin{aligned}\varphi(z) =& g(z+c_1)g(z+c_2)\cdots g(z+c_m)\\ & + b\left(\sum_{i=1}^{m}\frac{g(z+c_1)g(z+c_2)\cdots g(z+c_m)}{g(z+c_i)}\right)\\ & + \cdots + b^{m-1}\left(\sum_{i=1}^{m} g(z+c_i)\right) + b^m\\ & - \alpha(z)(g^n(z) + nbg^{n-1}(z) + \cdots + nb^{n-1}g(z) + b^n)\\ =& g^m(z)\left(\frac{g(z+c_1)g(z+c_2)\cdots g(z+c_m)}{g^m(z)}\right)\\ & + bg^{m-1}(z)\left(\sum_{i=1}^{m}\frac{g(z+c_1)g(z+c_2)\cdots g(z+c_m)}{g(z+c_i)g^{m-1}(z)}\right)\end{aligned}$$

$$
\begin{aligned}
&+\cdots+b^{m-1}g(z)\left(\sum_{i=1}^{m}\frac{g(z+c_i)}{g(z)}\right)+b^m\\
&-\alpha(z)(g^n(z)+nbg^{n-1}(z)+\cdots+nb^{n-1}g(z)+b^n)\\
=&b_0(z)g^m(z)+b_1(z)g^{m-1}(z)+\cdots+b_{m-1}(z)g(z)+b^m\\
&-\alpha(z)(g^n(z)+nbg^{n-1}(z)+\cdots+nb^{n-1}g(z)+b^n),
\end{aligned}
\tag{3.17}
$$

where $b_0(z)=\frac{g(z+c_1)g(z+c_2)\cdots g(z+c_m)}{g^m(z)},\cdots,\quad b_{m-2}(z)=\sum_{i\neq j}b^{m-2}\frac{g(z+c_i)g(z+c_j)}{g^2(z)}$, $b_{m-1}(z)=\sum_{i=1}^{m}b^{m-1}\frac{g(z+c_i)}{g(z)}(i,j=1,2,\cdots,m)$.

By (3.16), Lemma 2.1 and Lemma 2.2, we obtain

$$
\begin{aligned}
T(r,b_0)&=m(r,b_0)+N(r,b_0)\\
&\leq N\left(r,\frac{g(z+c_1)}{g(z)}\frac{g(z+c_2)}{g(z)}\cdots\frac{g(z+c_m)}{g(z)}\right)+S(r,g)\\
&\leq\sum_{i=1}^{m}N(r,g(z+c_i))+mN\left(r,\frac{1}{g(z)}\right)+S(r,g)\\
&\leq mN(r,g)+mN\left(r,\frac{1}{g}\right)+S(r,g)\leq S(r,g).
\end{aligned}
\tag{3.18}
$$

Thus, b_0 is a small function of g. Similarly, we deduce that $b_j(j=1,2,\cdots,m-1)$ are small functions of g.

Since $\varphi\not\equiv b^m-\alpha b^n$, then by (3.17) and (3.18), we have

$$
\varphi(z)=\sum_{i=1}^{s}d_i(z)g^{m_i}(z)+b^m-\alpha(z)b^n,
\tag{3.19}
$$

where $s(\leq\max\{m,n\})$ is a positive integer, $m_i(i=1,2,\cdots,s)$ are positive integers with $m_1<m_2<\cdots<m_s$, and $d_i(z)(\not\equiv 0)(i=1,2,\cdots,s)$ are small functions of f such that $\sum_{i=1}^{s}d_i(z)g^{m_i}(z)\not\equiv 0$.

By (3.19) and Lemma 2.4, we obtain

$$
T(r,\varphi)=m_sT(r,g)+S(r,g).
\tag{3.20}
$$

Next, we prove conclusion (i). It follows from $n\neq m$ that $2\leq s\leq\max\{m,n\}$. In the following, we consider two subcases.

Case 2.1 $b^m-\alpha(z)b^n\equiv 0$. By (3.19), we have

$$
\begin{aligned}
\varphi(z)=&\sum_{i=1}^{s}d_i(z)g^{m_i}(z)\\
=&g^{m_1}(z)(d_s(z)g^{m_s-m_1}(z)+\cdots+d_2(z)g^{m_2-m_1}+d_1(z)).
\end{aligned}
\tag{3.21}
$$

It follows from (3.21), Lemma 2.3 and Lemma 2.4 that

$$
\begin{aligned}
&(m_s-m_1)T(r,g)\\
=&T(r,d_sg^{m_s-m_1}+\cdots+d_2g^{m_2-m_1}+d_1)\\
\leq&\overline{N}(r,d_sg^{m_s-m_1}+\cdots+d_2g^{m_2-m_1}+d_1)
\end{aligned}
$$

$$
\begin{aligned}
&+\overline{N}\left(r,\frac{1}{d_sg^{m_s-m_1}+\cdots+d_2g^{m_2-m_1}+d_1-d_1}\right)\\
&+\overline{N}\left(r,\frac{1}{d_sg^{m_s-m_1}+\cdots+d_2g^{m_2-m_1}+d_1}\right)+S(r,g)\\
\le&\overline{N}\left(r,\frac{1}{g^{m_2-m_1}(d_sg^{m_s-m_2}+\cdots+d_2)}\right)\\
&+\overline{N}\left(r,\frac{1}{d_sg^{m_s-m_1}+\cdots+d_2g^{m_2-m_1}+d_1}\right)+S(r,g)\\
\le&(m_s-m_2)T(r,g)+\overline{N}\left(r,\frac{1}{d_sg^{m_s-m_1}+\cdots+d_2g^{m_2-m_1}+d_1}\right)+S(r,g)\\
\le&(m_s-m_2)T(r,g)+\overline{N}\left(r,\frac{1}{\varphi}\right)+S(r,g). \qquad (3.22)
\end{aligned}
$$

By (3.15), (3.20) and (3.22), we have

$$
T(r,\varphi)=m_sT(r,f)+S(r,f)\le\frac{m_s}{m_2-m_1}\overline{N}\left(r,\frac{1}{\varphi}\right)+S(r,f).
$$

It follows from $m_2>m_1$ that φ has infinitely many zeros and $\lambda(\varphi)=\rho(f)$.

Case 2.2. $b^m-\alpha(z)b^n\not\equiv 0$. It follows from (3.19) and Lemma 2.4 that

$$
\begin{aligned}
m_sT(r,g)=&T\left(r,\sum_{i=1}^{s}d_ig^{m_i}+b^m-\alpha b^n\right)\\
\le&\overline{N}\left(r,\sum_{i=1}^{s}d_ig^{m_i}+b^m-\alpha b^n\right)+\overline{N}\left(r,\frac{1}{\sum_{i=1}^{s}d_ig^{m_i}+b^m-\alpha b^n}\right)\\
&+\overline{N}\left(r,\frac{1}{\sum_{i=1}^{s}d_ig^{m_i}+b^m-\alpha b^n-(b^m-\alpha b^n)}\right)+S(r,g)\\
\le&(m_s-m_1)T(r,g)+\overline{N}\left(r,\frac{1}{\sum_{i=1}^{s}d_ig^{m_i}+b^m-\alpha b^n}\right)+S(r,g)\\
\le&(m_s-m_1)T(r,g)+\overline{N}\left(r,\frac{1}{\varphi}\right)+S(r,g). \qquad (3.23)
\end{aligned}
$$

By (3.15), (3.20) and (3.23), we have

$$
T(r,\varphi)\le m_sT(r,f)+S(r,f)\le\frac{m_s}{m_1}\overline{N}\left(r,\frac{1}{\varphi}\right)+S(r,f).
$$

It follows that φ has infinitely many zeros and $\lambda(\varphi)=\rho(f)$.

Thus, conclusion (i) is proved for Case 2. Next, we prove conclusion (ii).

Let $c(\not\equiv b^m-\alpha b^n)$ be a small function of f, then by (3.15) and (3.20), we know that c is a small function of φ. Hence, by (3.16), (3.19) and Lemma 2.3, we have

$$
\begin{aligned}
T(r,\varphi)\le&\overline{N}(r,\varphi)+\overline{N}\left(r,\frac{1}{\varphi-(b^m-\alpha b^n)}\right)+\overline{N}\left(r,\frac{1}{\varphi-c}\right)+S(r,\varphi)\\
\le&\overline{N}\left(r,\frac{1}{d_1g^{m_1}+d_2g^{m_2}+\cdots+d_sg^{m_s}}\right)+\overline{N}\left(r,\frac{1}{\varphi-c}\right)+S(r,\varphi)
\end{aligned}
$$

$$\leq \overline{N}\left(r, \frac{1}{g^{m_1}(d_1+d_2g^{m_2-m_1}+\cdots+d_sg^{m_s-m_1})}\right)+\overline{N}\left(r, \frac{1}{\varphi - c}\right)+S(r,\varphi)$$

$$\leq (m_s - m_1)T(r,g) + \overline{N}\left(r, \frac{1}{\varphi - c}\right) + S(r,\varphi). \tag{3.24}$$

By (3.20) and (3.24), we have

$$T(r,\varphi) \leq \frac{m_s}{m_1}\overline{N}\left(r, \frac{1}{\varphi - c}\right) + S(r,\varphi). \tag{3.25}$$

It follow that $\varphi - c$ has infinitely many zeros.

By (3.15), (3.20) and (3.25), we have

$$T(r,f) \leq \frac{1}{m_1}\overline{N}\left(r, \frac{1}{\varphi - c}\right) + S(r,f).$$

Hence, we obtain $\lambda(\varphi - c) = \rho(f)$. Thus the conclusion (ii) is proved.

This completes the proof of Theorem 1.4.

4. Proof of Theorem 1.10

Now, we prove the case of $n = 1$.

We assume that $a, b \in \mathbb{C}, a^2 - \alpha \not\equiv 0$ and $b^2 - \alpha \not\equiv 0$.

Since a, b are two distinct Borel exceptional values of f, then by Lemma 2.5, we obtain

$$N\left(r, \frac{1}{f-a}\right) = S(r,f), N\left(r, \frac{1}{f-b}\right) = S(r,f). \tag{4.1}$$

Set

$$H(z) = \frac{f(z) - a}{f(z) - b}. \tag{4.2}$$

By (4.2), we obtain

$$f(z) = \frac{a - bH(z)}{1 - H(z)}. \tag{4.3}$$

Thus, we have

$$T(r,f) = T(r,H) + S(r,H). \tag{4.4}$$

Obviously, $H \not\equiv 0, 1, \infty$. It follows from (4.1), (4.2) and (4.4) that

$$N(r,H) = S(r,H), N\left(r, \frac{1}{H}\right) = S(r,H). \tag{4.5}$$

By (4.3), we have

$$\begin{aligned}\varphi_1(z) - \alpha(z) &= \frac{a - bH(z)}{1 - H(z)}\frac{a - bH(z+c)}{1 - H(z+c)} - \alpha(z) \\ &= \frac{(b^2 - \alpha(z))A(z)H^2(z) - (ab - \alpha(z))(A(z)+1)H(z) + a^2 - \alpha(z)}{A(z)H^2(z) - (A(z)+1)H(z) + 1},\end{aligned} \tag{4.6}$$

where $A(z) = \frac{H(z+c)}{H(z)}$. By Lemma 2.1 and (4.5), we obtain that A is a small function of H.

Next, we consider two cases.

Case 1. $(b^2-\alpha(z))A(z)H^2(z)-(ab-\alpha(z))(A(z)+1)H(z)+a^2-\alpha(z)$ and $A(z)H^2(z)-(A(z)+1)H(z)+1$ are two mutually prime polynomials.

By (4.6) and Lemma 2.4, we have

$$T(r,\varphi_1) = T\left(r, \frac{b^2AH^2-ab(A+1)H+a^2}{AH^2-(A+1)H+1}\right) = 2T(r,H)+S(r,H). \tag{4.7}$$

Since $b^2 \not\equiv \alpha$, $a^2 \not\equiv \alpha$, then by (4.5), Lemma 2.3 and Lemma 2.4, we obtain

$$\begin{aligned}
2T(r,H) =& T\left(r,(b^2-\alpha)AH^2-(ab-\alpha)(A+1)H+a^2-\alpha\right)+S(r,H)\\
\le& \overline{N}(r,(b^2-\alpha)AH^2-(ab-\alpha)(A+1)H+a^2-\alpha)\\
&+\overline{N}\left(r,\frac{1}{(b^2-\alpha)AH^2-(ab-\alpha)(A+1)H+a^2-\alpha-(a^2-\alpha)}\right)\\
&+\overline{N}\left(r,\frac{1}{(b^2-\alpha)AH^2-(ab-\alpha)(A+1)H+a^2-\alpha}\right)+S(r,H)\\
\le& \overline{N}\left(r,\frac{1}{H\left[(b^2-\alpha)AH-(ab-\alpha)(A+1)\right]}\right)\\
&+\overline{N}\left(r,\frac{1}{(b^2-\alpha)AH^2-(ab-\alpha)(A+1)H+a^2-\alpha}\right)+S(r,H)\\
\le& T(r,H)+\overline{N}\left(r,\frac{1}{(b^2-\alpha)AH^2-(ab-\alpha)(A+1)H+a^2-\alpha}\right)+S(r,H)\\
\le& T(r,H)+\overline{N}\left(r,\frac{1}{\varphi_1-\alpha}\right)+S(r,H).
\end{aligned} \tag{4.8}$$

By (4.7) and (4.8), we have

$$T(r,\varphi_1) \le 2\overline{N}\left(r,\frac{1}{\varphi_1-\alpha}\right)+S(r,\varphi_1).$$

It follows that $\varphi_1-\alpha$ has infinitely many zeros.

By (4.4) and (4.8), we obtain

$$T(r,f) \le \overline{N}\left(r,\frac{1}{\varphi_1-\alpha}\right)+S(r,f).$$

Hence, we obtain $\lambda(\varphi_1-\alpha)=\rho(f)$.

Case 2. $[b^2-\alpha(z)]A(z)H^2(z)-[(ab-\alpha)(A(z)+1)]H(z)+[a^2-\alpha(z)]$ and $A(z)H^2(z)-(A(z)+1)H(z)+1$ have common factor $\gamma(z)$.

In the following, we consider two subcases.

Case 2.1. γ is a polynomial of H with $\deg\gamma = 1$.

By (4.6), we have

$$\varphi_1(z)-\alpha(z) = \frac{C_1(z)H(z)+D_1(z)}{A_1(z)H(z)+B_1(z)}, \tag{4.9}$$

where A_1, B_1, C_1, D_1 are small functions of H.

Since $a^2 \not\equiv \alpha$, then by (4.6) and (4.9), we deduce that $D_1 \not\equiv 0$. Similarly, we obtain $C_1 \not\equiv 0$. By using the same argument as used in Case 1, we prove that $\varphi_1 - \alpha$ has infinitely many zeros and $\lambda(\varphi_1 - \alpha) = \rho(f)$.

Case 2.2. γ is a polynomial of H with $\deg \gamma = 2$.

By (4.6), we obtain

$$\varphi_1(z) - \alpha(z) = B(z), \tag{4.10}$$

where B is a small function of H.

It follows

$$\varphi_1(z) = B_2(z), \tag{4.11}$$

where $B_2 = B + \alpha$ is a small function of H.

We claim that $B_2 \not\equiv 0$. Otherwise, it follows from (4.10) that $f \equiv 0$, a contradiction.

By (4.3), we have

$$\varphi_1(z) = \frac{a^2 - abH(z) - abH(z+c) + b^2 H(z)H(z+c)}{1 - H(z) - H(z+c) + H(z)H(z+c)}. \tag{4.12}$$

It follows from (4.11) and (4.12) that

$$\begin{aligned} & a^2 - \left(ab + ab\frac{H(z+c)}{H(z)}\right) H(z) + b^2 \frac{H(z+c)}{H(z)} H^2(z) \\ =& B_2(z) - \left(B_2(z) + B_2(z)\frac{H(z+c)}{H(z)}\right) H(z) + B_2(z)\frac{H(z+c)}{H(z)} H^2(z). \end{aligned} \tag{4.13}$$

Thus, we obtain

$$a^2 = B_2(z), \tag{4.14}$$

$$ab = B_2(z), \tag{4.15}$$

$$b^2 = B_2(z). \tag{4.16}$$

By (4.14) and (4.15), we have $a = b$, a contradiction. Hence we prove that Theorem 1.10 is valid for $a, b \in \mathbb{C}, a^2 - \alpha \not\equiv 0$ and $b^2 - \alpha \not\equiv 0$.

Next, we assume that $a \in \mathbb{C}, b = \infty, a^2 - \alpha \not\equiv 0$. Then, we have

$$\varphi_1(z) = [(f(z) - a) + a][(f(z+c) - a) + a]. \tag{4.17}$$

Set

$$g(z) = f(z) - a. \tag{4.18}$$

Thus, we obtain

$$T(r, f) = T(r, g) + S(r, g). \tag{4.19}$$

Since a, ∞ are two distinct Borel exceptional values of f, then by Lemma 2.5, we obtain

$$N\left(r, \frac{1}{g}\right) = S(r, g), N(r, g) = S(r, g). \tag{4.20}$$

By (4.17) and (4.18), we have

$$\begin{aligned} \varphi_1(z) &= [g(z) + a][g(z+c) + a] \\ &= g(z)g(z+c) + ag(z) + ag(z+c) + a^2 \end{aligned}$$

$$= \frac{g(z+c)}{g(z)} g^2(z) + a\left(\frac{g(z+c)}{g(z)} + 1\right) g(z) + a^2. \tag{4.21}$$

It follows from (4.20), Lemma 2.1 and Lemma 2.2 that

$$T\left(r, \frac{g(z+c)}{g(z)}\right) = m\left(r, \frac{g(z+c)}{g(z)}\right) + N\left(r, \frac{g(z+c)}{g(z)}\right) \leq S(r, g). \tag{4.22}$$

By (4.21) and (4.22), we have

$$\varphi_1(z) = \alpha_2(z) g^2(z) + \alpha_1(z) g(z) + \alpha_0, \tag{4.23}$$

where $\alpha_2(z) = \frac{g(z+c)}{g(z)}$, $\alpha_1(z) = a\left(\frac{g(z+c)}{g(z)} + 1\right)$, $\alpha_0(z) = a^2$ are small functions of $g(z)$.

By (4.23) and Lemma 2.4, we obtain

$$T(r, \varphi_1) = 2T(r, g) + S(r, g). \tag{4.24}$$

Since $\alpha \not\equiv a^2$, then by (4.20), (4.23) and Lemma 2.3, we obtain

$$\begin{aligned} T(r, \varphi_1) &\leq \overline{N}(r, \varphi_1) + \overline{N}\left(r, \frac{1}{\varphi_1 - \alpha_0}\right) + \overline{N}\left(r, \frac{1}{\varphi_1 - \alpha}\right) + S(r, \varphi_1) \\ &\leq \overline{N}\left(r, \frac{1}{\alpha_2 g^2 + \alpha_1 g}\right) + \overline{N}\left(r, \frac{1}{\varphi_1 - \alpha}\right) + S(r, \varphi_1) \\ &\leq \overline{N}\left(r, \frac{1}{g(\alpha_2 g + \alpha_1)}\right) + \overline{N}\left(r, \frac{1}{\varphi_1 - \alpha}\right) + S(r, \varphi_1) \\ &\leq T(r, g) + \overline{N}\left(r, \frac{1}{\varphi_1 - \alpha}\right) + S(r, \varphi_1). \end{aligned} \tag{4.25}$$

By (4.24) and (4.25), we have

$$T(r, \varphi_1) \leq 2\overline{N}\left(r, \frac{1}{\varphi_1 - \alpha}\right) + S(r, \varphi_1). \tag{4.26}$$

Thus, we deduce that $\varphi_1 - \alpha$ has infinitely many zeros.

By (4.19), (4.24) and (4.26), we obtain

$$T(r, f) \leq \overline{N}\left(r, \frac{1}{\varphi_1 - \alpha}\right) + S(r, f).$$

It follow $\lambda(\varphi_1 - \alpha) = \rho(f)$. Hence we prove that Theorem 1.10 is valid for $a \in \mathbb{C}, b = \infty, a^2 - \alpha \not\equiv 0$.

Similarly, we prove that Theorem 1.10 is valid for $n \geq 2$.

This completes the proof of Theorem 1.10.

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