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SECOND ORDER MELNIKOV FUNCTIONS FOR PLANAR PIECEWISE SMOOTH INTEGRABLE NON-HAMILTONIAN SYSTEMS WITH MULTIPLE ZONES AND APPLICATION

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Abstract In this paper, we study the expressions of the second order Melnikov functions for planar piecewise smooth integrable non-Hamiltonian systems with $m(\in \mathbb{N})$ zones separated by half straight lines through the origin under the piecewise perturbations. As an application, it is proved that there exists a planar piecewise quadratic system having at least 8 limit cycles near the origin.

Keywords Piecewise smooth integrable non-Hamiltonian system, second order Melnikov function, limit cycle, bifurcation.

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1. Introduction

In the qualitative theory of real planar polynomial differential systems, a very important topic is to determine the number and distribution of limit cycles. Recently, the interest in non-smooth differential systems has grown rapidly since a lot of problems on engineering, physical, biological, and real processes are naturally modeled by this class of differential systems. As in the smooth case, a very important problem is to determine the number of limit cycles and their distributions in the study of non-smooth differential systems. One can see [1–5, 7–32] and references therein.

For the piecewise smooth systems, the averaging theory and the Melnikov function theory are two main methodologies, which are used to study the number of limit cycles.

We shall recall the developments on the Melnikov function theory for planar piecewise smooth system. Let us consider the following system:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{cases} \begin{pmatrix} \frac{H_y^+(x,y)}{\mu_+(x,y)} + \epsilon f_1^+(x,y) + \epsilon^2 f_2^+(x,y) \\ -\frac{H_x^+(x,y)}{\mu_+(x,y)} + \epsilon g_1^+(x,y) + \epsilon^2 g_2^+(x,y) \end{pmatrix}, & x > 0, \\ \begin{pmatrix} \frac{H_y^-(x,y)}{\mu_-(x,y)} + \epsilon f_1^-(x,y) + \epsilon^2 f_2^-(x,y) \\ -\frac{H_x^-(x,y)}{\mu_-(x,y)} + \epsilon g_1^-(x,y) + \epsilon^2 g_2^-(x,y) \end{pmatrix}, & x \le 0, \end{cases} \tag{1.1}$$

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where $0 < |\epsilon| \ll 1$, $H^{\pm}(x,y)$, $H_y^{\pm}(x,y)$, $H_x^{\pm}(x,y)$, $f_i^{\pm}(x,y)$ and $g_i^{\pm}(x,y)$ are C^{∞} functions with $i = 1,2$, and $\mu_{\pm}(x,y)$ are the integrating factors for the right subsystem and the left subsystem respectively with $\mu_{\pm}(0,0) \neq 0$. We make the following three assumptions [25].

($\hat{\mathbf{A}}_{\mathbf{1}}$) There exists an open nonempty interval $J = (\alpha, \beta)$, and two points $A_1(h) = (0, a_1(h))$ and $A_3(h) = (0, a_3(h))$ such that for $h \in J$,

$$H^+(A_1(h)) = H^+(A_3(h)) = h, H^-(A_1(h)) = H^-(A_3(h)),$$

where $a_1(h) < 0$ and $a_3(h) > 0$.

($\hat{\mathbf{A}}_{\mathbf{2}}$) For $h \in J$, system $(1.1)_{\epsilon=0}$ has a family of periodic orbits surrounding the origin orientated counterclockwise and denoted by $L_h := L_h^+ \cup L_h^-$, where L_h^+ is an orbital arc in right half plane starting from $A_1(h)$ to $A_3(h)$ and L_h^- is an orbital arc in left half plane starting from $A_3(h)$ to $A_1(h)$.

($\hat{\mathbf{A}}_{\mathbf{3}}$) The curves $L^{\pm}(h)$, $h \in J$, are not tangent to the switch line $x = 0$ at points $A_1(h)$ and $A_3(h)$. In other words, $H_y^{\pm}(A_1(h))H_y^{\pm}(A_3(h)) \neq 0$ for $h \in J$.

Under the assumptions $(\hat{\mathbf{A}}_{\mathbf{1}})-(\hat{\mathbf{A}}_{\mathbf{3}})$, system (1.1) has a periodic annulus around the origin, and it has a displacement function of the form (1.2),

$$d(h,\epsilon) = \epsilon M_1(h) + \epsilon^2 M_2(h) + \epsilon^3 M_3(h) + \cdots, h \in (\alpha, \beta), \tag{1.2}$$

where $M_i(h)$ is called the i-order Melnikov function of system (1.1)$(i = 1, 2, \cdots)$.

For the first time, Liu et al. [19] proposed the assumptions $(\hat{\mathbf{A}}_{\mathbf{1}}) - (\hat{\mathbf{A}}_{\mathbf{3}})$ and obtained the expression of the first order Melnikov function for the near-Hamiltonian system (1.1)(that is $\mu_{\pm}(x,y) = 1$). Later, J. Yang et al. [29] and S. Li et al. [16] applied the result of [19] to the the integrable non-Hamiltonian system (1.1) and get the expression of the first order Melnikov function for system (1.1).

H. Tian and M. Han in [25] studied the problem of periodic orbit bifurcations for n-dimensional ($n \geq 3$) piecewise smooth near-integrable system, and they got the first order Melnikov vector function as follows,

$$M(h) = \int_{\widehat{AB}} D\mathbf{H}^+ g^+ dt + \overline{D\mathbf{H}^+(A)} \left[\overline{D\mathbf{H}^-(A)}\right]^{-1} \int_{\widehat{BA}} D\mathbf{H}^- g^- dt,$$

where $\mathbf{H}^{\pm}(\mathbf{x}) = (\mathbf{H}_1^{\pm}(\mathbf{x}), \mathbf{H}_2^{\pm}(\mathbf{x}), \cdots, \mathbf{H}_{n-1}^{\pm}(\mathbf{x}))^T$, for more details to see [25].

If $M_1(h) \equiv 0$ for $h \in (\alpha, \beta)$, we have to estimate the numbers of zeros of the second order Melnikov function $M_2(h)$ to obtain the number of zeros of the displacement function $d(h,\epsilon)$, which corresponds to the number of limit cycles of system (1.1). Recently, P. Yang et al. [30] obtained the expressions of the second order $M_2(h)$ for system (1.1) with $\mu_{\pm}(x,y) = 1$. They got the second order $M_2(h)$ as follows,

$$\begin{aligned} M_2(h) =& \frac{H_y^+(A_1)}{H_y^-(A_1)} \left[2\left(\int_{\widehat{A_3A_1}} \psi_- \omega_1^- - \omega_2^- + \frac{H_y^-(A_3)}{H_y^+(A_3)} \psi_-(A_3) \int_{\widehat{A_1A_3}} \omega_1^+ \right.\right. \\ & \left.\left. - \frac{P_0^-(A_3)}{H_y^+(A_3)} \int_{\widehat{A_1A_3}} \omega_1^+ \right) + \frac{H_{yy}^-(A_3)}{H_y^{-2}(A_3)} \left(\int_{\widehat{A_3A_1}} \omega_1^- \right)^2 \right] \\ & + \frac{H_y^+(A_1)}{H_y^-(A_1)} \frac{H_y^-(A_3)}{H_y^+(A_3)} \left[2\left(\int_{\widehat{A_1A_3}} \psi_+ \omega_1^+ - \omega_2^+ + \frac{P_0^+(A_3)}{H_y^+(A_3)} \int_{\widehat{A_1A_3}} \omega_1^+ \right.\right. \\ & \left.\left. - \psi_+(A_3) \int_{\widehat{A_1A_3}} \omega_1^+ \right) - \frac{H_{yy}^+(A_3)}{H_y^{+2}(A_3)} \left(\int_{\widehat{A_1A_3}} \omega_1^+ \right)^2 \right], \end{aligned} \tag{1.3}$$

where $\omega_i^{\pm}=f_i^{\pm}dy-g_i^{\pm}dx$ for $i=1,2$, $\psi_{\pm}(x,y)=\int_0^{t_{\pm}(x,y)} div\,(\chi_{\pm})\circ\left(\varphi_t^{\pm}\left(x_0^{\pm},y_0^{\pm}\right)\right)dt$, $\chi_{\pm}=\left(f_1^{\pm},g_1^{\pm}\right)$, and $\varphi_t^{\pm}(x_0,y_0)$ are the solutions of system $(1.1)_{\epsilon=0}$ passing through $\left(x_0^{\pm},y_0^{\pm}\right)$ at $t=0$ and ending at (x,y) with $t=t_{\pm}(x,y)$.

P. Yang et al. [31] gave the expression of $M_2(h)$ for piecewise Hamiltonian system with non-regular switching lines, and proved that there exists a linear piecewise Hamiltonian system with non-regular switching lines such that it has at least 5 limit cycles.

Recently, there are many results about the number of limit cycles by averaging method [10, 14, 17, 21, 24] and references therein.

Li et al. [14] studied the following piecewise smooth integrable non-Hamiltonian system

$$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix}=\begin{cases}\begin{pmatrix} y(1+ax)+\epsilon P_n^{(1)+}(x,y)+\epsilon^2P_n^{(2)+}(x,y)\\ -x(1+ax)+\epsilon Q_n^{(1)+}(x,y)+\epsilon^2Q_n^{(2)+}(x,y)\end{pmatrix}, & x\geq 0,\\ \begin{pmatrix} y(1+bx)+\epsilon P_n^{(1)-}(x,y)+\epsilon^2P_n^{(2)-}(x,y)\\ -x(1+bx)+\epsilon Q_n^{(1)-}(x,y)+\epsilon^2Q_n^{(2)-}(x,y)\end{pmatrix}, & x<0,\end{cases}\tag{1.4}$$

where $0<|\epsilon|\ll 1$, $a,b\in R$, $P_n^{(i)\pm}(x,y)$ and $Q_n^{(i)\pm}(x,y)$ are polynomial of x,y with degree n. Let $H(n)$ denote the maximum number of limit cycles bifurcating from any compact region of the periodic annulus of system (1.4) up to the **first order** averaging method.

Proposition 1.1 ([14]). *For system* (1.4)*, we have the following results.*

(i) *If $a=b=0$, then $H(n)=n$.*

(i) *If $a=0,b\neq 0$ or $a\neq 0,b=0$, then $H(n)=\left[\frac{n+1}{2}\right]+n+1$.*

(iii) *If $ab\neq 0$ and $a=-b$, then $H(n)=\left[\frac{n+1}{2}\right]+n$.*

(iv) *If $ab\neq 0$ and $a\neq -b$, then $H(n)=2\left[\frac{n+1}{2}\right]+n+1$.*

People also pay attention to the multiple zones of the piecewise smooth systems. We can see [15, 18, 27, 32] and the references therein. W. Liu and M. Han in [18] devoted to the study of limit cycle bifurcations in piecewise smooth near-Hamiltonian systems with multiple switching curves, obtaining a formula of the first order Melnikov function in general case.

Motivated by [13, 14, 16, 18, 27, 30] and [31], in this paper, we will study the expressions of the second order Melnikov functions for planar piecewise smooth integrable near-Hamiltonian systems with $m(\in\mathbb{N})$ zones separated by half straight lines through the origin. As an application, we will study the number of limit cycle of system (1.4) with $n=2$ and $a=b=1$ by the second order Melnikov function.

For system (1.1), we have the following result.

Theorem 1.1. *Suppose that system* (1.1) *satisfies the assumptions* $(\hat{\mathbf{A}}_\mathbf{1})-(\hat{\mathbf{A}}_\mathbf{3})$*. If $M_1\equiv 0$ for $h\in(\alpha,\beta)$, then the second order Melnikov function can be expressed as*

$$\begin{aligned}M_2(h)=&\frac{H_y^+}{H_y^-}(A_1)\Bigg\{2\left[\frac{H_y^-}{H_y^+}(A_3)\left(\int_{\widehat{A_1A_3}}\mu_+\psi_+\omega_1^+-\mu_+\omega_2^+\right)\right.\\ &\left.+\int_{\widehat{A_3A_1}}\mu_-\psi_-\omega_1^--\mu_-\omega_2^-\right]+\hat{M}_2^{(21)}(h)+\hat{M}_2^{(22)}(h)\Bigg\},\end{aligned}$$

where

$$\hat{M}_2^{(21)}(h)=2\left[\frac{H_y^-}{H_y^+}(A_3)\left(\frac{f_1^+}{H_y^+}\mu_+(A_3)-\mu_+(A_3)\psi_+(A_3)\right)-\frac{f_1^-}{H_y^+}\mu_-(A_3)\right]\int_{\widehat{A_1A_3}}\mu_+\omega_1^+,$$

$$\hat{M}_2^{(22)}(h)=\frac{H_{yy}^-}{\left(H_y^-\right)^2}(A_3)\left(\int_{\widehat{A_3A_1}}\mu_-\omega_1^-\right)^2-\frac{H_y^-}{H_y^+}(A_3)\frac{H_{yy}^+}{\left(H_y^+\right)^2}(A_3)\left(\int_{\widehat{A_1A_3}}\mu_+\omega_1^+\right)^2,$$

$\omega_i^\pm = f_i^\pm dy - g_i^\pm dx$ *for* $i = 1, 2$, $\psi_\pm(x, y) = \int_0^{t_\pm(x,y)} div\,(\chi_\pm) \circ \left(\varphi_t^\pm\left(x_0^\pm, y_0^\pm\right)\right) dt$, $\chi_\pm = \left(\mu_\pm f_1^\pm, \mu_\pm g_1^\pm\right)$, *and* $\varphi_t^\pm(x_0, y_0)$ *are the solutions of system* $(1.1)_{\epsilon=0}$ *passing through* $\left(x_0^\pm, y_0^\pm\right)$ *at* $t = 0$ *and ending at* (x, y) *with* $t = t_\pm(x, y)$.

As an application of Theorem 1.1, we obtain the following Theorem.

Theorem 1.2. *There exists* $\delta \in \mathbb{R}^9$ *such that the following system has at least 8 limit cycles near the origin:*

$$\begin{pmatrix}\dot{x}\\ \\ \dot{y}\end{pmatrix}=\begin{cases}\begin{pmatrix} y(1+x)+\epsilon(1+\frac{1}{2}x-\frac{1}{2}x^2-y^2+a_{0,1}^{*+}y)\\ +\epsilon^2(c_{0,0}^{*+}+c_{1,0}^{*+}x+c_{2,0}^{*+}x^2+c_{0,2}^{*+}y^2),\\ x(1+x)+\epsilon(xy+b_{0,2}^{*+}y^2),\end{pmatrix}, & x>0,\\ \begin{pmatrix} -y(1+x)+\epsilon(1+x-\frac{1}{2}y^2)+\epsilon^2(c_{0,0}^{*-}+c_{0,2}^{*-}y^2),\\ x(1+x)+\epsilon(\frac{1}{2}y+xy+b_{2,0}^{*-}x^2),\end{pmatrix}, & x\le 0,\end{cases} \qquad (E_1^*)$$

where $0 < |\epsilon| \ll 1$ *and* $\delta = (a_{0,1}^{*+}, b_{0,2}^{*+}, b_{2,0}^{*-}, c_{0,0}^{*+}, c_{0,0}^{*-}, c_{1,0}^{*+}, c_{2,0}^{*+}, c_{0,2}^{*+}, c_{0,2}^{*-})$. *The limit cycles are obtained by the second order of Melnikov function* $M_2(h)$ *if* $M_2(h) \not\equiv 0$ *near the origin.*

Remark 1.1.

(i) For the Theorem 1.1, if $\mu_\pm = 1$, the formula was obtained by Corollary 2.1 of [30] shown in Eq.(1.3).

(ii) By applying the **second order** averaging method to system

$$(\dot{x},\dot{y})=\begin{cases}\left(-y(1+x)+\epsilon F^+(x,y),x(1+x)+\epsilon G^+(x,y)\right), & \text{if } x\ge 0,\\ \left(-y(1+x)+\epsilon F^-(x,y),x(1+x)+\epsilon G^-(x,y)\right), & \text{if } x<0,\end{cases} \qquad (1.5)$$

Jiang [13] proved that the maximal lower bound of the number of limit cycles of system (1.5) is **5**, where $F^\pm(x, y)$ and $G^\pm(x, y)$ are quadratic polynomials with $F^\pm(0, 0) = G^\pm(0, 0) = 0$.

The rest of the paper is organized as follows. In § 2, we shall give the formulas of the second order Melnikov function for perturbations of piecewise smooth integrable non-Hamiltonian systems with multiple zones. The main results are stated in Theorem 2.1 and Theorem 2.2. In § 3, we shall study the number of limit cycle of system (1.4) with $n = 2$, $a = b = 1$ by the second order Melnikov function.

2. The second order Melnikov function with multiple zones

In this section, we will provide an expression of the second order Melnikov function for piecewise smooth integrable non-Hamiltonian systems with multiple zones. We begin with the case of four zones separated by both x-axis and y-axis.

2.1. Second order Melnikov function with four zones separated by both x-axis and y-axis

For $0 < |\epsilon| \ll 1$, let us consider the following perturbed piecewise smooth integrable differential system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{cases} \begin{pmatrix} \dfrac{H_{1y}(x,y)}{\mu_1(x,y)} + \epsilon\, f_1(x,y) \\ -\dfrac{H_{1x}(x,y)}{\mu_1(x,y)} + \epsilon\, g_1(x,y) \end{pmatrix}, & (x,y) \in \Omega_1, \\ \begin{pmatrix} \dfrac{H_{2y}(x,y)}{\mu_2(x,y)} + \epsilon\, f_2(x,y) \\ -\dfrac{H_{2x}(x,y)}{\mu_2(x,y)} + \epsilon\, g_2(x,y) \end{pmatrix}, & (x,y) \in \Omega_2, \\ \begin{pmatrix} \dfrac{H_{3y}(x,y)}{\mu_3(x,y)} + \epsilon\, f_3(x,y) \\ -\dfrac{H_{3x}(x,y)}{\mu_3(x,y)} + \epsilon\, g_3(x,y) \end{pmatrix}, & (x,y) \in \Omega_3, \\ \begin{pmatrix} \dfrac{H_{4y}(x,y)}{\mu_4(x,y)} + \epsilon\, f_4(x,y) \\ -\dfrac{H_{4x}(x,y)}{\mu_4(x,y)} + \epsilon\, g_4(x,y) \end{pmatrix}, & (x,y) \in \Omega_4, \end{cases} \tag{2.1}$$

where, for $i = 1,2,3,4$, $H_i(x,y)$, $f_i(x,y)$, $g_i(x,y)$ and $\mu_i(x,y) \in C^\infty$ with $\mu_i(0,0) \neq 0$, and

$$\Omega_1 = \{(x,y) \mid x > 0, y < 0\}, \qquad \Omega_2 = \{(x,y) \mid x > 0, y > 0\}, \tag{2.2}$$

$$\Omega_3 = \{(x,y) \mid x < 0, y > 0\}, \qquad \Omega_4 = \{(x,y) \mid x < 0, y < 0\} \tag{2.3}$$

are four zones separated by both x-axis and y-axis. It is easy to see that $\mu_i(x,y)$ are the integrating factors for the subsystems in the regions Ω_i respectively($i = 1,2,3,4$).

We follow the notations usually used in the literatures:

$$H_{iy}(x,y) = \frac{\partial H_i(x,y)}{\partial y}, \qquad H_{ix}(x,y) = \frac{\partial H_i(x,y)}{\partial x}, \tag{2.4}$$

$$H_{iyy}(x,y) = \frac{\partial^2 H_i(x,y)}{\partial y^2}, \qquad H_{ixx}(x,y) = \frac{\partial^2 H_i(x,y)}{\partial x^2}, \tag{2.5}$$

$$a_{i\epsilon}(0,h) = \frac{\partial a_i}{\partial \epsilon}(\epsilon,h)|_{\epsilon=0}, \qquad a_{i\epsilon\epsilon}(0,h) = \frac{\partial^2 a_i}{\partial \epsilon^2}(\epsilon,h)|_{\epsilon=0}. \tag{2.6}$$

From now on, for simplicity, we will write $\mu_i(x, y)$ as μ_i and so on.

For system $(2.1)_{\epsilon=0}$, we assume that there exists an open nonempty interval $J = (\alpha, \beta)$ satisfying the following assumptions [27].

$(\mathbf{A_1})$ For each $h \in J$, there are four points $A_i(h) = (0, a_i(h))(i = 1, 3)$ and $A_j(h) = (a_j(h), 0)(j = 2, 4)$ satisfying

$$H_1(A_1(h)) = H_1(A_2(h)) = h, \quad H_i(A_i(h)) = H_i(A_{i+1}(h))$$

for $i = 2, 3, 4$ with $A_5(h) = A_1(h)$, where $a_2(h),\ a_3(h) > 0$ and $a_1(h),\ a_4(h) < 0$.

$(\mathbf{A_2})$ For $h \in J$, system $(2.1)_{\epsilon=0}$ has a family of periodic orbits surrounding the origin orientated counterclockwise, which are denoted by $L_h := L_h^1 \cup L_h^2 \cup L_h^3 \cup L_h^4$, where L_h^i is an orbital arc in Ω_i starting from A_i to A_{i+1} for $i = 1, 2, 3, 4$ with $A_5 = A_1$.

$(\mathbf{A_3})$ The curves L_h^i, $h \in J$, are not tangent to the switching lines $x = 0$ and $y = 0$ at points $A_i(h)$ for $i = 1, 2, 3, 4$. In other words,

$$\begin{aligned} &H_{ix}(A_2) \neq 0, (i = 1, 2), \qquad H_{ix}(A_4) \neq 0, (i = 3, 4); \\ &H_{iy}(A_1) \neq 0, (i = 1, 4), \qquad H_{iy}(A_3) \neq 0, (i = 2, 3). \end{aligned}$$

Under the above Assumptions $(\mathbf{A_1}) - (\mathbf{A_3})$, system $(2.1)_{\epsilon=0}$ has an orbital arc for $h \in (\alpha, \beta)$, and we can define the bifurcation function of system (2.1). Let us consider the orbit L_h starting from the point $A_1(h)$ at the negative y-axis. Let $A_{j\epsilon}$ be the intersection points of L_h with the positive x-axis, positive y-axis, negative x-axis for $j = 2, 3, 4$ respectively, and B_ϵ be the first returning point on the negative y-axis(see Figure 1). Set $B_\epsilon(h) = (0, a_1(\epsilon, h))$, $A_{j\epsilon}(h) = (a_j(\epsilon, h), 0)(j = 2, 4)$

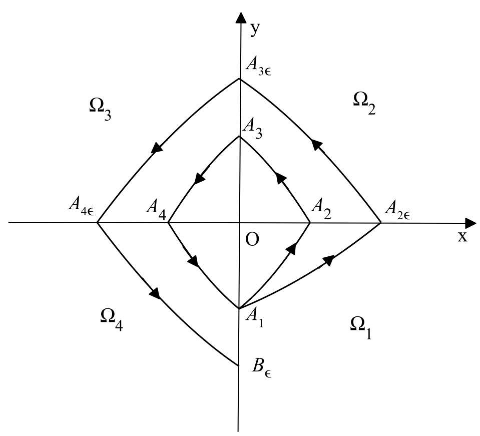

Figure 1. The period orbits of system (2.1).

and $A_{3\epsilon}(h) = (0, a_3(\epsilon, h))$. We know $A_{j\epsilon}(h)$ and $B_\epsilon(h)$ are C^∞ with respect to ϵ satisfying

$$B_\epsilon(h)|_{\epsilon=0} = A_1(h), \qquad A_{j\epsilon}(h)|_{\epsilon=0} = A_j(h),\ j = 2, 3, 4.$$

We define displacement function as

$$H_1\left(B_\epsilon\right)-H_1\left(A_1\right)=\epsilon M_1(h)+\epsilon^2 M_2(h)+\epsilon^3 M_3(h)+\cdots, \tag{2.7}$$

the function M_i is called Melnikov function of order $i(i=1,2,\cdots)$.

Theorem 2.1. *Consider system* (2.1) *with the assumptions* $(\mathbf{A_1})-(\mathbf{A_3})$. *For* $h\in J$, *if* $M_1\equiv 0$, *then the second order Melnikov function is expressed as*

$$\begin{aligned}M_2(h)=\frac{H_{1y}}{H_{4y}}(A_1)\Bigg[2\Bigg(&\frac{H_{4x}}{H_{3x}}(A_4)\frac{H_{3y}}{H_{2y}}(A_3)\frac{H_{2x}}{H_{1x}}(A_2)\int_{\widehat{A_1A_2}}\mu_1\psi_1\omega_1\\&+\frac{H_{4x}}{H_{3x}}(A_4)\frac{H_{3y}}{H_{2y}}(A_3)\int_{\widehat{A_2A_3}}\mu_2\psi_2\omega_2+\frac{H_{4x}}{H_{3x}}(A_4)\int_{\widehat{A_3A_4}}\mu_3\psi_3\omega_3\\&+\int_{\widehat{A_4A_1}}\mu_4\psi_4\omega_4\Bigg)+M_2^{(21)}(h)+M_2^{(22)}(h)\Bigg],\end{aligned}$$

where, for $i=1,2,3,4$, $\psi_i(x,y)=\int_0^{t_i(x,y)}div\left(\chi_i\right)\circ\varphi_i^t\left(x_0^i,y_0^i\right)dt$, $\chi_i=(\mu_if_i,\mu_ig_i)$, $\varphi_i^t(x_0,y_0)$ *are the solutions of system* $(2.1)_{\epsilon=0}$ *passing through* $\left(x_0^i,y_0^i\right)$ *at* $t=0$ *and ending at* (x,y) *with* $t=t_i(x,y)$, *and*

$$\begin{aligned}M_2^{(21)}(h)=&2\Bigg[a_{4\epsilon}(0,h)\big(R_{4x}(A_4)-\frac{H_{4x}}{H_{3x}}(A_4)R_{3x}(A_4)\big)\\&+a_{3\epsilon}(0,h)\frac{H_{4x}}{H_{3x}}(A_3)\big(R_{3y}(A_3)-\frac{H_{3y}}{H_{2y}}(A_3)R_{2y}(A_3)\big)\\&+a_{2\epsilon}(0,h)\frac{H_{4x}}{H_{3x}}(A_4)\frac{H_{3y}}{H_{2y}}(A_3)\big(R_{2x}(A_2)-\frac{H_{2x}}{H_{1x}}(A_2)R_{1x}(A_2)\big)\Bigg],\end{aligned} \tag{2.8}$$

$$\begin{aligned}M_2^{(22)}(h)=&a_{4\epsilon}^2(0,h)\big(H_{4xx}(A_4)-\frac{H_{4x}}{H_{3x}}(A_4)H_{3xx}(A_4)\big)\\&+a_{3\epsilon}^2(0,h)\frac{H_{4x}}{H_{3x}}(A_4)\big(H_{3yy}(A_3)-\frac{H_{3y}}{H_{2y}}(A_3)H_{2yy}(A_3)\big)\\&+a_{2\epsilon}^2(0,h)\frac{H_{4x}}{H_{3x}}(A_4)\frac{H_{3y}}{H_{2y}}(A_3)\big(H_{2xx}(A_2)-\frac{H_{2x}}{H_{1x}}(A_2)H_{1xx}(A_2)\big)\end{aligned} \tag{2.9}$$

with $R_{iy}=\mu_if_i-\psi_iH_{iy}$, $R_{ix}=-\mu_ig_i-\psi_iH_{ix}$ *and* $\omega_i=f_idy-g_idx$.

Proof. By the definitions of $\psi_i(x,y)$, we have $\psi_i(A_i)=0(i=1,2,3,4)$. We write system (2.1) as the corresponding Pfaffian form

$$\frac{dH_i(x,y)}{\mu_i(x,y)}+\epsilon\omega_i=0,\quad (x,y)\in\Omega_i,$$

where $\omega_i=f_i(x,y)dy-g_i(x,y)dx$ $(i=1,2,3,4)$. Set

$$\begin{aligned}&d_{i+1,i}(\epsilon,h)=H_{i+1}\left(A_{i+1,\epsilon}\right)-H_i\left(A_{i+1,\epsilon}\right),\ i=1,2,3,4,\\&d_{i,i}(\epsilon,h)=H_i\left(A_{i+1,\epsilon}\right)-H_i\left(A_{i\epsilon}\right),\ i=2,3,4,\\&d_{i,i}(\epsilon,h)=H_i\left(A_{i+1,\epsilon}\right)-H_i\left(A_i\right),\ i=1\end{aligned}$$

with $H_5(x,y)=H_1(x,y)$ and $A_{5\epsilon}=B_\epsilon$. Following [27], we have

$$d(\epsilon,h)=H_1(B_\epsilon)-H_1(A_1)=\sum_{i=1}^4 d_{i+1,i}(\epsilon,h)+\sum_{i=1}^4 d_{i,i}(\epsilon,h). \tag{2.10}$$

For $0<|\epsilon|\ll 1$, the function $d_{1,1}(\epsilon,h)$ can be written in power series of ϵ as

$$\begin{aligned}d_{1,1}(\epsilon,h)=&(H_1(A_{2\epsilon})-H_1(A_2))+(H_1(A_2)-H_1(A_1))\\=&H_1(A_2)-H_1(A_1)+\epsilon H_{1x}(A_2)a_{2\epsilon}(0,h)+\frac{1}{2}\epsilon^2H_{1x}(A_2)a_{2\epsilon\epsilon}(0,h)\\&+\frac{1}{2}\epsilon^2H_{1xx}(A_2)a_{2\epsilon}^2(0,h)+O(\epsilon^3).\end{aligned}$$

Then we have

$$\begin{aligned}&\frac{\partial d_{1,1}}{\partial\epsilon}(\epsilon,h)|_{\epsilon=0}=a_{2\epsilon}(0,h)H_{1x}(A_2),\\&\frac{\partial^2 d_{1,1}}{\partial\epsilon^2}(\epsilon,h)|_{\epsilon=0}=a_{2\epsilon\epsilon}(0,h)H_{1x}(A_2)+a_{2\epsilon}^2(0,h)H_{1xx}(A_2).\end{aligned}\tag{2.11}$$

Similarly, for $i=2,4$, we can get

$$\begin{aligned}\frac{\partial d_{i,i}}{\partial\epsilon}(\epsilon,h)|_{\epsilon=0}=&a_{i+1,\epsilon}(0,h)H_{iy}(A_{i+1})-a_{i,\epsilon}(0,h)H_{ix}(A_i),\\\frac{\partial^2 d_{i,i}}{\partial\epsilon^2}(\epsilon,h)|_{\epsilon=0}=&a_{i+1,\epsilon\epsilon}(0,h)H_{iy}(A_{i+1})+a_{i+1,\epsilon}^2(0,h)H_{iyy}(A_{i+1})\\&-a_{i,\epsilon\epsilon}(0,h)H_{ix}(A_i)-a_{i,\epsilon}^2(0,h)H_{ixx}(A_i);\end{aligned}\tag{2.12}$$

and for $i=3$, we get

$$\begin{aligned}\frac{\partial d_{3,3}}{\partial\epsilon}(\epsilon,h)|_{\epsilon=0}=&a_{4\epsilon}(0,h)H_{3x}(A_4)-a_{3\epsilon}(0,h)H_{3y}(A_3),\\\frac{\partial^2 d_{3,3}}{\partial\epsilon^2}(\epsilon,h)|_{\epsilon=0}=&a_{4\epsilon\epsilon}(0,h)H_{3x}(A_4)+a_{4\epsilon}^2(0,h)H_{3xx}(A_4)\\&-a_{3\epsilon\epsilon}(0,h)H_{3y}(A_3)-a_{3\epsilon}^2(0,h)H_{3yy}(A_3).\end{aligned}\tag{2.13}$$

For $0<|\epsilon|\ll 1$, the functions $d_{i+1,i}(\epsilon,h)$ $(i=1,2,3,4)$ can be written in power series of ϵ as follows: for $i=2,4$,

$$\begin{aligned}\frac{\partial d_{i+1,i}}{\partial\epsilon}(\epsilon,h)|_{\epsilon=0}=&a_{i+1,\epsilon}(0,h)H_{i+1,y}(A_{i+1})-a_{i+1,\epsilon}(0,h)H_{iy}(A_{i+1}),\\\frac{\partial^2 d_{i+1,i}}{\partial\epsilon^2}(\epsilon,h)|_{\epsilon=0}=&a_{i+1,\epsilon\epsilon}(0,h)H_{i+1,y}(A_{i+1})+a_{i+1,\epsilon}^2(0,h)H_{i+1,yy}(A_{i+1})\\&-a_{i+1,\epsilon\epsilon}(0,h)H_{iy}(A_{i+1})-a_{i+1,\epsilon}^2(0,h)H_{iyy}(A_{i+1});\end{aligned}\tag{2.14}$$

and for $i=1,3$,

$$\begin{aligned}\frac{\partial d_{i+1,i}}{\partial\epsilon}(\epsilon,h)|_{\epsilon=0}=&a_{i+1,\epsilon}(0,h)H_{i+1,x}(A_{i+1})-a_{i+1,\epsilon}(0,h)H_{ix}(A_{i+1}),\\\frac{\partial^2 d_{i+1,i}}{\partial\epsilon^2}(\epsilon,h)|_{\epsilon=0}=&a_{i+1,\epsilon\epsilon}(0,h)H_{i+1,x}(A_{i+1})+a_{i+1,\epsilon}^2(0,h)H_{i+1,xx}(A_{i+1})\\&-a_{i+1,\epsilon\epsilon}(0,h)H_{ix}(A_{i+1})-a_{i+1,\epsilon}^2(0,h)H_{ixx}(A_{i+1}).\end{aligned}\tag{2.15}$$

On the other hand, for $0<|\epsilon|\ll 1$, $d_{1,1}(\epsilon,h)$ can be rewritten as

$$d_{1,1}(\epsilon,h)=H_1(A_{2\epsilon})-H_1(A_1)=\int_{\widehat{A_1A_{2\epsilon}}}dH_1=\int_{\widehat{A_1A_{2\epsilon}}}H_{1x}dx+H_{1y}dy$$

$$
\begin{aligned}
&= \int_{\widehat{A_1A_{2\epsilon}}} H_{1x}\left(\frac{H_{1y}}{\mu_1}+\epsilon f_1\right)dt + H_{1y}\left(-\frac{H_{1x}}{\mu_1}+\epsilon g_1\right)dt\\
&= \epsilon\int_{\widehat{A_1A_{2\epsilon}}} \left(H_{1x}f_1 + H_{1y}g_1\right)dt\\
&= \epsilon\int_{\widehat{A_1A_{2\epsilon}}} f_1\left(\epsilon\mu_1 g_1 dt - \mu_1 dy\right) + g_1\left(\mu_1 dx - \epsilon\mu_1 f_1 dt\right)\\
&= -\epsilon\int_{\widehat{A_1A_2}} \mu_1\omega_1 + O(\epsilon^2).
\end{aligned}
$$

Hence we have

$$
\frac{\partial d_{1,1}}{\partial\epsilon}(\epsilon,h)|_{\epsilon=0} = -\int_{\widehat{A_1A_2}} \mu_1\omega_1. \tag{2.16}
$$

Following the same processes, we have

$$
\frac{\partial d_{i,i}}{\partial\epsilon}(\epsilon,h)|_{\epsilon=0} = -\int_{\widehat{A_iA_{i+1}}} \mu_i\omega_i,\ \ i=2,3,4, A_5=A_1. \tag{2.17}
$$

Combining (2.11) and (2.16), we get

$$
a_{2\epsilon}(0,h) = \frac{-1}{H_{1x}(A_2)}\int_{\widehat{A_1A_2}} \mu_1\omega_1. \tag{2.18}
$$

Noting (2.12)-(2.15) and (2.17), we can get

$$
\begin{aligned}
a_{3\epsilon}(0,h) =& \frac{-1}{H_{2y}(A_3)}\left(\frac{H_{2x}}{H_{1x}}(A_2)\int_{\widehat{A_1A_2}}\mu_1\omega_1 + \int_{\widehat{A_2A_3}}\mu_2\omega_2\right),\\
a_{4\epsilon}(0,h) =& \frac{-1}{H_{3x}(A_4)}\left(\frac{H_{3y}}{H_{2y}}(A_3)\frac{H_{2x}}{H_{1x}}(A_2)\int_{\widehat{A_1A_2}}\mu_1\omega_1\right.\\
&\left. + \frac{H_{3y}}{H_{2y}}(A_3)\int_{\widehat{A_2A_3}}\mu_2\omega_2 + \int_{\widehat{A_3A_4}}\mu_3\omega_3\right),\\
a_{1\epsilon}(0,h) =& \frac{-1}{H_{4y}(A_1)}\left(\frac{H_{4x}}{H_{3x}}(A_4)\frac{H_{3y}}{H_{2y}}(A_3)\frac{H_{2x}}{H_{1x}}(A_2)\int_{\widehat{A_1A_2}}\mu_1\omega_1\right.\\
&\left. + \frac{H_{4x}}{H_{3x}}(A_4)\frac{H_{3y}}{H_{2y}}(A_3)\int_{\widehat{A_2A_3}}\mu_2\omega_2 + \frac{H_{4x}}{H_{3x}}(A_4)\int_{\widehat{A_3A_4}}\mu_3\omega_3 + \int_{\widehat{A_4A_1}}\mu_4\omega_4\right).
\end{aligned} \tag{2.19}
$$

Substituting (2.18)-(2.19) into (2.11)-(2.15), we can get the first order Melnikov function

$$
\begin{aligned}
M_1(h) =& \sum_{i=1}^{4}\frac{\partial d_{i,i}}{\partial\epsilon}(\epsilon,h)|_{\epsilon=0} + \sum_{i=1}^{4}\frac{\partial d_{i+1,i}}{\partial\epsilon}(\epsilon,h)|_{\epsilon=0}\\
=& -\frac{H_{1y}}{H_{4y}}(A_1)\left[\frac{H_{4x}}{H_{3x}}(A_4)\frac{H_{3y}}{H_{2y}}(A_3)\frac{H_{2x}}{H_{1x}}(A_2)\int_{\widehat{A_1A_2}}\mu_1\omega_1\right.\\
&\left. + \frac{H_{4x}}{H_{3x}}(A_4)\frac{H_{3y}}{H_{2y}}(A_3)\int_{\widehat{A_2A_3}}\mu_2\omega_2 + \frac{H_{4x}}{H_{3x}}(A_4)\int_{\widehat{A_3A_4}}\mu_3\omega_3 + \int_{\widehat{A_4A_1}}\mu_4\omega_4\right].
\end{aligned} \tag{2.20}
$$

Next, we will consider the second order Melnikov function. Suppose that $M_1(h) \equiv 0$. Then we get

$$\frac{H_{4x}}{H_{3x}}(A_4)\frac{H_{3y}}{H_{2y}}(A_3)\frac{H_{2x}}{H_{1x}}(A_2)\int_{\widehat{A_1A_2}}\mu_1\omega_1 + \frac{H_{4x}}{H_{3x}}(A_4)\frac{H_{3y}}{H_{2y}}(A_3)\int_{\widehat{A_2A_3}}\mu_2\omega_2 + \frac{H_{4x}}{H_{3x}}(A_4)\int_{\widehat{A_3A_4}}\mu_3\omega_3 + \int_{\widehat{A_4A_1}}\mu_4\omega_4 \equiv 0. \tag{2.21}$$

Hence, for $h \in J$,

$$M_1(h) \equiv 0 \Longleftrightarrow a_{1\epsilon}(0,h) \equiv 0. \tag{2.22}$$

For $i = 1,2,3,4$, we denote by $\varphi_t^i = \varphi^i(t,x,y)$ the solution of subsystem $(2.1)_{\epsilon=0}$ on the region Ω_i starting from $\left(x_0^i, y_0^i\right) = A_i(h)$ and define a function $\psi_i(x,y)$ as

$$\psi_i(x,y) = \int_0^{t_i(x,y)} \operatorname{div}(\chi_i) \circ \left(\varphi_t^i\left(x_0^i, y_0^i\right)\right) dt, \tag{2.23}$$

where $\chi_i = (\mu_i f_i, \mu_i g_i)$ and $(x,y) = \varphi_i\left(t_i(x,y), x_0^i, y_0^i\right)$.

Set $\chi_0^i = \frac{H_{iy}}{\mu_i}\frac{\partial}{\partial x} - \frac{H_{ix}}{\mu_i}\frac{\partial}{\partial y}$. Then we have $\chi_0^i \cdot (\psi_i) = \frac{\operatorname{div}(\chi_i)}{\mu_i}$, which implies

$$\frac{1}{\mu_i} dH_i \wedge d\psi_i = -\frac{1}{\mu_i}\operatorname{div}(\chi_i)\, dx \wedge dy.$$

Note that by direct calculations, $\operatorname{div}(\chi_i)\, dx \wedge dy = d(\mu_i\omega_i)$. Then

$$\frac{1}{\mu_i} dH_i \wedge d\psi_i = -\frac{1}{\mu_i}\operatorname{div}(\chi_i)\, dx \wedge dy = -\frac{1}{\mu_i}\left(d\left(\psi_i dH_i\right)\right) = -\frac{1}{\mu_i} d(\mu_i\omega_i),$$

which gives

$$\mu_i\omega_i = \psi_i dH_i + dR_i. \tag{2.24}$$

Consider the following equation,

$$(1-\epsilon\psi_i)\left(\frac{dH_i}{\mu_i} + \epsilon\omega_i\right) = 0,$$

namely,

$$(1-\epsilon\psi_i)\left(dH_i + \epsilon\mu_i\omega_i\right) = 0. \tag{2.25}$$

In the following, we choose the subsystem on the region of Ω_1 to show how to do. By integrating (2.25) along $\widehat{A_1A_{2\epsilon}}$ and combining with (2.24), we have

$$\int_{\widehat{A_1A_{2\epsilon}}} d\left(H_1 + \epsilon R_1\right) = \epsilon^2 \int_{\widehat{A_1A_{2\epsilon}}} \mu_1\psi_1\omega_1,$$

which implies

$$H_1(A_{2\epsilon}) - H_1(A_1) + \epsilon R_1(A_{2\epsilon}) - \epsilon R_1(A_1) = \epsilon^2 \int_{\widehat{A_1A_2}} \mu_1\psi_1\omega_1 + O(\epsilon^3). \tag{2.26}$$

Due to $d_{1,1}(\epsilon,h)=H_1(A_{2\epsilon})-H_1(A_1)$, taking a second order partial derivative on both sides of (2.26) with respect to ϵ, we can obtain

$$\frac{\partial^2 d_{1,1}}{\partial\epsilon^2}(\epsilon,h)|_{\epsilon=0}=-2R_{1x}(A_2)a_{2\epsilon}(0,h)+2\int_{\widehat{A_1A_2}}\mu_1\psi_1\omega_1. \tag{2.27}$$

Combining (2.11) and (2.27), we have

$$a_{2\epsilon\epsilon}(0,h)=\frac{1}{H_{1x}(A_2)}\left[2\left(\int_{\widehat{A_1A_2}}\mu_1\psi_1\omega_1-R_{1x}(A_2)a_{2\epsilon}(0,h)\right)-H_{1xx}(A_2)a_{2\epsilon}^2(0,h)\right]. \tag{2.28}$$

For $d_{i,i}(\epsilon,h)(i=2,3,4)$, following the similar processes, we can obtain

$$\begin{aligned}
a_{3\epsilon\epsilon}(0,h)=&\frac{1}{H_{2y}(A_3)}\left[2\left(\int_{\widehat{A_2A_3}}\mu_2\psi_2\omega_2+R_{2x}(A_2)a_{2\epsilon}(0,h)-R_{2y}(A_3)a_{3\epsilon}(0,h)\right)\right.\\
&\left.+H_{2x}(A_2)a_{2\epsilon\epsilon}(0,h)+H_{2xx}(A_2)a_{2\epsilon}^2(0,h)-H_{2yy}(A_3)a_{3\epsilon}^2(0,h)\right],\\
a_{4\epsilon\epsilon}(0,h)=&\frac{1}{H_{3x}(A_4)}\left[2\left(\int_{\widehat{A_3A_4}}\mu_3\psi_3\omega_3+R_{3y}(A_3)a_{3\epsilon}(0,h)-R_{3x}(A_4)a_{4\epsilon}(0,h)\right)\right.\\
&\left.+H_{3y}(A_3)a_{3\epsilon\epsilon}(0,h)+H_{3yy}(A_3)a_{3\epsilon}^2(0,h)-H_{2xx}(A_4)a_{4\epsilon}^2(0,h)\right],\\
a_{1\epsilon\epsilon}(0,h)=&\frac{1}{H_{4y}(A_1)}\left[2\left(\int_{\widehat{A_4A_1}}\mu_4\psi_4\omega_4+R_{4x}(A_4)a_{4\epsilon}(0,h)\right)\right.\\
&\left.+H_{4x}(A_4)a_{4\epsilon\epsilon}(0,h)+H_{4xx}(A_4)a_{4\epsilon}^2(0,h)\right].
\end{aligned} \tag{2.29}$$

Thus we have

$$M_2(h)=\sum_{i=1}^{4}\frac{\partial^2 d_{i,i}}{\partial\epsilon^2}(\epsilon,h)|_{\epsilon=0}+\sum_{i=1}^{4}\frac{\partial^2 d_{i+1,i}}{\partial\epsilon^2}(\epsilon,h)|_{\epsilon=0}. \tag{2.30}$$

Substituting (2.11)-(2.15), (2.18)-(2.19), (2.28)-(2.29) into (2.30), after some simplifications, we can get the second order Melnikov function given in Theorem 2.1. This ends the proof. □

For $0<|\epsilon|\ll 1$, consider the system with the perturbations up to second order in ϵ, that is,

$$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix}=\left\{\begin{pmatrix}\dfrac{H_{iy}(x,y)}{\mu_i(x,y)}+\epsilon f_i(x,y)+\epsilon^2\hat{f}_i(x,y),\\ -\dfrac{H_{ix}(x,y)}{\mu_i(x,y)}+\epsilon g_i(x,y)+\epsilon^2\hat{g}_i(x,y),\end{pmatrix}\right.,\ (x,y)\in\Omega_i, \tag{2.1*}$$

where, for $i=1,2,3,4$, $\hat{f}_i(x,y)$, $\hat{g}_i(x,y)\in C^\infty$, $H_i(x,y)$, $H_{iy}(x,y)$, $H_{ix}(x,y)$, $f_i(x,y)$, $g_i(x,y)$, $\mu_i(x,y)$ and Ω_i are defined the same as for system (2.1). Then we have the following corollary.

Corollary 2.1. *Consider system* (2.1)* *with the assumptions* ($\mathbf{A_1}$)–($\mathbf{A_3}$). *If* $M_1\equiv 0$*, the second order Melnikov function of system* (2.1)* *is expressed as*

$$M_2(h)=\frac{H_{1y}}{H_{4y}}(A_1)\left\{2\left[\frac{H_{4x}}{H_{3x}}(A_4)\frac{H_{3y}}{H_{2y}}(A_3)\frac{H_{2x}}{H_{1x}}(A_2)\int_{\widehat{A_1A_2}}\mu_1\psi_1\omega_1-\mu_1\hat{\omega}_1\right.\right.$$

$$+\frac{H_{4x}}{H_{3x}}(A_4)\frac{H_{3y}}{H_{2y}}(A_3)\int_{\widehat{A_2A_3}}\mu_2\psi_2\omega_2-\mu_2\hat{\omega}_2 \;+\; \frac{H_{4x}}{H_{3x}}(A_4)\int_{\widehat{A_3A_4}}\mu_3\psi_3\omega_3$$
$$\left.\left.-\mu_3\hat{\omega}_3+\int_{\widehat{A_4A_1}}\mu_4\psi_4\omega_4-\mu_4\hat{\omega}_4\right]+M_2^{(21)}(h)+M_2^{(22)}(h)\right\},$$

where $M_2^{(21)}(h)$ and $M_2^{(22)}(h)$ are given by (2.8) *and* (2.9) *respectively, ψ_i, ω_i are defined the same as for Theorem 2.1, and $\hat{\omega}_i=\hat{f}_idy-\hat{g}_idx$, $i=1,2,3,4$.*

Remark 2.1. The formula (2.20) can be obtained directly by Lemma 2.1 of [27] by the time rescaling as follows

$$M_1(h)=-\frac{H_{1y}}{H_{4y}}(A_1)\left[\frac{H_{4x}}{H_{3x}}(A_4)\frac{H_{3y}}{H_{2y}}(A_3)\frac{H_{2x}}{H_{1x}}(A_2)\int_{\widehat{A_1A_2}}\omega_1\right.$$
$$\left.+\frac{H_{4x}}{H_{3x}}(A_4)\frac{H_{3y}}{H_{2y}}(A_3)\int_{\widehat{A_2A_3}}\omega_2+\frac{H_{4x}}{H_{3x}}(A_4)\int_{\widehat{A_3A_4}}\omega_3+\int_{\widehat{A_4A_1}}\omega_4\right].$$

2.2. Second order Melnikov function with multiple zones.

In this section, we shall consider the second order Melnikov function for piecewise smooth integrable non-Hamiltonian systems with multiple zones.

Let us consider the piecewise systems with m zones. For $i=1,2,\cdots,m\in\mathbb{N}$, we set m switching lines l_i as

$$l_i=\{(x,y)|\ y=k_ix,\ x\in[0,+\infty)\},k_i\in\mathbb{R},\ i=1,2,\cdots,m_1,$$
$$l_i=\{(x,y)|\ y=k_ix,,\ x\in(-\infty,0]\},k_i\in\mathbb{R},\ i=m_1+1,\cdots,m,$$

with $1\le m_1\le m$, $k_1<k_2<\cdots<k_{m_1}$, $k_{m_1+1}<k_{m_1+2}<\cdots<k_m$. We denote D_i be the open set between l_i and $l_{i+1}(i=1,2,\cdots,m)$ with $l_{m+1}=l_1$, and consider the following system:

$$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix}=\left\{\begin{pmatrix}\dfrac{H_{iy}(x,y)}{\mu_i(x,y)}+\epsilon f_i(x,y)+\epsilon^2\hat{f}_i(x,y),\\ -\dfrac{H_{ix}(x,y)}{\mu_i(x,y)}+\epsilon g_i(x,y)+\epsilon^2\hat{g}_i(x,y),\end{pmatrix}\right.,\ (x,y)\in D_i,\ i=1,2,\cdots,m, \tag{X_ϵ}$$

where $0<|\epsilon|\ll 1$, $H_i(x,y)$, $\mu_i(x,y)$, $f_i(x,y)$, $g_i(x,y)$, $\hat{f}_i(x,y)$, $\hat{g}_i(x,y)\in C^\infty$ with $\mu_i(0,0)\ne 0(i=1,2,\cdots,m)$. Suppose that (X_0) satisfies the following assumptions [18].

($\mathbf{H_1}$) There exists an open nonempty interval $J=(\alpha,\beta)$ such that for each $h\in J$, there are m points $A_i(h)=(a_i(h),k_ia_i(h))\in l_i$, $i=1,2,\cdots,m$, satisfying

$$H_1(A_1(h))=H_1(A_2(h))=h,\quad H_i(A_i(h))=H_i(A_{i+1}(h)),\ i=2,3,\cdots,m$$

with $H_{m+1}=H_1$, $A_{m+1}(h)=A_1(h)$.

($\mathbf{H_2}$) For $h\in J$, system (X_0) has a family of periodic orbits surrounding the origin orientated counterclockwise, which are denoted by $L_h=L_h^1\cup L_h^2\cup\cdots\cup L_h^m$ where L_h^i is an orbital arc in D_i starting from A_i to A_{i+1} for $i=1,2,3,\cdots,m$ with $A_{m+1}=A_1$.

($\mathbf{H_3}$) The arcs $L_h^i, h\in J$, are not tangent to the switching lines l_i and l_{i+1} at points $A_i(h)$ and $A_{i+1}(h)$ for $i=1,2,\cdots,m$. In other words, for each $h\in J$,

$$H_{ix}(A_i)+k_iH_{iy}(A_i)\ne 0,\ i=1,2,3,\cdots,m,$$

$$H_{ix}\left(A_{i+1}\right)+k_{i+1}H_{iy}\left(A_{i+1}\right)\neq 0,\ \ i=1,2,3,\cdots,m.$$

Following the idea of the proof of Theorem 2.1 and using mathematical induction, we can prove the following theorem.

Theorem 2.2. *Consider system* (X_ϵ) *with the assumptions* $(\mathbf{H_1})$–$(\mathbf{H_3})$. *If* $M_1\equiv 0$, *then the second order Melnikov function is expressed as*

$$\begin{aligned}M_2(h)=&2\sum_{k=1}^{m}\prod_{i=k}^{m}\frac{K_{i+1,1}}{S_{i1}}\int_{L_h^k}\mu_k\psi_k\omega_k-\mu_k\hat{\omega}_k\\&+2\sum_{k=2}^{m}\prod_{i=k}^{m}\frac{K_{i+1,1}}{S_{i1}}\left(N_{k1}-\frac{K_{k1}}{S_{k-1,1}}T_{k-1,1}\right)\\&\times a_{k\epsilon}(0,h)+\sum_{k=2}^{m}\prod_{i=k}^{m}\frac{K_{i+1,1}}{S_{i1}}\left(K_{k2}-\frac{K_{k1}}{S_{k-1,1}}S_{k-1,2}\right)a_{k\epsilon}^2(0,h),\end{aligned}$$

where

$$\begin{cases}K_{i1}=H_{ix}\left(A_i\right)+k_iH_{iy}\left(A_i\right),\\K_{i2}=H_{ixx}\left(A_i\right)+2k_iH_{ixy}\left(A_i\right)+k_i^2H_{iyy}\left(A_i\right),\\S_{i1}=H_{ix}\left(A_{i+1}\right)+k_{i+1}H_{iy}\left(A_{i+1}\right),\\S_{i2}=H_{ixx}\left(A_{i+1}\right)+2k_{i+1}H_{ixy}\left(A_{i+1}\right)+k_{i+1}^2H_{iyy}\left(A_{i+1}\right),\\T_{i1}=R_{ix}\left(A_{i+1}\right)+k_{i+1}R_{iy}\left(A_{i+1}\right),\\N_{i1}=R_{ix}\left(A_i\right)+k_iR_{iy}\left(A_i\right)\end{cases}\tag{2.31}$$

and

$$\begin{aligned}a_{j\epsilon}(0,h)=&\frac{\partial a_j}{\partial\epsilon}(\epsilon,h)|_{\epsilon=0}\\=&\frac{-1}{S_{j-1,1}}\left(\sum_{k=1}^{j-1}\prod_{i=k}^{j-2}\frac{K_{i+1,1}}{S_{i1}}\int_{L_h^k}\mu_k\omega_k\right),\ \ 2\leq j\leq m+1\end{aligned}\tag{2.32}$$

with $\prod_{i=k}^{j-2}\frac{K_{i+1,1}}{S_{i1}}=1$ *if* $k>j-2$, $a_{1\epsilon}(0,h)=a_{m+1,\epsilon}(0,h)$, $\omega_k=f_kdy-g_kdx$, $\hat{\omega_k}=\hat{f}_kdy-\hat{g}_kdx$, $\psi_k(x,y)=\int_0^{t_k(x,y)}div\left(\chi_k\right)\circ\varphi_k^t\left(x_0^k,y_0^k\right)dt$, $\chi_k=(\mu_kf_k,\mu_kg_k)$, $\varphi_k^t(x_0,y_0)$ *are the solutions of system* (X_0) *passing through* $\left(x_0^k,y_0^k\right)$ *at* $t=0$ *and ending at* (x,y) *with* $t=t_k(x,y)$, $R_{ky}=\mu_kf_k-\psi_kH_{ky}$, *and* $R_{kx}=-\mu_kg_k-\psi_kH_{kx}$.

Remark 2.2.

(i) If $m=2$, $\mu_i=1(i=1,2)$, the Theorem 2.2 was obtained in [31].

(ii) Consider system (X_ϵ) with the assumptions $(\mathbf{H_1})-(\mathbf{H_3})$. The first order Melnikov function is expressed as follows

$$M_1(h)=-\sum_{k=1}^{m}\prod_{i=k}^{m}\frac{K_{i+1,1}}{S_{i1}}\int_{L_h^k}\mu_k\omega_k,\tag{2.33}$$

where K_{i1} and S_{i1} are same as (2.31) showing. The formula (2.33) can be obtained by Remark 2.3 of [27] by the time rescaling.

(iii) If the i-th ray from the origin is the positive y-axis or the negative y-axis, the condition (2.31) has the following forms

$$\begin{aligned} K_{i1} &= H_{iy}(A_i), & K_{i2} &= H_{iyy}(A_i), \\ S_{i-1,1} &= H_{i-1,y}(A_i), & S_{i-1,2} &= H_{i-1,yy}(A_i), \\ T_{i-1,1} &= R_{i-1,y}(A_i), & N_{i1} &= R_{iy}(A_i). \end{aligned}$$

Theorem 2.2 and Remark 2.2(ii) also hold. Thus, we can obtain Theorem 1.1.

3. The proof of Theorem 1.2

In this section, we will give an example to illustrate our results.

The following lemma is helpful for estimating the number of zeros of Melnikov functions.

Lemma 3.1 (Lemma 5, [3]; Lemma 4.5 [6]). *Consider $p+1$ linearly independent functions $f_i : U \subset \mathbb{R} \to \mathbb{R}, i = 0, 1, \ldots, p$, where $U \subseteq R$ is an interval.*

(i) Given p arbitrary values $x_i \in U, i = 1, 2, \ldots, p$, there exist $p+1$ constants $C_i, i = 0, 1, \ldots, p$ such that

$$f(x) := \sum_{i=0}^{p} C_i f_i(x) \tag{3.1}$$

is not the zero function and $f(x_i) = 0$ for $i = 1, 2, \ldots, p$.

(ii) Furthermore, if all f_i are analytical functions on U, then for any nonempty open interval U_0 of U, there exists a function f of the form (3.1)*, which has at least n simple zeros in U_0.*

As an application, let's prove Theorem 1.2.

Proof. Suppose that $0 < |\epsilon| \ll 1$, $a^{\pm}_{i,j}$, $b^{\pm}_{i,j}$, $c^{\pm}_{i,j}$ and $d^{\pm}_{i,j}$ are arbitrary real numbers. Let us consider the following piecewise planar differential system with two zones separated by y-axis:

$$\begin{pmatrix} \dot{x} \\ \\ \dot{y} \end{pmatrix} = \begin{cases} \begin{pmatrix} -y(1+x) + \epsilon f_1^{+}(x,y) + \epsilon^2 f_2^{+}(x,y), \\ x(1+x) + \epsilon g_1^{+}(x,y) + \epsilon^2 g_2^{+}(x,y), \end{pmatrix}, & x > 0, \\ \begin{pmatrix} -y(1+x) + \epsilon f_1^{-}(x,y) + \epsilon^2 f_2^{-}(x,y), \\ x(1+x) + \epsilon g_1^{-}(x,y) + \epsilon^2 g_2^{-}(x,y), \end{pmatrix}, & x \le 0, \end{cases} \tag{E_1}$$

where

$$f_1^{\pm}(x,y) = \sum_{i+j=0}^{2} a^{\pm}_{i,j} x^i y^j,\ g_1^{\pm}(x,y) = \sum_{i+j=0}^{2} b^{\pm}_{i,j} x^i y^j,$$
$$f_2^{\pm}(x,y) = \sum_{i+j=0}^{2} c^{\pm}_{i,j} x^i y^j,\ g_2^{\pm}(x,y) = \sum_{i+j=0}^{2} d^{\pm}_{i,j} x^i y^j.$$

The unperturbed system (E_1) has a family of closed orbits $L_h : x^2 + y^2 = h^2$ for $0 < h < 1$, and satisfies the assumptions $(\hat{\mathbf{A}}_{\mathbf{1}}) - (\hat{\mathbf{A}}_{\mathbf{3}})$. It follows from [14] that the first order Melnikov function $M_1(h) \equiv 0$ for $h \in (0, 1)$ if and only if

$$\begin{cases} a^-_{0,0} = a^+_{0,0}, \\ a^-_{2,0} = a^-_{1,0} - a^+_{0,0}, \\ a^-_{0,2} = b^-_{0,1} - b^-_{1,1}, \\ a^+_{0,2} = b^+_{0,1} - b^+_{1,1}, \\ a^+_{1,0} = 2a^+_{0,0} - a^-_{1,0} - b^+_{0,1} - b^-_{0,1}, \\ a^+_{2,0} = a^+_{0,0} - a^-_{1,0} - b^+_{0,1} - b^-_{0,1}. \end{cases} \tag{3.2}$$

In the following, we will investigate the numbers of zeros of $M_2(h)$ under $M_1(h) \equiv 0$. We shall first obtain an expression of $M_2(h)$ without the condition (3.2).

Since $H^\pm(x, y) = x^2 + y^2 = h^2$, we have $A_1(h) = (0, -h)$, $A_3(h) = (0, h)$ and $H^+_y(A_3(h)) = 2h$, $\mu_\pm(A_3(h)) = 1$ and $f^\pm_1(A_3(h)) = a^\pm_{0,0} + a^\pm_{0,1}h + a^\pm_{0,2}h^2$. By Theorem 1.1, noting $\mu_\pm(x, y) = \frac{1}{1+x}$, we can simplify the formula as follows

$$\begin{aligned} M_2(h) = {} & 2\left(\int_{\widehat{A_1A_3}} \frac{\boldsymbol{\psi}_+\omega^+_1}{1+x} + \int_{\widehat{A_3A_1}} \frac{\boldsymbol{\psi}_-\omega^-_1}{1+x}\right) - 2\left(\int_{\widehat{A_1A_3}} \frac{\omega^+_2}{1+x} + \int_{\widehat{A_3A_1}} \frac{\omega^-_2}{1+x}\right) \\ & + \frac{2}{H^+_y(A_3)}\left[f^+_1(0, h) - f^-_1(0, h) \ -H^+_y(A_3)\psi_+(A_3)\right] \int_{\widehat{A_1A_3}} \frac{\omega^+_1}{1+x} \\ =: {} & N_1(h) + N_2(h) + N_3(h). \end{aligned} \tag{3.3}$$

By calculations, we have

$$\begin{aligned} N_3(h) = {} & \frac{2}{H^+_y(A_3)}\left[f^+_1(0, h) - f^-_1(0, h) \ -H^+_y(A_3)\psi_+(A_3)\right] \int_{\widehat{A_1A_3}} \frac{\omega^+_1}{1+x} \\ = {} & \frac{1}{h}\left[(a^+_{0,0} - a^-_{0,0}) + (a^+_{0,1} - a^-_{0,1})h + (a^+_{0,2} - a^-_{0,2})h^2 - 2h\boldsymbol{\psi}_+(A_3)\right] \\ & \times \sum_{i+j=0}^{2} h^{i+j+1} \int_{-\pi/2}^{\pi/2} \frac{1}{1+h\cos\theta}\left[a^+_{i,j}\cos^{i+1}\theta\sin^j\theta + b^+_{i,j}\cos^i\theta\sin^{j+1}\theta\right] d\theta; \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} N_2(h) = {} & -\left(\int_{\widehat{A_1A_3}} \frac{\omega^+_2}{1+x} + \int_{\widehat{A_3A_1}} \frac{\omega^-_2}{1+x}\right) \\ = {} & (c^+_{0,0} + d^+_{0,1} - c^+_{1,0} + c^+_{2,0} - c^+_{0,2} - d^+_{1,1})(I_{0,0}(h) - \pi) \\ & + (-d^+_{0,1} + c^+_{0,2} + d^+_{1,1})h^2 I_{0,0}(h) \\ & + (c^-_{0,0} + d^-_{0,1} - c^-_{1,0} + c^-_{2,0} - c^-_{0,2} - d^-_{1,1})(J_{0,0}(h) - \pi) \\ & + (-d^-_{0,1} + c^-_{0,2} + d^-_{1,1})h^2 J_{0,0}(h) \\ & + 2(d^+_{0,1} - c^+_{1,0} + c^-_{1,0} - d^-_{0,1} + c^+_{2,0} - c^+_{0,2} - d^+_{1,1} + d^-_{1,1} + c^-_{0,2} - c^-_{2,0})h \\ & - \frac{\pi}{2}(c^+_{2,0} + d^+_{1,1} + c^+_{0,2} + c^-_{2,0} + d^-_{1,1} + c^-_{0,2})h^2, \end{aligned} \tag{3.5}$$

where

$$I_{0,0}(h) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{1 + h\cos\theta} d\theta, \qquad J_{0,0}(h) = \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{1}{1 + h\cos\theta} d\theta. \tag{3.6}$$

By [14] or direct calculations, we can obtain that

$$I_{0,0}(h) = \frac{4}{\sqrt{1-h^2}} \arctan\sqrt{\frac{1-h}{1+h}}, \tag{3.7}$$
$$J_{0,0}(h) = \frac{4}{\sqrt{1-h^2}} \left(\frac{\pi}{2} - \arctan\sqrt{\frac{1-h}{1+h}}\right).$$

Next, we will give an expression of $\boldsymbol{\psi}_\pm(h,\theta)$.

For $h \in (0,1)$, let $(x,y) = (h\cos\alpha, h\sin\alpha)(\alpha \in [-\pi/2, \theta])$. Noting that $\mu_\pm(x,y) = \frac{1}{1+x}$, we get

$$\begin{aligned}
\boldsymbol{\psi}_+(h,\theta) =& \int_0^{t_+(x,y)} \left[\left(\mu_+ f_1^+\right)_x + \left(\mu_+ g_1^+\right)_y\right] dt \\
=& \int_{-\frac{\pi}{2}}^{\theta} \frac{1}{(1+h\cos\alpha)^2} \left[(a_{2,0}^+\cos^2\alpha - a_{0,2}^+\sin^2\alpha)h^2 + 2a_{2,0}^+ h\cos\alpha\right. \\
&\left. +(a_{1,1}^+ - a_{0,1}^+)h\sin\alpha + (a_{1,0}^+ - a_{0,0}^+)\right] d\alpha \\
&+ \int_{-\frac{\pi}{2}}^{\theta} \frac{1}{1+h\cos\alpha}\left[(2b_{0,2}^+\sin\alpha + b_{1,1}^+\cos\alpha)h + b_{0,1}^+\right] d\alpha \\
:=& a_{2,0}^+\boldsymbol{\psi}_1^+ + 2a_{2,0}^+ h\boldsymbol{\psi}_2^+ - a_{0,2}^+(h^2\boldsymbol{\psi}_7^+ - \boldsymbol{\psi}_1^+) + (a_{1,1}^+ - a_{0,1}^+)\boldsymbol{\psi}_3^+ \\
&+ (-a_{0,0}^+ + a_{1,0}^+)\boldsymbol{\psi}_7^+ + 2b_{0,2}^+\boldsymbol{\psi}_4^+ + b_{1,1}^+\boldsymbol{\psi}_5^+ + b_{0,1}^+\boldsymbol{\psi}_6^+,
\end{aligned} \tag{3.8}$$

where $\boldsymbol{\psi}_i^+ = \boldsymbol{\psi}_i^+(h,\theta)$ and the expressions of $\boldsymbol{\psi}_i^+(h,\theta)$ are as follows

$$\begin{aligned}
&\boldsymbol{\psi}_1^+(h,\theta) = \int_{-\frac{\pi}{2}}^{\theta} \frac{h^2\cos^2\alpha}{(1+h\cos\alpha)^2} d\alpha, \ \boldsymbol{\psi}_2^+(h,\theta) = \int_{-\frac{\pi}{2}}^{\theta} \frac{\cos\alpha}{(1+h\cos\alpha)^2} d\alpha, \\
&\boldsymbol{\psi}_3^+(h,\theta) = \int_{-\frac{\pi}{2}}^{\theta} \frac{h\sin\alpha}{(1+h\cos\alpha)^2} d\alpha, \ \boldsymbol{\psi}_4^+(h,\theta) = \int_{-\frac{\pi}{2}}^{\theta} \frac{h\sin\alpha}{1+h\cos\alpha} d\alpha, \\
&\boldsymbol{\psi}_5^+(h,\theta) = \int_{-\frac{\pi}{2}}^{\theta} \frac{h\cos\alpha}{1+h\cos\alpha} d\alpha, \quad \boldsymbol{\psi}_6^+(h,\theta) = \int_{-\frac{\pi}{2}}^{\theta} \frac{1}{1+h\cos\alpha} d\alpha, \\
&\boldsymbol{\psi}_7^+(h,\theta) = \int_{-\frac{\pi}{2}}^{\theta} \frac{1}{(1+h\cos\alpha)^2} d\alpha.
\end{aligned}$$

By direct calculations, we can get

$$\begin{aligned}
\boldsymbol{\psi}_1^+(h,\theta) =& \frac{\pi}{2} + \theta + \left(\frac{2}{(\sqrt{1-h^2}\ )^3} - \frac{4}{\sqrt{1-h^2}}\right)\lambda(h,\theta) \\
&+ \frac{1}{h^2-1}\left(\frac{h\sin\theta}{1+h\cos\theta} + h\right), \\
\boldsymbol{\psi}_2^+(h,\theta) =& \frac{1}{1-h^2} - \frac{2h}{(\sqrt{1-h^2}\ )^3}\lambda(h,\theta) + \frac{\sin\theta}{(1-h^2)(1+h\cos\theta)}, \\
\boldsymbol{\psi}_3^+(h,\theta) =& -\frac{h\cos\theta}{1+h\cos\theta}, \\
\boldsymbol{\psi}_4^+(h,\theta) =& -\ln(1+h\cos\theta),
\end{aligned}$$

$$\begin{aligned}
\boldsymbol{\psi}_5^+(h,\theta) =& \frac{\pi}{2}+\theta-\frac{2}{\sqrt{1-h^2}}\lambda(h,\theta),\\
\boldsymbol{\psi}_6^+(h,\theta) =& \frac{2}{\sqrt{1-h^2}}\lambda(h,\theta),\\
\boldsymbol{\psi}_7^+(h,\theta) =& \frac{2}{(\sqrt{1-h^2}\ )^3}\lambda(h,\theta)+\frac{1}{h^2-1}\left(\frac{h\sin\theta}{1+h\cos\theta}+h\right),
\end{aligned}$$

where

$$\lambda(h,\theta)=\arctan\left(\sqrt{\frac{1-h}{1+h}}\tan\frac{\theta}{2}\right)+\arctan\left(\sqrt{\frac{1-h}{1+h}}\right).$$

We can obtain

$$\begin{aligned}
\boldsymbol{\psi}_+(A_3) =& \frac{1}{1-h^2}\left[a_{0,2}^+h^2I_{0,0}+\left(-a_{0,0}^++a_{1,0}^+-a_{0,2}^+-a_{2,0}^+\right)I_{0,0}\right]\\
&+\frac{2}{1-h^2}\left[(a_{0,0}^+-a_{1,0}^++a_{2,0}^+-a_{0,2}^+)h+a_{0,2}^+h^3\right]\\
&+\left(a_{0,2}^++a_{2,0}^++b_{1,1}^+\right)\pi+\left(b_{0,1}^+-b_{1,1}^+\right)I_{0,0}(h).
\end{aligned}\tag{3.9}$$

For $h\in(0,1)$, let $(x,y)=(h\cos\alpha,h\sin\alpha)(\alpha\in[\pi/2,\theta])$. For $\boldsymbol{\psi}_-(h,\theta)$, we have

$$\begin{aligned}
\boldsymbol{\psi}_-(h,\theta) =& \int_0^{t_-(x,y)}\left[\left(\mu_-f_1^-\right)_x+\left(\mu_-g_1^-\right)_y\right]dt\\
=& \int_{\frac{\pi}{2}}^{\theta}\frac{1}{(1+h\cos\alpha)^2}\left[(a_{2,0}^-\cos^2\alpha-a_{0,2}^-\sin^2\alpha)h^2+2a_{2,0}^-h\cos\alpha\right.\\
&\left.+(a_{1,1}^--a_{0,1}^-)h\sin\alpha+(a_{1,0}^--a_{0,0}^-)\right]d\alpha\\
&+\int_{\frac{\pi}{2}}^{\theta}\frac{1}{1+h\cos\alpha}\left[(2b_{0,2}^-\sin\alpha+b_{1,1}^-\cos\alpha)h+b_{0,1}^-\right]d\alpha\\
:=& a_{2,0}^-\boldsymbol{\psi}_1^-+2a_{2,0}^-h\boldsymbol{\psi}_2^--a_{0,2}^-(h^2\boldsymbol{\psi}_7^--\boldsymbol{\psi}_1^-)+(a_{1,1}^--a_{0,1}^-)\boldsymbol{\psi}_3^-\\
&+(-a_{0,0}^-+a_{1,0}^-)\boldsymbol{\psi}_7^-+2b_{0,2}^-\boldsymbol{\psi}_4^-+b_{1,1}^-\boldsymbol{\psi}_5^-+b_{0,1}^-\boldsymbol{\psi}_6^-,
\end{aligned}\tag{3.10}$$

where $\boldsymbol{\psi}_i^-=\boldsymbol{\psi}_i^-(h,\theta)$ and the expressions of $\boldsymbol{\psi}_i^-(h,\theta)$ are as follows

$$\begin{aligned}
&\boldsymbol{\psi}_1^-(h,\theta)=\int_{\frac{\pi}{2}}^{\theta}\frac{h^2\cos^2\alpha}{(1+h\cos\alpha)^2}d\alpha,\ \boldsymbol{\psi}_2^-(h,\theta)=\int_{\frac{\pi}{2}}^{\theta}\frac{\cos\alpha}{(1+h\cos\alpha)^2}d\alpha,\\
&\boldsymbol{\psi}_3^-(h,\theta)=\int_{\frac{\pi}{2}}^{\theta}\frac{h\sin\alpha}{(1+h\cos\alpha)^2}d\alpha,\ \boldsymbol{\psi}_4^-(h,\theta)=\int_{\frac{\pi}{2}}^{\theta}\frac{h\sin\alpha}{1+h\cos\alpha}d\alpha,\\
&\boldsymbol{\psi}_5^-(h,\theta)=\int_{\frac{\pi}{2}}^{\theta}\frac{h\cos\alpha}{1+h\cos\alpha}d\alpha,\quad \boldsymbol{\psi}_6^-(h,\theta)=\int_{\frac{\pi}{2}}^{\theta}\frac{1}{1+h\cos\alpha}d\alpha,\\
&\boldsymbol{\psi}_7^-(h,\theta)=\int_{\frac{\pi}{2}}^{\theta}\frac{1}{(1+h\cos\alpha)^2}d\alpha.
\end{aligned}$$

Let $\beta=\alpha-\pi$. Then $\beta\in[-\pi/2,\theta-\pi/2]\subseteq[-\pi/2,\pi/2]$, and we can obtain

$$\boldsymbol{\psi}_1^-(h,\ \theta)=-\frac{\pi}{2}+\theta+\left(\frac{2}{(\sqrt{1-h^2})^3}-\frac{4}{\sqrt{1-h^2}}\right)\rho(h,\ \theta)$$

$$+\frac{1}{h^2-1}\left(\frac{h\sin\theta}{1+h\cos\theta}-h\right),$$

$$\boldsymbol{\psi}_2^-(h,\ \theta)=-\frac{1}{1-h^2}+\frac{2h}{(\sqrt{1-h^2})^3}\rho(h,\ \theta)+\frac{\sin\theta}{(1-h^2)\,(1+h\cos\theta)},$$

$$\boldsymbol{\psi}_3^-(h,\ \theta)=\frac{-h\cos\theta}{1+h\cos\theta},$$

$$\boldsymbol{\psi}_4^-(h,\ \theta)=-\ln(1+h\cos\theta),$$

$$\boldsymbol{\psi}_5^-(h,\ \theta)=-\frac{\pi}{2}+\theta-\frac{2}{\sqrt{1-h^2}}\rho(h,\ \theta),$$

$$\boldsymbol{\psi}_6^-(h,\ \theta)=\frac{2}{\sqrt{1-h^2}}\rho(h,\ \theta),$$

$$\boldsymbol{\psi}_7^-(h,\ \theta)=\frac{2}{(\sqrt{1-h^2})^3}\rho(h,\ \theta)+\frac{1}{h^2-1}\left(\frac{h\sin\theta}{1+h\cos\theta}-h\right),$$

where

$$\rho(h,\ \theta)=\arctan\left(\sqrt{\frac{1-h}{1+h}}\tan\frac{\theta}{2}\right)-\arctan\sqrt{\frac{1-h}{1+h}}.$$

Combing the above expressions, we obtain that

$$\begin{aligned}N_1(h)&=2\left(\int_{\widehat{A_1A_3}}\frac{\boldsymbol{\psi}_+\omega_1^+}{1+x}+\int_{\widehat{A_3A_1}}\frac{\boldsymbol{\psi}_-\omega_1^-}{1+x}\right),\\&=2\sum_{i+j=0}^{2}h^{i+j+1}\Bigg(\int_{-\pi/2}^{\pi/2}\frac{\boldsymbol{\psi}_+(h,\theta)}{1+h\cos\theta}\left[a_{i,j}^+\cos^{i+1}\theta\sin^j\theta+b_{i,j}^+\cos^i\theta\sin^{j+1}\theta\right]d\theta\\&\quad+\int_{\pi/2}^{3\pi/2}\frac{\boldsymbol{\psi}_-(h,\theta)}{1+h\cos\theta}\left[a_{i,j}^-\cos^{i+1}\theta\sin^j\theta+b_{i,j}^-\cos^i\theta\sin^{j+1}\theta\right]d\theta\Bigg).\end{aligned}\qquad(3.11)$$

We know that $\boldsymbol{\psi}_+(h,\theta)$ is analytic for $(h,\theta)\in[0,1)\times[-\pi/2,\pi/2]$, and it is a linear combination of $\boldsymbol{\psi}_i^+(h,\theta)$ for $i=1,\cdots,7$ with coefficients $a_{i,j}^+$ and $b_{i,j}^+$ for $0\le i+j\le 2$. Also the function $\boldsymbol{\psi}_-(h,\theta)$ is analytic for $(h,\theta)\in[0,1)\times[\pi/2,3\pi/2]$, and it is a linear combination of $\boldsymbol{\psi}_i^-(h,\theta)$ for $i=1,\cdots,7$ with coefficients $a_{i,j}^-$ and $b_{i,j}^-$ for $0\le i+j\le 2$. These mean that $N_1(h)$ is analytic on $h\in[0,1)$.

With the help of Maple software and taking Taylor series expansion at $h=0$ for $\boldsymbol{\psi}_\pm(h,\ \theta)$, $I_{0,0}(h)$ and $J_{0,0}(h)$ respectively, then we have

$$\boldsymbol{\psi}_+(h,\theta)=\sum_{i=0}^{8}\gamma_i(\theta)h^i+\gamma_9h^9(1+\xi_1(h)),\ 0\le h\ll 1,\qquad(3.12)$$

$$\boldsymbol{\psi}_-(h,\theta)=\sum_{i=0}^{8}\tau_i(\theta)h^i+\tau_9h^9(1+\xi_2(h)),\ 0\le h\ll 1,\qquad(3.13)$$

$$\begin{aligned}I_{0,0}(h)&=\pi-2\,h+\frac{\pi}{2}\ h^2-\frac{4}{3}h^3+\frac{3\pi}{8}\ h^4-\frac{16}{15}h^5+\frac{5\,\pi}{16}h^6-\frac{32}{35}h^7+\frac{35\,\pi}{128}h^8\\&\quad-\frac{256}{315}h^9(1+\xi_3(h)),\\J_{0,0}(h)&=\pi+2\,h+\frac{\pi}{2}\,h^2+\frac{4}{3}h^3+\frac{3\pi}{8}\ h^4+\frac{16}{15}h^5+\frac{5\,\pi}{16}h^6+\frac{32}{35}h^7+\frac{35\,\pi}{128}h^8\end{aligned}$$

$$+ \frac{256}{315} h^9(1 + \xi_4(h)), \tag{3.14}$$

where the expressions of $\gamma_i(\theta)$ and $\tau_i(\theta)$ are listed Appendix $\mathcal{A}$, and $\xi_i(h) = o(1)(i = 1, 2, 3, 4)$ is analytic $h \in [0, 1)$.

With the help of Maple software and substituting (3.4)-(3.14) into (3.3), for $0 \le h \ll 1$, we have

$$\begin{aligned} M_2(h) &= N_1(h) + N_2(h) + N_3(h) \\ &= \sum_{i=1}^{8} K_i h^i + K_9 h^9(1 + \xi_5(h)), \end{aligned} \tag{3.15}$$

where $\xi_5(h) = o(1)$ is analytic $h \in [0, 1)$ and the expression of $K_i(i = 1, 2, \cdots, 9)$ are listed in Appendix $\mathcal{B}$.

It is easy to see that the functions $h, h^2, \cdots, h^8, h^9(1 + \xi_5(h))$ are linearly independent for $0 \le h \ll 1$. Hence, in view of Lemma 3.1, there exist $K_i^*(i = 1, 2, \cdots, 9)$ such that $M_2(h) = \sum_{i=1}^{8} K_i h^i + K_9 h^9(1 + \xi_5(h))$ with the condition (3.2) has at least 8 zeros for $0 \le h \ll 1$. Set $a_{i,j}^{\pm} = a_{i,j}^{*\pm}$, $b_{i,j}^{\pm} = b_{i,j}^{*\pm}$, $c_{i,j}^{\pm} = c_{i,j}^{*\pm}$ and $d_{i,j}^{\pm} = d_{i,j}^{*\pm}$. Meanwhile, set $a_{i,j}^{*\pm} = 0$, $b_{i,j}^{*\pm} = 0$, $c_{i,j}^{*\pm} = 0$ and $d_{i,j}^{*\pm} = 0$ for $i, j = 0, 1, 2$ except $\{a_{0,0}^{*\pm} = 1,\ b_{1,1}^{*\pm} = 1,\ a_{0,2}^{*+} = -1,\ a_{0,2}^{*-} = -\frac{1}{2},\ a_{1,0}^{*+} = \frac{1}{2},\ a_{1,0}^{*-} = 1,\ a_{2,0}^{*+} = -\frac{1}{2},\ b_{0,1}^{*-} = \frac{1}{2}\}$ and $\delta = (a_{0,1}^{*+}, b_{0,2}^{*+}, b_{2,0}^{*-}, c_{0,0}^{*+}, c_{0,0}^{*-}, c_{1,0}^{*+}, c_{0,2}^{*+}, c_{2,0}^{*+}, c_{0,2}^{*-})$. Thus, we get the system (E_1^*). For system (E_1^*), we have

$$M_2(h) = \sum_{i=1}^{8} K_i^* h^i + K_9^* h^9(1 + \xi_5(h)),$$

where $K_i^*(i = 1, 2, \cdots, 9)$ has the following form

$$\begin{aligned}
K_1^* =& 2\pi - 2a_{0,1}^{*+} + 4c_{0,0}^{*-} - 4c_{0,0}^{*+}; \\
K_2^* =& 1/4\pi^2 + \pi a_{0,1}^{*+} - 2\pi b_{0,2}^{*+} + \pi c_{0,0}^{*+} - \pi c_{1,0}^{*+} + \pi c_{0,0}^{*-} + 11; \\
K_3^* =& -11\pi/8 + 1/9(4b_{2,0}^{*-} - 20a_{0,1}^{*+} + 34b_{0,2}^{*+}) \\
& + 4/3(-2c_{0,0}^{*+} - c_{0,2}^{*+} + 2c_{1,0}^{*+} - 2c_{2,0}^{*+} + 2c_{0,0}^{*-} + c_{0,2}^{*-}); \\
K_4^* =& \pi/32(17a_{0,1}^{*+} - b_{2,0}^{*-} + b_{0,2}^{*+}) + \pi/4(3c_{0,0}^{*-} + c_{0,2}^{*-} + 3c_{0,0}^{*+} + c_{0,2}^{*+} - 3c_{1,0}^{*+} + 3c_{2,0}^{*+}); \\
K_5^* =& 1/75(-104a_{0,1}^{*+} - 24b_{2,0}^{*-} - 4b_{0,2}^{*+}) + 8/15(-4c_{0,0}^{*+} - c_{0,2}^{*+} + 4c_{1,0}^{*+} - 4c_{2,0}^{*+} + 4c_{0,0}^{*-} + c_{0,2}^{*-}); \\
K_6^* =& \pi/96(-13b_{2,0}^{*-} + 37a_{0,1}^{*+} + b_{0,2}^{*+}) + \pi/8(5c_{0,0}^{*-} + c_{0,2}^{*-} + 5c_{0,0}^{*+} + c_{0,2}^{*+} - 5c_{1,0}^{*+} + 5c_{2,0}^{*+}); \\
K_7^* =& 1/735(-800a_{0,1}^{*+} - 352b_{2,0}^{*-} - 16b_{0,2}^{*+}) + 32/105(c_{0,2}^{*-} - c_{0,2}^{*+}) \\
& + 64/35(-c_{0,0}^{*+} + c_{1,0}^{*+} - c_{2,0}^{*+} + c_{0,0}^{*-}); \\
K_8^* =& \pi/1024(-165b_{2,0}^{*-} + 325a_{0,1}^{*+} + 5b_{0,2}^{*+}) \\
& + 5\pi/64(7c_{0,0}^{*-} + c_{0,2}^{*-} + 7c_{0,0}^{*+} + c_{0,2}^{*+} - 7c_{1,0}^{*+} + 7c_{2,0}^{*+}); \\
K_9^* =& 1/2835(-2624a_{0,1}^{*+} - 1472b_{2,0}^{*-} - 32b_{0,2}^{*+}) \\
& + 64/315(-8c_{0,0}^{*+} - c_{0,2}^{*+} + 8c_{1,0}^{+} - 8c_{2,0}^{*+} + 8c_{0,0}^{*-} + c_{0,2}^{*-}).
\end{aligned}$$

Further, we can get

$$\det \frac{D(K_1^*,\ K_2^*,\ K_3^*,\ K_4^*,\ K_5^*,\ K_6^*,\ K_7^*,\ K_8^*,\ K_9^*)}{D(a_{0,1}^{*+},\ b_{0,2}^{*+},\ b_{2,0}^{*-}, c_{0,0}^{*+},\ c_{0,0}^{*-},\ c_{1,0}^{*+},\ c_{0,2}^{*+},\ c_{2,0}^{*+},\ c_{0,2}^{*-})} = \frac{1024\pi^4}{93767625} \neq 0. \tag{3.16}$$

Hence we can conclude that there exists a system of the form (E_1^*), which has at least 8 limit cycles by Remark 2.1 of [12], multiplicity taken into account. This ends the proof. □

Declarations

Conflict of Interest. The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data Availability Statement. Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Appendix $\mathcal{A}$

The expressions of $\gamma_i(\theta)$ and $\tau_i(\theta)(i = 1, \cdots, 9)$ in formula (3.12)-(3.13)

$$
\begin{aligned}
\gamma_0(\theta) =& 1/2\left(4a_{0,0}^{+} + 4a_{0,2}^{+} - 4a_{1,0}^{+} + 4a_{2,0}^{+} - 4b_{0,1}^{+} + 4b_{1,1}^{+}\right)\arctan\left(\frac{-1+\cos\theta}{\sin\theta}\right)\\
&+ 1/2\left(-a_{0,0}^{+} + a_{1,0}^{+} + b_{0,1}^{+}\right)\pi + \theta\left(b_{1,1}^{+} + a_{0,2}^{+} + a_{2,0}^{+}\right)\\
\gamma_1(\theta) =& \left(2a_{0,0}^{+} - 2a_{1,0}^{+} + 2a_{2,0}^{+} - b_{0,1}^{+} + b_{1,1}^{+}\right)\sin\theta\\
&+ \left(291/8b_{0,2}^{+} - 3465\pi/256 - a_{1,1}^{+}\right)\cos\theta\\
&+ 2a_{0,0}^{+} - 2a_{1,0}^{+} + 2a_{2,0}^{+} - b_{0,1}^{+} + b_{1,1}^{+}\\
\gamma_2(\theta) =& 1/4\left(4b_{1,1}^{+} + 12a_{0,0}^{+} + 4a_{0,2}^{+} - 12a_{1,0}^{+} + 12a_{2,0}^{+} - 4b_{0,1}^{+}\right)\arctan\left(\frac{-1+\cos\theta}{\sin\theta}\right)\\
&+ 1/4\left(-299/2b_{0,2}^{+} + 3465\pi/64 + 4a_{1,1}^{+}\right)(\cos\theta)^2\\
&- 1/2\left(b_{1,1}^{+} + 3a_{0,0}^{+} - a_{0,2}^{+} - 3a_{1,0}^{+} + 3a_{2,0}^{+} - b_{0,1}^{+}\right)\cos\theta\sin(x)\\
&- 1/4\pi\left(b_{1,1}^{+} + 3a_{0,0}^{+} + a_{0,2}^{+} - 3a_{1,0}^{+} + 3a_{2,0}^{+} - b_{0,1}^{+}\right)\\
\gamma_3(\theta) =& 1/3\Big(\left(b_{1,1}^{+} + 4a_{0,0}^{+} - 2a_{0,2}^{+} - 4a_{1,0}^{+} + 4a_{2,0}^{+} - b_{0,1}^{+}\right)(\cos\theta)^2\\
&+2b_{1,1}^{+} + 8a_{0,0}^{+} + 2a_{0,2}^{+} - 8a_{1,0}^{+} + 8a_{2,0}^{+} - 2b_{0,1}^{+}\Big)\sin\theta\\
&+ 1/3\left(905/8b_{0,2}^{+} - 10395\pi/256 - 3a_{1,1}^{+}\right)(\cos\theta)^3\\
&+ 1/3(2b_{1,1}^{+} + 8a_{0,0}^{+} + 2a_{0,2}^{+} - 8a_{1,0}^{+} + 8a_{2,0}^{+} - 2b_{0,1}^{+})\\
\gamma_4(\theta) =& 1/16\left(12b_{1,1}^{+} + 60a_{0,0}^{+} + 12a_{0,2}^{+} - 60a_{1,0}^{+} + 60a_{2,0}^{+} - 12b_{0,1}^{+}\right)\\
&\times\arctan\left(\frac{-1+\cos\theta}{\sin\theta}\right) + 1/16\left(-606b_{0,2}^{+} + 3465\pi/16 + 16a_{1,1}^{+}\right)(\cos\theta)^4\\
&- 1/4\left(b_{1,1}^{+} + 5a_{0,0}^{+} - 3a_{0,2}^{+} - 5a_{1,0}^{+} + 5a_{2,0}^{+} - b_{0,1}^{+}\right)(\cos\theta)^3\sin\theta\\
&- 3/8\left(b_{1,1}^{+} + 5a_{0,0}^{+} + a_{0,2}^{+} - 5a_{1,0}^{+} + 5a_{2,0}^{+} - b_{0,1}^{+}\right)\cos\theta\sin\theta\\
&- 3\pi/16\left(b_{1,1}^{+} + 5a_{0,0}^{+} + a_{0,2}^{+} - 5a_{1,0}^{+} + 5a_{2,0}^{+} - b_{0,1}^{+}\right)\\
\gamma_5(\theta) =& 1/15\Big(\left(3b_{1,1}^{+} + 18a_{0,0}^{+} - 12a_{0,2}^{+} - 18a_{1,0}^{+} + 18a_{2,0}^{+} - 3b_{0,1}^{+}\right)(\cos\theta)^4\\
&+ \left(4b_{1,1}^{+} + 24a_{0,0}^{+} + 4a_{0,2}^{+} - 24a_{1,0}^{+} + 24a_{2,0}^{+} - 4b_{0,1}^{+}\right)(\cos(x))^2\\
&+8b_{1,1}^{+} + 48a_{0,0}^{+} + 8a_{0,2}^{+} - 48a_{1,0}^{+} + 48a_{2,0}^{+} - 8b_{0,1}^{+}\Big)\sin\theta
\end{aligned}
$$

$$
\begin{aligned}
&+ 1/15\left(4557b^{+}_{0,2}/8 - 51975\pi/256 - 15a^{+}_{1,1}\right)(\cos\theta)^5\\
&+ 1/15(8b^{+}_{1,1} + 48a^{+}_{0,0} + 8a^{+}_{0,2} - 48a^{+}_{1,0} + 48a^{+}_{2,0} - 8b^{+}_{0,1})\\
\gamma_6(\theta) =& 1/96(60b^{+}_{1,1} + 420a^{+}_{0,0} + 60a^{+}_{0,2} - 420a^{+}_{1,0} + 420a^{+}_{2,0} - 60b^{+}_{0,1})\\
&\times \arctan\left(\frac{-1+\cos\theta}{\sin\theta}\right) + 1/96\left(-3652b^{+}_{0,2} + 10395\pi/8 + 96a^{+}_{1,1}\right)(\cos\theta)^6\\
&- 1/6\left(b^{+}_{1,1} + 7a^{+}_{0,0} - 5a^{+}_{0,2} - 7a^{+}_{1,0} + 7a^{+}_{2,0} - b^{+}_{0,1}\right)(\cos\theta)^5\sin\theta\\
&- 5/24\left(b^{+}_{1,1} + 7a^{+}_{0,0} + a^{+}_{0,2} - 7a^{+}_{1,0} + 7a^{+}_{2,0} - b^{+}_{0,1}\right)(\cos\theta)^3\sin\theta\\
&- 5/16\left(b^{+}_{1,1} + 7a^{+}_{0,0} + a^{+}_{0,2} - 7a^{+}_{1,0} + 7a^{+}_{2,0} - b^{+}_{0,1}\right)\cos\theta\sin\theta\\
&- 5\pi/32\left(b^{+}_{1,1} + 7a^{+}_{0,0} + a^{+}_{0,2} - 7a^{+}_{1,0} + 7a^{+}_{2,0} - b^{+}_{0,1}\right)\\
\gamma_7(\theta) =& 1/35\Big(\left(5b^{+}_{1,1} + 40a^{+}_{0,0} - 30a^{+}_{0,2} - 40a^{+}_{1,0} + 40a^{+}_{2,0} - 5b^{+}_{0,1}\right)(\cos\theta)^6\\
&+ \left(6b^{+}_{1,1} + 48a^{+}_{0,0} + 6a^{+}_{0,2} - 48a^{+}_{1,0} + 48a^{+}_{2,0} - 6b^{+}_{0,1}\right)(\cos\theta)^4\\
&+ \left(8b^{+}_{1,1} + 64a^{+}_{0,0} + 8a^{+}_{0,2} - 64a^{+}_{1,0} + 64a^{+}_{2,0} - 8b^{+}_{0,1}\right)(\cos\theta)^2\\
&+16b^{+}_{1,1} + 128a^{+}_{0,0} + 16a^{+}_{0,2} - 128a^{+}_{1,0} + 128a^{+}_{2,0} - 16b^{+}_{0,1}\Big)\sin\theta\\
&+ 1/35\left(10665/8b^{+}_{0,2} - 121275\pi/256 - 35a^{+}_{1,1}\right)(\cos\theta)^7\\
&+ 1/35(16b^{+}_{1,1} + 128a^{+}_{0,0} + 16a^{+}_{0,2} - 128a^{+}_{1,0} + 128a^{+}_{2,0} - 16b^{+}_{0,1}),\\
\gamma_8(\theta) =& 1/768(420b^{+}_{1,1} + 3780a^{+}_{0,0} + 420a^{+}_{0,2} - 3780a^{+}_{1,0} + 3780a^{+}_{2,0} - 420b^{+}_{0,1})\\
&\times \arctan\left(\frac{-1+\cos\theta}{\sin\theta}\right) + 1/768\big(-29280b^{+}_{0,2} + 10395\pi + 768a^{+}_{1,1}\big)(\cos\theta)^8\\
&- 1/8\left(b^{+}_{1,1} + 9a^{+}_{0,0} - 7a^{+}_{0,2} - 9a^{+}_{1,0} + 9a^{+}_{2,0} - b^{+}_{0,1}\right)(\cos\theta)^7\sin\theta\\
&- 7/48\left(b^{+}_{1,1} + 9a^{+}_{0,0} + a^{+}_{0,2} - 9a^{+}_{1,0} + 9a^{+}_{2,0} - b^{+}_{0,1}\right)(\cos\theta)^5\sin\theta\\
&- 35/192\left(b^{+}_{1,1} + 9a^{+}_{0,0} + a^{+}_{0,2} - 9a^{+}_{1,0} + 9a^{+}_{2,0} - b^{+}_{0,1}\right)(\cos\theta)^3\sin\theta\\
&- 35/128\left(b^{+}_{1,1} + 9a^{+}_{0,0} + a^{+}_{0,2} - 9a^{+}_{1,0} + 9a^{+}_{2,0} - b^{+}_{0,1}\right)\cos\theta\sin\theta\\
&- 35\pi/256\left(b^{+}_{1,1} + 9a^{+}_{0,0} + a^{+}_{0,2} - 9a^{+}_{1,0} + 9a^{+}_{2,0} - b^{+}_{0,1}\right)\\
\gamma_9(\theta) =& 1/315\Big(\left(35b^{+}_{1,1} + 350a^{+}_{0,0} - 280a^{+}_{0,2} - 350a^{+}_{1,0} + 350a^{+}_{2,0} - 35b^{+}_{0,1}\right)(\cos\theta)^8\\
&+ \left(40b^{+}_{1,1} + 400a^{+}_{0,0} + 40a^{+}_{0,2} - 400a^{+}_{1,0} + 400a^{+}_{2,0} - 40b^{+}_{0,1}\right)(\cos\theta)^6\\
&+ \left(48b^{+}_{1,1} + 480a^{+}_{0,0} + 48a^{+}_{0,2} - 480a^{+}_{1,0} + 480a^{+}_{2,0} - 48b^{+}_{0,1}\right)(\cos\theta)^4\\
&+ \left(64b^{+}_{1,1} + 640a^{+}_{0,0} + 64a^{+}_{0,2} - 640a^{+}_{1,0} + 640a^{+}_{2,0} - 64b^{+}_{0,1}\right)(\cos\theta)^2\\
&+128b^{+}_{1,1} + 1280a^{+}_{0,0} + 128a^{+}_{0,2} - 1280a^{+}_{1,0} + 1280a^{+}_{2,0} - 128b^{+}_{0,1}\Big)\sin\theta\\
&+ 1/315\left(96145b^{+}_{0,2}/8 - 1091475\pi/256 - 315a^{+}_{1,1}\right)(\cos\theta)^9\\
&+ 128/315(b^{+}_{1,1} + a^{+}_{0,2} - b^{+}_{0,1}) + 256/63(a^{+}_{0,0} - a^{+}_{1,0} + a^{+}_{2,0})\\
\tau_0(\theta) =& 1/2\left(4a^{-}_{0,0} + 4a^{-}_{0,2} - 4a^{-}_{1,0} + 4a^{-}_{2,0} - 4b^{-}_{0,1} + 4b^{-}_{1,1}\right)\arctan\left(\frac{-1+\cos\theta}{\sin\theta}\right)\\
&+ 1/2\left(a^{-}_{0,0} - a^{-}_{1,0} - b^{-}_{0,1}\right)\pi + \theta\left(b^{-}_{1,1} + a^{-}_{0,2} + a^{-}_{2,0}\right);\\
\tau_1(\theta) =& \left(2a^{-}_{0,0} - 2a^{-}_{1,0} + 2a^{-}_{2,0} - b^{-}_{0,1} + b^{-}_{1,1}\right)\sin\theta + \left(a^{-}_{0,1} - a^{-}_{1,1} - 2b^{-}_{0,2}\right)\cos\theta\\
&- 2a^{-}_{0,0} + 2a^{-}_{1,0} - 2a^{-}_{2,0} + b^{-}_{0,1} - b^{-}_{1,1};
\end{aligned}
$$

$$
\begin{aligned}
\tau_2(\theta) =& 1/4\left(4b_{1,1}^- + 12a_{0,0}^- + 4a_{0,2}^- - 12a_{1,0}^- - 20a_{2,0}^- - 4b_{0,1}^-\right)\arctan\left(\frac{-1+\cos\theta}{\sin\theta}\right)\\
&+ 1/4\left(4b_{0,2}^- - 4a_{0,1}^- + 4a_{1,1}^-\right)(\cos(x))^2\\
&- 1/2\left(b_{1,1}^- + 3a_{0,0}^- - a_{0,2}^- - 3a_{1,0}^- + 3a_{2,0}^- - b_{0,1}^-\right)\cos\theta\sin\theta\\
&+ \pi/4\left(3a_{0,0}^- + a_{0,2}^- - 3a_{1,0}^- - 5a_{2,0}^- - b_{0,1}^- + b_{1,1}^-\right);\\
\tau_3(\theta) =& 1/3\Big(\left(b_{1,1}^- + 4a_{0,0}^- - 2a_{0,2}^- - 4a_{1,0}^- + 4a_{2,0}^- - b_{0,1}^-\right)(\cos\theta)^2\\
&+2b_{1,1}^- + 8a_{0,0}^- + 2a_{0,2}^- - 8a_{1,0}^- - 4a_{2,0}^- - 2b_{0,1}^-\Big)\sin\theta\\
&+ 1/3\left(3a_{0,1}^- - 3a_{1,1}^- - 2b_{0,2}^-\right)(\cos\theta)^3\\
&1/3(-2b_{1,1}^- - 8a_{0,0}^- - 2a_{0,2}^- + 8a_{1,0}^- + 4a_{2,0}^- + 2b_{0,1}^-);\\
\tau_4(\theta) =& 1/16\left(12b_{1,1}^- + 60a_{0,0}^- + 12a_{0,2}^- - 60a_{1,0}^- - 132a_{2,0}^- - 12b_{0,1}^-\right)\\
&\times\arctan\left(\frac{-1+\cos\theta}{\sin\theta}\right) + 1/16\left(8b_{0,2}^- - 16a_{0,1}^- + 16a_{1,1}^-\right)(\cos\theta)^4\\
&- 1/4\left(b_{1,1}^- + 5a_{0,0}^- - 3a_{0,2}^- - 5a_{1,0}^- + 5a_{2,0}^- - b_{0,1}^-\right)\sin\theta(\cos\theta)^3\\
&- 3/8\left(b_{1,1}^- + 5a_{0,0}^- + a_{0,2}^- - 5a_{1,0}^- - a_{2,0}^-/3 - b_{0,1}^-\right)\sin\theta\cos\theta\\
&+ 3\pi/16\left(b_{1,1}^- + 5a_{0,0}^- + a_{0,2}^- - 5a_{1,0}^- - 11a_{2,0}^- - b_{0,1}^-\right);\\
\tau_5(\theta) =& 1/15\Big(\left(3b_{1,1}^- + 18a_{0,0}^- - 12a_{0,2}^- - 18a_{1,0}^- + 18a_{2,0}^- - 3b_{0,1}^-\right)(\cos\theta)^4\\
&+ \left(4b_{1,1}^- + 24a_{0,0}^- + 4a_{0,2}^- - 24a_{1,0}^- + 4a_{2,0}^- - 4b_{0,1}^-\right)(\cos\theta)^2\\
&+8b_{1,1}^- + 48a_{0,0}^- + 8a_{0,2}^- - 48a_{1,0}^- - 52a_{2,0}^- - 8b_{0,1}^-\Big)\sin\theta\\
&+ 1/15\left(15a_{0,1}^- - 15a_{1,1}^- - 6b_{0,2}^-\right)(\cos\theta)^5\\
&+ 1/15(-8b_{1,1}^- - 80a_{0,0}^- - 8a_{0,2}^- + 80a_{1,0}^- + 52a_{2,0}^- + 8b_{0,1}^-15);\\
\tau_6(\theta) =& 1/96(60b_{1,1}^- + 420a_{0,0}^- + 60a_{0,2}^- - 420a_{1,0}^- - 1020a_{2,0}^- - 60b_{0,1}^-)\\
&\times\arctan\left(\frac{-1+\cos\theta}{\sin\theta}\right) + 1/96\big(32b_{0,2}^- - 96a_{0,1}^- + 96a_{1,1}^-\big)(\cos\theta)^6\\
&- 1/6\left(b_{1,1}^- + 7a_{0,0}^- - 5a_{0,2}^- - 7a_{1,0}^- + 7a_{2,0}^- - b_{0,1}^-\right)\sin\theta(\cos\theta)^5\\
&- 5/24\left(b_{1,1}^- + 7a_{0,0}^- + a_{0,2}^- - 7a_{1,0}^- + 11/5a_{2,0}^- - b_{0,1}^-\right)\sin\theta(\cos\theta)^3\\
&- 5/16\left(b_{1,1}^- + 7a_{0,0}^- + a_{0,2}^- - 7a_{1,0}^- - 21/5a_{2,0}^- - b_{0,1}^-\right)\sin\theta\cos\theta\\
&+ 5\pi/32\left(b_{1,1}^- + 7a_{0,0}^- + a_{0,2}^- - 7a_{1,0}^- - 17a_{2,0}^- - b_{0,1}^-\right);\\
\tau_7(\theta) =& 1/35\Big(\left(5b_{1,1}^- + 40a_{0,0}^- - 30a_{0,2}^- - 40a_{1,0}^- + 40a_{2,0}^- - 5b_{0,1}^-\right)(\cos\theta)^6\\
&+ \left(6b_{1,1}^- + 48a_{0,0}^- + 6a_{0,2}^- - 48a_{1,0}^- + 20a_{2,0}^- - 6b_{0,1}^-\right)(\cos\theta)^4\\
&+ \left(8b_{1,1}^- + 64a_{0,0}^- + 8a_{0,2}^- - 64a_{1,0}^- - 20a_{2,0}^- - 8b_{0,1}^-\right)(\cos\theta)^2\\
&+16b_{1,1}^- + 128a_{0,0}^- + 16a_{0,2}^- - 128a_{1,0}^- - 180a_{2,0}^- - 16b_{0,1}^-\Big)\sin\theta\\
&+ 1/35\left(35a_{0,1}^- - 35a_{1,1}^- - 10b_{0,2}^-\right)(\cos\theta)^7\\
&+ 1/35(-16b_{1,1}^- - 128a_{0,0}^- - 16a_{0,2}^- + 128a_{1,0}^- + 36a_{2,0}^- + 16b_{0,1}^-);\\
\tau_8(\theta) =& 1/768(420b_{1,1}^- + 3780a_{0,0}^- + 420a_{0,2}^- - 3780a_{1,0}^- - 9660a_{2,0}^- - 420b_{0,1}^-)
\end{aligned}
$$

$$\times \arctan\left(\frac{-1+\cos\theta}{\sin\theta}\right) + 1/768\big(192b_{0,2}^- - 768a_{0,1}^- + 768a_{1,1}^-\big)(\cos\theta)^8$$
$$- 1/8\left(b_{1,1}^- + 9a_{0,0}^- - 7a_{0,2}^- - 9a_{1,0}^- + 9a_{2,0}^- - b_{0,1}^-\right)\sin\theta(\cos\theta)^7$$
$$- 7/48\left(b_{1,1}^- + 9a_{0,0}^- + a_{0,2}^- - 9a_{1,0}^- + 31/7a_{2,0}^- - b_{0,1}^-\right)\sin\theta(\cos\theta)^5$$
$$- 35/192\left(b_{1,1}^- + 9a_{0,0}^- + a_{0,2}^- - 9a_{1,0}^- - 37/35a_{2,0}^- - b_{0,1}^-\right)\sin\theta(\cos\theta)^3$$
$$- 35/128\left(b_{1,1}^- + 9a_{0,0}^- + a_{0,2}^- - 9a_{1,0}^- - 293/35a_{2,0}^- - b_{0,1}^-\right)\sin\theta\cos\theta$$
$$+ 35\pi/256\left(b_{1,1}^- + 9a_{0,0}^- + a_{0,2}^- - 9a_{1,0}^- - 23a_{2,0}^- - b_{0,1}^-\right);$$

$$\tau_9(\theta) = 1/315\Big(\big(35b_{1,1}^- + 350a_{0,0}^- - 280a_{0,2}^- - 350a_{1,0}^- + 350a_{2,0}^- - 35b_{0,1}^-\big)(\cos\theta)^8$$
$$+ \left(40b_{1,1}^- + 400a_{0,0}^- + 40a_{0,2}^- - 400a_{1,0}^- + 220a_{2,0}^- - 40b_{0,1}^-\right)(\cos\theta)^6$$
$$+ \left(48b_{1,1}^- + 480a_{0,0}^- + 48a_{0,2}^- - 480a_{1,0}^- + 12a_{2,0}^- - 48b_{0,1}^-\right)(\cos\theta)^4$$
$$+ \left(64b_{1,1}^- + 640a_{0,0}^- + 64a_{0,2}^- - 640a_{1,0}^- - 404a_{2,0}^- - 64b_{0,1}^-\right)(\cos\theta)^2$$
$$+ 128b_{1,1}^- + 1280a_{0,0}^- + 128a_{0,2}^- - 1280a_{1,0}^- - 2068a_{2,0}^- - 128b_{0,1}^-\Big)\sin\theta$$
$$+ 1/315\big(315a_{0,1}^- - 315a_{1,1}^- - 70b_{0,2}^-\big)(\cos\theta)^9$$
$$+ 256/63(a_{1,0}^- - a_{0,0}^-) + 1/315(2068a_{2,0}^- + 128b_{0,1}^- - 128b_{1,1}^- - 128a_{0,2}^-).$$

Appendix $\mathcal{B}$

The expressions of $K_i (i = 1, \cdots, 9)$ in formula (3.15)

$$K_1 = (4{a_{1,0}^+}^2 + (-2a_{0,0}^+ + 4a_{0,2}^+ - 4a_{2,0}^+ + 4b_{1,1}^+)a_{1,0}^+ - 4{b_{0,1}^+}^2 + (6a_{0,0}^+ + 4a_{0,2}^+ - 4a_{2,0}^+$$
$$+ 4b_{1,1}^+)b_{0,1}^+ - 2{a_{0,0}^+}^2 + (-4a_{0,2}^+ + 4a_{2,0}^+ - 4b_{1,1}^+)a_{0,0}^+ - 2a_{0,0}^-(-a_{1,0}^- - b_{0,1}^-$$
$$+ a_{0,0}^-))\pi + (2a_{0,1}^- - 2a_{0,1}^+ + 4b_{0,0}^+)a_{1,0}^+ + (-2a_{0,1}^- + 2a_{0,1}^+ + 4b_{0,0}^+)b_{0,1}^+$$
$$- 4a_{0,0}^+b_{0,0}^+ + (-2a_{0,2}^+ + 2a_{2,0}^+ - 2b_{1,1}^+)a_{0,1}^+ + 2a_{0,2}^+a_{0,1}^- - 2a_{2,0}^+a_{0,1}^-$$
$$+ 2b_{1,1}^+a_{0,1}^- - 4a_{0,0}^-b_{0,0}^- + (4a_{1,0}^- + 4b_{0,1}^-)b_{0,0}^- - 4c_{0,0}^+ + 4c_{0,0}^-;$$

$$K_2 = 1/2({a_{0,0}^+}^2 + (-2b_{1,1}^+ - 2a_{0,2}^+ - 2a_{1,0}^+ - 2a_{2,0}^+ - 2b_{0,1}^+)a_{0,0}^+ + {a_{1,0}^+}^2$$
$$+ (2b_{1,1}^+ + 2a_{0,2}^+ + 2a_{2,0}^+ + 2b_{0,1}^+)a_{1,0}^+ + {b_{0,1}^+}^2 + (2b_{1,1}^+ + 2a_{0,2}^+ + 2a_{2,0}^+)b_{0,1}^+$$
$$- (-a_{1,0}^- - b_{0,1}^- + a_{0,0}^-)^2)\pi^2 + 1/2((-4b_{0,2}^+ - b_{1,0}^+ + a_{0,1}^+ - 2a_{1,1}^+ + 5b_{0,0}^+)a_{0,0}^+$$
$$+ (b_{1,0}^+ + a_{0,1}^+ - 5b_{0,0}^+)a_{1,0}^+ + (b_{1,0}^+ + a_{0,1}^+ - 3b_{0,0}^+)b_{0,1}^+ + (a_{0,1}^- - a_{0,1}^+ + 4b_{0,0}^+)a_{2,0}^+$$
$$+ (a_{0,1}^- - a_{0,1}^+ + 2b_{0,0}^+)b_{1,1}^+ + (-a_{0,1}^- + 2a_{1,1}^- - 5b_{0,0}^- + 4b_{0,2}^- + b_{1,0}^-)a_{0,0}^-$$
$$+ (a_{0,1}^- - a_{0,1}^+)a_{0,2}^+ + (-a_{0,1}^- + 5b_{0,0}^- - b_{1,0}^-)a_{1,0}^- + (-a_{0,1}^- + 3b_{0,0}^- - b_{1,0}^-)b_{0,1}^-$$
$$+ (-4a_{2,0}^- - 2b_{1,1}^-)b_{0,0}^- + 2c_{0,0}^+ - 2c_{1,0}^+ - 2d_{0,1}^+ + 2c_{0,0}^- - 2c_{1,0}^- - 2d_{0,1}^-)\pi$$
$$+ 8{a_{0,0}^+}^2 + 1/2(40b_{1,1}^+ + 32a_{0,2}^+ + 16a_{1,0}^+ - 16a_{2,0}^+ - 40b_{0,1}^+)a_{0,0}^+ - 16{a_{1,0}^+}^2$$
$$+ 1/2(-16b_{1,1}^+ + 4a_{0,2}^- - 36a_{0,2}^+ + 64a_{2,0}^+ + 16b_{0,1}^+)a_{1,0}^+ + 8{b_{0,1}^+}^2$$
$$+ 1/2(-32b_{1,1}^+ - 4a_{0,2}^- - 12a_{0,2}^+ - 16a_{2,0}^+)b_{0,1}^+ - 16{a_{2,0}^+}^2$$
$$+ 1/2(16b_{1,1}^+ - 4a_{0,2}^- + 36a_{0,2}^+)a_{2,0}^+ + 8{b_{1,1}^+}^2 + 1/2(4a_{0,2}^- + 12a_{0,2}^+)b_{1,1}^+$$
$$- 8{a_{0,0}^-}^2 + 1/2(16a_{1,0}^- - 16a_{2,0}^- + 8b_{0,1}^- - 8b_{1,1}^-)a_{0,0}^- + 2a_{0,2}^+(a_{0,2}^- - a_{0,2}^+);$$

$$
\begin{aligned}
K_3 =& -8/3c^+_{0,0}+8/3c^-_{0,0}+8/3c^+_{1,0}+4/3d^+_{0,1}-8/3c^-_{1,0}-4/3d^-_{0,1}-8/3c^+_{2,0}+8/3c^-_{2,0}\\
&+4/3c^-_{0,2}-4/3d^+_{1,1}-4/3c^+_{0,2}+4/3d^-_{1,1}-4/3a^-_{0,2}b^-_{0,0}-4/3a^+_{0,2}b^+_{0,0}\\
&+1/18(-16a^+_{0,1}+208b^+_{0,0}-40b^+_{0,2}+16b^+_{2,0}-32a^+_{1,1}-64b^+_{1,0})a^+_{1,0}\\
&+1/18(-16a^+_{0,1}+88b^+_{0,0}+8b^+_{0,2}+16b^+_{2,0}-8a^+_{1,1}-40b^+_{1,0})b^+_{0,1}\\
&+1/18(-114{a^+_{0,0}}^2+(-84b^+_{1,1}-66a^+_{0,2}+66a^+_{1,0}+66a^+_{2,0}+216b^+_{0,1})a^+_{0,0}\\
&+48{a^+_{1,0}}^2+(30b^+_{1,1}+12a^+_{0,2}-228a^+_{2,0}-162b^+_{0,1})a^+_{1,0}-66{b^+_{0,1}}^2\\
&+(66b^+_{1,1}+48a^+_{0,2}+96a^+_{2,0})b^+_{0,1}-114{a^-_{0,0}}^2\\
&+(-30a^-_{0,2}+174a^-_{1,0}-114a^-_{2,0}+108b^-_{0,1}-48b^-_{1,1})a^-_{0,0}-60{a^-_{1,0}}^2\\
&+(12a^-_{0,2}+60a^-_{2,0}-90b^-_{0,1}+30b^-_{1,1})a^-_{1,0}-30{b^-_{0,1}}^2+(12a^-_{0,2}+60a^-_{2,0}+30b^-_{1,1})b^-_{0,1}\\
&+180{a^+_{2,0}}^2+(36b^+_{1,1}+9a^-_{0,2}-9a^+_{0,2})a^+_{2,0}+9(a^-_{0,2}-5a^+_{0,2})(a^+_{0,2}+b^+_{1,1}))\pi\\
&+1/18(48a^-_{0,1}-192b^-_{0,0}+48b^-_{1,0})a^-_{2,0}+1/18(24a^-_{0,1}+24b^-_{1,0})b^-_{1,1}\\
&+1/18(48b^+_{1,0}+48a^+_{0,1}-192b^+_{0,0})a^+_{2,0}\\
&+1/18(24b^+_{1,0}+24a^+_{0,1}-72b^+_{0,0})b^+_{1,1}+1/18(-16a^-_{0,1}-32a^-_{1,1}+208b^-_{0,0}-40b^-_{0,2}\\
&-64b^-_{1,0}+16b^-_{2,0})a^-_{1,0}+1/18(-16a^-_{0,1}-8a^-_{1,1}+88b^-_{0,0}+8b^-_{0,2}-40b^-_{1,0}+16b^-_{2,0})b^-_{0,1}\\
&+1/18(-208b^+_{0,0}-16b^+_{2,0}+88b^+_{0,2}+64b^+_{1,0}-32a^+_{0,1}+80a^+_{1,1})a^+_{0,0}\\
&+1/18(-32a^-_{0,1}+80a^-_{1,1}-208b^-_{0,0}+88b^-_{0,2}+64b^-_{1,0}-16b^-_{2,0})a^-_{0,0}-4b^-_{0,0}b^-_{1,1};\\
K_4 =&1/48(18{b^+_{0,1}}^2+(6b^+_{1,1}-72a^+_{0,0}+6a^+_{0,2}+72a^+_{1,0}-30a^+_{2,0})b^+_{0,1}-54{a^-_{0,0}}^2\\
&+(-18a^-_{0,2}+108a^-_{1,0}-54a^-_{2,0}+72b^-_{0,1}-18b^-_{1,1})a^-_{0,0}-54{a^-_{1,0}}^2\\
&+(18a^-_{0,2}+54a^-_{2,0}-72b^-_{0,1}+18b^-_{1,1})a^-_{1,0}-18{b^-_{0,1}}^2+(18a^-_{0,2}+54a^-_{2,0}+18b^-_{1,1})b^-_{0,1}\\
&-24(b^+_{1,1}-3/4a^+_{0,0}+a^+_{0,2}+3/4a^+_{1,0}+a^+_{2,0})(b^+_{1,1}+3a^+_{0,0}+a^+_{0,2}-3a^+_{1,0}+3a^+_{2,0}))\pi^2\\
&+1/48((15a^+_{0,1}-75b^+_{0,0}-15b^+_{0,2}-21b^+_{2,0}+3a^+_{1,1}+39b^+_{1,0})b^+_{0,1}\\
&+(-231b^-_{0,0}-33b^-_{2,0}+45b^-_{0,2}+87b^-_{1,0}-21a^-_{0,1}+75a^-_{1,1})a^-_{0,0}\\
&+(231b^-_{0,0}-21b^-_{0,2}+33b^-_{2,0}-39a^-_{1,1}-87b^-_{1,0}-15a^-_{0,1})a^-_{1,0}\\
&+(75b^-_{0,0}+15b^-_{0,2}+21b^-_{2,0}-3a^-_{1,1}-39b^-_{1,0}-15a^-_{0,1})b^-_{0,1}\\
&+(-24b^+_{0,2}-6b^+_{1,0}-12a^+_{1,1}+30b^+_{0,0}+6a^+_{0,1})a^+_{0,2}\\
&+(21b^+_{0,2}+87b^+_{1,0}-33b^+_{2,0}+15a^+_{0,1}+39a^+_{1,1}-231b^+_{0,0})a^+_{1,0}\\
&+(12b^+_{0,2}-30b^+_{1,0}+12b^+_{2,0}-18a^+_{0,1}+66b^+_{0,0})b^+_{1,1}\\
&+(-45b^+_{0,2}-87b^+_{1,0}+33b^+_{2,0}+21a^+_{0,1}-75a^+_{1,1}+231b^+_{0,0})a^+_{0,0}\\
&+(-78b^+_{1,0}+24b^+_{2,0}-42a^+_{0,1}-12a^+_{1,1}+222b^+_{0,0})a^+_{2,0}\\
&+(42a^-_{0,1}+12a^-_{1,1}-222b^-_{0,0}+78b^-_{1,0}-24b^-_{2,0})a^-_{2,0}\\
&+(18a^-_{0,1}-66b^-_{0,0}-12b^-_{0,2}+30b^-_{1,0}-12b^-_{2,0})b^-_{1,1}\\
&+(-6a^-_{0,1}+12a^-_{1,1}-30b^-_{0,0}+24b^-_{0,2}+6b^-_{1,0})a^-_{0,2}\\
&+36c^+_{0,0}+12c^+_{0,2}-36c^+_{1,0}+36c^+_{2,0}-12d^+_{0,1}+12d^+_{1,1}+36c^-_{0,0}\\
&+12c^-_{0,2}-36c^-_{1,0}+36c^-_{2,0}-12d^-_{0,1}+12d^-_{1,1})\pi+20/3{b^+_{0,1}}^2
\end{aligned}
$$

$$+ 1/48(-640b_{1,1}^{+} - 1408a_{0,0}^{+} - 576a_{0,2}^{+} + 1024a_{1,0}^{+} - 1024a_{2,0}^{+})b_{0,1}^{+} - 16{a_{0,0}^{-}}^{2}$$
$$+ 1/48(-256a_{0,2}^{-} + 1024a_{1,0}^{-} - 1024a_{2,0}^{-} + 384b_{0,1}^{-} - 384b_{1,1}^{-})a_{0,0}^{-} - 16/3{a_{1,0}^{-}}^{2}$$
$$+ 1/48(128a_{0,2}^{-} + 512a_{2,0}^{-} - 256b_{0,1}^{-} + 256b_{1,1}^{-})a_{1,0}^{-} - 4/3{b_{0,1}^{-}}^{2}$$
$$+ 1/48(64a_{0,2}^{-} + 256a_{2,0}^{-} + 128b_{1,1}^{-})b_{0,1}^{-} + 16/3{a_{0,2}^{+}}^{2}$$
$$+ 1/48(576b_{1,1}^{+} + 1280a_{0,0}^{+} - 896a_{1,0}^{+} + 896a_{2,0}^{+})a_{0,2}^{+} - 16{a_{1,0}^{+}}^{2}$$
$$+1/48(-1024b_{1,1}^{+}+1536a_{2,0}^{+})a_{1,0}^{+}+20/3{b_{1,1}^{+}}^{2}+1/48(1408a_{0,0}^{+}+1024a_{2,0}^{+})b_{1,1}^{+}$$
$$+ 16{a_{0,0}^{+}}^{2} - 16{a_{2,0}^{+}}^{2} - 8/3(a_{2,0}^{-} + b_{1,1}^{-}/2)(2a_{2,0}^{-} + a_{0,2}^{-} + b_{1,1}^{-});$$

$$K_5 =1/15(32c_{0,0}^{-} - 32c_{0,0}^{+} + 32c_{1,0}^{+} + 8d_{0,1}^{+} - 32c_{1,0}^{-} - 8d_{0,1}^{-} - 32c_{2,0}^{+}$$
$$+ 32c_{2,0}^{-} + 8c_{0,2}^{-} - 8d_{1,1}^{+} - 8c_{0,2}^{+} + 8d_{1,1}^{-}) + \pi/900(-9885{a_{0,0}^{+}}^{2}$$
$$+ (-6090b_{1,1}^{+} - 5415a_{0,2}^{+} + 9645a_{1,0}^{+} - 1485a_{2,0}^{+} + 14250b_{0,1}^{+})a_{0,0}^{+}$$
$$- 450{a_{0,2}^{+}}^{2} + (-1125b_{1,1}^{+} + 3390a_{1,0}^{+} - 1350a_{2,0}^{+} + 3165b_{0,1}^{+})a_{0,2}^{+} + 240{a_{1,0}^{+}}^{2}$$
$$+ (4065b_{1,1}^{+} - 8640a_{2,0}^{+} - 12225b_{0,1}^{+})a_{1,0}^{+} + 8400{a_{2,0}^{+}}^{2} + (-2025b_{1,1}^{+}$$
$$+ 10185b_{0,1}^{+})a_{2,0}^{+} - 2715{b_{0,1}^{+}}^{2} + 3390b_{0,1}^{+}b_{1,1}^{+} - 675{b_{1,1}^{+}}^{2} - 9885{a_{0,0}^{-}}^{2}$$
$$+ (-3465a_{0,2}^{-} + 16395a_{1,0}^{-} - 13035a_{2,0}^{-} + 7500b_{0,1}^{-} - 4140b_{1,1}^{-})a_{0,0}^{-}$$
$$- 300{a_{0,2}^{-}}^{2} + (2790a_{1,0}^{-} - 1950a_{2,0}^{-} + 1665b_{0,1}^{-} - 825b_{1,1}^{-})a_{0,2}^{-} - 6510{a_{1,0}^{-}}^{2}$$
$$+ (9660a_{2,0}^{-} - 6825b_{0,1}^{-} + 3465b_{1,1}^{-})a_{1,0}^{-} - 3150{a_{2,0}^{-}}^{2}$$
$$+ (5985b_{0,1}^{-} - 2625b_{1,1}^{-})a_{2,0}^{-} - 1365(b_{0,1}^{-} - 5/13b_{1,1}^{-})(b_{0,1}^{-} - b_{1,1}^{-}))$$
$$+ 1/900((-16704b_{0,0}^{+}-3264b_{2,0}^{+}+864b_{0,2}^{+}+7104b_{1,0}^{+}-576a_{0,1}^{+}+4416a_{1,1}^{+})a_{0,0}^{+}$$
$$+ (400b_{0,2}^{+} + 640b_{1,0}^{+} + 800a_{1,1}^{+} - 2080b_{0,0}^{+} - 320a_{0,1}^{+} - 160b_{2,0}^{+})a_{0,2}^{+}$$
$$+ (-1344a_{0,1}^{+} + 16704b_{0,0}^{+} + 96b_{0,2}^{+} + 3264b_{2,0}^{+} - 2496a_{1,1}^{+} - 7104b_{1,0}^{+})a_{1,0}^{+}$$
$$+ (-2880b_{2,0}^{+} + 960a_{1,1}^{+} - 960b_{0,2}^{+} + 2880a_{0,1}^{+} + 6720b_{1,0}^{+} - 16320b_{0,0}^{+})a_{2,0}^{+}$$
$$+ (-1024a_{0,1}^{+} + 4384b_{0,0}^{+} + 1136b_{0,2}^{+} + 1504b_{2,0}^{+} + 64a_{1,1}^{+} - 2464b_{1,0}^{+})b_{0,1}^{+}$$
$$+ (-1040b_{0,2}^{+} + 2080b_{1,0}^{+} - 1120b_{2,0}^{+} + 1120a_{0,1}^{+} - 160a_{1,1}^{+} - 4000b_{0,0}^{+})b_{1,1}^{+}$$
$$+ (-576a_{0,1}^{-} + 4416a_{1,1}^{-} - 16704b_{0,0}^{-} + 864b_{0,2}^{-} + 7104b_{1,0}^{-} - 3264b_{2,0}^{-})a_{0,0}^{-}$$
$$+ (-320a_{0,1}^{-} + 800a_{1,1}^{-} - 2080b_{0,0}^{-} + 400b_{0,2}^{-} + 640b_{1,0}^{-} - 160b_{2,0}^{-})a_{0,2}^{-}$$
$$+ (-1344a_{0,1}^{-} - 2496a_{1,1}^{-} + 16704b_{0,0}^{-} + 96b_{0,2}^{-} - 7104b_{1,0}^{-} + 3264b_{2,0}^{-})a_{1,0}^{-}$$
$$+ (2880a_{0,1}^{-} + 960a_{1,1}^{-} - 16320b_{0,0}^{-} - 960b_{0,2}^{-} + 6720b_{1,0}^{-} - 2880b_{2,0}^{-})a_{2,0}^{-}$$
$$+ (-1024a_{0,1}^{-} + 64a_{1,1}^{-} + 4384b_{0,0}^{-} + 1136b_{0,2}^{-} - 2464b_{1,0}^{-} + 1504b_{2,0}^{-})b_{0,1}^{-}$$
$$+ (1120a_{0,1}^{-} - 160a_{1,1}^{-} - 4000b_{0,0}^{-} - 1040b_{0,2}^{-} + 2080b_{1,0}^{-} - 1120b_{2,0}^{-})b_{1,1}^{-});$$

$$K_6 =\pi^2/1440(2610{a_{0,0}^{+}}^{2}+(-1800b_{1,1}^{+}-1800a_{0,2}^{+}-5220a_{1,0}^{+}+720a_{2,0}^{+}-2700b_{0,1}^{+})a_{0,0}^{+}$$
$$+ 2610{a_{1,0}^{+}}^{2} + (1800b_{1,1}^{+} + 1800a_{0,2}^{+} - 720a_{2,0}^{+} + 2700b_{0,1}^{+})a_{1,0}^{+} - 1890{a_{2,0}^{+}}^{2}$$
$$+ (-2700b_{1,1}^{+} - 2700a_{0,2}^{+} - 1800b_{0,1}^{+})a_{2,0}^{+} - 2610{a_{0,0}^{-}}^{2} + (-900a_{0,2}^{-} + 5220a_{1,0}^{-}$$
$$- 3420a_{2,0}^{-}+2700b_{0,1}^{-}-900b_{1,1}^{-})a_{0,0}^{-}-90{a_{0,2}^{-}}^{2}+(900a_{1,0}^{-}-540a_{2,0}^{-}+540b_{0,1}^{-}$$

$$
\begin{aligned}
&- 180b_{1,1}^-)a_{0,2}^- - 2610{a_{1,0}^-}^2 + (3420a_{2,0}^- - 2700b_{0,1}^- + 900b_{1,1}^-)a_{1,0}^- - 810{a_{2,0}^-}^2\\
&+ (2340b_{0,1}^- - 540b_{1,1}^-)a_{2,0}^- - 450{b_{0,1}^-}^2 + 540b_{0,1}^-b_{1,1}^- - 90{b_{1,1}^-}^2\\
&- 450(b_{1,1}^+ + a_{0,2}^+ + b_{0,1}^+)(b_{1,1}^+ + a_{0,2}^+ - b_{0,1}^+))\\
&+ \pi/1440((10035b_{0,0}^+ + 2385b_{2,0}^+ + 435b_{0,2}^+ - 4635b_{1,0}^+ - 135a_{0,1}^+ - 2115a_{1,1}^+)a_{0,0}^+\\
&+ (1035a_{0,1}^+ - 10035b_{0,0}^+ - 795b_{0,2}^+ - 2385b_{2,0}^+ + 1215a_{1,1}^+ + 4635b_{1,0}^+)a_{1,0}^+\\
&+ (1125b_{0,2}^+ + 2235b_{2,0}^+ - 465a_{1,1}^+ - 1785a_{0,1}^+ - 4485b_{1,0}^+ + 9885b_{0,0}^+)a_{2,0}^+\\
&+ (-10035b_{0,0}^- - 2385b_{2,0}^- - 435b_{0,2}^- + 4635b_{1,0}^- + 135a_{0,1}^- + 2115a_{1,1}^-)a_{0,0}^-\\
&+ (-165b_{2,0}^- - 1155b_{0,0}^- - 75b_{0,2}^- + 435b_{1,0}^- - 105a_{0,1}^- + 375a_{1,1}^-)a_{0,2}^-\\
&+ (-1035a_{0,1}^- - 1215a_{1,1}^- + 10035b_{0,0}^- + 795b_{0,2}^- - 4635b_{1,0}^- + 2385b_{2,0}^-)a_{1,0}^-\\
&+ (1785a_{0,1}^- + 465a_{1,1}^- - 9885b_{0,0}^- - 1125b_{0,2}^- + 4485b_{1,0}^- - 2235b_{2,0}^-)a_{2,0}^-\\
&+ (-585a_{0,1}^- + 135a_{1,1}^- + 2205b_{0,0}^- + 645b_{0,2}^- - 1305b_{1,0}^- + 855b_{2,0}^-)b_{0,1}^-\\
&+ (615a_{0,1}^- - 165a_{1,1}^- - 2055b_{0,0}^- - 615b_{0,2}^- + 1155b_{1,0}^- - 705b_{2,0}^-)b_{1,1}^-\\
&+ (75b_{0,2}^+ - 435b_{1,0}^+ + 165b_{2,0}^+ + 105a_{0,1}^+ - 375a_{1,1}^+ + 1155b_{0,0}^+)a_{0,2}^+\\
&+ (615b_{0,2}^+ - 1155b_{1,0}^+ + 705b_{2,0}^+ - 615a_{0,1}^+ + 165a_{1,1}^+ + 2055b_{0,0}^+)b_{1,1}^+\\
&+ (-645b_{0,2}^+ + 1305b_{1,0}^+ - 855b_{2,0}^+ + 585a_{0,1}^+ - 135a_{1,1}^+ - 2205b_{0,0}^+)b_{0,1}^+\\
&+ 900c_{0,0}^+ + 180c_{0,2}^+ - 900c_{1,0}^+ + 900c_{2,0}^+ - 180d_{0,1}^+ + 180d_{1,1}^+ + 900c_{0,0}^-\\
&+ 180c_{0,2}^- - 900c_{1,0}^- + 900c_{2,0}^- - 180d_{0,1}^- + 180d_{1,1}^-)\\
&+ 1088/45{a_{0,0}^+}^2 - 128/9{a_{1,0}^+}^2 - 128/9{a_{2,0}^+}^2 - 8/9{a_{0,2}^-}^2 - 512/45{a_{1,0}^-}^2\\
&+ 1/1440(52224b_{1,1}^+ + 49152a_{0,2}^+ - 14336a_{1,0}^+ + 14336a_{2,0}^+ - 52224b_{0,1}^+)a_{0,0}^+\\
&+ (-43008b_{1,1}^+ - 39936a_{0,2}^+ + 40960a_{2,0}^+ + 43008b_{0,1}^+)a_{1,0}^+ + (43008b_{1,1}^+\\
&+ 39936a_{0,2}^+ - 43008b_{0,1}^+)a_{2,0}^+ + (-12288a_{0,2}^- + 51200a_{1,0}^- - 51200a_{2,0}^-\\
&+ 15360b_{0,1}^- - 15360b_{1,1}^-)a_{0,0}^- + (9216a_{1,0}^- - 9216a_{2,0}^- + 3328b_{0,1}^- - 3328b_{1,1}^-)a_{0,2}^-\\
&+ (32768a_{2,0}^- - 12288b_{0,1}^- + 12288b_{1,1}^-)a_{1,0}^- + (12288b_{0,1}^- - 12288b_{1,1}^-)a_{2,0}^-\\
&+ 1/45(128b_{0,1}^-b_{1,1}^- - 64{b_{0,1}^-}^2 - 512{a_{2,0}^-}^2 - 64{b_{1,1}^-}^2 - 1088{a_{0,0}^-}^2\\
&+ 256b_{1,1}^+ + 256a_{0,2}^+ - 256b_{0,1}^+(b_{1,1}^+ + 29/32a_{0,2}^+ - b_{0,1}^+));
\end{aligned}
$$

$$
\begin{aligned}
K_7 =\,&1/35(64c_{0,0}^- - 64c_{0,0}^+ + 64c_{1,0}^+ - 64c_{1,0}^- - 64c_{2,0}^+ + 64c_{2,0}^-) + 1/105(32d_{0,1}^+ - 32d_{1,1}^+\\
&- 32c_{0,2}^+ + 32d_{1,1}^- - 32d_{0,1}^- + 32c_{0,2}^-) + \frac{\pi}{88200}(-1397025{a_{0,0}^+}^2 + (-739620b_{1,1}^+\\
&- 684495a_{0,2}^+ + 1636425a_{1,0}^+ - 709065a_{2,0}^+ + 1666980b_{0,1}^+)a_{0,0}^+ - 64680{a_{0,2}^+}^2\\
&+ (-140385b_{1,1}^+ + 519120a_{1,0}^+ - 364560a_{2,0}^+ + 294945b_{0,1}^+)a_{0,2}^+\\
&- 239400{a_{1,0}^+}^2 + (574245b_{1,1}^+ - 448560a_{2,0}^+ - 1501605b_{0,1}^+)a_{1,0}^+\\
&+ 687960{a_{2,0}^+}^2 + (-419685b_{1,1}^+ + 1347045b_{0,1}^+)a_{2,0}^+ - 230265{b_{0,1}^+}^2\\
&+ 305970b_{0,1}^+b_{1,1}^+ - 75705{b_{1,1}^+}^2 - 1397025{a_{0,0}^-}^2 + (-477225a_{0,2}^-\\
&+ 2408175a_{1,0}^- - 2045295a_{2,0}^- + 895230b_{0,1}^- - 532350b_{1,1}^-)a_{0,0}^- - 48510{a_{0,2}^-}^2\\
&+ (422100a_{1,0}^- - 361620a_{2,0}^- + 168525b_{0,1}^- - 108045b_{1,1}^-)a_{0,2}^- - 1011150{a_{1,0}^-}^2
\end{aligned}
$$

$$
\begin{aligned}
&+(1659420a_{2,0}^{-}-840105b_{0,1}^{-}+477225b_{1,1}^{-})a_{1,0}^{-}-648270{a_{2,0}^{-}}^{2}\\
&+(779625b_{0,1}^{-}-416745b_{1,1}^{-})a_{2,0}^{-}-120015(b_{0,1}^{-}-63/127b_{1,1}^{-})(b_{0,1}^{-}-b_{1,1}^{-}))\\
&+1/88200((126336a_{0,1}^{-}-45696a_{1,1}^{-}-395136b_{0,0}^{-}-126784b_{0,2}^{-}+233856b_{1,0}^{-}\\
&-153216b_{2,0}^{-})b_{1,1}^{-}+(-122496a_{0,1}^{-}+41856a_{1,1}^{-}+418176b_{0,0}^{-}+130624b_{0,2}^{-}\\
&-256896b_{1,0}^{-}+176256b_{2,0}^{-})b_{0,1}^{-}+(419328a_{0,1}^{-}+64512a_{1,1}^{-}-2193408b_{0,0}^{-}\\
&-346752b_{0,2}^{-}+1064448b_{1,0}^{-}-580608b_{2,0}^{-})a_{2,0}^{-}+(-281088a_{0,1}^{-}-202752a_{1,1}^{-}\\
&+2216448b_{0,0}^{-}+296832b_{0,2}^{-}-1087488b_{1,0}^{-}+603648b_{2,0}^{-})a_{1,0}^{-}\\
&+(-8064a_{0,1}^{-}+61824a_{1,1}^{-}-233856b_{0,0}^{-}-46144b_{0,2}^{-}+99456b_{1,0}^{-}-45696b_{2,0}^{-})a_{0,2}^{-}\\
&+(119808a_{0,1}^{-}+364032a_{1,1}^{-}-2216448b_{0,0}^{-}-243072b_{0,2}^{-}+1087488b_{1,0}^{-}\\
&-603648b_{2,0}^{-})a_{0,0}^{-}+(-126784b_{0,2}^{+}+233856b_{1,0}^{+}-153216b_{2,0}^{+}+126336a_{0,1}^{+}\\
&-45696a_{1,1}^{+}-395136b_{0,0}^{+})b_{1,1}^{+}+(-122496a_{0,1}^{+}+418176b_{0,0}^{+}+130624b_{0,2}^{+}\\
&+176256b_{2,0}^{+}+41856a_{1,1}^{+}-256896b_{1,0}^{+})b_{0,1}^{+}+(-580608b_{2,0}^{+}+64512a_{1,1}^{+}\\
&-346752b_{0,2}^{+}+419328a_{0,1}^{+}+1064448b_{1,0}^{+}-2193408b_{0,0}^{+})a_{2,0}^{+}+(-281088a_{0,1}^{+}\\
&+2216448b_{0,0}^{+}+296832b_{0,2}^{+}+603648b_{2,0}^{+}-202752a_{1,1}^{+}-1087488b_{1,0}^{+})a_{1,0}^{+}\\
&+(-46144b_{0,2}^{+}+99456b_{1,0}^{+}+61824a_{1,1}^{+}-233856b_{0,0}^{+}-8064a_{0,1}^{+}-45696b_{2,0}^{+})a_{0,2}^{+}\\
&+(-2216448b_{0,0}^{+}-603648b_{2,0}^{+}-243072b_{0,2}^{+}+1087488b_{1,0}^{+}\\
&+119808a_{0,1}^{+}+364032a_{1,1}^{+})a_{0,0}^{+});
\end{aligned}
$$

$$
\begin{aligned}
K_8 =&\pi^2/161280(409500{a_{0,0}^{+}}^{2}+(-220500b_{1,1}^{+}-220500a_{0,2}^{+}-819000a_{1,0}^{+}+245700a_{2,0}^{+}\\
&-352800b_{0,1}^{+})a_{0,0}^{+}-37800{a_{0,2}^{+}}^{2}+(-75600b_{1,1}^{+}+220500a_{1,0}^{+}-302400a_{2,0}^{+}\\
&-6300b_{0,1}^{+})a_{0,2}^{+}+409500{a_{1,0}^{+}}^{2}+(220500b_{1,1}^{+}-245700a_{2,0}^{+}+352800b_{0,1}^{+})a_{1,0}^{+}\\
&-163800{a_{2,0}^{+}}^{2}+(-302400b_{1,1}^{+}-270900b_{0,1}^{+})a_{2,0}^{+}+44100{b_{0,1}^{+}}^{2}-6300b_{0,1}^{+}b_{1,1}^{+}\\
&-37800{b_{1,1}^{+}}^{2}-409500{a_{0,0}^{-}}^{2}+(-132300a_{0,2}^{-}+819000a_{1,0}^{-}-598500a_{2,0}^{-}\\
&+352800b_{0,1}^{-}-132300b_{1,1}^{-})a_{0,0}^{-}-12600{a_{0,2}^{-}}^{2}+(132300a_{1,0}^{-}-100800a_{2,0}^{-}\\
&+56700b_{0,1}^{-}-25200b_{1,1}^{-})a_{0,2}^{-}-409500{a_{1,0}^{-}}^{2}+(598500a_{2,0}^{-}-352800b_{0,1}^{-}\\
&+132300b_{1,1}^{-})a_{1,0}^{-}-189000{a_{2,0}^{-}}^{2}+(321300b_{0,1}^{-}-100800b_{1,1}^{-})a_{2,0}^{-}\\
&-44100(b_{0,1}^{-}-2/7b_{1,1}^{-})(b_{0,1}^{-}-b_{1,1}^{-}))+\pi/161280((1451835b_{0,0}^{+}+437535b_{2,0}^{+}\\
&+229005b_{0,2}^{+}-746235b_{1,0}^{+}-128835a_{0,1}^{+}-179865a_{1,1}^{+})a_{0,0}^{+}+(38430b_{0,2}^{+}\\
&-64890b_{1,0}^{+}-29610a_{1,1}^{+}+140490b_{0,0}^{+}-1890a_{0,1}^{+}+33390b_{2,0}^{+})a_{0,2}^{+}\\
&+(217035a_{0,1}^{+}-1451835b_{0,0}^{+}-254205b_{0,2}^{+}-437535b_{2,0}^{+}+91665a_{1,1}^{+}\\
&+746235b_{1,0}^{+})a_{1,0}^{+}+(426510b_{2,0}^{+}-14490a_{1,1}^{+}+277830b_{0,2}^{+}-294210a_{0,1}^{+}\\
&-735210b_{1,0}^{+}+1440810b_{0,0}^{+})a_{2,0}^{+}+(75915a_{0,1}^{+}-239715b_{0,0}^{+}-77805b_{0,2}^{+}\\
&-107415b_{2,0}^{+}-31815a_{1,1}^{+}+151515b_{1,0}^{+})b_{0,1}^{+}+(76230b_{0,2}^{+}-140490b_{1,0}^{+}\\
&+96390b_{2,0}^{+}-77490a_{0,1}^{+}+33390a_{1,1}^{+}+228690b_{0,0}^{+})b_{1,1}^{+}+(128835a_{0,1}^{-}\\
&+179865a_{1,1}^{-}-1451835b_{0,0}^{-}-229005b_{0,2}^{-}+746235b_{1,0}^{-}-437535b_{2,0}^{-})a_{0,0}^{-}
\end{aligned}
$$

$$
\begin{aligned}
&+(1890a_{0,1}^{-}+29610a_{1,1}^{-}-140490b_{0,0}^{-}-38430b_{0,2}^{-}+64890b_{1,0}^{-}-33390b_{2,0}^{-})a_{0,2}^{-}\\
&+(-217035a_{0,1}^{-}-91665a_{1,1}^{-}+1451835b_{0,0}^{-}+254205b_{0,2}^{-}-746235b_{1,0}^{-}\\
&+437535b_{2,0}^{-})a_{1,0}^{-}+(294210a_{0,1}^{-}+14490a_{1,1}^{-}-1440810b_{0,0}^{-}-277830b_{0,2}^{-}\\
&+735210b_{1,0}^{-}-426510b_{2,0}^{-})a_{2,0}^{-}+(-75915a_{0,1}^{-}+31815a_{1,1}^{-}+239715b_{0,0}^{-}\\
&+77805b_{0,2}^{-}-151515b_{1,0}^{-}+107415b_{2,0}^{-})b_{0,1}^{-}+(77490a_{0,1}^{-}-33390a_{1,1}^{-}\\
&-228690b_{0,0}^{-}-76230b_{0,2}^{-}+140490b_{1,0}^{-}-96390b_{2,0}^{-})b_{1,1}^{-}+88200c_{0,0}^{+}\\
&+12600c_{0,2}^{+}-88200c_{1,0}^{+}+88200c_{2,0}^{+}-12600d_{0,1}^{+}\\
&+12600d_{1,1}^{+}+88200c_{0,0}^{-}+12600c_{0,2}^{-}-88200c_{1,0}^{-}+88200c_{2,0}^{-}-12600d_{0,1}^{-}\\
&+12600d_{1,1}^{-})+2048/63{a_{0,0}^{+}}^{2}+1/161280(6782976b_{1,1}^{+}+6488064a_{0,2}^{+}\\
&-3407872a_{1,0}^{+}+3407872a_{2,0}^{+}-6782976b_{0,1}^{+})a_{0,0}^{+}+496/105{a_{0,2}^{+}}^{2}\\
&+1/161280(1572864b_{1,1}^{+}-5603328a_{1,0}^{+}+5603328a_{2,0}^{+}-1572864b_{0,1}^{+})a_{0,2}^{+}\\
&-512/45{a_{1,0}^{+}}^{2}+1/161280(-5898240b_{1,1}^{+}+3670016a_{2,0}^{+}+5898240b_{0,1}^{+})a_{1,0}^{+}\\
&-512/45{a_{2,0}^{+}}^{2}+1/161280(5898240b_{1,1}^{+}-5898240b_{0,1}^{+})a_{2,0}^{+}\\
&+1/35(176{b_{0,1}^{+}}^{2}-352b_{0,1}^{+}b_{1,1}^{+}+176{b_{1,1}^{+}}^{2})-2048/63{a_{0,0}^{-}}^{2}-5632/315{a_{2,0}^{-}}^{2}\\
&+1/161280(-1769472a_{0,2}^{-}+8126464a_{1,0}^{-}-8126464a_{2,0}^{-}+2064384b_{0,1}^{-}\\
&-2064384b_{1,1}^{-})a_{0,0}^{-}-16/15{a_{0,2}^{-}}^{2}+1/161280(1474560a_{1,0}^{-}-1474560a_{2,0}^{-}\\
&+393216b_{0,1}^{-}-393216b_{1,1}^{-})a_{0,2}^{-}-5632/315{a_{1,0}^{-}}^{2}+1/161280(5767168a_{2,0}^{-}\\
&-1769472b_{0,1}^{-}+1769472b_{1,1}^{-})a_{1,0}^{-}+1/161280(1769472b_{0,1}^{-}-1769472b_{1,1}^{-})a_{2,0}^{-}\\
&-48/35(b_{0,1}^{-}-b_{1,1}^{-})^{2};
\end{aligned}
$$

$$
\begin{aligned}
K_9 =&1/315(512c_{0,0}^{-}-512c_{0,0}^{+}+512c_{1,0}^{+}+64d_{0,1}^{+}-512c_{1,0}^{-}-64d_{0,1}^{-}-512c_{2,0}^{+}+512c_{2,0}^{-}\\
&+64c_{0,2}^{-}-64d_{1,1}^{+}-64c_{0,2}^{+}+64d_{1,1}^{-})+\pi/6350400(-132343785{a_{0,0}^{+}}^{2}\\
&+(-61909470b_{1,1}^{+}-58436595a_{0,2}^{+}+170919945a_{1,0}^{+}-96086025a_{2,0}^{+}\\
&+136743390b_{0,1}^{+})a_{0,0}^{+}-4957470{a_{0,2}^{+}}^{2}+(-10411065b_{1,1}^{+}+48017970a_{1,0}^{+}\\
&-38663730a_{2,0}^{+}+19765305b_{0,1}^{+})a_{0,2}^{+}-38576160{a_{1,0}^{+}}^{2}+(51490845b_{1,1}^{+}\\
&+2318400a_{2,0}^{+}-126324765b_{0,1}^{+})a_{1,0}^{+}+36257760{a_{2,0}^{+}}^{2}+(-42136605b_{1,1}^{+}\\
&+116970525b_{0,1}^{+})a_{2,0}^{+}-148078335{b_{0,1}^{+}}^{2}+20261430b_{0,1}^{+}b_{1,1}^{+}-54533595{b_{1,1}^{+}}^{2}\\
&-132343785{a_{0,0}^{-}}^{2}+(-42373485a_{0,2}^{-}+233431695a_{1,0}^{-}-205046415a_{2,0}^{-}\\
&+74231640b_{0,1}^{-}-45846360b_{1,1}^{-})a_{0,0}^{-}-3817800{a_{0,2}^{-}}^{2}+(38900610a_{1,0}^{-}\\
&-35352450a_{2,0}^{-}+11679885b_{0,1}^{-}-8131725b_{1,1}^{-})a_{0,2}^{-}-101087910{a_{1,0}^{-}}^{2}\\
&+(173790540a_{2,0}^{-}-70758765b_{0,1}^{-}+42373485b_{1,1}^{-})a_{1,0}^{-}-72702630{a_{2,0}^{-}}^{2}\\
&+(67210605b_{0,1}^{-}-38825325b_{1,1}^{-})a_{2,0}^{-}-7862085(b_{0,1}^{-}-1245/2269b_{1,1}^{-})(b_{0,1}^{-}-b_{1,1}^{-}))\\
&+1/6350400((-5686272b_{0,2}^{+}+8699904b_{1,0}^{+}+2912256a_{1,1}^{+}-17731584b_{0,0}^{+}\\
&+958464a_{0,1}^{+}-4829184b_{2,0}^{+})a_{0,2}^{+}+(-9556992b_{0,2}^{+}+17731584b_{1,0}^{+}\\
&-12570624b_{2,0}^{+}+9990144a_{0,1}^{+}-4829184a_{1,1}^{+}-28053504b_{0,0}^{+})b_{1,1}^{+}
\end{aligned}
$$

$$
\begin{aligned}
&+ (23633920a_{0,1}^- + 17653760a_{1,1}^- - 199106560b_{0,0}^- - 38555648b_{0,2}^-\\
&+ 106209280b_{1,0}^- - 64921600b_{2,0}^-)a_{0,0}^- + (-9846784a_{0,1}^+ + 29200384b_{0,0}^+\\
&+ 9700352b_{0,2}^+ + 137717504b_{2,0}^+ + 4685824a_{1,1}^+ - 188878464b_{1,0}^+)b_{0,1}^+\\
&+ (-63774720b_{2,0}^+ - 1843200a_{1,1}^+ - 435573248b_{0,2}^+ + 43130880a_{0,1}^+\\
&+ 105062400b_{1,0}^+ - 197959680b_{0,0}^+)a_{2,0}^+ + (-199106560b_{0,0}^+ - 64921600b_{2,0}^+\\
&- 38555648b_{0,2}^+ + 106209280b_{1,0}^+ + 23633920a_{0,1}^+ + 17653760a_{1,1}^+)a_{0,0}^+\\
&+ (9990144a_{0,1}^- - 4829184a_{1,1}^- - 28053504b_{0,0}^- - 9556992b_{0,2}^- + 177731584b_{1,0}^-\\
&- 125706624b_{2,0}^-)b_{1,1}^- + (-33955840a_{0,1}^+ + 199106560b_{0,0}^+ + 41136128b_{0,2}^+\\
&+ 64921600b_{2,0}^+ - 7331840a_{1,1}^+ - 106209280b_{1,0}^+)a_{1,0}^+ + (958464a_{0,1}^- + 2912256a_{1,1}^-\\
&- 177731584b_{0,0}^- - 5686272b_{0,2}^- + 8699904b_{1,0}^- - 4829184b_{2,0}^-)a_{0,2}^-\\
&+ (-33955840a_{0,1}^- - 7331840a_{1,1}^- + 199106560b_{0,0}^- + 41136128b_{0,2}^-\\
&- 106209280b_{1,0}^- + 64921600b_{2,0}^-)a_{1,0}^- + (43130880a_{0,1}^- - 1843200a_{1,1}^-\\
&- 197959680b_{0,0}^- - 435573248b_{0,2}^- + 105062400b_{1,0}^- - 63774720b_{2,0}^-)a_{2,0}^-\\
&+ (-9846784a_{0,1}^- + 4685824a_{1,1}^- + 29200384b_{0,0}^- + 9700352b_{0,2}^- - 188878464b_{1,0}^-\\
&+ 137717504b_{2,0}^-)b_{0,1}^-).
\end{aligned}
$$

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