

TRANSMISSION DYNAMICS OF A CHAGAS DISEASE MODEL WITH STANDARD INCIDENCE INFECTION

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Abstract In this paper, an insect-parasite-host model with Ricker's type reproduction of triatomines and the standard incidence rate of the interaction between insects and hosts is formulated to study the transmission dynamics of Chagas disease. Two thresholds of the ecological basic reproduction number of triatomines and the epidemiological basic reproduction number of Chagas disease are derived, which determine the dynamics of this model. As a result, the existence of equilibria and the local/global stabilities of the equilibrium are accordingly obtained. Moreover, backward bifurcation, forward bifurcation and saddle-node bifurcation are also shown analytically and numerically. Biologically speaking, Chagas disease may undergo outbreak if the number of bites of per triatomine bug per unit time or the transmission probability from infected bugs to susceptible competent hosts per bite increase.

Keywords Chagas disease, Ricker's type reproduction, standard incidence rate, backward/forward bifurcation.

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1. Introduction

Chagas disease, known as American trypanosomiasis, which is spread through bites by triatomine bugs infected with *T. cruzi* or transmission from mother to child. It is early discovered in 1908 by Doctor Chagas and is considered as a neglected disease while still attracts public health attention. It is reported that about 13% of the Latin American population is at risk of Chagas disease infection [23], and it also has been observed considerable spread of this disease due to the convenient transportation and the widespread development of globalization [6, 25, 27–29]. The disease is transmitted by a protozoan parasite called by *Trypanosoma cruzi* (*T. cruzi*) by invading the lymphatic system and the blood stream. It is mainly prevalent in Central and South America, such as Argentina, Bolivia, Brazil, Chile, etc. The disease can cause serious clinical symptoms involving cardiomyopathy and heart disease, which may cause many infected people to die due to these symptoms [4]. People infecting with *T. cruzi* generally passes through acute and chronic stages.

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The former includes fever, facial edema, anemia and so on and the latter often has myocarditis, cerebral embolism, sudden death and so on [21]. There are an estimated 6-8 million persons in the Americas with Chagas disease, causing a burden of 29,000,000 DALYs and a health care cost of 24.73 billion.

In order to fully know about the measures of precaution, therapy and control, many mathematical models have studied the dynamics of Chagas disease from different angles, such as the different transmission routes of the interaction between hosts and vectors [5, 21, 34–36], disease transmission in host movement and host community composition [1, 12, 18, 24, 33], the triatomine population from temporal or spatial variations [3, 10, 11, 16, 30, 32], and the control optimization of Chagas disease [2, 9, 13, 15, 17, 19, 20, 22].

Recently, Wu et al. [36] formulated a new model (1) considering the Ricker's type function growth of triatomine bugs and pathogenic effect on triatomine bugs to study the dynamics of interaction between triatomine bugs and *T. rangeli*, which is relevant to Chagas disease. The model is

$$\begin{aligned} S'_h(t) &= \Lambda_h - \tilde{\beta}_h I_v(t) S_h(t) - \mu_1 S_h(t), \\ I'_h(t) &= \tilde{\beta}_h I_v(t) S_h(t) - \mu_1 I_h(t), \\ S'_v(t) &= r(S_v(t) + \theta I_v(t)) e^{-\sigma(S_v(t) + I_v(t))} - \tilde{\beta}_v S_v(t) I_h(t) - \beta_c S_v(t) I_v(t) - \mu_2 S_v(t), \\ I'_v(t) &= \tilde{\beta}_v S_v(t) I_h(t) + \beta_c S_v(t) I_v(t) - d_v I_v(t) - \mu_2 I_v(t), \end{aligned} \quad (1.1)$$

where the population is divided into four components: susceptible and infected competent hosts, susceptible and infected triatomine bugs, denoted by S_h, I_h, S_v, I_v , respectively. Λ_h is the constant recruitment rate of susceptible competent host per unit time. The transmission rate from infected bugs to susceptible competent hosts is denoted by $\tilde{\beta}_h = \frac{ba}{N_c + \alpha N_q}$, where b is the transmission probability from infected bugs to susceptible competent hosts per bite, a is the number of bites of per triatomine bug per unit time, α is the biting preference of quasi-competent hosts to competent hosts, N_c is the total number of competent hosts, N_q is the total number of quasi-competent hosts. The transmission rate from infected competent hosts to susceptible bugs is denoted by $\tilde{\beta}_v = \frac{ca}{N_c + \alpha N_q}$, where c is the transmission probability from infected hosts to susceptible triatomine bug per bite. The total infection rate through co-feeding transmission between susceptible and infected bugs is β_c , which is transmitted by both the competent and quasi-competent hosts. The Ricker's type function $b(x) = rx e^{-\sigma x}$ was chosen to model the reproduction rate of *R. prolixus*. Integrating the pathogenic effect, the growth rate of triatomine bugs is modeled as $r(S_v + \theta I_v) e^{-\sigma(S_v + I_v)}$, where r is the maximal number of offsprings that a triatomine bug can produce per unit time, and $\theta \in [0, 1]$ is the reproduction reduction of bugs due to the pathogenic effect of *T. rangeli* on bugs, σ is the density-dependency strength measuring the reproduction of bugs. μ_1 and μ_2 are the natural death rates of competent hosts and triatomine bugs, respectively. d_v is the death rate of infected vectors induced by pathogenic effect.

Model (1.1) was used to develop the systemic and co-feeding transmission routes among vectors and hosts, and two thresholds was derived to characterize the dynamical behavior of this model. Interestingly, sustained oscillations of model solutions were observed numerically by altering parameter d_v and θ and the results have shown that the oscillation amplitude will be larger if d_v is larger or θ is smaller.

As we know, *Trypanosoma rangeli* (*T. rangeli*) is a kind of parasite which is pathogenic to some vector species including triatomine bug, although it can infect

Table 1. Parameter description of system (2.1)

Parameter	Range / value	Description
Λ_h	Varied	a constant recruitment rate per unit time of susceptible host
a	[0.2, 33]/day	number of bites of per triatomine bug per unit time [26, 36]
b	[0.00271, 0.06]	transmission probability from infected bugs to susceptible competent hosts per bite [21, 33, 36]
c	[0.00026, 0.49]	transmission probability from infected hosts to susceptible triatomine bug per bite [1, 33]
r	[0.0274, 0.7714]/day	the maximal number of offsprings that a triatomine bug can produce per unit time [1, 21]
σ	(0, ∞)	density-dependency strength measuring the reproduction of bugs [36]
μ_1	[0.000038, 0.0025]/day	natural death rate of hosts [1, 33]
μ_2	[0.0045, 0.0083]/day	natural death rate of triatomine bugs [1, 33]
d	Varied	<i>T. cruzi</i> -induced death rate of hosts

mammals through the same triatomines, it is not pathogenic to human. Based on model (1.1), in this paper, the *T. cruzi*-induced death rate of hosts and the standard incidence rate are introduced in order to study the transmission dynamics of chagas disease.

The rest of the paper is organized as follows. In the next section, a new model with the death rate of the infected hosts, the standard incidence rate are formulated. The corresponding basic reproduction number is given. In section 3, the existence, and local and global stability of equilibria are obtained. In section 4, the forward bifurcation and the backward bifurcation are derived by using the center-manifold theorem. Numerical simulations are also performed to illustrate the obtained results. Finally, conclusion and discussion are also given.

2. Model description

Motivated by Wu et al. [36], in this paper, a new chagas model involved by the effect of the *T. cruzi*-induced death rate of hosts and the standard incidence rate between hosts and triatomine bugs is formulated. We assume the generation rate of triatomine bugs follows Ricker's type function instead of the logistic growth, and further study the dynamical behavior of triatomine-host transmission. The interaction among hosts, *T. cruzi* parasite and triatomine bugs are described by the following model with Ricker's type reproduction function of triatomine bugs:

$$\begin{aligned}
 S'_h(t) &= \Lambda_h - ab \frac{I_v(t)}{S_h(t) + I_h(t)} S_h(t) - \mu_1 S_h(t), \\
 I'_h(t) &= ab \frac{I_v(t)}{S_h(t) + I_h(t)} S_h(t) - dI_h(t) - \mu_1 I_h(t), \\
 S'_v(t) &= r(S_v(t) + I_v(t))e^{-\sigma(S_v(t) + I_v(t))} - ac \frac{S_v(t)}{S_h(t) + I_h(t)} I_h(t) - \mu_2 S_v(t),
 \end{aligned} \tag{2.1}$$

$$I_v'(t) = ac \frac{S_v(t)}{S_h(t) + I_h(t)} I_h(t) - \mu_2 I_v(t),$$

where all the parameters are non-negative, and its biological meanings and ranges are given in Table 1.

Let $N_h(t) = S_h(t) + I_h(t)$. Summing up the first and the second equations of system (2.1), we obtain $N_h'(t) = \Lambda_h - \mu_1 N_h - dI_h$, then we have the feasible region of system (2.1) is

$$D = \{(S_h, I_h, S_v, I_v) | S_h \geq 0, I_h \geq 0, S_v \geq 0, I_v \geq 0, S_h + I_h \leq \frac{\Lambda_h}{\mu_1}\}. \quad (2.2)$$

Theorem 2.1. *System (2.1) with initial value lying in the region D has nonnegative, bounded solution which exists for all $t \geq 0$.*

Proof. The above claim directly follows from Theorem 5.2.1 in Smith [31] that system (2.1) admits a nonnegative solution $(S_h(t), I_h(t), S_v(t), I_v(t))$ through an initial value $(S_h(0), I_h(0), S_v(0), I_v(0)) \in D$ with the maximal interval of existence $[0, T)$ for some $T > 0$. It remains to prove the boundedness. We know that $S_h + I_h \leq \frac{\Lambda_h}{\mu_1}$. Summing up the third and the fourth equations of system (2.1) and using $xe^{-\sigma x} \leq \frac{1}{\sigma e}$ if $x > 0$, we have

$$\begin{aligned} (S_v + I_v)'(t) &= r(S_v(t) + I_v(t))e^{-\sigma(S_v(t) + I_v(t))} - \mu_2(S_v(t) + I_v(t)) \\ &\leq \frac{r}{\sigma e} - \mu_2(S_v(t) + I_v(t)), \end{aligned}$$

so that we can immediately obtain

$$\limsup_{t \rightarrow \infty} (S_v(t) + I_v(t)) \leq \frac{r}{\mu_2 \sigma e},$$

which indicates the boundedness of S_v and I_v mentioned above. \square

Now, let us calculate the basic reproduction number of system (2.1) by the method of Dreessche and Watmough [8].

Define

$$F = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & ab \\ 0 & 0 & 0 & 0 \\ 0 & \frac{acS_v^0}{S_h^0} & 0 & 0 \end{pmatrix}, \quad (2.3)$$

$$V = \begin{pmatrix} \mu_1 & 0 & 0 & ab \\ 0 & d + \mu_1 & 0 & 0 \\ 0 & \frac{acS_v^0}{S_h^0} & e^{-S_v^0 \sigma} r(S_v^0 \sigma - 1) + \mu_2 e^{-S_v^0 \sigma} r(S_v^0 \sigma - 1) & \\ 0 & 0 & 0 & \mu_2 \end{pmatrix}. \quad (2.4)$$

Then the *T. cruzi* basic reproduction number of system (2.1) is given by the spectral radius of the next generation matrix FV^{-1} , which is

$$R_0 = \rho(FV^{-1}) = \sqrt{\frac{a^2 bc S_v^0}{S_h^0 \mu_2 (d + \mu_1)}} = \sqrt{\frac{a^2 bc \mu_1 \ln R_v}{\Lambda_h \mu_2 \sigma (d + \mu_1)}}, \quad (2.5)$$

where $R_v = \frac{r}{\mu_2}$.

3. Existence and stability of equilibria

3.1. Existence of equilibria

Obviously, system (2.1) has a vector-free equilibrium $E_0(S_h^0, 0, 0, 0)$ and a parasite-free equilibrium $E_S(S_h^0, 0, S_v^0, 0)$. Now we turn to analyse the existence of positive equilibria $E^*(S_h^*, I_h^*, S_v^*, I_v^*)$ of system (2.1).

For any positive equilibrium $E^*(S_h^*, I_h^*, S_v^*, I_v^*)$ of system (2.1), its coordinates satisfy

$$\begin{aligned} I_h^* &= \frac{\Lambda_h - \mu_1 S_h^*}{d + \mu_1}, \\ S_v^* &= \frac{\mu_2(\Lambda_h + dS_h^*)^2}{a^2 bc S_h^*(d + \mu_1)}, \\ I_v^* &= \frac{(\Lambda_h + dS_h^*)(\Lambda_h - \mu_1 S_h^*)}{ab S_h^*(d + \mu_1)}, \end{aligned} \quad (3.1)$$

and it is easy to see that I_h^* , S_v^* and I_v^* are greater than zero if S_h^* is greater than zero. The coordinate of S_h^* should be the positive root of the following cubic equation:

$$f_1(S_h) = AS_h^3 + BS_h^2 + CS_h + D, \quad (3.2)$$

where

$$\begin{aligned} A &= d\mu_1\sigma(ac\mu_1 - d\mu_2), \\ B &= a^2bc \ln R_v \mu_1(d + \mu_1) + ac\Lambda_h\mu_1\sigma(\mu_1 - 2d) + d\Lambda_h\mu_2\sigma(d - 2\mu_1), \\ C &= \Lambda_h(ac\Lambda_h\sigma(d - 2\mu_1) + \Lambda_h\mu_2\sigma(2d - \mu_1) - a^2bc \ln R_v(d + \mu_1)), \\ D &= \Lambda_h^3\sigma(ac + \mu_2). \end{aligned} \quad (3.3)$$

For simplicity, we assume that $A \neq 0$, then we can rewrite equation (2.1) as

$$f(S_h) = S_h^3 + B_1S_h^2 + B_2S_h + B_3, \quad (3.4)$$

where

$$\begin{aligned} B_1 &= \frac{a^2bc \ln R_v \mu_1(d + \mu_1) + ac\Lambda_h\mu_1\sigma(\mu_1 - 2d) + d\Lambda_h\mu_2\sigma(d - 2\mu_1)}{d\mu_1\sigma(ac\mu_1 - d\mu_2)}, \\ B_2 &= \frac{\Lambda_h(ac\Lambda_h\sigma(d - 2\mu_1) + \Lambda_h\mu_2\sigma(2d - \mu_1) - a^2bc \ln R_v(d + \mu_1))}{d\mu_1\sigma(ac\mu_1 - d\mu_2)}, \\ B_3 &= \frac{\Lambda_h^3(ac + \mu_2)}{d\mu_1(ac\mu_1 - d\mu_2)}. \end{aligned} \quad (3.5)$$

The derivative of equation (3.4) is

$$f'(S_h) = 3S_h^2 + 2B_1S_h + B_2, \quad (3.6)$$

which has two real roots $\eta_{\pm} = \frac{-B_1 \pm \sqrt{B_1^2 - 3B_2}}{3}$ as $B_1^2 - 3B_2 > 0$.

Define $\Delta_3 = \frac{4B_1^3B_3 - B_1^2B_2^2 + 4B_2^3 - 18B_1B_2B_3 + 27B_3^2}{108}$. According to system (2.1), we have

$$\begin{aligned} \Delta_3 = & -\frac{1}{108d^4\mu_1^4\sigma^4(ac\mu_1 - d\mu_2)^4}(a^2c\Lambda_h^2(d + \mu_1)^3(a^2bc \ln R_v\mu_1 - \Lambda_h(d + \mu_1)\mu_2\sigma)^2 \\ & \times (a^2b^2c \ln^2 R_v(d + \mu_1) + 2abc \ln R_v\Lambda_h(\mu_1 - d)\sigma \\ & + \Lambda_h\sigma(-4bd \ln R_v\mu_2 + c\Lambda_h(d + \mu_1)\sigma)), \end{aligned} \quad (3.7)$$

and more details are provided in Appendix I.

Then the following theorems are obtained by applying Cardano formula [14, 37]:

Theorem 3.1. *If $ac\mu_1 > d\mu_2$, system (2.1) has at most two positive equilibria. Moreover, system (2.1) has*

- (1) *two positive equilibria if*
 - (i) $\Delta_3 < 0, B_1 \leq 0, B_3 > 0$;
 - (ii) $\Delta_3 < 0, B_1 > 0, B_2 < 0, B_3 > 0$;
- (2) *one positive equilibrium if*
 - (i) $B_1^2 = -B_2, B_1B_2 = B_3 > 0$;
 - (ii) $\Delta_3 = 0, B_3 > \max\{B_1B_2, 0\}$;
- (3) *no positive equilibrium in other cases.*

Proof. Obviously, the number of positive equilibrium of system (2.1) is corresponding to the number of different positive roots of cubic equation (3.4). We can see that $f(0) = B_3 > 0$ and $f(\pm\infty) = \pm\infty$ when $ac\mu_1 > d\mu_2$, which implies that system (2.1) has at most two positive equilibria.

When condition (1) holds, namely, $\Delta_3 < 0, B_1 \leq 0, B_3 > 0$ or $\Delta_3 < 0, B_1 > 0, B_2 < 0, B_3 > 0$, we know that the cubic it is impossible for equation (3.4) to have two negative roots by Veda's theorem, which means that there is at least one positive root η_+ of equation (3.4). By case III3 and III4 in Table 2 in Appendix I and $B_3 > 0$, we conclude that $f(S_h)$ has two different positive roots if $\Delta_3 < 0$, which is described in Figure 1 (a). Then we proved case (1).

By cases II2 and II5 in Table 2 in Appendix I, we can know that it is a root of multiplicity 2 when equation (3.4) has one positive root and $B_3 > 0$. It is described in Figure 1 (b). This completes the proof. \square

Theorem 3.2. *If $ac\mu_1 < d\mu_2$, system (2.1) has at least one positive equilibrium and at most three positive equilibria. Moreover, system (2.1) has*

- (1) *three positive equilibria if $\Delta_3 < 0, B_1 < 0, B_2 > 0, B_3 < 0$;*
- (2) *two positive equilibria if $\Delta_3 = 0, B_1B_2 < B_3 < 0$ and $B_1^2 - 3B_2 > 0$;*
- (3) *one positive equilibrium in other cases.*

Proof. The number of positive equilibrium of system (2.1) is corresponding to the number of positive root of cubic equation (3.4). When $ac\mu_1 < d\mu_2$, $f(0) = B_3 < 0$ and $f(\pm\infty) = \pm\infty$, $f(S_h)$ has at least one positive equilibrium and at most three positive equilibria.

When $\Delta_3 < 0, B_1 < 0, B_2 > 0, B_3 < 0$, there are three different positive roots of the cubic equation (3.4) by case III4 of Table 2. When condition (1) holds, $f(S_h)$ has three different positive roots by Descartes' rule of sign which is described in Figure 2 (a).

If $f(S_h)$ has two different positive roots, one of the two roots is of multiplicity 2, which implies that $f(\eta_+) = 0$ or $f(\eta_-) = 0$, see Figure 2 (b). By case II4 of Table 2, we know that $f(S_h)$ has two different positive roots if $\Delta_3 = 0, B_1 B_2 < B_3 < 0$. In addition, we add a constraint which is $B_1^2 - 3B_2 > 0$ when $f(S_h)$ has a root of multiply 3 to ensure that $f(S_h)$ has two different positive roots. Then case (2) is proved. From previous analysis, we know that $f(S_h)$ has at least one positive equilibrium due to $B_3 < 0$, which means that there is one positive equilibrium in other cases in Table 2 when $B_3 < 0$. Thus, we complete the proof. \square

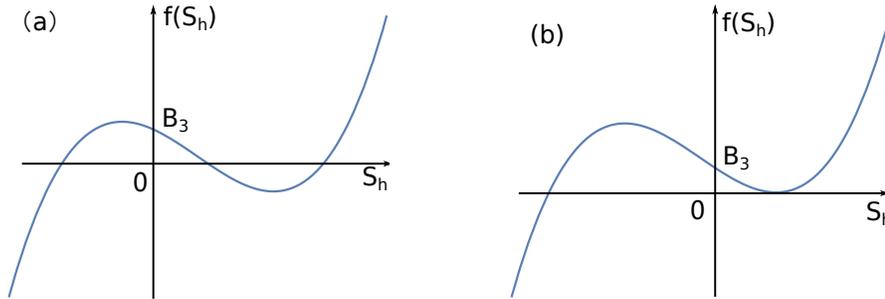


Figure 1. The number of root for $f(S_h) = 0$ when $ac\mu_1 > d\mu_2$. (a) Two different positive roots. (b) One positive root.

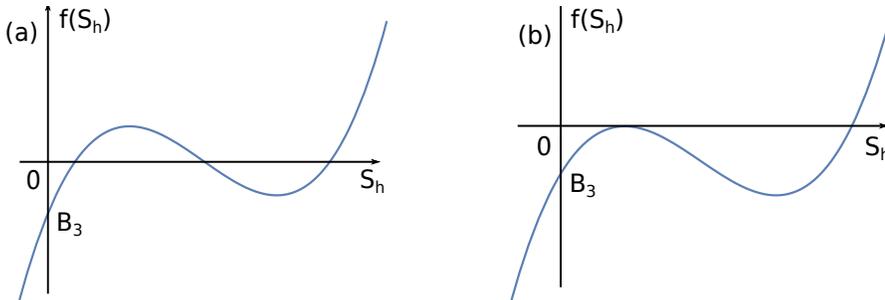


Figure 2. The number of root for $f(S_h) = 0$ when $ac\mu_1 < d\mu_2$. (a) Three different positive roots. (b) Two different positive roots.

Theorem 3.3. *If $ac\mu_1 = d\mu_2$, system (2.1) has at most three positive equilibria. Moreover, system (2.1) has*

- (1) *two positive equilibria if $ab \ln R_v > \Lambda_h \sigma$ and $C' > 0$ (where C' is defined as (3.9));*
- (2) *one positive equilibrium if $ab \ln R_v < \Lambda_h \sigma$;*
- (3) *no positive equilibrium in other cases.*

Proof. If $ac\mu_1 = d\mu_2$, then equation (3.2) becomes the following quadratic equation:

$$f_2(S_h) = B'S_h^2 + C'S_h + D', \tag{3.8}$$

where

$$\begin{aligned}
 B' &= d\mu_2(d + \mu_1)(ab \ln R_v - \Lambda_h \sigma), \\
 C' &= -\frac{1}{\mu_1} \Lambda_h \mu_2 (d + \mu_1) (abd \ln R_v + \Lambda_h \sigma (-d + \mu_1)), \\
 D' &= \frac{1}{\mu_1} \Lambda_h^3 \mu_2 \sigma (d + \mu_1), \\
 \Delta &= \frac{1}{\mu_1^2} \Lambda_h^2 \mu_2^2 (d + \mu_1)^2 (abd \ln R_v - \Lambda_h (d + \mu_1) \sigma)^2.
 \end{aligned} \tag{3.9}$$

It is easy to see that D' and Δ are greater than zero.

If $ab \ln R_v > \Lambda_h \sigma$, then $B' > 0$. Therefore, there are two positive equilibria as long as $C' < 0$, otherwise, there is no positive equilibrium. We complete the proof of case (1).

If $ab \ln R_v < \Lambda_h \sigma$, then $B' < 0$. According to the diagram of the quadratic equation (3.9) under these conditions, we can know that there is only one equilibrium. We complete the proof of case (2). \square

3.2. Stability of equilibria

3.2.1. Stability of boundary equilibrium

Theorem 3.4. *The vector-free equilibrium $E_0(S_h^0, 0, 0, 0)$ of system (2.1) is locally asymptotically stable when $R_v < 1$, where $S_h^0 = \frac{\Lambda_h}{\mu_1}$.*

Proof. It is easy to calculate the following Jacobian matrix $J(E_0)$ at E_0 of system (2.1),

$$J(E_0) = \begin{pmatrix} -\mu_1 & 0 & 0 & -ab \\ 0 & -(d + \mu_1) & 0 & ab \\ 0 & 0 & r - \mu_2 & r \\ 0 & 0 & 0 & -\mu_2 \end{pmatrix}. \tag{3.10}$$

Obviously, the characteristic values of matrix $J(E_0)$ are $-\mu_1, -(d + \mu_1), r - \mu_2$ and $-\mu_2$. In view of the condition $R_v < 1$, all the characteristic values are negative. Therefore, the vector-free equilibrium $E_0(S_h^0, 0, 0, 0)$ of system (2.1) is locally asymptotically stable. \square

Theorem 3.5. *The vector-free equilibrium $E_0(S_h^0, 0, 0, 0)$ of system (2.1) is globally asymptotically stable if $R_v < 1$ and unstable if $R_v > 1$.*

Proof. Let $N_v = S_v + I_v$ and sum up the third and fourth equations of system (2.1), we have

$$\begin{aligned}
 N_v' &= rN_v e^{-\sigma N_v} - \mu_2 N_v \\
 &\leq (r - \mu_2) N_v.
 \end{aligned} \tag{3.11}$$

In view of $N_v(0) = S_v(0) + I_v(0)$, we have

$$\limsup_{t \rightarrow \infty} N_v(t) \leq \lim_{t \rightarrow \infty} N_v(0) e^{(r - \mu_2)t} = 0,$$

when $R_v < 1$. It means that the solutions of S_v and I_v with feasible initial condition tend to zero if $R_v < 1$. For N_h , it is cooperative with its positive invariance set $[0, \frac{\Lambda_h}{\mu_1}]$. We also know that E_0 is a unique equilibrium of system (2.1) when $R_v < 1$. Thus, we know that E_0 is globally asymptotically stable if $R_v < 1$, and unstable if $R_v > 1$. \square

Theorem 3.6. *The parasite-free equilibrium $E_S(S_h^0, 0, S_v^0, 0)$ of system (2.1) is*

- (1) *unstable if $R_0 > 1$;*
- (2) *locally asymptotically stable if $R_0 < 1$;*
- (3) *a saddle-node point if $R_0 = 1$.*

Proof. System (2.1) has a parasite-free equilibrium $E_S(S_h^0, 0, S_v^0, 0)$ when $R_v > 1$. It is easy to obtain the following Jacobian matrix $J(E_S)$ of system (2.1) at E_S ,

$$J(E_S) = \begin{pmatrix} -\mu_1 & 0 & 0 & -ab \\ 0 & -(d + \mu_1) & 0 & ab \\ 0 & -\frac{ac\mu_1 \ln R_v}{\Lambda_h \sigma} & -\mu_2 \ln R_v & \mu_2 - \mu_2 \ln R_v \\ 0 & \frac{ac\mu_1 \ln R_v}{\Lambda_h \sigma} & 0 & -\mu_2 \end{pmatrix}. \quad (3.12)$$

Then we have the following characteristic polynomial

$$P(\lambda) = (\lambda + \mu_1)(\lambda + \mu_2 \ln R_v)(\lambda^2 + b_0\lambda + c_0), \quad (3.13)$$

where

$$b_0 = d + \mu_1 + \mu_2, c_0 = (d + \mu_1)\mu_2 - \frac{a^2bc\mu_1 \ln R_v}{\Lambda_h \sigma}.$$

The eigenvalues of Jacobian matrix (3.12) are denoted by

$$\lambda_1 = -\mu_1, \lambda_2 = -\mu_2 \ln R_v, \lambda_3 = \frac{-b_0 + \sqrt{\Delta_0}}{2}, \lambda_4 = \frac{-b_0 - \sqrt{\Delta_0}}{2},$$

where

$$\Delta_0 = b_0^2 - 4c_0 = (d + \mu_1 - \mu_2)^2 + \frac{4a^2bc\mu_1 \ln R_v}{\Lambda_h \sigma} > 0.$$

If $R_0 > 1$, then $\lambda_1 < 0, \lambda_2 < 0, \lambda_3 > 0, \lambda_4 < 0$, i.e., E_S is unstable.

If $R_0 < 1$, then $\lambda_1 < 0, \lambda_2 < 0, \lambda_3 < 0, \lambda_4 < 0$, i.e., E_S is locally asymptotically stable.

If $R_0 = 1$, we have $\lambda_3 = 0$, so that we need to illustrate the type of E_S .

Firstly, let $S_h = x_1 + \frac{\Lambda_h}{\mu_1}, I_h = x_2, S_v = x_3 + \frac{1}{\sigma} \ln R_v, I_v = x_4$ to move the equilibrium E_S to the origin, then system (2.1) becomes

$$\begin{aligned} x_1' &= -\frac{1}{\Lambda_h^3}(\Lambda_h^3\mu_1x_1 + abx_4(\Lambda_h + \mu_1x_1)(\Lambda_h^2 - \Lambda_h\mu_1(x_1 + x_2) + \mu_1^2(x_1 + x_2)^2)), \\ x_2' &= \frac{1}{\Lambda_h^3}(-\Lambda_h^3x_2(d + \mu_1) + abx_4(\Lambda_h + \mu_1x_1)(\Lambda_h^2 - \Lambda_h\mu_1(x_1 + x_2) + \mu_1^2(x_1 + x_2)^2)), \\ x_3' &= \frac{1}{2\Lambda_h^3\sigma}(\sigma(-2ac\mu_1x_2x_3(\Lambda_h^2 - \Lambda_h\mu_1(x_1 + x_2) + \mu_1^2(x_1 + x_2)^2) + \Lambda_h^3\mu_2(\sigma^2x_4^3 \end{aligned} \quad (3.14)$$

$$\begin{aligned}
 & + \sigma x_3^2(\sigma x_3 - 2) + \sigma x_4^2(3\sigma x_3 - 2) + x_4(2 - 4\sigma x_3 + 3\sigma^2 x_3^2)) + (-2ac\mu_1 x_2(\Lambda_h^2 \\
 & - \Lambda_h \mu_1(x_1 + x_2) + \mu_1^2(x_1 + x_2)^2) + \Lambda_h^3 \mu_2 \sigma(x_3 + x_4)(\sigma x_3 + \sigma x_4 - 2)) \ln R_v), \\
 x_4' = & - \frac{1}{\Lambda_h^3 \sigma} (\Lambda_h^2 - \Lambda_h \mu_1(x_1 + x_2) + \mu_1^2(x_1 + x_2)^2)((-ac\mu_1 x_2 x_3 + x_4(\Lambda_h \\
 & + (x_1 + x_2)\mu_1)\mu_2)\sigma - ac\mu_1 \ln R_v x_2).
 \end{aligned}$$

By using the transformation

$$x_1 = m_{11}X_1 + X_2 + m_{13}X_3, x_2 = m_{21}X_1 + m_{23}X_3, x_3 = -X_1 - X_3 + X_4, x_4 = X_1 + X_3,$$

where

$$m_{11} = -\frac{ab}{\mu_1}, m_{13} = \frac{ab}{d + \mu_2}, m_{21} = \frac{ab}{d + \mu_1}, m_{23} = -\frac{ab}{\mu_2}.$$

Then we obtain

$$\begin{aligned}
 X_1' &= \frac{ab((d - \mu_1)\mu_2 - ac\mu_1)}{\Lambda_h(d + \mu_1 + \mu_2)} X_1^2 + X_1 \mathcal{O}(|X_2, X_3, X_4|) + \mathcal{O}(|X_2, X_3, X_4|^2), \\
 X_2' &= -\mu_1 X_2 + \mathcal{O}(|X_1, X_2, X_3, X_4|^2), \\
 X_3' &= -(d + \mu_1 + \mu_2)X_3 + \mathcal{O}(|X_1, X_2, X_3, X_4|^2), \\
 X_4' &= -\mu_2 \ln R_v X_4 + \mathcal{O}(|X_1, X_2, X_3, X_4|^2),
 \end{aligned} \tag{3.15}$$

which shows that E_S is a saddle-node point when $R_0 = 1$. □

3.2.2. Stability of positive equilibrium

In this subsection, we will study the local stability of positive equilibrium $E^*(S_h^*, I_h^*, S_v^*, I_v^*)$.

The Jacobian Matrix of system (2.1) at E^* is

$$J(E^*) = \begin{pmatrix} m - \frac{abI_v^*}{I_h^* + S_h^*} - \mu_1 & m & 0 & -\frac{abS_h^*}{I_h^* + S_h^*} \\ -m + \frac{abI_v^*}{I_h^* + S_h^*} & -d - \mu_1 - m & 0 & \frac{abS_h^*}{I_h^* + S_h^*} \\ n & n - \frac{acS_v^*}{I_h^* + S_h^*} & J_{33} & J_{33} + \frac{acI_h^*}{I_h^* + S_h^*} + \mu_2 \\ -n & -n + \frac{acS_v^*}{I_h^* + S_h^*} & \frac{acI_h^*}{I_h^* + S_h^*} & -\mu_2 \end{pmatrix},$$

where $m = \frac{abI_v^* S_h^*}{(I_h^* + S_h^*)^2}$, $n = \frac{acI_h^* S_v^*}{(I_h^* + S_h^*)^2}$, and $J_{33} = re^{-(I_v^* + S_v^*)\sigma} - re^{-(I_v^* + S_v^*)\sigma} (I_v^* + S_v^*)\sigma - \frac{acI_h^*}{I_h^* + S_h^*} - \mu_2$. It is easy to obtain the characteristic polynomial of $J(E^*)$:

$$P(\lambda) = \lambda^4 + a_1\lambda^3 + a_2\lambda^2 + a_3\lambda + a_4, \tag{3.16}$$

$$P_1(\xi; S_h^*, I_h^*, S_v^*, I_v^*) = \xi^4 + a_1\xi^3 + a_2\xi^2 + a_3\xi + a_4, \tag{3.17}$$

where

$$\begin{aligned}
 a_1 &= \frac{abI_v^* + d(I_h^* + S_h^*) - (I_h^* + S_h^*)(J_{33} - 2\mu_1 - \mu_2)}{I_h^* + S_h^*}, \\
 a_2 &= -\frac{1}{(I_h^* + S_h^*)^2} (a^2c(bS_h^*S_v^* + cI_h^{*2}) + a(I_h^* + S_h^*)(cI_h^*(J_{33} + \mu_2))
 \end{aligned}$$

$$\begin{aligned}
 & -bI_v^*(d - J_{33} + \mu_1 + \mu_2) + (I_h^* + S_h^*)^2(d(J_{33} - \mu_1 - \mu_2 + m) + J_{33}(2\mu_1 + \mu_2) \\
 & - \mu_1(\mu_1 + 2\mu_2))), \\
 a_3 = & \frac{1}{(I_h^* + S_h^*)^3}(a^3bc^2I_h^*(S_h^*S_v^* - I_h^*I_v^*) - a^2c(I_h^* + S_h^*)(b(I_h^*I_v^*(J_{33} + \mu_2) \\
 & + S_h^*S_v^*(\mu_1 - J_{33})) + cI_h^{*2}(d + 2\mu_1)) + (I_h^* + S_h^*)^3(d(J_{33}(-\mu_1 - \mu_2 + m) \\
 & + \mu_2(\mu_1 - m)) + \mu_1(\mu_1\mu_2 - J_{33}(\mu_1 + 2\mu_2))) \\
 & - a(I_h^* + S_h^*)^2(b(d(I_v^*(J_{33} - \mu_2) + nS_h^*) \\
 & + I_v^*(J_{33}(\mu_1 + \mu_2) - \mu_1\mu_2)) + cI_h^*(d + 2\mu_1)(J_{33} + \mu_2))), \\
 a_4 = & -\frac{1}{(I_h^* + S_h^*)^3}(acI_h^* + J_{33}(I_h^* + S_h^*))(a^2bc(dI_h^*I_v^* + I_h^*I_v^*\mu_1 - \mu_1S_h^*S_v^*) \\
 & - a(I_h^* + S_h^*)(b(-dI_v^*\mu_2 + dnS_h^* - I_v^*\mu_1\mu_2) + cI_h^*(d(m - \mu_1) - \mu_1^2) \\
 & - \mu_2(I_h^* + S_h^*)^2(d(m - \mu_1) - \mu_1^2)). \tag{3.18}
 \end{aligned}$$

Then by using Routh-Hurwitz criteria, we have the following theorem:

Theorem 3.7. *The positive equilibrium $E^*(S_h^*, I_h^*, S_v^*, I_v^*)$ is locally asymptotically stable if $a_1a_2a_3 - a_3^2 - a_1^2a_4 > 0$, and $a_i > 0, i = 1, 2, 3, 4$ when*

- (1) $ac\mu_1 > d\mu_2$, and
 - (i) $B_1^2 = -B_2, B_1B_2 = B_3 > 0$;
 - (ii) $\Delta_3 = 0, B_3 > \max\{B_1B_2, 0\}$;
- (2) $ac\mu_1 = d\mu_2$, and $ab \ln R_v < \Lambda_h\sigma$.

4. Backward bifurcation and forward bifurcation

Now, we will study the forward bifurcation and the backward bifurcation of system (2.1) by calculating the direction of transcritical bifurcation point in the following.

Theorem 4.1. *When $R_0 = 1$, system (2.1) undergoes*

- (1) *backward bifurcation if $\Lambda_h < \Lambda_h^*$;*
- (2) *forward bifurcation if $\Lambda_h > \Lambda_h^*$ where $\Lambda_h^* = \frac{ab \ln R_v(d-\mu_1)}{\sigma(d+\mu_1)}$.*

Proof. The Jacobian matrix at $E_S(S_h^0, 0, S_v^0, 0)$ of system (2.1) is

$$J(E_S) = \begin{pmatrix} -\mu_1 & 0 & 0 & -ab \\ 0 & -(d + \mu_1) & 0 & ab \\ 0 & -\frac{ac\mu_1 \ln R_v}{\Lambda_h\sigma} & -\mu_2 \ln R_v & \mu_2 - \mu_2 \ln R_v \\ 0 & \frac{ac\mu_1 \ln R_v}{\Lambda_h\sigma} & 0 & -\mu_2 \end{pmatrix}. \tag{4.1}$$

Then the corresponding eigenvalues of the Jacobian matrix $J(E_S)$ are, respectively,

$$\lambda_1 = -\mu_1, \lambda_2 = -\mu_2 \ln R_v, \lambda_3 = \frac{-b_0 + \sqrt{\Delta_0}}{2}, \lambda_4 = \frac{-b_0 - \sqrt{\Delta_0}}{2},$$

where

$$b_0 = d + \mu_1 + \mu_2, c_0 = (d + \mu_1)\mu_2 - \frac{a^2bc\mu_1 \ln R_v}{\Lambda_h\sigma},$$

$$\Delta_0 = b_0^2 - 4c_0 = (d + \mu_1 - \mu_2)^2 + \frac{4a^2bc\mu_1 \ln R_v}{\Lambda_h\sigma} > 0.$$

Because of the biological significance, we know that all parameter values are non-negative. Obviously, eigenvalues λ_1 and λ_2 are negative.

Let $R_0 = 1$, then we have

$$c^* \triangleq \frac{\Lambda_h\mu_2\sigma(d + \mu_1)}{a^2b\mu_1 \ln R_v}.$$

Replacing c in λ_3 and λ_4 with c^* , we obtain that $\lambda_3 = 0, \lambda_4 < 0$. From Theorem 3.6, we know that parasite-free equilibrium $E_S(S_h^0, 0, S_v^0, 0)$ of system (2.1) is unstable if $c > c^*$, and locally asymptotically stable if $c < c^*$ which means that $c = c^*$ is a bifurcation value.

Through calculation, we get a right eigenvector u and a left eigenvector v associated with the zero eigenvalue as follows:

$$u = (u_1, u_2, u_3, u_4)^T = (-ab(d + \mu_1)S_h, ab\mu_1 S_h, -\mu_1(d + \mu_1)S_h, \mu_1(d + \mu_1)S_h)^T, \quad (4.2)$$

$$v = (v_1, v_2, v_3, v_4) = (0, \mu_2, 0, ab).$$

By using orthogonal condition $\langle u, v \rangle = 1$, one yields

$$S_h^* = \frac{1}{ab\mu_1(d + \mu_1 + \mu_2)}.$$

Next, set

$$S_h = x_1, I_h = x_2, S_v = x_3, I_v = x_4,$$

then system (2.1) has the form $\frac{dx}{dt} = f$, where $x = (x_1, x_2, x_3, x_4)^T$, $f = (f_1, f_2, f_3, f_4)^T$. Then we have

$$x_1'(t) = \Lambda_h - ab \frac{x_4(t)}{x_1(t) + x_2(t)} x_1(t) - \mu_1 x_1(t) := f_1,$$

$$x_2'(t) = ab \frac{x_4(t)}{x_1(t) + x_2(t)} x_1(t) - dx_2(t) - \mu_1 x_2(t) := f_2, \quad (4.3)$$

$$x_3'(t) = r(x_3(t) + x_4(t))e^{-\sigma(x_3(t) + x_4(t))} - ac \frac{x_3(t)}{x_1(t) + x_2(t)} x_2(t) - \mu_2 x_3(t) := f_3,$$

$$x_4'(t) = ac \frac{x_3(t)}{x_1(t) + x_2(t)} x_2(t) - \mu_2 x_4(t) := f_4.$$

The bifurcation coefficients in system (2.1) at E_S are as follows:

$$a^* = \sum_{i,j,k=1}^4 v_i u_j u_k \frac{\partial^2 f_i}{\partial x_j \partial x_k}(E_S, c^*), b^* = \sum_{i,j=1}^4 v_i u_j \frac{\partial^2 f_i}{\partial x_j \partial c}(E_S, c^*).$$

Obviously, since $v_1 = v_3 = 0$, we only consider the cross derivatives of f_2 and f_4 in system (4.3) at E_S . Then we can obtain non-zero terms as follows:

$$\begin{aligned} \frac{\partial^2 f_2}{\partial x_2 \partial x_4} &= \frac{\partial^2 f_2}{\partial x_4 \partial x_2} = -\frac{ab\mu_1}{\Lambda_h}, & \frac{\partial^2 f_4}{\partial x_1 \partial x_2} &= \frac{\partial^2 f_4}{\partial x_2 \partial x_1} = -\frac{\mu_1\mu_2(d + \mu_1)}{ab\Lambda_h}, \\ \frac{\partial^2 f_4}{\partial x_2 \partial x_3} &= \frac{\partial^2 f_4}{\partial x_3 \partial x_2} = \frac{\mu_2\sigma(d + \mu_1)}{ab \ln R_v}, & \frac{\partial^2 f_4}{\partial x_2 \partial x_2} &= -\frac{2\mu_1\mu_2(d + \mu_1)}{ab\Lambda_h}, \\ \frac{\partial^2 f_4}{\partial x_2 \partial c} &= \frac{\partial^2 f_4}{\partial c \partial x_2} = \frac{a\mu_1 \ln R_v}{\sigma\Lambda_h}. \end{aligned} \quad (4.4)$$

The corresponding values of a^* and b^* are as follows:

$$\begin{aligned} a^* &= \sum_{i,j,k=1}^4 v_i u_j u_k \frac{\partial^2 f_i}{\partial x_j \partial x_k}(E_S, c^*) \\ &= v_2 \sum_{j,k=1}^4 u_j u_k \frac{\partial^2 f_2}{\partial x_j \partial x_k}(E_S, c^*) + v_4 \sum_{j,k=1}^4 u_j u_k \frac{\partial^2 f_4}{\partial x_j \partial x_k}(E_S, c^*) \\ &= -\frac{2ab\mu_1^2\mu_2(d + \mu_1)}{\Lambda_h \ln R_v} (ab \ln R_v (\mu_1 - d) + \Lambda_h \sigma (d + \mu_1)) S_h^{*2}, \\ b^* &= \sum_{i,j=1}^4 v_i u_j \frac{\partial^2 f_i}{\partial x_j \partial c}(E_S, c^*) \\ &= v_4 u_2 \frac{\partial^2 f_4}{\partial x_2 \partial c}(E_S, c^*) \\ &= \frac{a^3 b^2 \mu_1^2 \ln R_v}{\sigma \Lambda_h} S_h^* > 0. \end{aligned} \quad (4.5)$$

From [37, 38], we realized that the signs of a^* and b^* determine the local dynamical behaviour around parasite-free equilibrium E_S of system (2.1). It is easy to see that $b^* > 0$ since S_h^* are great than zero. There are an unstable equilibrium exhibiting backward bifurcation near E_S if $a^* > 0$, and a locally asymptotically stable equilibrium undergoing forward bifurcation if $a^* < 0$. Then we will discuss the sign of a^* to study the dynamics of system (2.1).

Define

$$\Lambda_h^* = \frac{ab \ln R_v (d - \mu_1)}{\sigma (d + \mu_1)}. \quad (4.6)$$

Then we can draw conclusions that system (2.1) undergoes a backward bifurcation if $\Lambda_h < \Lambda_h^*$, i.e., $a^* > 0$; If $\Lambda_h > \Lambda_h^*$, i.e., $a^* < 0$, system (2.1) undergoes a forward bifurcation. This completes the proof. \square

5. Numerical simulations

In this section, we will illustrate numerically the results of system (2.1) with the following parameter values in Wu et al. [36] by using Matlab and Auto07P [7]:

$$\Lambda_h = 5, \quad a = 0.6, \quad b = 0.06, \quad c = 0.49, \quad r = 0.0274, \quad \sigma = 0.0002,$$

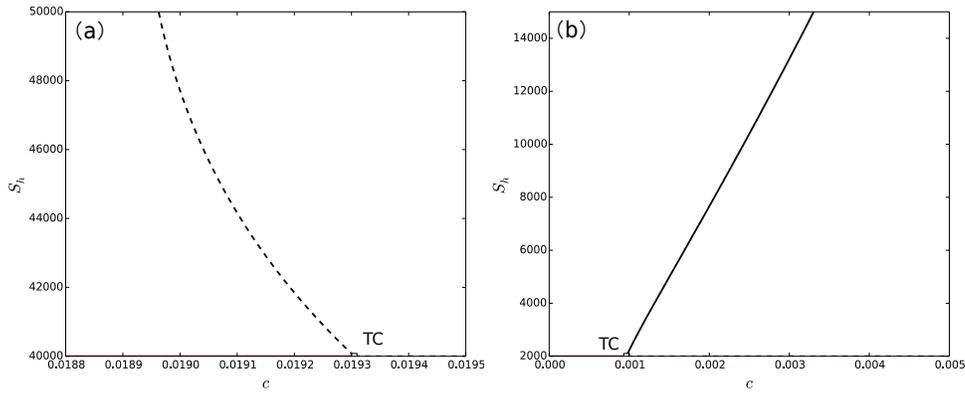


Figure 3. (a) Backward bifurcation diagram of system (2.1) showing c vs S_h with $\Lambda_h = 100$; (b) Forward bifurcation diagram of system (2.1) showing c vs S_h with $\Lambda_h = 5$.

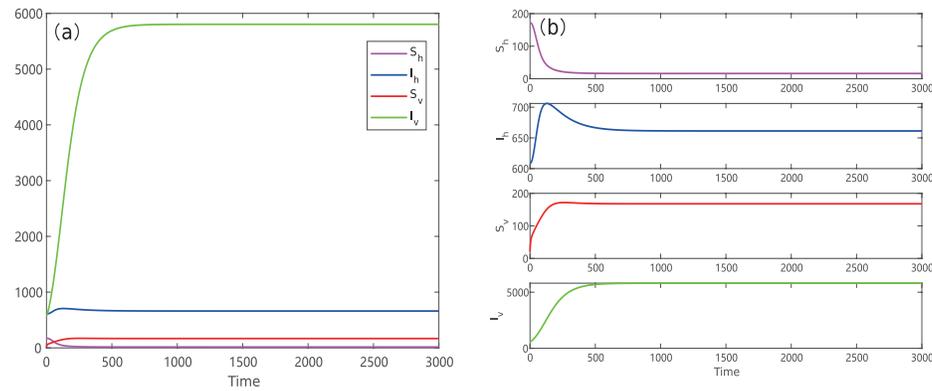


Figure 4. Numerical solution of system (2.1) tends to a stable equilibrium when the time tends to infinity, where $\Lambda_h = 5$, $r = 0.0274$, $\sigma = 0.0002$, $\mu_1 = 0.0025$, $\mu_2 = 0.0083$, $d = 0.0083$. The initial solution is $(S_h, I_h, S_v, I_v) = (172, 609, 21, 576)$.

$$\mu_1 = 0.0025, \mu_2 = 0.0083, d = 0.005. \tag{5.1}$$

Using the above parameter values, it is easy to calculate that there is a vector-free equilibrium $(2000, 0, 0, 0)$, a parasite-free equilibrium $(2000, 0, 5971.437, 0)$ and a parasite-positive equilibrium $(16.081, 661.306, 167.827, 5803.610)$.

Firstly, we simulate the backward and forward bifurcation to verify the results in the previous sections. We choose transmission probability from infected hosts to susceptible triatomine bug per bite (c) as the primary bifurcation parameter and keep the other parameters fixed as (5.1), then we get the one-parameter bifurcation diagram, shown in Figure 3. In Figure 3 (a), there is a transcritical bifurcation point $TC(4 \times 10^4, 0, 5.97144 \times 10^3, 0)$ showing backward bifurcation with $\Lambda_h = 100$ when $c = 0.0193049$, and in Figure 3 (b), there is a transcritical bifurcation point $TC(2 \times 10^3, 0, 5.97144 \times 10^3, 0)$ showing forward bifurcation with $\Lambda_h = 5$ when $c = 0.0009652$. A numerical solution of system (2.1) tending to a stable equilibrium when the time tends to infinity with initial solution $(S_h, I_h, S_v, I_v) = (172, 609, 21, 576)$ is shown in Figure 4, where $\Lambda_h = 5$, $r = 0.0274$, $\sigma = 0.0002$, $\mu_1 =$

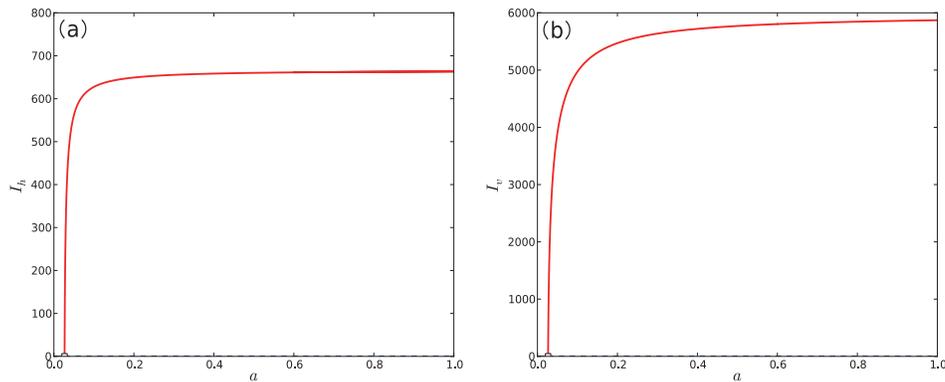


Figure 5. One-parameter bifurcation diagram of system (2.1) with respect to a . (a) a vs. I_h ; (b) a vs. I_v .

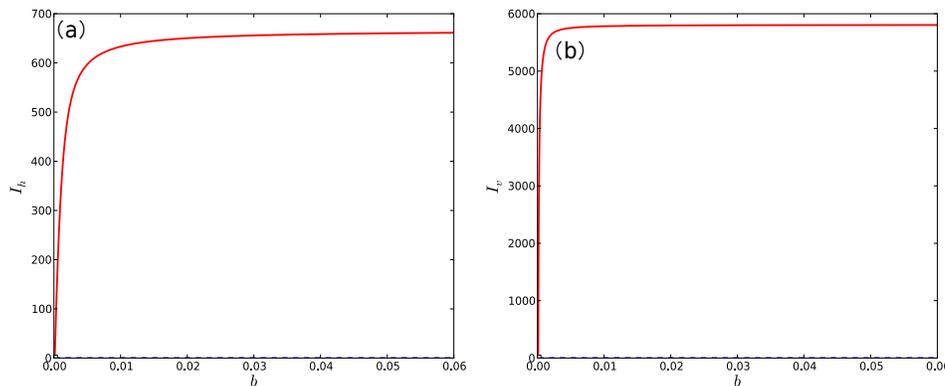


Figure 6. One-parameter bifurcation diagram of system (2.1) with respect to b . (a) b vs. I_h ; (b) b vs. I_v .

$0.0025, \mu_2 = 0.0083, d = 0.0083$.

Secondly, when the number of bites of per triatomine bug per unit time (a) is used as the primary bifurcation parameter, then we obtain the bifurcation diagram, shown in Figure 5. There is a transcritical bifurcation point $TC(2000, 0, 5971.4377253, 0)$ showing forward bifurcation with $a = 0.02663$. It is obvious that infected hosts (I_h) and infected vectors (I_v) will increase as the parameter a increases, which indicates that chagas disease may go through outbreak.

Thirdly, the transmission probability from infected bugs to susceptible competent hosts per bite (b) is used as the primary bifurcation parameter, we obtain the bifurcation diagram, shown in Figure 6. A transcritical bifurcation point $TC(2000, 0, 5971.4377253, 0)$ shows forward bifurcation as $b = 0.0001182$. The infected hosts (I_h) and infected vectors (I_v) will increase as the parameter b increases which indicates that chagas disease may go through outbreak.

Finally, we take the T. cruzi-induced death rate of hosts (d) as the primary bifurcation parameter, the bifurcation diagram shown in Figure 7 is obtained. The number of infected hosts (I_h) and infected vectors (I_v) will decrease as the parameter

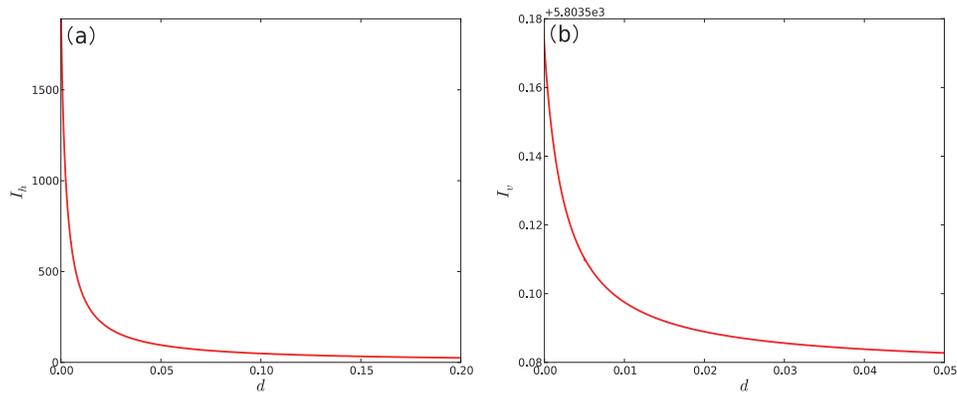


Figure 7. (a) One-parameter bifurcation diagram of system (2.1) showing d vs. I_h ; (b) One-parameter bifurcation diagram of system (2.1) showing d vs. I_v .

d increases. It means that the increase of death rate of hosts caused by *T. cruzi* will threaten the infected hosts and infected vectors.

Biologically speaking, chagas disease may undergo outbreak if the number of bites of per triatomine bug per unit time (a), the transmission probability from infected bugs to susceptible competent hosts per bite (b) increase while chagas disease may disappear if the *T. cruzi*-induced death rate of hosts (d) decreases.

6. Discussion and conclusion

In this paper, a chagas model (2.1) with Ricker's type function and the standard incidence rate is investigated by using the dynamical system approach, and the existence and stability of equilibria are obtained, the related bifurcations are given, which tells that the positive equilibrium of model (2.1) is locally asymptotically stable when the number of bites of per triatomine bug per unit time (a), the transmission probability from infected bugs to susceptible competent hosts per bite (b) and *T. cruzi*-induced death rate of hosts (d) increase. Once the standard incidence rate is considered, the positive equilibrium of system (2.1) will be stable, however, it still tells us that the outbreak of infected hosts and infected vectors without sustained oscillations.

7. Appendix

7.1. Appendix I

Consider the cubic equation

$$x^3 + b_1x^2 + b_2x + b_3 = 0, \quad (7.1)$$

where the coefficients b_1, b_2, b_3 are real. It is known that equation (7.1) has three roots in field C as following:

$$x_1 = \sqrt[3]{-\frac{Q}{2} + \sqrt{\Delta_3}} + \sqrt[3]{-\frac{Q}{2} - \sqrt{\Delta_3}} - \frac{b_1}{3},$$

$$x_2 = \sqrt[3]{-\frac{Q}{2} + \sqrt{\Delta_3}\omega^2} + \sqrt[3]{-\frac{Q}{2} - \sqrt{\Delta_3}\omega} - \frac{b_1}{3},$$

$$x_3 = \sqrt[3]{-\frac{Q}{2} + \sqrt{\Delta_3}\omega} + \sqrt[3]{-\frac{Q}{2} - \sqrt{\Delta_3}\omega^2} - \frac{b_1}{3}$$

where $\omega = \frac{(-1+\sqrt{3}i)}{2}$, $Q = \frac{2}{27}b_1^3 + b_3 - \frac{1}{3}b_1b_2$, and $\Delta_3 = \frac{4b_1^3b_3 - b_1^2b_2^2 + 4b_2^3 - 18b_1b_2b_3 + 27b_3^2}{108}$.

The following table is the distribution of roots of equation (7.1):

Table 2. Distribution of roots of equation (7.1)

Case	Possibilities of b_1, b_2, b_3	Distribution of roots
I1	$b_1^2 < 4b_2, b_1 < (>)b_3 = 0$	$x_1 = 0, \Re x_{2,3} > (<)0, \Im x_{2,3} \neq 0$
I2	$b_2 > 0, b_1b_2 = b_3 > (resp. =, <)0$	$x_1 < (resp. =, >)0, \Re x_{2,3} = 0, \Im x_{2,3} \neq 0$
I3	$\Delta_3 > 0, b_1b_2 > (<)b_3 > (<)0$	$x_1 < (>)0, \Re x_{2,3} < (>)0, \Im x_{2,3} \neq 0$
I4	$\Delta_3 > 0, b_3 > \max(< \min)\{b_1b_2, 0\}$	$x_1 < (>)0, \Re x_{2,3} > (<)0, \Im x_{2,3} \neq 0$
II1	$b_1^2 = 4b_2, b_1b_2 < (>)b_3 = 0$	$x_1 = 0, x_2 = x_3 > (<)0$
II2	$b_1^2 = -b_2, b_1b_2 = b_3 > (<)0$	$x_1 < (>)0, x_2 = x_3 > (<)0$
II3	$b_1 > (resp. =, <)b_2 = b_3 = 0$	$x_1 < (resp. =, >)x_2 = x_3 = 0$
II4	$\Delta_3 = 0, b_1b_2 < (>)b_3 < (>)0$	$x_1 > (<)0, x_2 = x_3 > (<)0$
II5	$\Delta_3 = 0, b_3 > \max(< \min)\{b_1b_2, 0\}$	$x_1 < (>)0, x_2 = x_3 > (<)0$
III1	$b_1^2 > 4b_2, b_1 < (>)0, b_2 > 0, b_3 = 0$	$x_1 > (=)0, x_2 > (<)0, x_3 = (<)0$
III2	$b_2 < 0, b_3 = 0$	$x_1 > 0, x_2 = 0, x_3 < 0$
III3	$\Delta_3 < 0, b_1 \geq (\leq)0, b_3 < (>)0$	$x_1 > 0, x_2 < (>)0, x_3 < 0$
III4	$\Delta_3 < 0, b_1 < (>)0, b_2 > (<)0, b_3 < (>)0$	$x_1 > 0, x_2 > 0, x_3 > (<)0$
III5	$\Delta_3 < 0, b_1 < (>)0, b_2 < (>)0, b_3 < (>)0$	$x_1 > (<)0, x_2 < 0, x_3 < 0$

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