Journal of Applied Analysis and Computation
Volume 14, Number 2, April 2024, 964–975
Website:http://www.jaac-online.com
DOI:10.11948/20230221

GENERALIZED CAPUTO–FABRIZIO FRACTIONAL DIFFERENTIAL EQUATION*

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Abstract In this paper, a generalization of the Caputo–Fabrizio fractional derivative is proposed. The purpose of this study is to derive a solution formula for ordinary differential equations with the generalized Caputo–Fabrizio fractional derivative. The main result can be applied to solve the Caputo–Fabrizio fractional differential equation $\mathcal{D}^{\alpha} y = f(y)$. That is, a new result even for common Caputo–Fabrizio fractional differential equations is obtained. The strength of the results obtained in this study is that the solution to the differential equation can be given using only the kernel included in the derivative and the right-hand side f of the equation. In other words, rather than providing a method to solve the solution, this study provides a formula for the solution. This study is proposed as a tool for solving many nonlinear equations, including the logistic type fractional differential equations.

Keywords Caputo–Fabrizio fractional derivative, fractional differential equation, fractional calculus, nonsingular kernel, logistic equation.

MSC(2010) 34A08, 26A33, 34A05.

1. Introduction

In this section, we first introduce the definitions of known fractional derivatives (Caputo–Fabrizio derivative and Caputo derivative), and then define a new fractional derivative that will be dealt with in this study. We call it a generalized Caputo–Fabrizio derivative. Next, we consider a fractional order differential equation involving the generalized Caputo–Fabrizio derivative and introduce logistic fractional differential equations. In addition, the solution formula, which is the main theorem of this study, is provided. Finally, we give remarks about the properties of solutions to nonlinear equations, which are important for understanding the main theorem.

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[*]The first author was supported by the Japan Society for the Promotion of Science (JSPS) KAKENHI (grant number JP20K03668).
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1.1. Fractional derivatives

In 2015, the fractional derivative $\mathcal{D}^\alpha\phi(x)$ with $0<\alpha<1$ was introduced by Caputo and Fabrizio [8]. The definition of $\mathcal{D}^\alpha\phi(x)$ is as follows.

Definition 1.1 (Caputo–Fabrizio derivative)**.** Define

$$\mathcal{D}^\alpha\phi(x) := \frac{1}{1-\alpha}\int_0^x e^{-\frac{\alpha}{1-\alpha}(x-\xi)}\phi'(\xi)d\xi, \quad x\geq 0,$$

with $0<\alpha<1$.

Needless to say, the Caputo–Fabrizio derivative $\mathcal{D}^\alpha\phi$ is a completely different derivative than the Caputo derivative $D^\alpha\phi(x)$ which is defined by the following.

Definition 1.2 (Caputo derivative)**.** Define

$$D^\alpha\phi(x) := \frac{1}{\Gamma(1-\alpha)}\int_0^x (x-\xi)^{-\alpha}\phi'(\xi)d\xi, \quad x\geq 0,$$

with $0<\alpha<1$.

Of course, there are many other fractional derivatives, but the kernel

$$\mathcal{K}_\alpha(x,\xi) := e^{-\frac{\alpha}{1-\alpha}(x-\xi)}$$

of the Caputo–Fabrizio derivative $\mathcal{D}^\alpha\phi$ is characterized by the absence of singularity. Although the Caputo–Fabrizio fractional derivative was recently proposed, it has been applied in various fields. For example, it is useful for elucidating various phenomena including infectious diseases [1, 4, 5, 11, 20, 26, 31]. In particular, Alinei-Poiana, Dulf, and Kovacs [1] have recently demonstrated the superiority of fractional order models in tumor growth modeling in mathematical oncology. In these applications, derivatives with singular kernels and derivatives with nonsingular kernels are compared in various situations and compete for superiority with each other. In many cases, the Caputo–Fabrizio fractional derivative is compared with the derivative with a singular kernel, as a representative of the derivative with a nonsingular kernel. However, it is possible that there are nonsingular kernels that can reproduce the phenomenon better than the Caputo–Fabrizio fractional derivative. Therefore, in this study, we propose the following new derivative to expand the class of derivatives with nonsingular kernels.

Definition 1.3 (Generalized Caputo–Fabrizio derivative)**.** Define

$${}^G\mathcal{D}^\alpha\phi(x) := \frac{1}{1-\alpha}\int_0^x \frac{\kappa_\alpha(\xi)}{\kappa_\alpha(x)}\phi'(\xi)d\xi, \quad x\geq 0,$$

with $0<\alpha<1$ for continuously differentiable function ϕ, where κ_α is a positive continuously differentiable function on $[0,\infty)$.

Note here that ${}^G\mathcal{D}^\alpha\phi$ has the nonsingular kernel

$${}^G\mathcal{K}_\alpha(x,\xi) := \frac{\kappa_\alpha(\xi)}{\kappa_\alpha(x)}.$$

Moreover, if $\kappa_\alpha(x)\equiv e^{\frac{\alpha}{1-\alpha}x}$ or ${}^G\mathcal{K}_\alpha=\mathcal{K}_\alpha$, then the derivative ${}^G\mathcal{D}^\alpha\phi$ is just the Caputo–Fabrizio derivative $\mathcal{D}^\alpha\phi$. Then we say that ${}^G\mathcal{D}^\alpha\phi$ is an α-th order generalized Caputo–Fabrizio fractional derivative with the kernel ${}^G\mathcal{K}_\alpha(x,\xi)$.

1.2. Generalized Caputo–Fabrizio fractional differential equation

This study deals with the generalized Caputo–Fabrizio fractional differential equation

$$ {}^{G}\mathcal{D}^{\alpha} y = f(y) \tag{1.1} $$

with the kernel ${}^{G}\mathcal{K}_{\alpha}(x,\xi)$. Here $f : Y \to \mathbb{R}$ is continuously differentiable, where Y is an interval of $\mathbb{R}$; that is, Y is the domain of definition of f. If ${}^{G}\mathcal{K}_{\alpha} = \mathcal{K}_{\alpha}$, then (1.1) becomes the Caputo–Fabrizio fractional differential equation

$$ \mathcal{D}^{\alpha} y = f(y). \tag{1.2} $$

Studies on fractional ordinary differential equations with the Caputo–Fabrizio derivative are in progress [6, 19, 21, 28, 33]. In addition, there are applications for the Caputo–Fabrizio derivative in the field of partial differential equations [9, 12]. Against this background, it is an important question to know whether it is possible to find an exact solution to the Caputo–Fabrizio fractional differential equations.

When $\alpha \to 1^-$ in (1.2), equation (1.2) is the well-known autonomous ordinary differential equation

$$ y' = f(y). $$

Needless to say, the method of solving this equation when the integral of $1/f(y)$ is explicitly given is widely known in many textbooks on ordinary differential equations. For example, using the method of separation of variables, the solution of the logistic equation

$$ y' = y(1-y) $$

with the initial condition $y(0) = y_0$ is displayed in the form

$$ y(x) = \frac{y_0}{y_0 + (1-y_0)e^{-x}}. $$

In 2022, Nieto [23] derived a formula for the implicit solution of the fractional version of the logistic differential equation

$$ \mathcal{D}^{\alpha} y = y(1-y) \tag{1.3} $$

with $y(0) = y_0$. The form of the solution given by Nieto is as follows:

$$ \frac{y(x) - y^2(x)}{(1-y(x))^{\frac{2}{\alpha}}} = \frac{y_0 - y_0^2}{(1-y_0)^{\frac{2}{\alpha}}} \cdot e^{x} \tag{1.4} $$

with $0 < \alpha < 1$ and $y(x) \neq 0,\ 1$ for all $x \in [0,\infty)$. We can find that other types of Caputo–Fabrizio fractional differential equations have also been solved recently [3, 10, 14, 32]. In particular, Cui [10] uses the same technique as Nieto's paper [23] to obtain implicit solutions to the equations

$$ \mathcal{D}^{\alpha} y = -y^2 + 1, \quad \mathcal{D}^{\alpha} y = 2y - y^2 + 1, \quad \mathcal{D}^{\alpha} y = -y + y^2, $$

and

$$ \mathcal{D}^{\alpha} y = y(1-y^2). $$

See also [7, 15–17, 22, 27] for solutions of fractional differential equations with another fractional derivatives. In particular, for the fractional logistic equation, see [2, 13, 24, 30]. Logistic-type differential equations are known to be extremely effective not only as models for representing population growth, but also for understanding the mechanisms of infectious diseases. For example, Nishiura, Tsuzuki, Yuan, Yamaguchi, and Asai [25] analyzes the dynamics of cholera in Yemen. Wang, Wu, and Yang [29] proved that the exponential term in a generalized logistic equation has a one-to-one nonlinear correspondence to the basic reproduction number of the SIR model. Therefore, it is very important to study logistic equations and their generalized equations in order to understand phenomena. In particular, the papers [1–4, 14, 23, 24] deal with fractional-order differential equations of the logistic type, mention the importance of fractional-order differential equations of the logistic type, and compare solutions of fractional-order differential equations with solutions of ordinary differential equations.

Inspired by the above results, in order to respond to the demands of society, the present study attempts to apply the technique used in [23] to more general equations. That is, we aim to derive a formula for the implicit solution of (1.1). The following theorem and corollary realize the expression of the non-trivial solution determined only by κ_α, which constitutes the kernel given to the fractional differential equation, and the form f on the right-hand side of the equation. This fact extracts the essence that cannot be found just by looking at the solution to a concrete equation. Our results have the strength that if $\int \frac{d\eta}{f(\eta)}$ is given explicitly, the differential equation can be solved. The obtained result is as follows.

Theorem 1.1. *Let $S := \{\eta \in Y : f(\eta) \neq 0\}$, and let $F : S \to \mathbb{R}$ be a function defined by $F(\eta) := \int \frac{d\eta}{f(\eta)}$. Then an implicit solution of* (1.1) *with $y(0) = y_0 \in Y$ is given by*

$$\frac{\kappa_\alpha(x) f(y(x))}{\kappa_\alpha(0) f(y_0)} \exp\left(\frac{F(y(x)) - F(y_0)}{\alpha - 1}\right) = 1$$

for all $x \in [0, T)$, with $0 < \alpha < 1$, whenever $y(x) \in S$ for all $x \in [0, T)$, where $0 < T \leq \infty$ is the smaller of the maximal existence time M of the solution $y(x)$ and $\inf\{x \in [0, \infty) : f(y(x)) = 0\}$ if $\{x \in [0, \infty) : f(y(x)) = 0\} \neq \emptyset$; M if $\{x \in [0, \infty) : f(y(x)) = 0\} = \emptyset$.

When $\kappa_\alpha(x) \equiv e^{\frac{\alpha}{1-\alpha}x}$, we can obtain the following result.

Corollary 1.1. *Let $S := \{\eta \in Y : f(\eta) \neq 0\}$, and let $F : S \to \mathbb{R}$ be a function defined by $F(\eta) := \int \frac{d\eta}{f(\eta)}$. Then an implicit solution of* (1.2) *with $y(0) = y_0 \in Y$ is given by*

$$\frac{f(y(x))}{f(y_0)} \exp\left(\frac{F(y(x)) - F(y_0) - \alpha x}{\alpha - 1}\right) = 1$$

for all $x \in [0, T)$, with $0 < \alpha < 1$, whenever $y(x) \in S$ for all $x \in [0, T)$, where $0 < T \leq \infty$ is the smaller of the maximal existence time M of the solution $y(x)$ and $\inf\{x \in [0, \infty) : f(y(x)) = 0\}$ if $\{x \in [0, \infty) : f(y(x)) = 0\} \neq \emptyset$; M if $\{x \in [0, \infty) : f(y(x)) = 0\} = \emptyset$.

1.3. Remarks

Remark 1.1. Throughout this paper, let $[0, M)$ be the maximum existence interval of the solution $y(x)$ of (1.1), and $0 < M \leq \infty$ be called the maximal existence time of $y(x)$. If there exists a constant $y_1 \in Y$ such that $f(y_1) = 0$, then we see that the constant function $y(x) = y_1$ for $x \in [0, M)$ is a trivial solution. Especially, M of this solution can be choose $M = \infty$. That is, the trivial solutions of (1.1) exists for all $x \in [0, \infty)$.

On the other hand, an implicit solution given in Theorem 1.1 can be said to be a formula for nontrivial solutions. There are some facts to note about $0 < T \leq \infty$ that appears in Theorem 1.1 because (1.1) is nonlinear. First, we note that there may be nontrivial solutions that blow up in finite time. For example, the equation

$$y' = -y(1-y)$$

with $y(0) = y_0$ has the solution

$$y(x) = \frac{y_0}{y_0 + (1-y_0)e^x}.$$

This implies that if $y_0 > 1$ holds, then the solution $y(x)$ blows up in $x = \ln\left(\frac{-y_0}{1-y_0}\right)$. That is, $M = \ln\left(\frac{-y_0}{1-y_0}\right)$ if $y_0 > 1$, but $M = \infty$ if $y_0 \leq 1$. In addition, notice that in the first case, M is depend on the initial value y_0. Next, we note that nontrivial solutions may match a trivial solution after a finite time. In other words, the uniqueness of the solution may be lost at some finite time. For example, the equation

$$y' = -\sqrt{y}$$

with $y(0) = \frac{1}{4}$ has the solution

$$y(x) = \begin{cases} \frac{(1-x)^2}{4} & \text{if } 0 \leq x < 1, \\ 0 & \text{if } x \geq 1. \end{cases}$$

This solution will match the trivial (constant) solution $y(x) \equiv 0$ after the finite time $x = 1$. This says that we can choose $M = \infty$, but we must choose $T = 1$ in Theorem 1.1. Of course, if all solutions exist for all $x \in [0, \infty)$ and the uniqueness of all solutions is guaranteed, then we can choose $T = \infty$.

Remark 1.2. Define $U := \{\eta \in \mathbb{R} : \eta \neq 0, 1\}$. Let $y(x) \in U$ for all $x \in [0, T)$ be a solution of (1.2) with $f(y) = y(1-y)$ and $y(0) = y_0$, where T is a suitable value or ∞. Since

$$\begin{aligned} \exp\left(\frac{F(y(x)) - F(y_0)}{\alpha - 1}\right) &= \exp\left(\frac{-1}{1-\alpha}\int_{y_0}^{y(x)} \frac{1}{f(\eta)} d\eta\right) \\ &= \exp\left(\frac{-1}{1-\alpha}\int_{y_0}^{y(x)} \left(\frac{1}{\eta} - \frac{-1}{1-\eta}\right) d\eta\right) \\ &= \left|\frac{(1-y_0)y(x)}{y_0(1-y(x))}\right|^{\frac{-1}{1-\alpha}} \\ &= \left|\frac{(1-y_0)y(x)}{y_0(1-y(x))}\right|^{\frac{-\alpha}{1-\alpha}-1} \end{aligned} \tag{1.5}$$

and

$$0 < \frac{f(y(x))}{f(y_0)} = \frac{y(x)(1-y(x))}{y_0(1-y_0)} = \left|\frac{y(x)(1-y(x))}{y_0(1-y_0)}\right| \tag{1.6}$$

hold for $x \in [0, T)$, by Corollary 1.1, we obtain

$$
\begin{aligned}
1 &= \frac{f(y(x))}{f(y_0)} \exp\left(\frac{F(y(x)) - F(y_0) - \alpha x}{\alpha - 1}\right) \\
&= \left|\frac{y_0(1-y(x))e^x}{(1-y_0)y(x)}\right|^{\frac{\alpha}{1-\alpha}} \left(\frac{1-y(x)}{1-y_0}\right)^2 \\
&= \left[\frac{y_0(1-y_0)e^x}{y(x)(1-y(x))}\left(\frac{1-y(x)}{1-y_0}\right)^{\frac{2}{\alpha}}\right]^{\frac{\alpha}{1-\alpha}}
\end{aligned}
$$

for $x \in [0, T)$. This equation implies (1.4). That is, we obtain an implicit solution of (1.3) with $y(x) \in U$ for all $x \in [0, T)$. In addition, we note that using Corollary 1.1, Cui's results [10] are also derivable.

This paper is organized as follows. In the next section, we will introduce some basic facts about a fractional integral. In Section 3, we prove the main theorem. In Section 4, we present important examples. In the last section, we will state our conclusions and future outlook.

2. Fractional integral

It is known that the fractional integral [18] corresponding to Caputo–Fabrizio derivative is given by

$$
\mathcal{I}^\alpha \phi(x) = (1-\alpha)(\phi(x) - \phi(0)) + \alpha \int_0^x \phi(\xi) d\xi. \tag{2.1}
$$

Then we see that

$$
\mathcal{I}^\alpha \mathcal{D}^\alpha \phi(x) = \phi(x) + C, \tag{2.2}
$$

where C is an arbitrary constant. But, note that

$$
\mathcal{D}^\alpha \mathcal{I}^\alpha \phi(x) = \phi(x) - \phi(0) e^{-\frac{\alpha}{1-\alpha}x}
$$

holds (see, [19]). Two properties of fractional integrals, (2.1) and (2.2), play an important role in this paper.

3. Proof of main theorem

We will prove the main theorem.

Proof. [Proof of Theorem 1.1] Let $S = \{\eta \in Y : f(\eta) \neq 0\}$, and let $y(x)$ for $x \in [0, M)$ be a solution of (1.1) with $y(0) = y_0 \in Y$, where Y is the domain of definition of f and M is the maximal existence time of $y(x)$. Define

$$
g(x) := \kappa_\alpha(x) e^{-\frac{\alpha}{1-\alpha}x},
$$

with $0 < \alpha < 1$. Then we see that

$$
{}^G\mathcal{D}^\alpha y(x) = \frac{1}{1-\alpha}\int_0^x \frac{g(\xi)}{g(x)} e^{-\frac{\alpha}{1-\alpha}(x-\xi)} y'(\xi) d\xi
$$

$$= \frac{1}{g(x)}\mathcal{D}^{\alpha}\left(\int g(x)y'(x)dx\right).$$

From this with (1.1), we obtain the equation

$$\mathcal{D}^{\alpha}\left(\int g(x)y'(x)dx\right) = g(x)f(y(x)).$$

Thus, by (2.1) and (2.2), the equality

$$\begin{aligned}\int g(x)y'(x)dx + C_1 =& \mathcal{I}^{\alpha}\mathcal{D}^{\alpha}\left(\int g(x)y'(x)dx\right) = \mathcal{I}^{\alpha}g(x)f(y(x)) \\ =&(1-\alpha)(g(x)f(y(x)) - g(0)f(y(0))) \\ &+ \alpha\int_0^x g(\xi)f(y(\xi))d\xi\end{aligned}$$

holds, where C_1 is an arbitrary constant. Since κ_α is continuously differentiable, g is also continuously differentiable. Differentiating both sides of the above equation with respect to x, we have

$$\begin{aligned}g(x)y'(x) =&(1-\alpha)g'(x)f(y(x)) + (1-\alpha)g(x)f'(y(x))y'(x) \\ &+ \alpha g(x)f(y(x))\end{aligned} \tag{3.1}$$

for $x \in [0, M)$.

Let T be the smaller of M and $\inf\{x \in [0,\infty) : f(y(x)) = 0\}$ if $\{x \in [0,\infty) : f(y(x)) = 0\} \neq \emptyset$; M if $\{x \in [0,\infty) : f(y(x)) = 0\} = \emptyset$. If $y_0 \in Y$ satisfies $f(y_0) = 0$, then $y(x) = y_0$ for all $x \in [0,\infty)$ is a trivial solution, and $T = 0$. Now we assume that $y(x) \in S$ for all $x \in [0, T)$, where T is positive by the definition of S and T. From this with the fact that $g(x) \neq 0$ holds, we obtain

$$\frac{1-(1-\alpha)f'(y(x))}{f(y(x))}y'(x) = (1-\alpha)\frac{g'(x)}{g(x)} + \alpha,$$

for $x \in [0, T)$. Integrating both sides by x and by the definition of F, we get

$$F(y(x)) - (1-\alpha)\ln|f(y(x))| = (1-\alpha)\ln|g(x)| + \alpha x + C_2,$$

where C_2 is an arbitrary constant. This implies that

$$|g(x)f(y(x))|^{-(1-\alpha)} = \exp\left(-F(y(x)) + \alpha x + C_2\right),$$

and so that

$$g(x)f(y(x)) = \pm\exp\left(\frac{-F(y(x)) + \alpha x + C_2}{-(1-\alpha)}\right).$$

Substituting $x = 0$, and using $y(0) = y_0$, we have

$$g(0)f(y_0) = \pm\exp\left(\frac{-F(y_0) + C_2}{-(1-\alpha)}\right).$$

Hence

$$\frac{g(x)f(y(x))}{g(0)f(y_0)}\exp\left(\frac{F(y(x)) - F(y_0) - \alpha x}{-(1-\alpha)}\right) = 1$$

holds for $x \in [0, T)$. Recalling that $g(x) = \kappa_\alpha(x)e^{-\frac{\alpha}{1-\alpha}x}$, we obtain the solution formula for (1.1). This completes the proof. □

Remark 3.1. If $f(y) = 0$ for all $y \in Y$ in (1.1), then we see that $y(x) = y_0$ for all $x \in [0, \infty)$ is the solution of (1.1) with $y(0) = y_0 \in Y$. In fact, in this case, we obtain

$$y'(x) = 0$$

from (3.1). Thus, $y(x)$ is a constant, and so that we have $y(x) = y_0$ for all $x \in [0, \infty)$ by $y(0) = y_0$.

4. Examples

Example 4.1. Consider the linear differential equation

$$^{G}\mathcal{D}^{\alpha} y = \lambda y \tag{4.1}$$

with the kernel $^{G}\mathcal{K}_{\alpha}(x, \xi)$, where $\lambda \neq 0$. By Theorem 1.1, we obtain the following implicit solution of (4.1) with $y(0) = y_0$:

$$\frac{\kappa_{\alpha}(x)}{\kappa_{\alpha}(0)} \left| \frac{y(x)}{y_0} \right|^{\frac{1+\lambda(\alpha-1)}{\lambda(\alpha-1)}} = 1$$

with $0 < \alpha < 1$, whenever $y(x) \in V_1 := \{\eta \in \mathbb{R} : \eta \neq 0\}$ for all $x \in [0, T)$, where T is a suitable value or ∞. Since this equation can be solved for $y(x)$, we can get the explicit solution

$$y(x) = y_0 \left(\frac{\kappa_{\alpha}(0)}{\kappa_{\alpha}(x)} \right)^{\frac{\lambda(\alpha-1)}{1+\lambda(\alpha-1)}} \tag{4.2}$$

with $0 < \alpha < 1$. Note here that we can choose $T = \infty$ because κ_{α} is positive and $y_0 \in V_1$. Moreover, when $y_0 = 0$ this function satisfies $y(x) = 0$ for all $x \in [0, \infty)$. Combining the above facts, we see that for any $y_0 \in \mathbb{R}$, the explicit solution of (4.1) with $y(0) = y_0$ is given by (4.2) for all $x \in [0, \infty)$.

In particular, if $\kappa_{\alpha}(x) \equiv e^{\frac{\alpha}{1-\alpha}x}$ or $^{G}\mathcal{K}_{\alpha} = \mathcal{K}_{\alpha}$, then (4.2) is

$$y(x) = y_0 e^{\frac{\lambda\alpha}{1+\lambda(\alpha-1)}x}$$

for all $x \in [0, \infty)$. When $\alpha \to 1^{-}$ it becomes

$$y(x) = y_0 e^{\lambda x}$$

for all $x \in [0, \infty)$. Clearly this is the solution of the first-order linear differential equation $y' = \lambda y$ with $y(0) = y_0 \in \mathbb{R}$.

Example 4.2. Consider the logistic differential equation

$$^{G}\mathcal{D}^{\alpha} y = y(1 - y) \tag{4.3}$$

with the kernel

$$^{G}\mathcal{K}_{\alpha}(x, \xi) = \frac{\xi + 1}{x + 1} e^{-\frac{\alpha}{1-\alpha}(x-\xi)}. \tag{4.4}$$

Since $\kappa_{\alpha}(x) = (x+1)e^{\frac{\alpha}{1-\alpha}x}$ holds, using Theorem 1.1 and (1.5) and (1.6), we obtain the following implicit solution of (4.3) with (4.4) and $y(0) = y_0$:

$$(x + 1) \left| \frac{y_0(1 - y(x))e^{x}}{(1 - y_0)y(x)} \right|^{\frac{\alpha}{1-\alpha}} \left(\frac{1 - y(x)}{1 - y_0} \right)^{2} = 1$$

with $0 < \alpha < 1$, whenever $y(x) \in U$ for all $x \in [0, T)$, where T is a suitable value or ∞.

Figure 1 shows that solution curves of classical logistic equation (black), Caputo–Fabrizio fractional logistic equation (1.3) (blue), and generalized Caputo–Fabrizio fractional logistic equation (4.3) with (4.4) (red) with $y(0) = 0.1$ and $\alpha = 0.1$. After intersecting the solution curve of classical logistic equation, the solution curve of equation (4.3) with (4.4) lies between solution curves of classical logistic equation and Caputo–Fabrizio fractional logistic equation (1.3). Figures 2 and 3 show the case where $\alpha = 0.5$ and 0.9. As $\alpha \to 1^-$, the graphs of (1.3) and (4.3) with (4.4) approach the graph of classical logistic equation.

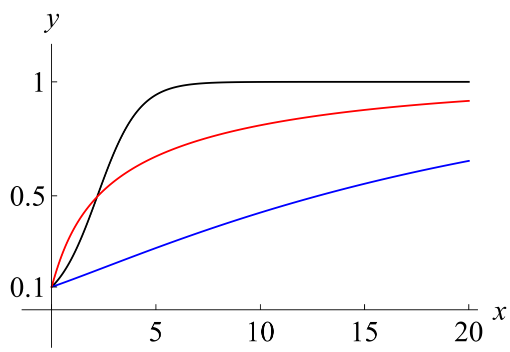

Figure 1. Solution curves of classical logistic equation (black), Caputo–Fabrizio fractional logistic equation (1.3) (blue), and generalized Caputo–Fabrizio fractional logistic equation (4.3) with (4.4) (red), with $y(0) = 0.1$ and $\alpha = 0.1$.

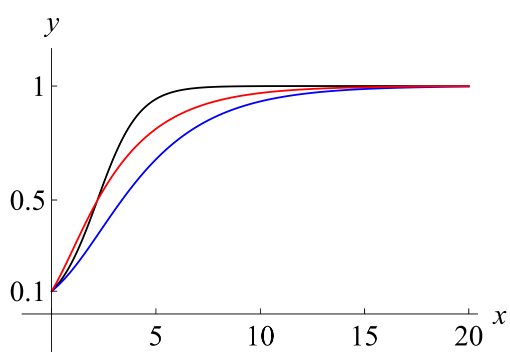

Figure 2. Solution curves of classical logistic equation (black), Caputo–Fabrizio fractional logistic equation (1.3) (blue), and generalized Caputo–Fabrizio fractional logistic equation (4.3) with (4.4) (red), with $y(0) = 0.1$ and $\alpha = 0.5$.

More generally, we get the following result.

Example 4.3. Consider equation (4.3) with the kernel

$$ {}^{G}\mathcal{K}_{\alpha}(x,\xi) = \frac{h(\xi)}{h(x)} e^{-\frac{\alpha}{1-\alpha}(x-\xi)}, \tag{4.5} $$

where $h(x)$ is a positive continuous function. Doing the same as in Example 4.1,

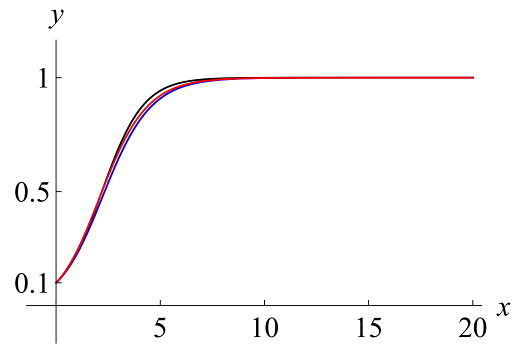

Figure 3. Solution curves of classical logistic equation (black), Caputo–Fabrizio fractional logistic equation (1.3) (blue), and generalized Caputo–Fabrizio fractional logistic equation (4.3) with (4.4) (red), with $y(0) = 0.1$ and $\alpha = 0.9$.

we obtain the following implicit solution of (4.3) with (4.5) and $y(0) = y_0$:

$$\frac{h(x)}{h(0)} \left| \frac{y_0(1-y(x))e^x}{(1-y_0)y(x)} \right|^{\frac{\alpha}{1-\alpha}} \left(\frac{1-y(x)}{1-y_0} \right)^2 = 1$$

with $0 < \alpha < 1$, whenever $y(x) \in U$ for all $x \in [0, T)$, where T is a suitable value or ∞.

Example 4.4. Consider the differential equation

$$^G\mathcal{D}^\alpha y = \frac{y}{y-1} \tag{4.6}$$

with the kernel $^G\mathcal{K}_\alpha(x, \xi)$, where $y \neq 1$ i.e., $y(x) \in Y_1 := \{\eta \in \mathbb{R} : \eta \neq 1\}$. By Theorem 1.1, we obtain the following implicit solution of (4.6) with $y(0) = y_0$:

$$\frac{\kappa_\alpha(x)y(x)(y_0-1)}{\kappa_\alpha(0)y_0(y(x)-1)} \left| \frac{y(x)}{y_0} \right|^{\frac{1}{1-\alpha}} \exp\left(\frac{y(x)-y_0}{\alpha-1} \right) = 1$$

with $0 < \alpha < 1$, whenever $y(x) \in V_2 := \{\eta \in Y_1 : \eta \neq 0\}$ for all $x \in [0, T)$, where T is a suitable value or ∞.

5. Conclusions and future outlook

This study introduces a novel generalized Caputo–Fabrizio fractional derivative and presents a versatile method for solving generalized Caputo–Fabrizio fractional differential equations. Our findings extend to encompass Caputo–Fabrizio fractional differential equations, offering a broad application range. Importantly, we provide a theorem that offers a versatile solution framework, not restricted to specific equations like the logistic or Riccati equations.

When $\kappa_\alpha(x) \equiv 1$, the expression

$$^G\mathcal{D}^\alpha \phi(x) = \frac{1}{1-\alpha}(\phi(x) - \phi(0))$$

reveals that the new derivative, $^G\mathcal{D}^\alpha \phi$, can be viewed as an extension of the Caputo–Fabrizio fractional derivative, albeit not necessarily as a traditional "derivative".

It's worth noting that further research could potentially lead to characterizing the generalized Caputo–Fabrizio fractional derivative by imposing specific constraints on the nonsingular kernel, ${}^{G}\mathcal{K}_{\alpha}(x,\xi)$.

References

[1] T. Alinei-Poiana, E. H. Dulf and L. Kovacs, *Fractional calculus in mathematical oncology*, Sci. Rep., 2023, 13, Paper No. 10083.

[2] I. Area and J. J. Nieto, *On the fractional Allee logistic equation in the Caputo sense*, Ex. Countex., 2023, 4, Paper No. 100121, 4 pp.

[3] I. Area and J. J. Nieto, *On a quadratic nonlinear fractional equation*, Fractal Fract., 2023, 7(6), Paper No. 469.

[4] N. Attia, A. Akğul, D. Seba and A. Nour, *On solutions of fractional logistic differential equations*, Prog. Fract. Differ. Appl., 2023, 9, 351–362.

[5] I. Baba and F. A. Rihan, *A fractional-order model with different strains of COVID-19*, Phys. A, 2022, 603, Paper No. 127813, 12 pp.

[6] Y. Basci, S. Ogrekci and A. Misir, *On Hyers–Ulam Stability for fractional differential equations including the new Caputo–Fabrizio fractional derivative*, Mediterr. J. Math., 2019, 16, Paper No. 131.

[7] M. Berman and L. S. Cederbaum, *Fractional driven-damped oscillator and its general closed form exact solution*, Phys. A, 2018, 505, 744–762.

[8] M. Caputo and M. Fabrizio, *A new definition of fractional derivative without singular kernel*, Prog. Fract. Differ. Appl., 2015, 1, 73–85.

[9] M. Caputo and M. Fabrizio, *On the singular kernels for fractional derivatives. Some applications to partial differential equations*, Prog. Fract. Differ. Appl., 2021, 7, 79–82.

[10] Z. Cui, *Solutions of some typical nonlinear differential equations with Caputo–Fabrizio fractional derivative*, AIMS Math., 2022, 7(8), 14139–14153.

[11] B. Dhar, P. K. Gupta and M. Sajid, *Solution of a dynamical memory effect COVID-19 infection system with leaky vaccination efficacy by non-singular kernel fractional derivatives*, Math. Biosci. Eng., 2022, 19(5), 4341–4367.

[12] J. Jia and H. Wang, *Analysis of asymptotic behavior of the Caputo–Fabrizio time-fractional diffusion equation*, Appl. Math. Lett., 2023, 136, Paper No. 108447, 7 pp.

[13] L. N. Kaharuddin, C. Phang and S. S. Jamaian, *Solution to the fractional logistic equation by modified Eulerian numbers*, Eur. Phys. J. Plus, 2020, 135(2), Paper No. 229.

[14] S. Khajanchi, M. Sardar and J. J. Nieto, *Application of non-singular kernel in a tumor model with strong Allee effect*, Differ. Equ. Dyn. Syst., 2023, 31(3), 687–692.

[15] L. Li and D. Li, *Exact solutions and numerical study of time fractional Burgers' equations*, Appl. Math. Lett., 2020, 100, Paper No. 106011, 7 pp.

[16] L. Liu, Q. Dong and G. Li, *Exact solutions and Hyers–Ulam stability for fractional oscillation equations with pure delay*, Appl. Math. Lett., 2021, 112, Paper No. 106666, 7 pp.

[17] L. Liu, Q. Dong and G. Li, *Exact solutions of fractional oscillation systems with pure delay*, Fract. Calc. Appl. Anal., 2022, 25(4), 1688–1712.

[18] J. Losada and J. J. Nieto, *Properties of a new fractional derivative without singular kernel*, Prog. Fract. Differ. Appl., 2015, 1, 87–92.

[19] J. Losada and J. J. Nieto, *Fractional integral associated to fractional derivatives with nonsingular kernels*, Prog. Fract. Differ. Appl., 2021, 7, 137–143.

[20] B. Melkamu and B. Mebrate, *A fractional model for the dynamics of smoking tobacco using Caputo–Fabrizio derivative*, J. Appl. Math., 2022, Paper No. 2009910, 10 pp.

[21] Z. Mokhtary, M. B. Ghaemi and S. Salahshour, *A new result for fractional differential equation with nonlocal initial value using Caputo-Fabrizio derivative*, Filomat, 2022, 36(9), 2881–2890.

[22] A. M. Nass, *Lie symmetry analysis and exact solutions of fractional ordinary differential equations with neutral delay*, Appl. Math. Comput., 2019, 347, 370–380.

[23] J. J. Nieto, *Solution of a fractional logistic ordinary differential equation*, Appl. Math. Lett., 2022, 123, Paper No. 107568, 5 pp.

[24] J. J. Nieto, *Fractional Euler numbers and generalized proportional fractional logistic differential equation*, Fract. Calc. Appl. Anal., 2022, 25(3), 876–886.

[25] H. Nishiura, S. Tsuzuki, B. Yuan, T. Yamaguchi and Y. Asai, *Transmission dynamics of cholera in Yemen, 2017: A real time forecasting*, Theor. Biol. Med. Model., 2017, 14, 14.

[26] A. Omame, M. Abbas and A. Abdel-Aty, *Assessing the impact of SARS-CoV-2 infection on the dynamics of dengue and HIV via fractional derivatives*, Chaos Solitons Fractals, 2022, 162, Paper No. 112427, 22 pp.

[27] D. Shi and Y. Zhang, *Diversity of exact solutions to the conformable space-time fractional MEW equation*, Appl. Math. Lett., 2020, 99, Paper No. 105994, 7 pp.

[28] F. Si Bachir, S. Abbas, M. Benbachir and M. Benchohra, *Successive approximations for Caputo–Fabrizio fractional differential equations*, Tatra Mt. Math. Publ., 2022, 81, 117–128.

[29] X.-S. Wang, J. Wu and Y. Yang, *Richards model revisited: Validation by and application to infection dynamics*, J. Theoret. Biol., 2012, 313, 12–19.

[30] B. J. West, *Exact solution to fractional logistic equation*, Phys. A, 2015, 429, 103–108.

[31] M. Xu and Y. Jian, *Unsteady rotating electroosmotic flow with time-fractional Caputo–Fabrizio derivative*, Appl. Math. Lett., 2020, 100, Paper No. 106015, 6 pp.

[32] B. Yang, Z. Luo, X. Zhang, Q. Tang and J. Liu, *Trajectories and singular points of two-dimensional fractional-order autonomous systems*, Adv. Math. Phys., 2022, Paper No. 3722011, 9 pp.

[33] T. Zhang and Y. Li, *Exponential Euler scheme of multi-delay Caputo–Fabrizio fractional-order differential equations*, Appl. Math. Lett., 2022, 124, Paper No. 107709, 8 pp.