Journal of Applied Analysis and Computation
Volume 14, Number 2, April 2024, 986–1005
Website:http://www.jaac-online.com
DOI:10.11948/20230246

GLOBAL EXISTENCE AND CONTINUOUS DEPENDENCE ON PARAMETERS OF CONFORMABLE PSEUDO-PARABOLIC INCLUSION

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Abstract In this paper, we establish global existence and continuous dependence on parameters for sets of solutions of the differential inclusion including self-adjoint operators with fractional order in the form

$$\begin{cases} \dfrac{{}^C\partial^\alpha}{\partial t^\alpha} u(t) + m\mathcal{L}^{s_1} \dfrac{{}^C\partial^\alpha}{\partial t^\alpha} u(t) + \mathcal{L}^{s_2} u(t) \in F(t, u(t), \eta), \ t \in [0, T), \\ u(0) = \varphi, \qquad \text{on } \mathcal{D}, \end{cases}$$

where $s_1, s_2 > 0$. We first use spectral theory on Hilbert spaces to obtain formulation for mild solutions. With this formulation, we use a measure of noncompactness with values in ordered space to construct useful bounds for solution operators. Then, we establish necessarily upper semicontinuous and condensing settings, which mainly help to obtain the global existence of mild solutions and the compactness of the mild solutions set. Before ending the article, we discuss the continuous dependence of the solution set when the input data contains the parameter η.

Keywords Multi-function, measure of compactness, differential inclusion, self-adjoint operator.

MSC(2010) 35R11, 35B65, 26A33.

1. Introduction

Let T be a positive number, $\mathcal{D}$ be a bounded domain with sufficiently smooth boundary $\partial\mathcal{D}$ in Euclid space $\mathbb{R}^N$ and $s_1, s_2 > 0$. We consider the problem of

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finding the function $u = u(x,t)$ satisfying the system

$$\begin{cases} \dfrac{{}^C\partial^\alpha}{\partial t^\alpha} u(x,t) + m\mathcal{L}^{s_1} \dfrac{{}^C\partial^\alpha}{\partial t^\alpha} u(x,t) + \mathcal{L}^{s_2} u(x,t) \in G(x,t,u(x,t)), & (x,t) \in \mathcal{D} \times (0,T], \\ u(x,0) = \varphi(x), & x \in \mathcal{D}, \end{cases} \tag{1.1}$$

where $\frac{{}^C\partial^\alpha}{\partial t^\alpha} u$ is the symbol for the derivative with respect to the variable t of the function u, m is a positive constant and $\mathcal{L}$ is a self-adjoint operator, regularity with fractional order $s \in \{s_1, s_2\}$ on the Hilbert space $L^2(\mathcal{D})$, this is, $\langle \mathcal{L}^s u, w\rangle = \langle u, \mathcal{L}^s w\rangle$ for all $s \in \{s_1, s_2\}$. Recently, differential equations and inclusions have gain much attention according to wide applications in physics, economic, control theory, etc, see e.g. [2, 4, 7, 11, 12, 15, 16, 21, 22, 26, 30, 35, 37–41, 44, 45]. There have been many studies on the existence and the stability of the solution of the problem with the source single-valued function or with non-integer order derivatives, for example [1, 3, 6, 13, 19, 36, 42, 43]. Recently, CoD$_t$ has been thoroughly investigated in many studies; see, eg. Baleanu [5, 17, 46, 47], Jaiswal and Bahuguna [17]. Fractional diffusion equations with CFD$_t$ and RLFD$_t$ have always been of great interest to researchers in many fields and in the context of determining universal parameters; see, eg. [31]. There are many studies on the properties of mixed derivatives, but its application to equations is not much (specifically, differential diffusion equations), some applications of partial derivatives with suitable derivatives in some references [14, 27]. Now we would like to introduce some studies on this derivative:

- In [10], Au et al studied a nonlinear diffusion equation: $\mathcal{D}_t^{(\alpha)} u - \Delta u = \mathcal{L}(x,t;u(x,t))$, where $0 < \alpha < 1$, $(x,t) \in \Omega \times (0,T)$. They considered both of the problems as follows:
  - Case 1 : the mild solution contains the integral $I = \int_0^t \tau^\gamma \mathrm{d}\tau$ (critical as $\gamma \leq -1$).
  - Case 2: The existence of a mild solution, for the Hadamard sense unstable final value problem, they proposeed two regularization methods to solve the nonlinear problem in case the source term is a Lipschitz function. They obtained the regularization problem, adding the existence, uniqueness and stability. To suggest a normal estimate of the solution, they also developed a number of new functional analysis techniques
- In [29], Tien considered the problem (1.2) for the conformable heat equation as follows

$$\begin{cases} \dfrac{\partial^{\alpha(t)}}{\partial t^{\alpha(t)}} z(x,t) + \mathcal{A} z(x,t) = F(x,t), & x \in \Omega, t \in (0,T), \\ z(x,t) = 0, & x \in \partial\Omega, t \in (0,T), \end{cases} \tag{1.2}$$

  where $\dfrac{\partial^{\alpha(t)}}{\partial t^{\alpha(t)}} z$ is defined in [29]. They are interested to study two following conditions

$$z(x,0) = z_0(x), \quad x \in \Omega, \tag{1.3}$$

  or nonlocal condition

$$z(x,0) + \gamma z(x,T) = z_0(x), \quad \gamma > 0, \quad x \in \Omega. \tag{1.4}$$

In this research work, they surveyed the well-posedness of the mild solution. They also proved the convergence of mild solution of the nonlocal problem to solutions of the initial problem throught the theory of Fourier series in combination with evaluation techniques for some generalized integrals.

- In [44], Tuan-Tien-Chao Yang studied the problem (1.5) with a conformable derivative as follows

$$\begin{cases} \dfrac{\partial^\alpha}{\partial t^\alpha} w(x,t) + (-\Delta)^\beta w(x,t) - m\dfrac{{}^c\partial^\alpha}{\partial t^\alpha}\Delta w = F(w(x,t)), & x \in \Omega, t \in (0,T), \\ w(x,t) = 0, & x \in \partial\Omega, t \in (0,T), \\ w(x,0) = w_0(x) & x \in \Omega. \end{cases} \tag{1.5}$$

In this research work, they focused on investigating the existence of the global solution and examining the derivative's regularity. In addition, they contributed two interesting results. Firstly, they proved the convergence of the mild solution of the pseudo-parabolic equation to the solution of the parabolic equation. Secondly, they examined the convergence of solution when the order of the derivative of the fractional operator approaches 1^-, based on the Banach fixed point theorem and Sobolev embedding.

Regarding our work, we can list some of the following results:

- In [9], Anh-Ke-Lan studied the following fractional differential inclusion

$$\partial_t^\alpha u(t) - Au(t) \in F(t, u, u_t), \quad t > 0,\ 0 < \alpha < 1, \tag{1.6}$$

involving impulsive effects. They proved the global solvability and weakly asymptotic stability for solutions by analyzing the behavior of its solutions on the half-line. This equation was also studied in [19]. In [28], Phong-Lan concerned with the retarded fractional evolution inclusion

$$\partial_t^\alpha u(t) - Au(t) \in F(t, u_t), \quad t > 0,\, 0 < \alpha < 1, \tag{1.7}$$

equipped with the history condition

$$u(s) = \varphi(s), \quad s \in [-h, 0],\, h > 0.$$

- In [35], Tri-Rezapour established a general existence result on the positive eigenvalue interval for multivalued operators in cones. The research team investigated the existence of a positive solution of the multi-point boundary value problem through the inclusion of multiple parameter values.
In [24], Ngoc-Tri discussed the existence and compactness of the solutions set of following fractional pseudo-parabolic inclusion

$$\begin{cases} \partial_t^\alpha u + \kappa(-\Delta)^{s_1}\partial_t^\alpha u + (-\Delta)^{s_2} u \in F(t,u), & 0 < t < T,\ x \in \Omega, \\ u(t,x) = 0, & 0 < t < T,\ x \in \partial\Omega, \\ u(0,x) = \varphi(x), & x \in \Omega, \end{cases} \tag{1.8}$$

where ∂_t^α signifies the Caputo time derivative of fractional order $\alpha \in (0,1)$. In this work, they constructed useful bounds for solution operators by basing on asymptotic behaviors of the Mittag-Leffler functions to prove the compactness and continuous dependence on parameters of solutions set of Problem (1.8).

- In [32], Tri established compactness and continuous dependence on the parameters for the solution set of the 2nd order differential including the auto-completion operator in the form

$$\begin{cases} \dfrac{\partial^2}{\partial t^2}\nu(t,x) + 2\mathcal{A}\dfrac{\partial}{\partial t}\nu(t,x) + \mathcal{A}^2\nu(t,x) \in F(t,\nu(t),\mu),\ (t,x) \in [0,T) \times \Omega, \\ \nu(0,x) = \frac{\partial}{\partial t}\nu(0,x) = 0, \qquad\qquad x \in \Omega. \end{cases} \tag{1.9}$$

  Based on Hilbert space spectral theory to obtain mild solution formula, using mild solution formula together with measure of in-compactness with values in ordered space, we construct useful limits for the experimental operator. Then they set condensing and semi-continuous settings on necessarily, we obtain the global existence of mild solutions and compactness of mild solution set and continuous dependence of solution set method into the μ parameter is considered.

- In [25], Ngoc-Tri-Hammouch-Can considered final value problems for time-space fractional pseudo-parabolic equations subject to the final value condition. The main purpose is to investigate solution continuity for fractional order $\alpha \in (0,1)$, thereby answering the question: $\rho_{\alpha_n} \to \rho_\alpha$ work? in the proper sense $\alpha_n \to \alpha$? First, a formula for the integral solution was established. Afterthat, the desired continuity is then obtained using the resolved representations of the Mittag-Leffler operators on Hankel's contour. Finally, they presented some numerical examples to illustrate the proposed theory.

- In [33], Tri presented some results on fixed point index calculations for multi-valued mappings and apply them to prove the existence of solutions to multivalued equations in ordered space, under flexible conditions for the positive eigenvalue.

- In [34], Tri's aim to present an advantage of using K-normed space (or called cone normed space) for investigating Cauchy problem in a scale of Banach spaces $\{(F_s, \|\|\|_s .) : s \in (0,1]\}$.

Let $Kv\left(L^2(\mathcal{D})\right)$ be the family of all non-empty convex and compact subsets of $L^2(\mathcal{D})$. Throughout this work, we will assume on F the following assumption, we denote

$$L^1((0,T);\mathbb{R}) \triangleq \Big\{\beta \in L^1((0,T);\mathbb{R}) \mid t \mapsto \beta(t) \in L^\infty((0,T);\mathbb{R})\Big\}. \tag{1.10}$$

Let $F : L^2(\mathcal{D}) \times [0,T] \to Kv\left(L^2(\mathcal{D})\right)$ be a multimap satisfying that:

- The multifunction $F(\cdot, v)$ has a strongly measurable selection for every $v \in L^2(\mathcal{D})$.
- The multimap $F(\cdot,t) : L^2(\mathcal{D}) \to Kv\left(L^2(\mathcal{D})\right)$ is upper semicontinuous (usc) for a.e. $t \in [0,T]$.
- There exists a function $\beta \in L^{1,\gamma}((0,T);\mathbb{R})$ such that

$$\|F(t,v)\| \le \beta(t)\big(1 + \|v\|_{L^2(\mathcal{D})}\big), \quad \text{a.e. } t \in (0,T), \forall v \in L^2(\mathcal{D}). \tag{1.11}$$

- (χ-regularity condition) There is $B \in L^1\big((0,T);\mathbb{R}\big)$ satisfying

$$\chi(F(t,D)) \leq B(t)\chi(D) \text{ for a.e. } t \in (0,T) \text{ for all } D \in b(\mathcal{H}),$$

here χ is MNC in $L^2(\mathcal{D})$ defined $\chi(D) = \inf\{\varepsilon > 0 : D \text{ has a finite } \varepsilon\text{-net}\}$.

Our work will be presented as follows. In section 2, we recall basic preliminaries containing multi operator. In Section 3, presents the global existence of mild solutions and compactness of the solution set of problem (1.1). Finally, we discuss on the continuous dependence on parameters η of the solution set of problem (1.1).

2. Preliminaries

Definition 2.1. The conformable fractional derivative starting from a function $f : [a,\infty) \mapsto \mathbb{R}$ of order α is defined by

$$T_\alpha f(t) = \lim_{\epsilon \to 0} \frac{f\left(t + \epsilon t^{1-\alpha}\right) - f(t)}{\epsilon}. \tag{2.1}$$

If $T_\alpha f(t)$ exists on $(0,b)$, then $T_\alpha f(0) = \lim_{t\to 0} T^0_\alpha f(t)$.

Definition 2.2. [18, Definition 2.1.1] Let E be a Banach space and $(C,\preceq)$ a partially ordered set. A map $\varphi : \mathcal{Y} \subset \mathcal{P}(E) \to C$ is said to be a *measure noncompactness* (MNC) in $\mathcal{Y}$ if $\varphi(\overline{co}(D)) = \varphi(D)$ for all $D \in \mathcal{Y}$. A multi-mapping $F : E \to \mathcal{Y}$ is called *condensing* to φ (in short, φ-condensing) if $D \in \mathcal{Y}$ with $\varphi(D) \preceq \varphi(F(D))$ then D is relative compact in E.

Let G be a subset of a metric (E,d) and a positive number ϵ. A subset A of E is said to be ϵ-net of G if $G \subset \bigcup_{x\in A} \{y \in E : d(x,y) < \epsilon\}$. If A is finite, A is called a *finite ϵ-net*. We need the Hausdorff measure χ which defined in [18, Definition 2.1.1], i.e., $\chi(G) = \inf\{\epsilon > 0 : G \text{ has a finite } \epsilon\text{-net}\}$.

Lemma 2.1. [18, Definition 2.1.1] *Let E be a Banach space and χ a Hausdorff MNC defined on family $\mathcal{F}$ of subsets of E. Then χ has the following properties:*

- *Monotone: if $D_1 \subset D_2$ implies $\chi(D_1) \leq \chi(D_2)$, for all $D_1, D_2 \in \mathcal{F}$.*
- *Algebraically semiadditive: if $\chi(D_1+D_2) \leq \chi(D_1)+\chi(D_2)$ for all $D_1, D_2 \in \mathcal{F}$.*
- *Nonsingular: if $\chi(\{a\} \cup D) = \chi(D)$ for all $a \in E, D \in \mathcal{F}$.*
- *Regular: $\chi(D) = 0$ if and only if D is relatively compact, $D \in \mathcal{F}$.*
- *Semi-homogeneity: that is $\chi(\lambda D) = |\lambda|\chi(D)$ for all $\lambda \in \mathbb{R}, D \in \mathcal{F}$.*

Lemma 2.2. *If M is a convex closed subset of E, and $F : M \to Kv(M)$ is a closed β-condensing multimap, where β is a nonsingular MNC defined on subsets of M, then* $\mathrm{Fix}(F) \neq \emptyset$.

Lemma 2.3. *Let X be a closed subset of a Banach space E and $F : X \to K(E)$a closed multimap, which is β-condensing on every bounded subset of X, where β is a monotone MNC in E. If the fixed points set* $\mathrm{Fix}(F)$ *is bounded then it is compact.*

Lemma 2.4. *Let X be a closed subset of a Banach space E, β be a monotone MNC in E, Y be a metric space, and $G : Y \times X \to K(E)$ be a closed multimap which is β-condensing in the second variable and such that $F(\lambda) := \mathrm{Fix}\, G(\lambda,\cdot) \neq \emptyset$, for all $\lambda \in Y$. Then the multimap $F : Y \to P(E)$ is u.s.c.*

In addition to the above mentioned basic properties of multivalued analysis we also use the Grönwall's inequality presented in the following lemma.

Lemma 2.5. *Let* $a \geq 0$, $0 < T \leq \infty$, *and continuous functions* $\beta, \mu : [0,T] \to \mathbb{R}_+$ *satisfying* $\int_0^T \beta(s)ds < \infty$, *and* $\sup_{t\in[0,T]} \mu(t) < \infty$, $0 \leq \gamma \leq \xi \leq T$ *and*

$$\mu(t) \leq a + \int_0^t \beta(s)\mu(s)ds, \quad t \in [0,T]. \tag{2.2}$$

Then $\mu(t) \leq a e^{\int_\gamma^t \beta(s)ds}$, $\forall t \in [0,T]$.

3. Main results

In the first part in this section, we present the mild solution for problem (1.1). In the next part, we establish the existence and compactness of the solutions set. In the final part, on the basis of these results we discuss the continuous dependence of the solution set of the inclusion (3.5) on the parameter.

3.1. Mild solution

For $u \in C([0,T]; L^2(\mathcal{D}))$, denoting

$$\mathcal{S}_F(u) = \Big\{ f \in L^1((0,T); L^2(\mathcal{D})) | f(\cdot,t) \in F(t,u), \text{ for a.e. } t \in (0,T) \Big\}. \tag{3.1}$$

It is easy to see that $u = u(\cdot,t)$ is a solution of Problem (1.1) iff there exists $f \in \mathcal{S}_F(u)$ satisfying

$$\begin{cases} \dfrac{{}^C\partial^\alpha}{\partial t^\alpha}\big(1 + m\mathcal{L}^{s_1}\big)u(x,t) + \mathcal{L}^{s_2}u(x,t) = f(x,t), \ (x,t) \in \mathcal{D} \times [0,T), \\ u(x,0) = \varphi(x), \qquad\qquad\qquad\qquad\qquad\qquad x \in \mathcal{D}. \end{cases} \tag{3.2}$$

Let assume that $e_k \in L^2(\mathcal{D})$ is the eigen-function corresponding to the eigenvalue λ_k of the operator $\mathcal{L}$, i.e. $\mathcal{L}^s(e_k) = \lambda_k^s e_k(x)$ for $s \in \{s_1, s_2\}$. Taking the inner product of both sides of (3.2) with $e_k(x)$, we receive

$$(1 + m\lambda_k^{s_1}) \frac{{}^C\partial^\alpha}{\partial t^\alpha} \underbrace{\langle u(\cdot,t), e_k \rangle}_{u_k(t)} + \lambda_k^{s_2} \underbrace{\langle u(\cdot,t), e_k \rangle}_{u_k(t)} = \underbrace{\langle f(\cdot,t), e_k(x) \rangle}_{f_k(t)}. \tag{3.3}$$

From (3.3), we have

$$\begin{aligned} u_k(t) = &\exp\Big(- \frac{\lambda_k^{s_2}}{1 + m\lambda_k^{s_1}} \frac{t^\alpha}{\alpha} \Big) \Big(\int_{\mathcal{D}} \varphi(x) e_k(x) dx \Big) \\ &+ \frac{1}{1 + m\lambda_k^{s_1}} \bigg(\int_0^t r^{\alpha-1} \exp\Big(\frac{\lambda_k^{s_2}}{1 + m\lambda_k^{s_1}} \frac{r^\alpha - t^\alpha}{\alpha} \Big) \\ &\times \Big(\int_{\mathcal{D}} f(x,s) e_k(x) dx \Big) dr \bigg) e_k(\cdot). \end{aligned} \tag{3.4}$$

Throughout this paper, let e_k, $k \in \mathbb{N}^*$, be the eigenfunction corresponding to the positive eigenvalues λ_n satisfying $0 < \lambda_1 < \lambda_2 < \dots$ and $\lim_{k\to\infty} \lambda_k = \infty$. Furthermore, assume that $\{e_k\}_{k\in\mathbb{N}^*}$ is an orthonormal basis of $L^2(\mathcal{D})$.

Definition 3.1. An element $u \in C([0,T];L^2(\mathcal{D}))$ is a mild solution of (1.1) if the following conditions are satisfied

- $u(.,0) = \varphi(x)$,
- there is $f \in \mathcal{T}_F(u)$ such that for any $t \in [0,T]$, we have

$$u(x,t) = \sum_{k=1}^{\infty} \exp\Big(-\frac{\lambda_k^{s_2}}{1+m\lambda_k^{s_1}}\frac{t^\alpha}{\alpha}\Big)\Big(\int_{\mathcal{D}} \varphi(x)e_k(x)dx\Big)e_k(x)$$
$$+\sum_{k=1}^{\infty}\frac{1}{1+m\lambda_k^{s_1}}\bigg(\int_0^t r^{\alpha-1}\exp\Big(\frac{\lambda_k^{s_2}}{1+m\lambda_k^{s_1}}\frac{r^\alpha-t^\alpha}{\alpha}\Big)$$
$$\times\Big(\int_{\mathcal{D}} f(x,r)e_k(x)dx\Big)dr\bigg)e_k(x). \tag{3.5}$$

By assumption $s_1 \geq s_2 > 0$, it is clear that $\frac{\lambda_k^{s_2}}{1+m\lambda_k^{s_1}} = \frac{1}{\lambda_k^{s_2}+m\lambda_k^{s_1-s_2}} \leq \frac{1}{\lambda_1^{s_2}+m\lambda_1^{s_1-s_2}}$. This leads to $\frac{\lambda_k^{s_2}}{1+m\lambda_k^{s_1}}$ is bounded sequences in $\mathbb{R}$. Hence, if $\varphi \in L^2(\mathcal{D})$ and $f \in L^1((0,T);L^2(\mathcal{D}))$ then (3.5) is well defined and $u(\cdot,t) \in L^\infty((0,T);L^2(\mathcal{D}))$ for a.e $t \in [0,T]$.

3.2. Upper semicontinuous and condensing settings

For $f \in L^1((0,T);L^2(\mathcal{D}))$, we define

$$\mathcal{Z}[f](x,t) \triangleq \sum_{k=1}^{\infty}\frac{1}{1+m\lambda_k^{s_1}}\bigg(\int_0^t r^{\alpha-1}\exp\Big(\frac{\lambda_k^{s_2}}{1+m\lambda_k^{s_1}}\frac{r^\alpha-t^\alpha}{\alpha}\Big)$$
$$\times\Big(\int_{\mathcal{D}} f(\cdot,r)e_k(x)dx\Big)dr\bigg)e_k(x). \tag{3.6}$$

We see that $\mathcal{Z}[f]$ is well defined. In this subsection our aim is to obtain the upper semicontinuous, χ-condensing properties of the multioperators $\mathcal{Z}\circ\mathcal{S}_F$. The following lemmas helps us to obtain the above properties.

Lemma 3.1. *For $\lambda > 0$, let $\mathbf{S}_{s_1,s_2}(\lambda) = \frac{\lambda^{s_2}}{1+m\lambda^{s_1}}$ is defined on $(0,\infty)$.*

- *If $s_1 \leq s_2$, then $\mathbf{S}_{s_1,s_2}(\lambda)$ is increasing.*
- *If $s_1 > s_2$, then $\mathbf{S}_{s_1,s_2}(\lambda)$ attains its maximum at $\overline{\lambda} = s_2^{\frac{1}{s_1}}(m(s_1-s_2))^{-\frac{1}{s_1}}$.*

Lemma 3.2. *Let $\alpha \in (0,1)$, for $s_1 > s_2$, we have*

$$\exp\Big(\frac{\lambda_k^{s_2}}{1+m\lambda_k^{s_1}}\frac{r^\alpha-t^\alpha}{\alpha}\Big) \leq \mathcal{C}_{\alpha,\gamma}\Big(\frac{\lambda_k^{s_2}}{1+m\lambda_k^{s_1}}\Big)^{-\gamma}\big(t^\alpha-r^\alpha\big)^{-\gamma}. \tag{3.7}$$

Lemma 3.3. *Let $\{f_k\} \subset L^1\big((0,T);L^2(\mathcal{D})\big)$ be a semicompact sequence such that and $f_k \rightharpoonup f_0$.*

Then the following statements hold.

- *The sequence $\{\mathcal{Z}[f_k]\}$ is equicontinuous.*

- *The sequence* $\{\mathcal{Z}[f_k]\}$ *is relatively compact in* $C\big([0,T];L^2(\mathcal{D})\big)$.
- $\mathcal{Z}\big[f_k\big] \to \mathcal{Z}\big[f_0\big]$.

Proof.

- Proving the assertion a). For any $t, t' \in [0,T]$ satisfying $0 \le t < t' \le T$, we have

$$\mathcal{Z}\big[f_k\big]\big(x,t'\big) - \mathcal{Z}\big[f_k\big](x,t) = \mathcal{C}_{1,k}\big(x,t,t'\big) + \mathcal{C}_{2,k}\big(x,t,t'\big), \tag{3.8}$$

whereby

$$\begin{aligned}\mathcal{C}_{1,k}\left(x,t,t'\right) &= (1+m\lambda_k^{s_1})^{-1}\int_t^{t'} r^{\alpha-1}\exp\Big(\frac{\lambda_k^{s_2}}{1+m\lambda_k^{s_1}}\frac{r^\alpha - t'^\alpha}{\alpha}\Big)\\ &\quad\times\Big(\int\limits_{\mathcal{D}} f(x,r)e_k(x)dx\Big)dr,\\ \mathcal{C}_{2,k}\left(x,t,t'\right) &= (1+m\lambda_k^{s_1})^{-1}\int_0^{t} r^{\alpha-1}\bigg(\exp\Big(\frac{\lambda_k^{s_2}}{1+m\lambda_k^{s_1}}\frac{r^\alpha - t'^\alpha}{\alpha}\Big)\\ &\quad-\exp\Big(\frac{\lambda_k^{s_2}}{1+m\lambda_k^{s_1}}\frac{r^\alpha - t^\alpha}{\alpha}\Big)\bigg)\Big(\int\limits_{\mathcal{D}} f(x,r)e_k(x)dx\Big)dr.\end{aligned} \tag{3.9}$$

Step 1. Estimate the term $\mathcal{C}_{1,k}(x,t,t')$ as follow:

$$\begin{aligned}&\big\|\mathcal{C}_{1,k}(\cdot,t,t')\big\|_{L^2(\Omega)}\\ =&\bigg\|\frac{1}{1+m\lambda_k^{s_1}}\int_t^{t'} r^{\alpha-1}\exp\Big(\frac{\lambda_k^{s_2}}{1+m\lambda_k^{s_1}}\frac{r^\alpha - t'^\alpha}{\alpha}\Big)\Big(\int\limits_{\mathcal{D}} f(x,r)e_k(x)dx\Big)dr\bigg\|_{L^2(\Omega)}\\ \le&\frac{1}{1+m\lambda_k^{s_1}}\int_t^{t'} r^{\alpha-1}\bigg\|\exp\big(\frac{\lambda_k^{s_2}}{1+m\lambda_k^{s_1}}\frac{r^\alpha - t'^\alpha}{\alpha}\big)\Big(\int\limits_{\mathcal{D}} f(x,r)e_k(x)dx\Big)\bigg\|_{L^2(\mathcal{D})}dr\\ \le&\frac{\exp\big(-\frac{\lambda_k^{s_2}}{1+m\lambda_k^{s_1}}\frac{t'^\alpha}{\alpha}\big)}{1+m\lambda_k^{s_1}}\int_t^{t'} r^{\alpha-1}\exp\Big(\frac{\lambda_k^{s_2}}{1+m\lambda_k^{s_1}}\frac{r^\alpha}{\alpha}\Big)\Big(\int_{\mathcal{D}} f(x,r)e_k(x)dx\Big)dr\\ \le&\frac{\exp\Big(-\frac{\lambda_k^{s_2}}{1+m\lambda_k^{s_1}}\frac{t'^\alpha}{\alpha}\Big)}{1+m\lambda_k^{s_1}}\big(\operatorname*{esssup}_{s\in[0,T]}\|f_k(\cdot,r)\|\big)\underbrace{\int_t^{t'} r^{\alpha-1}\exp\Big(\frac{\lambda_k^{s_2}}{1+m\lambda_k^{s_1}}\frac{r^\alpha}{\alpha}\Big)dr}_{:=\mathcal{I}_1}.\end{aligned} \tag{3.10}$$

Next, we will go to calculate the integral $\mathcal{I}_1$, the substitution rule for definite integrals, we get

$$\mathcal{I}_1 \le \Big(\frac{\lambda_k^{s_1}}{1+m\lambda_k^{s_1}}\Big)^{-1}\Big(\exp\Big(\frac{\lambda_k^{s_2}}{1+m\lambda_k^{s_1}}\frac{t'^\alpha}{\alpha}\Big) - \exp\Big(\frac{\lambda_k^{s_2}}{1+m\lambda_k^{s_1}}\frac{t^\alpha}{\alpha}\Big)\Big). \tag{3.11}$$

From (3.10) and (3.11), we receive

$$\begin{aligned}&\big\|\mathcal{C}_{1,k}(\cdot,t,t',)\big\|_{L^2(\Omega)}\\ \le&(\operatorname*{esssup}_{r\in[0,T]}\|f_k(\cdot,r)\|)\lambda_k^{-s_2}\Big(1-\exp\Big(-\frac{\lambda_k^{s_2}}{1+m\lambda_k^{s_1}}\frac{t'^\alpha - t^\alpha}{\alpha}\Big)\Big).\end{aligned} \tag{3.12}$$

Using the inequality $1-\exp(-z) \le z, \forall z \ge 0$, and for $0 < y_0 < y_1$, $|y_1^\sigma - y_0^\sigma| \le |y_1 - y_0|^\sigma$ for $0 < \sigma < 1$, it gives

$$\begin{aligned}\left\|\mathcal{C}_{1,k}(\cdot,t,t',)\right\|_{L^2(\mathcal{D})} &\le \Big(\operatorname*{esssup}_{r\in[0,T]}\|f_k(\cdot,r)\|\Big)\Big(\alpha(1+m\lambda_1^{s_1})\Big)^{-1}\big|t'^\alpha - t^\alpha\big| \\ &\le \Big(\operatorname*{esssup}_{r\in[0,T]}\|f_k(\cdot,r)\|\Big)\Big(\alpha(1+m\lambda_1^{s_1})\Big)^{-1}\big|t'-t\big|^\alpha. \end{aligned} \tag{3.13}$$

Step 2. Estimate of the term $\mathcal{C}_{2,k}(x,t,t')$:
Before going into the estimation to $\left\|\mathcal{C}_{2,k}(x,t',t)\right\|_{L^2(\mathcal{D})}$, we put

$$\begin{aligned}\mathcal{I}_2 &= \exp\Big(\frac{\lambda_k^{s_2}}{1+m\lambda_k^{s_1}}\frac{r^\alpha - t'^\alpha}{\alpha}\Big) - \exp\Big(\frac{\lambda_k^{s_2}}{1+m\lambda_k^{s_1}}\frac{r^\alpha - t^\alpha}{\alpha}\Big) \\ &= \exp\Big(\frac{\lambda_k^{s_2}}{1+m\lambda_k^{s_1}}\frac{r^\alpha}{\alpha}\Big)\Bigg(\exp\Big(-\frac{\lambda_k^{s_2}}{1+m\lambda_k^{s_1}}\frac{t'^\alpha}{\alpha}\Big) - \exp\Big(-\frac{\lambda_k^{s_2}}{1+m\lambda_k^{s_1}}\frac{t^\alpha}{\alpha}\Big)\Bigg).\end{aligned} \tag{3.14}$$

Since $0 < \sigma < 1$,there exists a constant $C_\sigma > 0$ such that for any $z > 0$, the basic inequality $e^{-z} < C_\sigma z^{\sigma-1}$ is correct. Thus for $0 < y_0 < y_1$, one has

$$\left|e^{-y_1} - e^{-y_0}\right| = \left|\int_{y_0}^{y_1} e^{-z}\ \mathrm{d}z\right| \le C_\sigma\left|\int_{y_0}^{y_1} z^{\sigma-1}\ \mathrm{d}z\right| = \frac{C_\sigma}{\sigma}\left|y_1^\sigma - y_0^\sigma\right|. \tag{3.15}$$

Here, we denote $\xi_{m,k}^{s_1,s_2} = \frac{\lambda_k^{s_2}}{1+m\lambda_k^{s_1}}$, and to get

$$\begin{aligned}\left|e^{-\xi_{m,k}^{s_1,s_2}\frac{t'^\alpha}{\alpha}} - e^{-\xi_{m,k}^{s_1,s_2}\frac{t^\alpha}{\alpha}}\right| &\le \frac{C_\sigma}{\sigma}\left|\Big(\xi_{m,k}^{s_1,s_2}\frac{t'^\alpha}{\alpha}\Big)^\sigma - \Big(\xi_{m,k}^{s_1,s_2}\frac{t^\alpha}{\alpha}\Big)^\sigma\right| \\ &\le \frac{C_\sigma}{\sigma}\big(\xi_{m,k}^{s_1,s_2}\big)^\sigma\left|\frac{t'^{\alpha\sigma} - t^{\alpha\sigma}}{\alpha^\sigma}\right| \\ &\le \frac{C_\sigma}{\sigma}\Big(\frac{\lambda_k^{s_2}}{1+m\lambda_k^{s_1}}\Big)^\sigma\frac{1}{\alpha^\sigma}\big|t'-t\big|^{\alpha\sigma}.\end{aligned} \tag{3.16}$$

From the condition $s_2 \le s_1$, from (3.16) it implies that

$$\left|e^{-\xi_{m,k}^{s_1,s_2}\frac{t'^\alpha}{\alpha}} - e^{-\xi_{m,k}^{s_1,s_2}\frac{t^\alpha}{\alpha}}\right| \le \frac{C_\sigma}{\sigma}\Big(\frac{1}{\alpha m\lambda_k^{s_1-s_2}}\Big)^\sigma\big|t'-t\big|^{\alpha\sigma}. \tag{3.17}$$

From the observed above, we have

$$\begin{aligned}&\left\|\mathcal{C}_{2,k}(x,t',t)\right\|_{L^2(\mathcal{D})} \\ \le&\frac{\Big(\operatorname*{esssup}_{s\in[0,T]}\|f_k(\cdot,r)\|\Big)}{1+m\lambda_k^{s_1}}\frac{C_\sigma}{\sigma}\Big(\frac{1}{\alpha m\lambda_k^{s_1-s_2}}\Big)^\sigma\big|t'-t\big|^{\alpha\sigma} \\ &\times\underbrace{\int_0^t r^{\alpha-1}\exp\Big(\frac{\lambda_k^{s_2}}{1+m\lambda_k^{s_1}}\frac{r^\alpha}{\alpha}\Big)ds}_{:=\mathcal{I}_2}.\end{aligned} \tag{3.18}$$

By the same argument as in the integral proof $\mathcal{I}_1$, one has

$$\mathcal{I}_2 \leq \Big(\frac{\lambda_k^{s_2}}{1+m\lambda_k^{s_1}}\Big)^{-1} \Big(\exp\Big(\frac{\lambda_k^{s_2}}{1+m\lambda_k^{s_1}}\frac{t^\alpha}{\alpha}\Big)-1\Big). \tag{3.19}$$

Because of $s_2 < s_1$, then $\xi_{m,k}^{s_1,s_2}$ is bounded sequence in $\mathbb{R}$, therefore the exits the constant $\mathcal{M}$ such that

$$\mathcal{I}_2 \leq \Big(\frac{\lambda_k^{s_2}}{1+m\lambda_k^{s_1}}\Big)^{-1}\mathcal{M} \leq \Big(\frac{1}{m\lambda_1^{s_1-s_2}}\Big)\mathcal{M}. \tag{3.20}$$

Combining (3.18) and (3.19), we have

$$\big\|\mathcal{C}_{2,k}(x,t',t)\big\|_{L^2(\mathcal{D})} \leq \big(\underset{r\in[0,T]}{\text{esssup}}\|f_k(\cdot,r)\|\big)\lambda_1^{-s_2}\frac{C_\sigma}{\sigma}\Big(\frac{1}{\alpha m\lambda_1^{s_1-s_2}}\Big)^\sigma |t'-t|^{\alpha\sigma}\mathcal{M}, \tag{3.21}$$

for all $0 < s < t \leq t'$. Combination of (3.13), (3.21), one has

$$\begin{aligned}\big\|\mathcal{Z}\big[f_k\big](x,t') - \mathcal{Z}\big[f_k\big](x,t)\big\|_{L^2(\mathcal{D})} \leq & \mathcal{C}_3\ |t'-t|^\alpha + \mathcal{C}_4\ |t'-t|^{\alpha\sigma} \\ & \text{for all } k=1,2,....\end{aligned} \tag{3.22}$$

This deduce the assertion a.

- Proving the assertion b.
  We will prove the set $\{\mathcal{Z}[f_k] : k\in\mathbb{N}^*\}$ is bounded at any point $t\in[0,T]$, we get

$$\begin{aligned}&\big\|\mathcal{Z}[f_k](\cdot,t)\big\|^2_{L^2(\mathcal{D})} \\ \leq&\frac{1}{(1+m\lambda_1^{s_1})^2}\Big(\int_0^t r^{\alpha-1}\exp\Big(\frac{\lambda_k^{s_2}}{1+m\lambda_k^{s_1}}\frac{r^\alpha-t^\alpha}{\alpha}\Big)\Big(\int_{\mathcal{D}} f(x,r)e_k(x)dx\Big)dr\Big)^2.\end{aligned}$$

We have

$$\begin{aligned}&\sum_{k=1}^\infty\Big[\int_0^t r^{\alpha-1}\exp\Big(\frac{\lambda_k^{s_1}}{1+m\lambda_k^{s_2}}\frac{r^\alpha-t^\alpha}{\alpha}\Big)\Big(\int_{\mathcal{D}} f(x,r)e_k(x)dx\Big)dr\Big]^2 \\ \leq& t\sum_{k=1}^\infty\int_0^t r^{2\alpha-2}\Big(\int_{\mathcal{D}} f(x,r)e_k(x)dx\Big)^2 dr \\ \leq& T\Big(\int_0^t r^{(2\alpha-2)p}dr\Big)^{\frac{1}{p}}\Big(\int_0^t \big\|f(.,r)\big\|^{\frac{2p}{p-1}}_{L^2(\mathcal{D})}dr\Big)^{\frac{p-1}{p}}, \quad \text{for } p<\frac{1}{2-2\alpha} \\ \leq&\frac{T^{2\alpha+\frac{1}{p}-1}}{\sqrt[p]{2\alpha p-2p+1}}\Big(\int_0^t(\beta(r))^{\frac{2p}{p-1}}dr\Big)^{\frac{p-1}{p}} \\ \leq&\frac{T^{2\alpha}}{\sqrt[p]{2\alpha p-2p+1}}\|\beta\|^2_{L^\infty(0,T;L^2(\mathcal{D}))}.\end{aligned} \tag{3.23}$$

Therefore, we have

$$\big\|\mathcal{Z}[f_k](\cdot,t)\big\|_{L^2(\mathcal{D})} \leq \mathcal{C}_5, \text{ for all } k=1,2,.... \tag{3.24}$$

This implies that $\{\mathcal{Z}[f_k] : k \in \mathbb{N}^*\}$ is relative compact in $C([0,T], L^2(\mathcal{D}))$ by Arzela-Ascoli theorem. Assertion c. is a consequence of b. with the note that $\mathcal{Z}[f_k]$ is bounded linear mapping from $L^1((0,T); L^2(\mathcal{D}))$ to $C([0,T]; L^2(\mathcal{D}))$ By the same argument, we can also prove the following lemma.

- Proving the assertion c Next, we proceed to prove Part b. Assertion c. is a consequence of b. with the note that Φ is bounded linear mapping from $L^1((0,T); L^2(\mathcal{D}))$ to $C([0,T]; L^2(\mathcal{D}))$.

Using the upper semicontinuous assumption (Hb) of F and applying Mazur's theorem we obtain the following lemma. □

Lemma 3.4. *Let the sequences* $\{v_k\}_{k\geq 1} \subset C\left([0,T]; L^2(\mathcal{D})\right), \{f_k\}_{k\geq 1} \subset L^1\big((0,T); L^2(\mathcal{D})\big)$ *such that* $f_k \in \mathcal{P}_{\mathcal{F}}(v_k)$ *for all* $k \geq 1$. *If* $v_k \to v_0$ *and* $f_k \rightharpoonup f_0$, *then* $f_0 \in \mathcal{P}_{\mathcal{F}}(v_0)$.

Upon Lemmas 3.3 and 3.4, the closed property of $\mathcal{Z} \circ \mathcal{P}_{\mathcal{F}}$ is obtained as the following lemma.

Lemma 3.5. *Under the assumption (H), the multioperator* $\mathcal{Z} \circ \mathcal{P}_{\mathcal{F}}$ *is closed.*

Proof. The proof of this lemma can be found in [23]. □

Lemma 3.6. *Assume that the assumption (H) holds. Then the multioperator* $\mathcal{Z}\circ\mathcal{P}_{\mathcal{F}}$ *is u.s.c.*

Next, we will construct the condensing property of the multioperator $\mathcal{Z} \circ \mathcal{P}_{\mathcal{F}}$ associated with a suitable MNC. For a positive constant L, let us consider the following MNC

$$\nu_L(D) \triangleq \max_{Q\in\Delta(D)} \left(\gamma_L(Q); \mathrm{mod}_{\mathrm{C}}(Q)\right), \tag{3.25}$$

where $\Delta(D)$ is the collection of all denumerable subsets of D, L is a positive constant, and

$$\gamma_L(Q) \triangleq \sup_{t\in[0,T]} e^{-L\frac{t^\alpha}{\alpha}} \chi(Q(t)), \quad \mathrm{mod}_{\mathrm{C}}(Q) \triangleq \lim_{\delta\to 0} \sup_{v\in D} \max_{|t'-t|\leq\delta} \|v(t') - v(t)\|, \tag{3.26}$$

see [[18], Example 2.1.4] for more details.

Lemma 3.7. *Let* $a_1 > -1, a_2 > -1$ *such that* $a_1 + a_2 \geq -1, \rho > 0$ *and* $t \in [0,T]$. *For* $h > 0$, *the following limit holds*

$$\lim_{\rho\to\infty} \Big(\sup_{t\in[0,T]} t^h \int_0^1 \nu^{a_1}(1-\nu)^{a_2} e^{-\rho t(1-\nu)} \mathrm{d}\nu \Big) = 0. \tag{3.27}$$

Lemma 3.8. *Let* $\mathcal{P}_{\mathcal{F}} : C\left([0,T]; L^2(\mathcal{D})\right) \to \mathcal{P}\left(L^p\big((0,T); L^2(\mathcal{D})\big)\right.$ *be the superposition multioperator generated by* F. *Then there exists* $L > 0$ *such that the multioperator* $\mathcal{Z} \circ \mathcal{P}_{\mathcal{F}}$ *is* ν_L*-condensing on bounded sets.*

Proof. This proof is obtained by following the proof of [18], Theorem 5.1.3, where we will also apply this Lemma to have an upper estimate for $\gamma_L\left(\mathcal{Z} \circ \mathcal{P}_{\mathcal{F}}(D)\right)$. We begin with taking a bounded set $D \subset C\left([0,T]; L^2(\mathcal{D})\right)$ such that

$$\nu_L(D) \leq \nu_L\big(\mathcal{Z} \circ \mathcal{P}_{\mathcal{F}}(D)\big), \tag{3.28}$$

where the order is taken in $\mathbb{R}^2$ induced by the positive cone $\mathbb{R}^2_+$. It is necessary to prove that D is relatively compact. Let $\nu_L\big(\mathcal{Z}\circ\mathcal{P}_{\mathcal{F}}(D)\big)$ be achieved on a sequence $\big\{g_n\big\}$, namely,

$$\nu_L\big(\big\{g_n\big\}\big)=\big(\gamma_L\big(\big\{g_n\big\}\big);\mathrm{mod}_C\left(\{g_n\}\right)\big), \tag{3.29}$$

with $g_k(\cdot,t)=\mathcal{Z}\big[f_k\big](\cdot,t), f_k\in\mathcal{P}_{\mathcal{F}}\big(v_k\big)$, and $\big\{v_k\big\}\subset D$. Then by making use of Lemma 3.3 similarly as (3.24), there hold that

$$\begin{aligned}
&e^{-L\frac{t^\alpha}{\alpha}}\chi\left(\{g_n(t,\cdot)\}\right)\\
=&e^{-L\frac{t^\alpha}{\alpha}}\chi\Bigg(\Bigg\{\frac{1}{1+m\lambda_k^{s_1}}\int_0^t r^{\alpha-1}\exp\Big(\frac{\lambda_k^{s_2}}{1+m\lambda_k^{s_1}}\frac{r^\alpha-t^\alpha}{\alpha}\Big)\\
&\times\Big(\int_{\mathcal{D}}f(x,r)e_k(x)dx\Big)dr\Bigg\}\Bigg)\\
\le&2e^{-L\frac{t^\alpha}{\alpha}}\int_0^t r^{\alpha-1}\chi\Bigg(\Bigg\{\frac{1}{1+m\lambda_k^{s_1}}\exp\Big(\frac{\lambda_k^{s_2}}{1+m\lambda_k^{s_1}}\frac{r^\alpha-t^\alpha}{\alpha}\Big)\\
&\times\Big(\int_{\mathcal{D}}f(x,r)e_k(x)dx\Big)\Bigg\}\Bigg)dr\\
\le&\mathcal{C}_{\alpha,\gamma}\Big(\sup_{r\in[0,T]}e^{-L\frac{r^\alpha}{\alpha}}\chi\big(\big\{v_n(s,\cdot)\big\}\big)\Big)\ t^b\int_0^t r^{\alpha-1-b}\Big(\frac{t^\alpha-r^\alpha}{\alpha}\Big)^{-\gamma}e^{-L\left(\frac{t^\alpha-r^\alpha}{\alpha}\right)}dr,
\end{aligned} \tag{3.30}$$

whereby $\mathcal{C}_{\alpha,\gamma}=\Big(\frac{1}{\lambda_k^{s_2}}\Big)^\gamma\Big(\frac{1}{1+m\lambda_k^{s_1}}\Big)^{1-\gamma}$. Next, let us control to the term

$$\mathcal{I}_L(t)=t^b\int_0^t r^{\alpha-1-b}\Big(\frac{t^\alpha-r^\alpha}{\alpha}\Big)^{-\gamma}e^{-L\left(\frac{t^\alpha-r^\alpha}{\alpha}\right)}dr. \tag{3.31}$$

To change variable $r=ts^{\frac{1}{\alpha}}$, then we get $dr=\frac{t}{\alpha}s^{\frac{1}{\alpha}-1}ds$. We have immediately that

$$\begin{aligned}
\mathcal{I}_L(t)&=t^b\int_0^t r^{\alpha-1-b}\Big(\frac{t^\alpha-r^\alpha}{\alpha}\Big)^{-\gamma}e^{-L\left(\frac{t^\alpha-r^\alpha}{\alpha}\right)}dr\\
&=\frac{1}{\alpha}t^{\alpha-\alpha\gamma}\int_0^1 s^{-\frac{b}{\alpha}}\Big(\frac{1-s}{\alpha}\Big)^{-\gamma}e^{-Lt^\alpha\left(\frac{1-s}{\alpha}\right)}ds.
\end{aligned} \tag{3.32}$$

Let us took at Lemma 3.6. Since $\gamma<1$, $\alpha-\alpha\gamma>0$, $-\frac{b}{\alpha}>-1$, we deduce that

$$\lim_{L\to+\infty}\Big[\sup_{0\le t\le T}t^b\int_0^t r^{\alpha-1-b}\Big(\frac{t^\alpha-r^\alpha}{\alpha}\Big)^{-\gamma}e^{-L\left(\frac{t^\alpha-r^\alpha}{\alpha}\right)}dr\Big]=0. \tag{3.33}$$

Hence, there exists a positive number L_0 such that

$$\sup_{0\le t\le T}t^b\int_0^t r^{\alpha-1-b}\Big(\frac{t^\alpha-r^\alpha}{\alpha}\Big)^{-\gamma}e^{-L_0\frac{(t^\alpha-r^\alpha)}{\alpha}}dr\le\frac{1}{4}, \tag{3.34}$$

where we have used χ-regularity condition (Hd) in the last estimate. By taking supremums with suitable order and using the definition of the MNC γ_L, we have

$$\gamma_L\left(\{g_n\}\right)\le 2\mathcal{C}_{\alpha,\gamma}\Bigg(\sup_{t\in[0,T]}t^b\int_0^t r^{\alpha-1-b}\Big(\frac{t^\alpha-r^\alpha}{\alpha}\Big)^{-\gamma}e^{-L\left(\frac{t^\alpha-r^\alpha}{\alpha}\right)}dr\Bigg)\gamma_L\big(\big\{v_n\big\}\big). \tag{3.35}$$

Due to Lemma 3.6, one can find a positive constant L_0 such that

$$\Big(\sup_{t\in[0,T]} t^b \int_0^t r^{\alpha-1-b}\Big(\frac{t^\alpha - r^\alpha}{\alpha}\Big)^{-\gamma} e^{-L\left(\frac{t^\alpha-r^\alpha}{\alpha}\right)} dr\Big) < \frac{1}{4\mathcal{C}_{\alpha,\gamma}}, \quad \forall L \geq L_0. \tag{3.36}$$

This consequently deduces that $\gamma_{L_0}\big(\{g_n\}\big) \leq \frac{1}{2}\gamma_{L_0}\big(\{v_n\}\big)$. In addition, the relation (3.28) yields $\gamma_{L_0}(\{g_n\}) \geq \gamma_{L_0}(\{v_n\})$. These together imply that $\gamma_{L_0}(\{v_n\}) = 0$ and hence $\chi(\{v_n(t,\cdot)\}) = 0$ for all $t \in [0,T]$. Due to conditions (Hc), (Hd), this deduces the semicompactness of $\{f_n\}$. Subsequently, Lemma 3.3 yields that $\{g_n\}$ is relatively compact. Therefore, $\nu_{L_0}(D) = (0,0)$ and conclude the proof. $\square$

Theorem 3.1. *Suppose the assumption (H) holds, then $\Sigma_\varphi^F[0,T]$ is a nonempty compact subset of $C\left([0,T];L^2(\mathcal{D})\right)$.*

Proof. We consider the multioperator

$$\mathcal{M} : C\left([0,T];L^2(\mathcal{D})\right) \to Kv\left(C\left([0,T];L^2(\mathcal{D})\right)\right), \tag{3.37}$$

defined by

$$\begin{aligned}\mathcal{M}(u) \triangleq \Big\{ & v \in C\left([0,T];L^2(\mathcal{D})\right) \mid v(\cdot,t) = \exp\Big(-\frac{\lambda_k^{s_2}}{1+m\lambda_k^{s_1}}\frac{t^\alpha}{\alpha}\Big)\varphi + \mathcal{Z}[f](\cdot,t), \\ & f \in \mathcal{P}_{\mathcal{F}}(u)\Big\}.\end{aligned} \tag{3.38}$$

Then by Lemma 3.6 and Lemma 3.7, this multioperator is u.s.c. and there exists $L_0 > 0$ largely enough such that $\mathcal{M}$ is ν_{L_0}-condensing on bounded sets. We recall that L_0 is chosen such that the condition (3.8) is satisfied. Let us introduce the temporally weighted space

$$\begin{aligned}& C_{L_0}\left([0,T];L^2(\mathcal{D})\right) \\ =& \Big\{ v \in C\left([0,T];L^2(\mathcal{D})\right) \mid \exists K > 0, \|v(t,\cdot)\| \leq K e^{L_0\frac{t^\alpha}{\alpha}}, \forall t \in [0,T]\Big\},\end{aligned} \tag{3.39}$$

endowed with the weighted norm

$$\|v\|_{C_{L_0}([0,T];L^2(\Omega))} = \sup_{t\in[0,T]} e^{-L_0\frac{t^\alpha}{\alpha}}\|v(t,\cdot)\|, \quad \forall v \in C_{L_0}\left([0,T];L^2(\mathcal{D})\right).$$

In this space, we denote by $\bar{B}_{L_0}(r)$ the closed ball centered at the zero function with the radius r. It is necessary to show that there exists $r > 0$ such that $\mathcal{M}$ maps the ball $\overline{B}_{L_0}(r)$ into itself. Indeed, let r be satisfied $r \geq \mathcal{C}_1\|\varphi\| + \dfrac{1+r}{4}$. For $u \in \bar{B}_{L_0}(r)$ and $v \in \mathcal{M}(u)$, there holds

$$\begin{aligned} & e^{-L_0\frac{t^\alpha}{\alpha}}\|v(\cdot,t)\| \\ \leq & e^{-L_0\frac{t^\alpha}{\alpha}}\Big\|\exp\Big(-\frac{\lambda_k^{s_2}}{1+m\lambda_k^{s_1}}\frac{t^\alpha}{\alpha}\Big)\Big(\int_{\mathcal{D}}\varphi(x)e_k(x)dx\Big)\Big\| + e^{-L_0\frac{t^\alpha}{\alpha}}\|\mathcal{Z}[f](\cdot,t)\| \\ \leq & \mathcal{C}_1 e^{-L_0\frac{t^\alpha}{\alpha}}\|\varphi\| + \frac{e^{-L_0\frac{t^\alpha}{\alpha}}}{1+m\lambda_k^{s_1}}\int_0^t r^{\alpha-1}\Big\|\exp\Big(\frac{\lambda_k^{s_2}}{1+m\lambda_k^{s_1}}\Big)\frac{r^\alpha - t^\alpha}{\alpha}f(\cdot,r)\Big\|dr \\ \leq & \mathcal{C}_1 e^{-L_0\frac{t^\alpha}{\alpha}}\|\varphi\| + \mathcal{C}_{\alpha,\gamma}t^b\int_0^t r^{\alpha-1-b}\Big(\frac{t^\alpha-r^\alpha}{\alpha}\Big)^{-\gamma}\Big(e^{-L_0\frac{t^\alpha}{\alpha}} + re^{-L_0\frac{(t^\alpha-r^\alpha)}{\alpha}}\Big)dr\end{aligned}$$

$$\leq \mathcal{C}_1 e^{-L_0 \frac{t^\alpha}{\alpha}} \|\varphi\| + \mathcal{C}_{\alpha,\gamma}(1+r)t^b \int_0^t r^{\alpha-1-b}\Big(\frac{t^\alpha - r^\alpha}{\alpha}\Big)^{-\gamma} e^{-L_0 \frac{(t^\alpha - r^\alpha)}{\alpha}} ds, \tag{3.40}$$

whereby the condition (H) with $\mathcal{A}(r) = 1$, this implies that

$$\big\|f(\cdot,r)\big\| \leq \mathcal{A}(r)(1 + \|u(\cdot,r)\|) \leq \big(1 + re^{L_0 \frac{r^\alpha}{\alpha}}\big), \forall f \in \mathrm{P}_F(u). \tag{3.41}$$

Due to the condition (3.36), we imply

$$e^{-L_0 \frac{t^\alpha}{\alpha}} \|v(\cdot,t)\| \leq \mathcal{C}_1 e^{-L_0 \frac{t^\alpha}{\alpha}} \|\varphi\| + \frac{1+r}{4} \leq r. \tag{3.42}$$

By applying Lemma 2.2, this deduces that $\Sigma_\varphi^F[0,T]$ is nonempty in $C\left([0,T]; L^2(\mathcal{D})\right)$. It is remain to prove that $\Sigma_\varphi^F[0,T]$ is a compact set. For $u \in \Sigma_\varphi^F[0,T]$ and $t \in [0,T]$, one can observe that

$$\begin{aligned}
\big\|u(\cdot,t)\big\| &\leq \left\|\exp\Big(-\frac{\lambda_k^{s_2}}{1+m\lambda_k^{s_1}}\frac{t^\alpha}{\alpha}\Big)\varphi\right\| + \big\|\mathcal{Z}[f](\cdot,t)\big\| \\
&\leq \mathcal{C}_1\|\varphi\| + \frac{1}{1+m\lambda_k}\int_0^t r^{\alpha-1}\exp\Big(\frac{\lambda_k^{s_2}}{1+m\lambda_k^{s_1}}\frac{r^\alpha - t^\alpha}{\alpha}\Big)\big(1+\big\|u(\cdot,r)\big\|\big)dr \\
&\leq \mathcal{C}_1\|\varphi\| + \frac{1}{1+m\lambda_k^{s_1}}\int_0^t r^{\alpha-1}dr + \frac{1}{1+m\lambda_k^{s_1}}\int_0^t r^{\alpha-1}\big\|u(\cdot,r)\big\|dr \\
&\leq \mathcal{C}_1\big\|\varphi\big\| + T^\alpha\big(\alpha(1+m\lambda_1^{s_1})\big)^{-1} + \int_0^t r^{\alpha-1}\big\|u(\cdot,r)\big\|dr.
\end{aligned} \tag{3.43}$$

Applying the Gronwall's inequality, we get $\|u(\cdot,)\| \leq \mathcal{C}_2 \exp\big(\frac{T^\alpha}{\alpha}\big)\mathcal{C}_3$, whereby $\mathcal{C}_2 = \mathcal{C}_1\big\|\varphi\big\| + T^\alpha\big(\alpha(1+m\lambda_1^{s_1})\big)^{-1}$ and $\mathcal{C}_3$ depends on T. We deduce from the above arguments that $\Sigma_\varphi^F[0,T]$ is bounded. Therefore, by applying Lemma 2.2, the compactness of $\Sigma_\varphi^F[0,T]$ is obtained. The proof is completed. □

4. Continuous dependence on parameters

In this section, we discuss on the dependence of the solutions of parameterized problem on the parameter η in a metric space (Y,d)

$$\begin{cases} \dfrac{{}^C\partial^\alpha}{\partial t^\alpha}u(t) + m\mathcal{L}^{s_1}\dfrac{{}^C\partial^\alpha}{\partial t^\alpha}u(t) + \mathcal{L}^{s_2}u(t) \in F(t,u(t),\eta), & t \in \mathcal{I}, \\ u(0) = \varphi, & \text{on } \mathcal{D}, \end{cases} \tag{4.1}$$

where the differential operators ∂_t^α and $(-\Delta)^s$ are defined by (1.2) for $0 < \alpha < 1$, and by (1.3) for $0 < s < 1$. For a given parameter $\mu_0 \in Y$, the main purpose of this part is to study the continuous of respectively integral solutions, namely, we will show that if $\mu \in Y$ satisfying $d(\mu,\mu_0)$ small enough, the solution set with respect to λ contains in a neighbourhood of the solution sets with respect to μ_0. In order to establish the continuous dependence on parameters, we will consider the following assumption on the nonlinearity. (Hμ) Let $F : [0,T] \times L^2(\mathcal{D}) \times Y \to Kv\left(L^2(\mathcal{D})\right)$ be a multimap satisfying that:

- For every $(v,\mu) \in L^2(\mathcal{D}) \times Y$, the multifunction $F(\cdot,v,\mu)$ has a strongly measurable selection.

- The multimap $F(\cdot,\cdot,t) : L^2(\mathcal{D}) \times Y \to Kv\left(L^2(\mathcal{D})\right)$ is usc for a.e. $t \in [0,T]$.
- There exists a function $A \in L^{1,\gamma}((0,T);\mathbb{R})$ for some $\gamma > -1$ such that

$$\|F(t,v,\mu)\| \le A(t), \quad \text{a.e. } t \in (0,T), \forall v \in L^2(\mathcal{D}), \forall \lambda \in Y.$$

- (χ-regularity condition) There exists a function $B \in L^{1,\gamma_2}((0,T);\mathbb{R})$ for some constant $\gamma_2 > -1$ such that

$$\chi(F(t,D,Y)) \le B(t)\chi(D), \quad \text{a.e. } t \in (0,T),$$

  for every nonempty bounded subset $D \subset L^2(\mathcal{D})$. Similarly as Theorem 3.10, we also denote by $M^\lambda\left(v_{\bar{T}}\right)$ the integral multioperator.

We also denote by $M^\mu\left(v_{\bar{T}}\right)$ the integral multioperator in $C\left([0,T];L^2(\mathcal{D})\right)$ corresponding to the nonlinearity $F(\cdot,\cdot,\mu)$ for each $\mu \in Y$. Let $R_\varphi^{F(\cdot,\cdot,\lambda_0)}$ be the set of all integral solutions of Problem (4.1), where $u \in R_\varphi^{F(\cdot,\cdot,\mu_0)}$ if there exists $\mathcal{T} > 0$ such that $\forall \bar{T} < \mathcal{T}$ and $v_{\bar{T}} = u|_{[0,\bar{T}]}$, it holds $v_{\bar{T}} \in \mathcal{M}^\lambda\left(v_{\bar{T}}\right)$. Moreover, we use "dist" to denote the distance between a point and a set, and B_ϵ to denote ϵ-neighbourhood of a point or a set.

Lemma 4.1. *Assume that $\varphi \in L^2(\mathcal{D})$, and $s_1 \ge s_2$. Then it holds*

$$\left\|\exp\Big(-\frac{\lambda_k^{s_2}}{1+m\lambda_k^{s_1}}\frac{t^\alpha}{\alpha}\Big)\varphi - \varphi\right\|_{L^2(\mathcal{D})} \le \|\varphi\|_{L^2(\mathcal{D})}, \quad \forall t \ge 0. \tag{4.2}$$

Denote by $R_h^{F,\varsigma}$ the family of all local mild solutions of Problem (4.1), i.e., $u \in R_\varphi^{F,\varsigma}$ iff there exists $\sigma \in (0,T]$ and $u \in C([0,T];L^2(\mathcal{D}))$ such that for all $\overline{\sigma} \in [0,\sigma]$ and $v_{\overline{\sigma}} = u|_{[0,\overline{\sigma}]}$, it gives

$$v_{\overline{\sigma}} \in \Big\{ y \in C([0,\overline{\sigma}];L^2(\mathcal{D})) : y(t) = \sum_{k=1}^\infty \exp\Big(-\frac{\lambda_k^{s_2}}{1+m\lambda_k^{s_1}}\frac{t^\alpha}{\alpha}\Big)\langle\varphi,e_k\rangle e_k + \mathcal{Z}[f](t), \\ f \in S_{F,\varsigma}(u)\Big\},$$

and $R_\varphi^{F,\varsigma}[0,T] := \left\{v \in R_\varphi^{F,\varsigma} : v \in M^\varsigma(v)\right\}$, whereby

$$M^\varsigma(u) := \Big\{ v \in C([0,T];L^2(\mathcal{D})) : v(t) = \sum_{k=1}^\infty \exp\Big(-\frac{\lambda_k^{s_2}}{1+m\lambda_k^{s_1}}\frac{t^\alpha}{\alpha}\Big)\langle\varphi,e_k\rangle e_k \\ + \mathcal{Z}[f](t), f \in S_{F,\varsigma}(u)\Big\}.$$

Theorem 4.1. *Assume that the condition* (H_ς) *holds, for some $\varsigma_0 \in E$ the set $R_h^{F,\varsigma_0}[0,T]$ is bounded and*

$$R_\varphi^{F,\varsigma_0}[0,\overline{\sigma}] = R_\varphi^{F,\varsigma_0}[0,T]|_{[0,\overline{\sigma})} \quad \textit{for all } \ \overline{\sigma} \in (0,T]. \tag{4.3}$$

Then, for every given $\varepsilon > 0$, there is $\delta_\varepsilon > 0$ satisfying

$$R_\varphi^{F,\varsigma}[0,T] \subset \mathcal{N}_\epsilon\big(R_\varphi^{F,\varsigma_0}[0,T]\big) \quad \textit{for all } \ \varsigma \in B_{\delta_\epsilon}(\varsigma_0). \tag{4.4}$$

Proof. Assume that $c > 0$ with $\left\|R_\varphi^{F,\varsigma}[0,T]\right\| < c$. Firstly, we will prove the following statement by contraction argument: There exists $\delta > 0$ such that if $\varsigma \in \mathcal{N}_\delta(\varsigma_0) \subset E$, then

$$\left\|R_\varphi^{F,\varsigma}(t)\right\| \le 3c \ \text{ for all } \ t \in [0,T]. \tag{4.5}$$

Indeed, we assume by contradiction that (4.5) fails. Then, taking sequences $\{\varsigma_k\} \subset E$, $\{t_k\} \subset [0,T]$, $\{u_k\} \subset C([0,T];L^2(\mathcal{D}))$, $\varsigma_k \to \varsigma_0$ such that $y_k \in M_{\varsigma_n}(y_k)$ and

$$\operatorname{dist}\big(y_k\big(t_k\big), R_\varphi^{F,\varsigma_0}\big(t_k\big)\big) \geq 2c, \quad \operatorname{dist}\big(y_k(t), R_h^{F,\varsigma_0}(t)\big) < 2c, \tag{4.6}$$

for all $t \in [0,t_k)$. We note $t_* = \underline{\lim}\{t_k\}$, proving that $t_* \in (0,T]$. Indeed, assume that $t_* = 0$. Let us choose a sub-sequence of $\big\{t_k\big\} \to 0$, which we also denote by $\{t_k\}$ for convenience. Since $R_\varphi^{F,\varsigma_0}$ is bounded and from (4.3) it follows that $R_\varphi^{F,\varsigma_0}$ is compact, and so the distance between φ and $R_\varphi^{F,\varsigma_0}(t_k)$ tends to zero. We observe that

$$\begin{aligned}
2c &\leq \operatorname{dist}\big(y_k(t_k), R_\varphi^{F,\varsigma_0}(t_k)\big) \\
&\leq \big\|y_k(t_k) - \varphi\big\|_{L^2(\mathcal{D})} + \operatorname{dist}\big(\varphi, R_\varphi^{F,\varsigma_0}(t_k)\big) \\
&\leq \Big\|\sum_{k=1}^{\infty} e^{-\frac{\lambda_k^{s_2}}{1+m\lambda_k^{s_1}}\frac{t_k^\alpha}{\alpha}} \langle \varphi, e_k\rangle e_k - \varphi\Big\|_{L^2(\mathcal{D})} + \Big\|\mathcal{Z}[f_k](t_k)\Big\|_{L^2(\mathcal{D})} + \operatorname{dist}\big(\varphi, R_\varphi^{F,\varsigma_0}(t_k)\big) \\
&\leq \frac{t_k^\alpha}{\alpha}\big(m\lambda_1^{s_1-s_2}\big)^{-1}\|\varphi\|_{L^2(\mathcal{D})} + \mathcal{C}_2 \exp\big(\frac{t_k^\alpha}{\alpha}\big)\mathcal{C}_3 + \operatorname{dist}\big(\varphi, R_\varphi^{F,\varsigma_0}(t_k)\big).
\end{aligned} \tag{4.7}$$

Letting $k \to \infty$ in (4.7) we derive the contradiction $2c \leq 0$. Summarily, we deduce that $t_* > 0$. By the definition of t_* there exists number ζ with $0 < \zeta < t_* \leq T$ such that all solution y_k are defined on $[0, t_* - \zeta]$. Afterthat, we prove that for every y_k, there exists $\sigma_k \in [0, t_* - \zeta] \subsetneq [0,T]$ satisfying

$$\operatorname{dist}\big(y_k(\sigma_k), R_\varphi^{F,\varsigma_0}(\sigma_k)\big) \geq \epsilon. \tag{4.8}$$

For every k, let any $t_\dagger \in [0,t_k)$, by the compactness of $R_\varphi^{F,\varsigma_0}$, we can assume that $\big\|y_k(t_\dagger) - y_\dagger(t_\dagger)\big\|_{L^2(\mathcal{D})} < \epsilon$ for some $y_\dagger \in R_\varphi^{F,\varsigma_0}$. Then we receive quadrilateral inequality

$$\begin{aligned}
&\big\|y_k(t_\dagger + t) - y_\dagger(t_\dagger + t)\big\|_{L^2(\mathcal{D})} \\
&\leq \underbrace{\big\|y_k(t_\dagger + t) - y_k(t_\dagger)\big\|_{L^2(\mathcal{D})}}_{\mathbb{E}_1} + \underbrace{\big\|y_\dagger(t_\dagger + t) - y_\dagger(t_\dagger)\big\|_{L^2(\mathcal{D})}}_{\mathbb{E}_2} + \big\|y_k(t_\dagger) - y_\dagger(t_\dagger)\big\|_{L^2(\mathcal{D})}.
\end{aligned} \tag{4.9}$$

With the same arguments as the first part of Lemma 3.3, one can choose t small enough such that both $\mathbb{E}_1$ and $\mathbb{E}_2$ are less than $\dfrac{\epsilon}{4}$. Hence, the norms $\|y_k(t_\dagger + t) - y_\dagger(t_\dagger + t)\|_{L^2(\mathcal{D})} \leq \dfrac{3\epsilon}{2}$, which contradicts (4.6). Hence, (4.8) is proved.

Now, by similar arguments as obtaining Lemma 3.8, we note that the multimap $M_* : E \times C([0, t_* - \zeta]; L^2(\mathcal{D})) \to Kv(C([0, t_* - \gamma]; L^2(\mathcal{D})))$, $M_*(\varsigma, u) = M^\mu(u)$, is ν_L-condensing for some $L > 0$. This ensures relative compactness of the sequence $\{y_k|_{[0,t_*-\zeta]}\}$. Let us take $y_* = \lim y_k|_{[0,\zeta - t_*]}$, which belongs to $M_*(\lambda_0, y_*)$ on $[0, t_* - \zeta]$. Thus, by passing to the limit in (4.8), we obtain

$$\operatorname{dist}\big(y_*(t_*), R_\varphi^{F,\varsigma_0}(t_*)\big) \geq \epsilon.$$

Consequently, the solution u_* cannot be extended to the interval $[0,T]$, which contradicts (4.3). Finally, we complete the proof by applying Lemma 2.4. $\square$

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