Journal of Applied Analysis and Computation
Volume 14, Number 3, June 2024, 1302–1329

Website:http://www.jaac-online.com
DOI:10.11948/20230125

A MIXED-TYPE PICARD-S ITERATIVE METHOD FOR ESTIMATING COMMON FIXED POINTS IN HYPERBOLIC SPACES

Austine Efut Ofem[1], Jacob Ashiwere Abuchu[1,2], Godwin Chidi Ugwunnadi[3,4], Ojen Kumar Narain[1], Hassen Aydi[5,6,7,†] and Choonkil Park[8,†]

Abstract This article presents a modified Picard-S iterative method in hyperbolic spaces. The proposed iterative method is used to approximate the common fixed point of two contractive-like mappings. We consider new concepts of data dependence and weak w^2-stability results of the proposed iterative scheme involving two contractive-like mappings in hyperbolic spaces. We prove the strong and $\triangle$-convergence results of our new algorithm for common fixed points of two mappings enriched with the condition (E). With numerical examples, we show the advantage and efficiency of the proposed method over some existing methods. Our results generalize and improve several results in the literature.

Keywords Weak w^2-stability, data dependence, strong and $\triangle$-convergence, common fixed point.

MSC(2010) 47H05, 47H09, 39B82.

[†]The corresponding author.
[1]School of Mathematics, Statistics and Computer Science, University of KwaZulu-Natal, Durban, South Africa
[2]Department of Mathematics, University of Calabar, Calabar, Nigeria
[3]Department of Mathematics, University of Eswatini, Kwaluseni, Eswatini
[4]Department of Mathematics and Applied Mathematics, Sefako Makgatho Health Sciences University, P.O. Box 94 Medunsa 0204, Pretoria, South Africa
[5]Université de Sousse, Institut Supérieur d'Informatique et des Techniques de Communication, H. Sousse 4000, Tunisia
[6]China Medical Univesity Hospital, China Medical University, Taichung 40402, Taiwan
[7]Department of Mathematics and Applied Mathematics, Sefako Makgatho Health Sciences University, Ga-Rankuwa, South Africa
[8]Research Institute for Convergence of Basic Science, Hanyang University, Seoul 04763, Korea
Email: 222128007@stu.ukzn.ac.za(A. E. Ofem), 221114675@stu.ukzn.ac.za(J. A. Abuchu), ugwunnadi4u@yahoo.com(G. C. Ugwunnadi), naraino@ukzn.ac.za(O. K. Narain), hassen.aydi@isima.rnu.tn(H. Aydi), baak@hanyang.ac.kr(C. Park)

1. Introduction

Several problems in diverse disciplines of applied sciences and engineering are nonlinear in nature. Fixed point theory studied in the framework of normed linear or Banach spaces enjoys the linear structure of the ambient spaces. A nonlinear setting for fixed theory is a metric space endowed with convex structure. It is well known that the class of hyperbolic spaces are nonlinear in nature and they are valuable among non-positive curved spaces. The class of hyperbolic spaces provides a rich geometrical structure for various results with applications in signal processing, wave propagation, robotics, telecommunications, system identification, biology, heat transfer, traffic systems, viscoelasticity, graph theory, topology, multivalued analysis, game theory and so on.

In this manuscript, we will consider the hyperbolic spaces studied of Kohlenbach [28]. The concept of the hyperbolic space of Kohlenbach [28] is more restrictive than that in [14] and more general than that in [47]. CAT(0) and Banach spaces are contained in hyperbolic spaces. Also, the class of hyperbolic spaces includes Cartesian product of Hilbert spaces, Hadamard manifolds, Hilbert ball endowed with hyperbolic metric [15] and $\mathbb{R}$-trees.

Definition 1.1. In the sense of Kohlenbach [28], a hyperbolic space $(\mathcal{S}, \rho, \mathcal{V})$ is a metric space $(\mathcal{S}, \rho)$ together with a convexity mapping $\mathcal{V} : \mathcal{S}^2 \times [0, 1] \to \mathcal{S}$ satisfying

(V_1) $\rho(w, \mathcal{V}(u, v, h)) \leq (1 - h)\rho(w, u) + h\rho(w, v)$;

(V_2) $\rho(\mathcal{V}(u, v, h), \mathcal{V}(u, v, \eta)) = |h - \eta|\rho(u, v)$;

(V_3) $\mathcal{V}(u, v, h) = \mathcal{V}(v, u, (1 - h))$;

(V_4) $\rho(\mathcal{V}(u, w, h), \mathcal{V}(v, s, h)) \leq (1 - h)\rho(u, v) + h\rho(w, s)$,

for all $u, v, w, s \in \mathcal{S}$ and $h, \eta \in [0, 1]$. A nonempty subset $\mathcal{G}$ of a hyperbolic space $\mathcal{S}$ is termed convex, if $\mathcal{V}(u, v, h) \in \mathcal{G}$, for all $u, v \in \mathcal{G}$ and $h \in [0, 1]$.

Suppose $u, v \in \mathcal{S}$ and $h \in [0, 1]$. The notation $(1-h)u \oplus hv$ is used for $\mathcal{V}(u, v, h)$. In a convex metric space, the following is also true [20]: for any $u, v \in \mathcal{S}$ and $h \in [0, 1]$, $\rho(u, (1 - h)u \oplus hv) = h\rho(u, v)$ and $\rho(v, (1 - h)u \oplus hv) = (1 - h)\rho(u, v)$. Consequently, $1u \oplus 0v = u$, $0u \oplus 1v = v$ and $(1 - h)u \oplus hu = hu \oplus (1 - h)u = u$.

If $(\mathcal{S}, \rho)$ is a metric space, then an element $u \in \mathcal{S}$ is said to be a fixed point of the mapping $M : \mathcal{S} \to \mathcal{S}$ if $Mu = u$. We denote the set of all fixed points of M by $F(M) = \{u \in \mathcal{S} : Mu = u\}$. There exist several recent results in the literature on complete metric spaces, see for example, [12, 27, 55].

Definition 1.2. A mapping $M : \mathcal{S} \to \mathcal{S}$ is called:

(a) a contraction if there exists a constant $k \in [0, 1)$ such that for all $u, v \in \mathcal{S}$, we have

$$\rho(Mu, Mv) \leq k\rho(u, v);$$

(b) an almost contraction if there exist some constants $k \in [0, 1)$ and $L \geq 0$ such that for all $u, v \in \mathcal{S}$, we have

$$\rho(Mu, Mv) \leq k\rho(u, v) + L\rho(u, Mu).$$

This class of mappings was introduced by Berinde [4] and has been studied recently by several authors (see [35] and the references therein).

Definition 1.3. A mapping $M : \mathcal{S} \to \mathcal{S}$ is called contractive-like if there exists $k \in [0, 1)$ and a strictly increasing continuous function $\Psi : [0, \infty) \to [0, \infty)$ with $\Psi(0) = 0$ such that for all $u, v \in \mathcal{S}$, we have

$$\rho(Mu, Mv) \leq k\rho(u, v) + \Psi(\rho(u, Mu)). \tag{1.1}$$

This class of mappings was introduced by Imoru and Olantiwo [21]. The class of contractive-like mappings includes the class of almost contraction mappings for $\Psi(u) = Lu$. There are several recent results on the studies of this class of mappings (see [13, 23, 35] and the references in them).

Definition 1.4. A mapping $M : \mathcal{S} \to \mathcal{S}$ is called:

(d) nonexpansive if for all $u, v \in \mathcal{S}$, we have

$$\rho(Mu, Mv) \leq \rho(u, v);$$

(e) quasi-nonexpansive if for all $u \in \mathcal{S}$ and $p^\dagger \in F(M) \neq \emptyset$, we have

$$\rho(Mu, p^\dagger) \leq \rho(u, p^\dagger);$$

(f) Suzuki generalized nonexpansive or having the condition (C) if for all $u, v \in \mathcal{S}$, we have

$$\frac{1}{2}\rho(u, Mu) \leq \rho(u, v) \;\Rightarrow\; \rho(Mu, Mv) \leq \rho(u, v).$$

This class of mappings was introduced in 2008 by Suzuki [52] as a generalization of the class of nonexpansive mappings. The author studied the existence and convergence analysis of such mappings.

In 2011, García-Falset et al. [13] introduced the notion of mappings having the condition (E) which are generally weaker than the class of nonexpansive mappings and mappings having the condition (C), but stronger than the class of quasi-nonexpansive mappings.

Definition 1.5. A mapping $M : \mathcal{S} \to \mathcal{S}$ is said to satisfy condition E_μ if there exists $\mu \geq 1$ such that

$$\rho(u, Mv) \leq \mu\rho(u, Mu) + \rho(u, v), \; \forall\, u, v \in \mathcal{S}. \tag{1.2}$$

Now, M is said to satisfy the condition (E), whenever M satisfies the condition E_μ for some $\mu \geq 1$.

In recent years, iterative methods have been considered as the main tool for fixed point analysis of nonlinear operators. In the past two decades or so, several iterative methods have been introduced for approximating fixed points of different classes of mappings. Some of these prominent iterative methods are: Mann [31], Ishikawa [22], Noor [32], S [2], Picard-S [17], Picard-Mann [25] and Abbas [1] iterative methods.

Very recently, Gursoy and Karakaya [17] introduced the Picard-S iterative method in Banach spaces as follows:

$$\begin{cases} u_1 \in \mathcal{G}, \\ w_\gamma = (1 - h_\gamma)u_\gamma + h_\gamma M u_\gamma, \\ v_\gamma = (1 - g_\gamma)Mu_\gamma + g_\gamma M w_\gamma, \\ u_{\gamma+1} = Mv_\gamma, \end{cases} \quad \gamma \geq 1, \tag{1.3}$$

where $\{g_\gamma\}$ and $\{h_\gamma\}$ are real sequences in (0,1). The authors [16] proved some fixed point results of the iterative method (1.3) for contraction mappings. They further showed that (1.3) converges faster than a number of existing iterative processes.

On the other hand, the notion of stability of iterative methods was initiated by Ostrowski [45]. The results of Ostrowski [45] were extended by Harder [18], Harder and Chicks [19] for contractive-type mappings. The results of Ostrowski [45], Harder [18], Harder and Chicks [19] were later improved and generalized by Roades [48], Osilike [40–44] and Berinde [3, 5].

In 2007, Timis [54] defined a more generalized and natural notion of stability known as weak w^2-stability.

Definition 1.6. [6] A sequence $\{u_\gamma\}$ is said to be equivalent to another $\{v_\gamma\}$, if

$$\rho(u_\gamma, v_\gamma) \to 0, \text{ as } \gamma \to +\infty.$$

Definition 1.7. [54] If $(\mathcal{S}, \rho)$ is a metric space and $M : \mathcal{S} \to \mathcal{S}$, then for an arbitrary $u_1 \in M$, $\{u_\gamma\}$ is the iterative algorithm defined by

$$u_{\gamma+1} = f(M, u_\gamma), \ \gamma \geq 0. \tag{1.4}$$

Assume that $u_\gamma \to p^\dagger$ as $\gamma \to +\infty$, for all $p^\dagger \in F(M)$ and given a sequence $\{a_\gamma\} \subset \mathcal{S}$ which is equivalent to $\{u_\gamma\}$, we get

$$\lim_{\gamma \to +\infty} \rho(a_{\gamma+1}, f(M, a_\gamma)) = 0 \implies \lim_{\gamma \to +\infty} a_\gamma = p^\dagger,$$

then the iterative procedure (1.4) is said to be weak w^2-stable with respect to M.

For recent results on weak w^2-stability, the reader can refer to [33, 34, 36–39] and the references therein.

Remark 1.1. The concept of stability studied in Definition 1.7 involves only one mapping and as far as we know, there are no results on weak w^2-stability of iterative algorithms involving two contractive-like mappings in hyperbolic spaces.

Another important aspect of fixed point theory is the data dependence results of iterative methods. In recent years, many valuable contributions to this regard have been made by prominent authors (see [35]).

Remark 1.2. The existing data dependence results in the literature have been achieved for iterative schemes with one mapping. To the best of our knowledge, the concept of data dependence results of iterative schemes involving two contractive-like mappings is yet to be studied in hyperbolic spaces.

The concept of iterative methods involving two mappings was initiated by Das and Debata [8]. The problems dealing with approximation of common fixed points of finitely many mappings play a significant role in applied mathematics, particularly in the minimization problems and the theory of evolution equations [9–11, 26, 30].

To fill the gaps in Remark 1.1 and Remark 1.2, we introduce the hyperbolic space version of Picard-S iterative method (1.3) which deals with two mappings as follows:

$$\begin{cases} u_1 \in \mathcal{G}, \\ w_\gamma = (1-h_\gamma)u_\gamma \oplus h_\gamma M_2 u_\gamma, \\ v_\gamma = (1-g_\gamma)M_2 u_\gamma \oplus g_\gamma M_1 w_\gamma, \\ u_{\gamma+1} = M_1 v_\gamma, \end{cases} \quad \gamma \geq 1, \tag{1.5}$$

where $\{g_\gamma\}$ and $\{h_\gamma\}$ are real sequences in (0,1).

The aim of this article is to prove the strong convergence of the mixed-type Picard-S iterative method (1.5) for common fixed points of contractive-like mappings in hyperbolic spaces. We present some examples of contractive-like mappings to test competence of the new iterative method with some existing iterative methods. We consider new notions of weak w^2-stability and data dependence of iterative methods. Precisely, we prove the stability and data dependence results using the mixed-type Picard-S iterative scheme (1.5) for contractive-like mappings. Several strong and $\triangle$-convergence theorems of (1.5) for approximations of common fixed points of mappings satisfying condition (E) are also obtained. We provide nontrivial examples to authenticate the mild conditions in convergence results and further use one of the numerical examples to test the applicability and efficiency of (1.5).

2. Preliminaries

A hyperbolic space $(\mathcal{S}, \rho, \mathcal{V})$ is called uniformly convex [20], if for any $k > 0$ and $\epsilon \in (0, 2]$, there exists $\nu \in (0, 1]$ such that for all $u, v, p \in \mathcal{S}$,

$$\rho(\frac{1}{2}u \oplus \frac{1}{2}v, p) \leq (1-\epsilon)k,$$

provided $\rho(u,p) \leq k, \rho(v,p) \leq k$ and $\rho(u,v) \geq \epsilon k$. A mapping $\theta : (0, \infty) \times (0, 2] \to (0, 1]$ is said to be modulus of uniform convexity, provided that $\nu = \theta(k, \epsilon)$ for any $k > 0$ and $\epsilon \in (0, 2]$. We say that θ is monotone if for fixed ϵ, it decreases with k, which implies that, $\theta(k_2, \epsilon) \leq \theta(k_1, \epsilon)$, for all $k_2 \geq k_1 > 0$.

In 2007, Leustean [29] showed that if the modulus of uniform convexity $\nu(s, \epsilon) = \frac{\epsilon^2}{8}$ quadratic in ϵ, then the CAT(0) space is a uniformly convex hyperbolic spaces. It therefore means that the class of uniformly convex hyperbolic spaces properly includes both CAT(0) space and a uniformly convex Banach space [20].

We now present the following concepts which will be useful in the definition of $\triangle$-convergence. Let $(\mathcal{S}, \rho)$ be a metric space and $\mathcal{G}$ be a nonempty subset of $\mathcal{S}$. If $\{u_\gamma\}$ is any sequence that is bounded in $\mathcal{S}$. For any $u \in \mathcal{S}$, we define:

- asymptotic radius of $\{u_\gamma\}$ at u as

$$r_a(\{u_\gamma\}, u) = \limsup_{\gamma \to \infty} d(u_\gamma, u);$$

- asymptotic radius of $\{u_\gamma\}$ relative to $\mathcal{G}$ as

$$r_a(\{u_\gamma\}, \mathcal{G}) = \inf\{r_a(\{u_\gamma\}, u); u \in \mathcal{G}\};$$

- asymptotic center of $\{u_\gamma\}$ relative to $\mathcal{G}$ as

$$AC(\{u_\gamma\}, \mathcal{G}) = \{u \in \mathcal{G}; r_a(\{u_\gamma\}, u) = r_a(\{u_\gamma\}, \mathcal{G})\}. \tag{2.1}$$

Every bounded sequence has a unique asymptotic center with respect to each closed convex subset in CAT(0) and Banach spaces. Suppose the asymptotic center is considered with respect to $\mathcal{S}$, then we simplify represent it by $AC(\{u_\gamma\})$.

In [29], Leustean showed that the above property is also true in a complete uniformly convex hyperbolic space as follows:

Lemma 2.1. [29] *Let ν be the monotone modulus of uniform convexity of the complete uniformly convex hyperbolic $(\mathcal{S}, \rho, \mathcal{V})$. Then any bounded sequence $\{u_\gamma\}$ in $\mathcal{S}$, has a unique asymptotic center with respect to any nonempty closed convex subset $\mathcal{G}$ of $\mathcal{S}$.*

Now, we give the following established results which will be used in the sequel.

Definition 2.1. If u is the unique asymptotic center of every subsequence $\{u_{\gamma_l}\}$ of $\{u_\gamma\}$ in $\mathcal{S}$, then $\{u_\gamma\}$ is said to be $\triangle$-convergent to an element u in $\mathcal{S}$. We write $\triangle - \lim\limits_{\gamma\to\infty} u_\gamma = u$ and say u the $\triangle$-limit of $\{u_\gamma\}$.

Lemma 2.2. [24] *Let ν be the monotone modulus of uniform convexity of a uniformly convex hyperbolic space $\mathcal{S}$. Let $u \in \mathcal{S}$ and $\{h_\gamma\}$ be a sequence in $[e, d]$ such that $e, d \in (0, 1)$. Assume that $\{u_\gamma\}$ and $\{v_\gamma\}$ are sequences in $\mathcal{S}$ with $\limsup\limits_{\gamma\to+\infty} \rho(u_\gamma, u) \leq c$, $\limsup\limits_{\gamma\to+\infty} \rho(v_\gamma, u) \leq c$ and $\lim\limits_{\gamma\to+\infty} \rho(h_\gamma u_\gamma \oplus (1 - h_\gamma)v_\gamma, u) = c$ for some $c \geq 0$, implies that $\lim\limits_{\gamma\to+\infty} \rho(u_\gamma, v_\gamma) = 0$.*

Lemma 2.3. [51] *Let $\{d_\gamma\}$ be a nonnegative sequence. Assume there exists a $\gamma_0 \in \mathbb{N}$ such that for any $\gamma \geq \gamma_0$, the following inequality holds:*

$$d_{\gamma+1} \leq (1 - \varphi_\gamma)d_\gamma + \varphi_\gamma\phi_\gamma$$

where $\varphi_\gamma \in (0, 1)$ for all $\gamma \in \mathbb{N}$, $\sum_{\gamma=0}^{+\infty} \varphi_\gamma = +\infty$ and $\phi_\gamma \geq 0$ $\forall\gamma \in \mathbb{N}$. Then the following inequality is true:

$$0 \leq \limsup_{\gamma\to+\infty} d_\gamma \leq \limsup_{\gamma\to+\infty} \phi_\gamma.$$

Definition 2.2. [51] Let M, $\tilde{M} : \mathcal{S} \to \mathcal{S}$. Then $\tilde{M}$ is an approximate operator of M if for any $\epsilon > 0$, it follows that $d(Mu, \tilde{M}u) \leq \epsilon$ holds for any $u \in \mathcal{S}$.

Proposition 2.1. [13] *Let $M : \mathcal{S} \to \mathcal{S}$ be a mapping which satisfies the condition (E) with $F(M) \neq \emptyset$, then M is quasi-nonexpansive.*

3. Convergence, weak w^2 stability and data dependence results

In this section, we show that the mixed-type Picard-S iterative method (1.5) converges to the common fixed points of two mappings satisfying (1.1). The convergence result will be useful in obtaining our data dependence and weak w^2-stability results for two mappings satisfying (1.1).

In the remaining part of this article, we use $\mathbb{R}$ to denote the set of all real numbers.

Theorem 3.1. *Let $(\mathcal{S}, \rho, \mathcal{V})$ be a hyperbolic space, $\mathcal{G}$ be a nonempty closed convex subset of $\mathcal{S}$ and $M_1, M_2 : \mathcal{G} \to \mathcal{G}$ be two contractive-like mappings with $\Omega = F(M_1) \cap F(M_2) \neq \emptyset$. If $\{u_\gamma\}$ is the sequence defined by (1.5), then $\{u_\gamma\}$ converges to a point in Ω.*

Proof. Suppose $p^\dagger \in F(M_1) \cap F(M_2)$. Using (1.5), we obtain

$$\begin{aligned}\rho(w_\gamma, p^\dagger) &= \rho(((1-h_\gamma)u_\gamma \oplus h_r M_2 u_\gamma), p^\dagger)\\ &\leq (1-h_\gamma)\rho(u_\gamma, p^\dagger) + h_\gamma \rho(M_2 u_\gamma, p^\dagger)\\ &\leq (1-h_\gamma)d(u_\gamma, p^\dagger) + h_\gamma k \rho(u_\gamma, p^\dagger)\\ &= (1-(1-k)h_\gamma)\rho(u_\gamma, p^\dagger). \end{aligned} \tag{3.1}$$

Using (1.5) and (3.1), we have

$$\begin{aligned}\rho(v_\gamma, p^\dagger) &= \rho(((1-g_\gamma)M_2 u_\gamma \oplus g_\gamma M_1 w_\gamma), p^\dagger)\\ &\leq (1-g_\gamma)\rho(M_2 u_\gamma, p^\dagger) + g_\gamma \rho(M_1 w_\gamma, p^\dagger)\\ &\leq k(1-g_\gamma)\rho(u_\gamma, p^\dagger) + k g_\gamma \rho(w_\gamma, p^\dagger)\\ &\leq k(1-g_\gamma)\rho(u_\gamma, p^\dagger) + k g_\gamma (1-(1-k)h_\gamma)\rho(u_\gamma, p^\dagger)\\ &= k(1-(1-k)h_\gamma g_\gamma)\rho(u_\gamma, p^\dagger). \end{aligned} \tag{3.2}$$

By (1.5) and (3.2), we obtain

$$\begin{aligned}\rho(u_{\gamma+1}, p^\dagger) &= \rho(M_1 v_\gamma, p^\dagger)\\ &\leq k\rho(v_\gamma, p^\dagger)\\ &\leq k^2(1-(1-k)h_\gamma g_\gamma)\rho(u_\gamma, p^\dagger). \end{aligned} \tag{3.3}$$

Since $0 \leq k < 1$ and $0 < h_\gamma, g_\gamma < 1$, we have $(1-(1-k)h_\gamma g_\gamma) < 1$. So, (3.3) becomes

$$\rho(u_{\gamma+1}, p^\dagger) \leq k^2 \rho(u_\gamma, p^\dagger). \tag{3.4}$$

Inductively, we obtain

$$\rho(u_{\gamma+1}, p^\dagger) \leq k^{2(\gamma+1)} \rho(u_0, p^\dagger).$$

Since $0 \leq k < 1$, it follows that $\lim_{\gamma \to +\infty} u_\gamma = p^\dagger$. This completes the proof. □

Now, we authenticate the results in Theorem 3.1 with the following examples of contractive-like mappings which will be used to test the efficiency of our new method (1.5) with some well known algorithms.

Example 3.1. Let $\mathcal{S} = \mathbb{R}^2$ and $\mathcal{G} = \{u = (u_1, u_2) : (u_1, u_2) \in [0, 10] \times [0, 10]\}$ be a subset of $\mathcal{W}$ with the taxi-cab metric

$$\rho((u_1, u_2), (v_1, v_2)) = |u_1 - v_1| + |u_2 - v_2|,$$

for all (u_1, u_2) and (v_1, v_2) in $\mathcal{G}$. Let $M_1, M_2 : \mathcal{G} \to \mathcal{G}$ be defined by

$$M_1(u_1, u_2) = \begin{cases} \left(\frac{u_1}{8}, \frac{u_2}{8}\right), & \text{if } (u_1, u_2) \in [0, 5) \times [0, 5),\\ \left(\frac{u_1}{16}, \frac{u_1}{16}\right), & \text{if } (u_1, u_2) \in [5, 10] \times [5, 10]; \end{cases}$$

and

$$M_2(u_1,u_2)=\begin{cases}\left(\frac{u_1}{6},\frac{u_2}{6}\right), & \text{if } (u_1,u_2)\in[0,5)\times[0,5),\\ \left(\frac{u_1}{12},\frac{u_2}{12}\right), & \text{if } (u_1,u_2)\in[5,10]\times[5,10].\end{cases}$$

Since every nonexpansive mapping is continuous, we know that M_1 and M_2 are not nonexpansive mappings because of their discontinuity at $5\in\mathcal{S}$ and hence, they are not contraction mappings.

Now, we show that M_1 satisfies (1.1). For this, we define $\Psi(u)=\frac{u}{14}$. It is easy to see that the function Ψ is strictly increasing and continuous such that $\Psi(0)=0$. It is worthy to note that for all $u=(u_1,u_2)\in[0,5)\times[0,5)$, we have

$$\begin{aligned}\rho(u,M_1u)&=\rho\left((u_1,u_2),\left(\frac{u_1}{8},\frac{u_2}{8}\right)\right)\\&=\left|u_1-\frac{u_1}{8}\right|+\left|u_2-\frac{u_2}{8}\right|\\&=\left|\frac{7u_1}{8}\right|+\left|\frac{7u_2}{8}\right|\end{aligned}$$

and

$$\Psi(\rho(u,Mu))=\left|\frac{u_1}{16}\right|+\left|\frac{u_2}{16}\right|. \tag{3.5}$$

Also, if $u=(u_1,u_2)\in[5,10]\times[5,10]$, then we have

$$\begin{aligned}\rho(u,M_1u)&=\rho\left((u_1,u_2),\left(\frac{u_1}{16},\frac{u_2}{16}\right)\right)\\&=\left|u_1-\frac{u_1}{16}\right|+\left|u_2-\frac{u_2}{16}\right|\\&=\left|\frac{15u_1}{16}\right|+\left|\frac{15u_2}{16}\right|\end{aligned}$$

and

$$\Psi(\rho(u,M_1u))=\left|\frac{15u_1}{224}\right|+\left|\frac{15u_2}{224}\right|. \tag{3.6}$$

Next, we verify the following cases:

Case A. If $u=(u_1,u_2),v=(v_1,v_2)\in[0,5)\times[0,5)$, then by (3.5), we have

$$\begin{aligned}\rho(M_1u,M_1v)&=\rho\left(\left(\frac{u_1}{8},\frac{u_2}{8}\right),\left(\frac{v_1}{8},\frac{v_2}{8}\right)\right)\\&=\left|\frac{u_1}{8}-\frac{v_1}{8}\right|+\left|\frac{u_2}{8}-\frac{v_2}{8}\right|\\&=\frac{1}{8}\left|u_1-v_1\right|+\frac{1}{8}\left|u_2-v_2\right|\\&=\frac{1}{8}\rho((u_1,v_2),(u_1,v_2))\\&\le\frac{1}{8}\rho(u,v)+\left|\frac{u_1}{16}\right|+\left|\frac{u_2}{16}\right|\\&=\frac{1}{8}\rho(u,v)+\Psi(\rho(u,M_1u)).\end{aligned}$$

Case B. If $u = (u_1, u_2), v = (v_1, v_2) \in [5, 10] \times [5, 10]$, then by (3.6), we get

$$\begin{aligned}
\rho(M_1u, M_1v) &= \rho\left(\left(\frac{u_1}{16}, \frac{u_2}{16}\right), \left(\frac{v_1}{16}, \frac{v_2}{12}\right)\right) \\
&= \left|\frac{u_1}{16} - \frac{v_1}{16}\right| + \left|\frac{u_2}{16} - \frac{v_2}{16}\right| \\
&= \frac{1}{16}|u_1 - v_1| + \frac{1}{16}|u_2 - v_2| \\
&= \frac{1}{16}\rho((u_1, v_2), (u_1, v_2)) \\
&\leq \frac{1}{8}\rho(u, v) + \left|\frac{15u_1}{224}\right| + \left|\frac{15u_2}{224}\right| \\
&= \frac{1}{8}\rho(u, v) + \psi(\rho(u, M_1u)).
\end{aligned}$$

Case C. If $u = (u_1, u_2) \in [0, 5) \times [0, 5)$ and $v = (v_1, v_2) \in [5, 10] \times [5, 10]$, then by (3.5), we have

$$\begin{aligned}
\rho(M_1u, M_1v) &= \rho\left(\left(\frac{u_1}{8}, \frac{u_2}{8}\right), \left(\frac{v_1}{16}, \frac{v_2}{16}\right)\right) \\
&= \left|\frac{u_1}{8} - \frac{v_1}{16}\right| + \left|\frac{u_2}{8} - \frac{v_2}{16}\right| \\
&= \left|\frac{u_1}{16} + \frac{u_1}{16} - \frac{v_1}{16}\right| + \left|\frac{u_2}{16} + \frac{u_2}{16} - \frac{v_2}{16}\right| \\
&\leq \left|\frac{u_1}{16}\right| + \left|\frac{u_2}{16}\right| + \left|\frac{u_1}{16} - \frac{v_1}{16}\right| + \left|\frac{u_2}{16} - \frac{v_2}{16}\right| \\
&= \frac{1}{16}(|u_1 - v_1| + |u_2 - v_2|) + \Psi(\rho(u, M_1u)) \\
&\leq \frac{1}{8}\rho((u_1, u_2), (v_1, v_2)) + \Psi(\rho(u, M_1u)) \\
&= \frac{1}{8}\rho(u, v) + \Psi(\rho(u, M_1u)).
\end{aligned}$$

Case D. If $u = (u_1, u_2) \in [5, 10] \times [5, 10]$ and $v = (v_1, v_2) \in [0, 5) \times [0, 5)$, then by (3.5), we have

$$\begin{aligned}
\rho(M_1u, M_1v) &= \rho\left(\left(\frac{u_1}{16}, \frac{u_2}{16}\right), \left(\frac{v_1}{8}, \frac{v_2}{8}\right)\right) \\
&= \left|\frac{u_1}{16} - \frac{v_1}{8}\right| + \left|\frac{u_2}{16} - \frac{v_2}{8}\right| \\
&= \left|\frac{u_1}{8} - \frac{u_1}{16} - \frac{v_1}{8}\right| + \left|\frac{u_2}{8} - \frac{u_2}{16} - \frac{v_2}{8}\right| \\
&\leq \left|\frac{u_1}{16}\right| + \left|\frac{u_2}{16}\right| + \left|\frac{u_1}{8} - \frac{v_1}{8}\right| + \left|\frac{u_2}{8} - \frac{v_2}{8}\right| \\
&= \frac{1}{8}(|u_1 - v_1| + |u_2 - v_2|) + \Psi(\rho(u, M_1u)) \\
&= \frac{1}{8}\rho((u_1, u_2), (v_1, v_2)) + \Psi(\rho(u, M_1u) \\
&= \frac{1}{8}\rho(u, v) + \Psi(\rho(u, M_1u)).
\end{aligned}$$

Thus, from all the above cases, it is shown that M_1 is a contractive-like mapping with $k = \frac{1}{8}$. The fixed point of M_1 is $(0, 0)$. Using a similar approach above, we can show that M_2 is a contractive-like mapping with $k = \frac{1}{6}$. The fixed point of M_2 is $(0, 0)$. Clearly, $F(M_1) \cap F(M_1)$={(0,0)}.

For all $\gamma \geq 1$, let $h_\gamma = h_\gamma = t_\gamma = \frac{3}{4}$ be the control parameters and $(2, 4)$ be the starting point. We use MATLAB R2015a to obtain the Table 1, Table 2, Figure 1 and Figure 2. It is not hard to see that Mixed-type Picard-S iterative converges faster to the common fixed point (0,0) than Man, Ishikawa, S, Noor, Abbas and Picard-Man iterative schemes.

Table 1. Convergence comparison of different iterative algorithms for contraction-like mappings.

u_γ	Mann	Ishikawa	S	Mixed-Type Picard-S
g_1	(2.000000, 4.000000)	(2.000000, 4.000000)	(2.000000, 4.000000)	(2.000000, 4.000000)
g_2	(0.687500, 1.375000)	(0.564453, 1.128906)	(0.126953, 0.253906)	(0.019206, 0.038411)
g_3	(0.236328, 0.472656)	(0.159304, 0.318607)	(0.008059, 0.016117)	(0.000184, 0.000369)
g_4	(0.081238, 0.162476)	(0.044960, 0.089919)	(0.000512, 0.001023)	(0.000002, 0.000004)
g_5	(0.027925, 0.055851)	(0.012689, 0.025378)	(0.000032, 0.000065)	(0.000000, 0.000000)
g_6	(0.009599, 0.019199)	(0.003581, 0.007162)	(0.000002, 0.000004)	(0.000000, 0.000000)
g_7	(0.003300, 0.006600)	(0.001011, 0.002021)	(0.000000, 0.000000)	(0.000000, 0.000000)
g_8	(0.001134, 0.002269)	(0.000285, 0.000570)	(0.000000, 0.000000)	(0.000000, 0.000000)
g_9	(0.000390, 0.000780)	(0.000081, 0.000161)	(0.000000, 0.000000)	(0.000000, 0.000000)
g_{10}	(0.000134, 0.000268)	(0.000023, 0.000045)	(0.000000, 0.000000)	(0.000000, 0.000000)
g_{11}	(0.000046, 0.000092)	(0.000006, 0.000013)	(0.000000, 0.000000)	(0.000000, 0.000000)
g_{12}	(0.000016, 0.000032)	(0.000002, 0.000004)	(0.000000, 0.000000)	(0.000000, 0.000000)
g_{13}	(0.000005, 0.000011)	(0.000001, 0.000001)	(0.000000, 0.000000)	(0.000000, 0.000000)
g_{14}	(0.000002, 0.000004)	(0.000000, 0.000000)	(0.000000, 0.000000)	(0.000000, 0.000000)
g_{15}	(0.000001, 0.000001)	(0.000000, 0.000000)	(0.000000, 0.000000)	(0.000000, 0.000000)
g_{16}	(0.000000, 0.000000)	(0.000000, 0.000000)	(0.000000, 0.000000)	(0.000000, 0.000000)

Table 2. Convergence comparison of different iterative algorithms for contraction-like mappings.

u_γ	Noor	Abbas	Picard-Man	Mixed-Type Picard-S
g_1	(2.000000, 4.000000)	(2.000000, 4.000000)	(2.000000, 4.000000)	(2.000000, 4.000000)
g_2	(0.552917, 1.105835)	(0.068420, 0.136841)	(0.085938, 0.171875)	(0.019206, 0.038411)
g_3	(0.152859, 0.305718)	(0.002341, 0.004681)	(0.003693, 0.007385)	(0.000184, 0.000369)
g_4	(0.042259, 0.084518)	(0.000080, 0.000160)	(0.000159, 0.000317)	(0.000002, 0.000004)
g_5	(0.011683, 0.023366)	(0.000003, 0.000005)	(0.000007, 0.000014)	(0.000000, 0.000000)
g_6	(0.003230, 0.006460)	(0.000000, 0.000000)	(0.000000, 0.000001)	(0.000000, 0.000000)
g_7	(0.000893, 0.001786)	(0.000000, 0.000000)	(0.000000, 0.000000)	(0.000000, 0.000000)
g_8	(0.000247, 0.000494)	(0.000000, 0.000000)	(0.000000, 0.000000)	(0.000000, 0.000000)
g_9	(0.000068, 0.000136)	(0.000000, 0.000000)	(0.000000, 0.000000)	(0.000000, 0.000000)
g_{10}	(0.000019, 0.000038)	(0.000000, 0.000000)	(0.000000, 0.000000)	(0.000000, 0.000000)
g_{11}	(0.000005, 0.000010)	(0.000000, 0.000000)	(0.000000, 0.000000)	(0.000000, 0.000000)
g_{12}	(0.000001, 0.000003)	(0.000000, 0.000000)	(0.000000, 0.000000)	(0.000000, 0.000000)
g_{13}	(0.000000, 0.000001)	(0.000000, 0.000000)	(0.000000, 0.000000)	(0.000000, 0.000000)
g_{14}	(0.000000, 0.000000)	(0.000000, 0.000000)	(0.000000, 0.000000)	(0.000000, 0.000000)

Now, we give the hyperbolic space version of the definition of weak w^2-stability involving two mappings as follows:

Definition 3.1. Let $(\mathcal{S}, \rho, \mathcal{V})$ be a hyperbolic space, $M_1, M_2 : \mathcal{S} \to \mathcal{S}$ and for

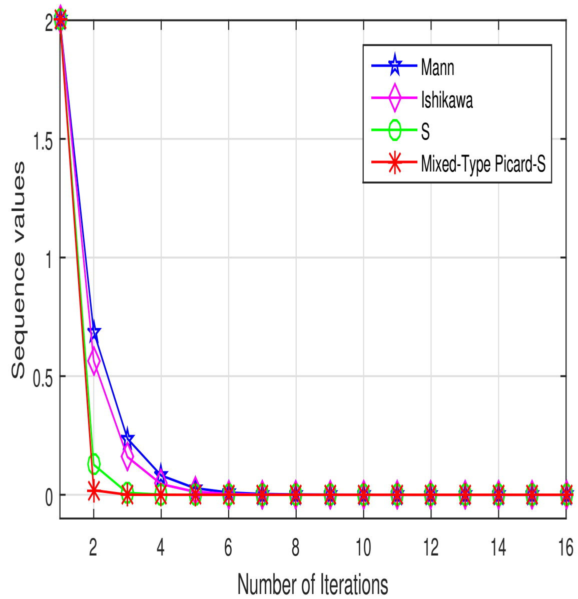

Figure 1. Graph corresponding to Table 1.

arbitrary $u_1 \in \mathcal{S}$, the sequence$\{u_\gamma\}$ be defined by

$$u_{\gamma+1} = f(M_i, u_\gamma) \ (i = 1, 2), \ \gamma \geq 1. \tag{3.7}$$

Assume that $u_\gamma \to p^\dagger$ as $\gamma \to +\infty$, for all $p^\dagger \in \Omega = F(M_1) \cap F(M_2)$ and given any sequence $\{x_\gamma\} \subset \mathcal{S}$ that is equivalent to $\{u_\gamma\}$, we obtain

$$\lim_{\gamma \to +\infty} \rho(x_{\gamma+1}, f(M_i, x_\gamma)) = 0 \implies \lim_{\gamma \to +\infty} x_\gamma = p^\dagger,$$

then we say that (3.7) is weak w^2-stable with respect to M_1 and M_2.

Theorem 3.2. *Assume that all the conditions in Theorem 3.1 hold. Then, the sequence $\{u_\gamma\}$ generated by (1.5) is weak w^2-stable with respect to M_1 and M_2.*

Proof. Let $\{u_\gamma\}$ be the sequence generated by (1.5) and $\{x_\gamma\} \subset \mathcal{G}$ be a sequence which is equivalent to $\{u_\gamma\}$. We define $\{\epsilon_\gamma\} \in [0, \infty)$ by

$$\begin{cases} u_1 \in \mathcal{G}, \\ z_\gamma = (1 - h_\gamma)x_\gamma \oplus h_\gamma M_2 x_\gamma, \\ y_\gamma = (1 - g_\gamma)M_2 x_\gamma \oplus g_\gamma M_1 z_\gamma, \\ \epsilon_m = \rho(x_{\gamma+1}, M_1 y_\gamma), \end{cases} \quad \gamma \geq 1, \tag{3.8}$$

where $\{g_\gamma\}$ and $\{h_\gamma\}$ are real sequences in $(0, 1)$.

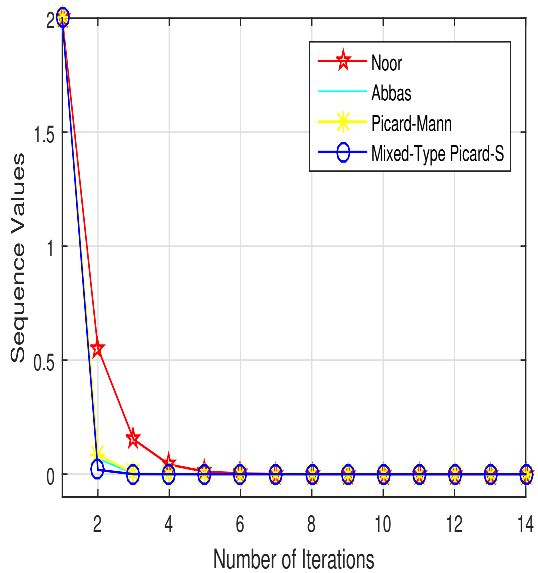

Figure 2. Graph corresponding to Table 2.

Suppose $\lim_{\gamma\to\infty}\epsilon_\gamma = 0$ and $p^\dagger \in F(M_1)\cap F(M_2)$. From (1.5) and (3.8), we have

$$\begin{aligned}\rho(w_\gamma, z_\gamma) &= \rho((1-h_\gamma)u_\gamma \oplus h_\gamma M_2 u_\gamma, (1-h_\gamma)x_\gamma \oplus h_\gamma M_2 x_\gamma)\\ &\le (1-h_\gamma)\rho(u_\gamma, x_\gamma) + h_\gamma \rho(M_2 u_\gamma, M_2 x_\gamma)\\ &\le (1-h_\gamma)\rho(u_\gamma, x_\gamma) + h_\gamma k\rho(u_\gamma, x_\gamma) + \Psi(\rho(u_\gamma, M_2 u_\gamma))\\ &= (1-(1-k)h_\gamma)\rho(u_\gamma, x_\gamma) + \Psi(\rho(u_\gamma, M_2 u_\gamma)). \end{aligned} \tag{3.9}$$

Since $0 \le k < 1$ and $0 < h_\gamma < 1$, we have $1-(1-k)h_\gamma < 1$. So (3.9) becomes

$$\rho(w_\gamma, z_\gamma) \le \rho(u_\gamma, x_\gamma) + \Psi(\rho(u_\gamma, M_2 u_\gamma)). \tag{3.10}$$

By (1.5), (3.8) and (3.10), we have

$$\begin{aligned}\rho(v_\gamma, y_\gamma) &= \rho((1-g_\gamma)M_2 u_\gamma \oplus g_\gamma M_1 w_\gamma, (1-g_\gamma)M_2 x_\gamma \oplus g_\gamma M_1 z_\gamma)\\ &\le (1-g_\gamma)\rho(M_2 u_\gamma, M_2 x_\gamma) + g_\gamma \rho(M_1 w_\gamma, M_1 z_\gamma) \qquad (3.11)\\ &\le (1-g_\gamma)[k\rho(u_\gamma, x_\gamma) + \Psi(\rho(u_\gamma, M u_\gamma))]\\ &\quad + g_\gamma[k\rho(w_\gamma, z_\gamma) + \Psi(\rho(w_\gamma, M w_\gamma))]\\ &\le (1-g_\gamma)[k\rho(u_\gamma, x_\gamma) + \Psi(\rho(u_\gamma, M u_\gamma))]\\ &\quad + g_\gamma[k\rho(u_\gamma, x_\gamma) + k\Psi(\rho(u_\gamma, M_2 u_\gamma)) + \Psi(\rho(w_\gamma, M_1 w_\gamma))]\\ &\le k\rho(u_\gamma, x_\gamma) + (1+g_\gamma k)\Psi(\rho(u_\gamma, M_2 u_\gamma)) + g_\gamma \Psi(\rho(w_\gamma, M_1 w_\gamma)). \end{aligned}$$

From (1.5), (3.8), (3.10) and (3.12), we obtain

$$\rho(x_{\gamma+1}, p^\dagger) \le \rho(x_{\gamma+1}, u_{\gamma+1}) + \rho(u_{\gamma+1}, p^\dagger)$$

$$\begin{aligned}
&\leq \rho(x_{\gamma+1}, M_1y_\gamma) + \rho(M_1y_\gamma, u_{\gamma+1}) + \rho(u_{\gamma+1}, p^\dagger) \\
&= \epsilon_\gamma + \rho(M_1v_\gamma, M_1y_\gamma) + \rho(u_{\gamma+1}, p^\dagger) \\
&\leq \epsilon_\gamma + k\rho(v_\gamma, y_\gamma) + \Psi(\rho(v_\gamma, M_1v_\gamma)) + \rho(u_{\gamma+1}, p^\dagger) \\
&\leq \epsilon_\gamma + k^2\rho(u_\gamma, x_\gamma) + k(1+g_\gamma k)\Psi(\rho(u_\gamma, M_2u_\gamma)) + kg_\gamma\Psi(\rho(w_\gamma, M_1w_\gamma)) \\
&\quad +\Psi(\rho(v_\gamma, M_1v_\gamma)) + \rho(u_{\gamma+1}, p^\dagger).
\end{aligned}$$

As established in Theorem 3.1, $\lim_{\gamma\to\infty} \rho(u_\gamma, p^\dagger) = 0$. Notice that

$$\begin{aligned}
\rho(u_\gamma, M_2u_\gamma) &\leq \rho(u_\gamma, p^\dagger) + \rho(p^\dagger, M_2u_\gamma) \\
&\leq \rho(u_\gamma, p^\dagger) + k\rho(p^\dagger, u_\gamma) \\
&= (1+k)\rho(u_\gamma, p^\dagger) \to 0 \text{ as } \gamma \to \infty.
\end{aligned}$$

Using a similar approach above, one can show that $\lim_{\gamma\to+\infty} \rho(w_\gamma, M_1w) = \lim_{\gamma\to+\infty} \rho(v_\gamma, M_1v_\gamma) = 0$. Since Ψ is a strictly increasing continuous self function defined on $[0, +\infty)$ such that $\Psi(0) = 0$, it follows that $\lim_{\gamma\to+\infty} \rho(u_\gamma, M_2u_\gamma) = \lim_{\gamma\to+\infty} \Psi(\rho(u_\gamma, M_2u_\gamma)) = \Psi(\lim_{\gamma\to+\infty} \rho(u_\gamma, M_2u_\gamma)) = 0$. Similar argument holds for others. Since $\lim_{\gamma\to+\infty} \rho(u_\gamma, p^\dagger) = 0$, we have $\lim_{\gamma\to+\infty} \rho(u_{\gamma+1}, p^\dagger) = 0$. Also, by the equivalence of $\{u_\gamma\}$ and $\{x_\gamma\}$, we have $\lim_{\gamma\to+\infty} \rho(u_\gamma, x_\gamma) = 0$.

Thus if we take the limit on both sides of (3.12), then we get

$$\lim_{\gamma\to+\infty} d(x_\gamma, p^\dagger) = 0.$$

This implies that (1.5) is weak w^2-stable with respect to M_1 and M_2. □

Next, we prove that the new method (1.5) is data dependent with respect to both M_1 and M_2 satisfying (1.1).

Theorem 3.3. *Let $(\mathcal{S}, \rho, \mathcal{V})$ be a hyperbolic space, $\mathcal{G}$ be a nonempty closed convex subset of $\mathcal{S}$ and $M_1, M_2 : \mathcal{G} \to \mathcal{G}$ be two mappings satisfying (1.1). Let $\tilde{M}_1, \tilde{M}_2 : \mathcal{G} \to \mathcal{G}$ be approximate operators of M_1 and M_2, respectively with $\rho(M_1u, \tilde{M}_1u) \leq \epsilon$ and $\rho(M_2u, \tilde{M}_2u) \leq \epsilon$ for all $u \in \mathcal{G}$. If $\{u_\gamma\}$ is the sequence generated by (1.5) for two mappings M_1 and M_2 satisfying (1.1). Let an iterative sequence $\{\tilde{u}_\gamma\}$ be defined as follows:*

$$\begin{cases}
\tilde{u_1} \in \mathcal{G}, \\
\tilde{w}_\gamma = (1-h_\gamma)\tilde{u}_\gamma \oplus h_\gamma\tilde{M}\tilde{u}_\gamma, \\
\tilde{v}_\gamma = (1-g_\gamma)\tilde{M}\tilde{u}_\gamma \oplus g_\gamma\tilde{M}\tilde{w}_\gamma, \\
\tilde{u}_{\gamma+1} = \tilde{M}\tilde{v}_\gamma,
\end{cases} \quad \gamma \geq 1, \tag{3.12}$$

where $\{g_\gamma\}$ and $\{h_\gamma\}$ are real sequences in (0,1) such that $\frac{1}{2} \leq h_\gamma g_\gamma$. Let $F(M_1) \cap F(M_2) \neq \emptyset$ and $F(\tilde{M_1}) \cap F(\tilde{M_2}) \neq \emptyset$. Then for each $p^\dagger \in F(M_1) \cap F(M_2)$ and $\tilde{p}^\dagger \in F(\tilde{M_1}) \cap F(\tilde{M_2})$ with $\tilde{u}_\gamma \to \tilde{p}^\dagger$ as $\gamma \to +\infty$, we have

$$\rho(p^\dagger, \tilde{p}^\dagger) \leq \frac{7\epsilon}{1-k},$$

where ϵ is a fixed number.

Proof. From (1.5) and (3.12), we have

$$\begin{aligned}\rho(w_\gamma,\tilde{w}_\gamma) &= \rho((1-h_\gamma)u_\gamma \oplus h_\gamma M_2u_\gamma,(1-h_\gamma)\tilde{u}_\gamma \oplus h_\gamma \tilde{M}_2\tilde{u}_\gamma)\\ &\leq (1-h_\gamma)\rho(u_\gamma,\tilde{u}_\gamma)+h_\gamma\rho(M_2u_\gamma,\tilde{M}_2\tilde{u}_\gamma)\\ &\leq (1-h_\gamma)\rho(u_\gamma,\tilde{u}_\gamma)+h_\gamma\rho(M_2u_\gamma,M_2\tilde{u}_\gamma)+h_\gamma\rho(M_2\tilde{u}_\gamma,\tilde{M}_2\tilde{u}_\gamma)\\ &\leq (1-h_\gamma)\rho(u_\gamma,\tilde{u}_\gamma)+h_\gamma k\rho(u_\gamma,\tilde{u}_\gamma)+h_\gamma\Psi(d(u_\gamma,M_2u_\gamma)+h_\gamma\epsilon\\ &= (1-(1-k)h_\gamma)\rho(u_\gamma,\tilde{u}_\gamma)+h_\gamma\Psi(\rho(u_\gamma,M_2u_\gamma))+h_\gamma\epsilon. \end{aligned} \tag{3.13}$$

From (1.5) and (3.12), we have

$$\begin{aligned}\rho(v_\gamma,\tilde{v}_\gamma) &= \rho((1-g_\gamma)M_2u_\gamma \oplus g_\gamma M_1w_\gamma,(1-g_\gamma)\tilde{M}_2\tilde{u}_\gamma \oplus g_\gamma\tilde{M}_1\tilde{w}_\gamma)\\ &\leq (1-g_\gamma)\rho(M_2u_\gamma,\tilde{M}_2\tilde{u}_\gamma)+g_\gamma\rho(M_1w_\gamma,\tilde{M}_1\tilde{w}_\gamma) \qquad (3.14)\\ &\leq (1-g_\gamma)\rho(M_2u_\gamma,M_2\tilde{u}_\gamma)+(1-g_\gamma)\rho(M_2\tilde{u}_\gamma,\tilde{M}_2\tilde{u}_\gamma)\\ &\quad +g_\gamma\rho(M_1w_\gamma,M_1\tilde{w}_\gamma)+h_\gamma\rho(M_1\tilde{w}_\gamma,\tilde{M}_1\tilde{w}_\gamma)\\ &\leq (1-g_\gamma)k\rho(u_\gamma,\tilde{u}_\gamma)+(1-g_\gamma)\Psi(d(u_\gamma,M_2u_\gamma)+(1-g_\gamma)\epsilon\\ &\quad +g_\gamma k\rho(w_\gamma,\tilde{w}_\gamma)+g_\gamma\Psi(d(w_\gamma,M_1w_\gamma)+g_\gamma\epsilon\\ &\leq (1-g_\gamma)k\rho(u_\gamma,\tilde{u}_\gamma)+\Psi(d(u_\gamma,M_2u_\gamma)+\epsilon\\ &\quad +g_\gamma k\rho(w_\gamma,\tilde{w}_\gamma)+g_\gamma\Psi(d(w_\gamma,M_1w_\gamma)+g_\gamma\epsilon\\ &\leq (1-g_\gamma)k\rho(u_\gamma,\tilde{u}_\gamma)+\Psi(d(u_\gamma,M_2u_\gamma)+\epsilon\\ &\quad +g_\gamma k[(1-(1-k)h_\gamma)\rho(u_\gamma,\tilde{u}_\gamma)+h_\gamma\Psi(\rho(u_\gamma,M_2u_\gamma))+h_\gamma\epsilon]\\ &\quad +g_\gamma\Psi(d(w_\gamma,M_1w_\gamma)+g_\gamma\epsilon\\ &\leq k(1-(1-k)h_\gamma g_\gamma)\rho(u_\gamma,\tilde{u}_\gamma)+\Psi(d(u_\gamma,M_2u_\gamma))+\epsilon\\ &\quad +kg_\gamma h_\gamma\Psi(\rho(u_\gamma,M_2u_\gamma))+kg_\gamma h_\gamma\epsilon+g_\gamma\Psi(d(w_\gamma,M_1w_\gamma)+g_\gamma\epsilon. \end{aligned}$$

From (1.5), (3.12) and (3.15), we have

$$\begin{aligned}\rho(u_{\gamma+1},\tilde{u}_{\gamma+1}) &= d(M_1v_\gamma,\tilde{M}_1\tilde{v}_\gamma) \qquad (3.15)\\ &\leq \rho(M_1v_\gamma,M_1\tilde{v}_\gamma)+\rho(M_1\tilde{v}_\gamma,\tilde{M}_1\tilde{v}_\gamma)\\ &\leq k\rho(v_\gamma,\tilde{v}_\gamma)+\Psi(\rho(v_\gamma,M_1v_\gamma))+\epsilon\\ &\leq k^2(1-(1-k)h_\gamma g_\gamma)\rho(u_\gamma,\tilde{u}_\gamma)+k\Psi(d(u_\gamma,M_2u_\gamma))+k\epsilon\\ &\quad +k^2g_\gamma h_\gamma\Psi(\rho(u_\gamma,M_2u_\gamma))+k^2g_\gamma h_\gamma\epsilon+kg_\gamma\Psi(d(w_\gamma,M_1w_\gamma)\\ &\quad +kg_\gamma\epsilon+\Psi(\rho(v_\gamma,M_1v_\gamma))+\epsilon. \qquad (3.16)\end{aligned}$$

Since $0 \leq k < 1$ and $0 < h_\gamma, g_\gamma < 1$, (3.16) becomes

$$\begin{aligned}\rho(u_{\gamma+1},\tilde{u}_{\gamma+1}) &\leq (1-(1-k)h_\gamma g_\gamma)\rho(u_\gamma,\tilde{u}_\gamma)+\Psi(\rho(u_\gamma,M_2u_\gamma))\\ &\quad +g_\gamma h_\gamma\Psi(\rho(u_\gamma,M_2u_\gamma))+g_\gamma h_\gamma\epsilon+\Psi(\rho(w_\gamma,M_1w_\gamma)\\ &\quad +\Psi(\rho(v_\gamma,M_1v_\gamma))+3\epsilon. \end{aligned} \tag{3.17}$$

Since $\frac{1}{2} \leq h_\gamma g_\gamma$, $\forall\gamma \geq 1$, $1 \leq 2h_\gamma g_\gamma$, $\forall\gamma \geq 1$, (3.17) becomes

$$\begin{aligned}&\rho(u_{\gamma+1},\tilde{u}_{\gamma+1})\\ &\leq (1-(1-k)h_\gamma g_\gamma)\rho(u_\gamma,\tilde{u}_\gamma)+2h_\gamma g_\gamma\Psi(\rho(u_\gamma,M_2u_\gamma)) \qquad (3.18)\\ &\quad +g_\gamma h_\gamma\Psi(\rho(u_\gamma,M_2u_\gamma))+2h_\gamma g_\gamma\Psi(\rho(w_\gamma,M_1w_\gamma)\end{aligned}$$

$$+2h_\gamma g_\gamma \Psi(\rho(v_\gamma, M_1 v_\gamma)) + 7h_\gamma g_\gamma \epsilon$$
$$= (1-(1-k)h_\gamma g_\gamma)\rho(u_\gamma, \tilde{u}_\gamma) + (1-k)h_\gamma g_\gamma$$
$$\times \left[\frac{3\Psi(\rho(u_\gamma, M_2 u_\gamma)) + 2\Psi(\rho(w_\gamma, M_1 w_\gamma) + 2\Psi(\rho(v_\gamma, M_1 v_\gamma)) + 7\epsilon}{(1-k)}\right].$$

Therefore,

$$d_{\gamma+1} = (1-\varphi_\gamma)d_\gamma + \varphi_\gamma \phi_\gamma,$$

where

$$d_{\gamma+1} = \rho(u_{\gamma+1}, \tilde{u}_{\gamma+1}),$$
$$\varphi_\gamma = (1-k)h_\gamma g_\gamma \in (0,1),$$
$$\phi_\gamma = \left[\frac{3\Psi(\rho(u_\gamma, M_2 u_\gamma)) + 2\Psi(\rho(w_\gamma, M_1 w_\gamma) + 2\Psi(\rho(v_\gamma, M_1 v_\gamma)) + 7\epsilon}{(1-k)}\right] \geq 0.$$

Again, following a similar argument in Theorem 3.2, we can show that

$$\lim_{\gamma\to+\infty} \Psi(\rho(u_\gamma, M_2 u_\gamma)) = \lim_{\gamma\to+\infty} \Psi(\rho(w_\gamma, M_1 w_\gamma)) = \lim_{\gamma\to+\infty} \Psi(\rho(v_\gamma, M_1 v_\gamma)) = 0.$$

By the hypothesis $\tilde{u}_\gamma \to \tilde{p}^\dagger$ as $\gamma \to +\infty$ and Lemma 2.3, we obtain

$$\rho(p^\dagger, \tilde{p}^\dagger) \leq \frac{7\epsilon}{1-k}.$$

This completes the proof. □

4. Convergence analysis for two mappings satisfying condition (E)

In part of the article, we prove △-convergence and strong convergence results of our new method (1.5) for common fixed points of two mappings enriched with the condition (E). Throughout the remaining part of this article, let $(\mathcal{S}, \rho, \mathcal{V})$ be a complete uniformly convex hyperbolic space with a monotone modulus of convexity ν.

Theorem 4.1. *Let $\mathcal{G}$ be a nonempty closed convex subset of $\mathcal{S}$ and $M_1, M_2 : \mathcal{G} \to \mathcal{G}$ be two mappings enriched with the condition (E). If $\Omega = F(M_1) \cap F(M_2) \neq \emptyset$ and $\{u_\gamma\}$ is the sequence generated by* (3.12)*. Then $\{u_\gamma\}$ △-converges to an element in Ω.*

Proof. We will divide the proof into three steps as follows:

Step a. Firstly, we prove that $\lim_{\gamma\to+\infty} \rho(u_\gamma, p^\dagger)$ exists for each $p^\dagger \in F(M_1) \cap F(M_2)$. By Proposition 2.1, we know that M_1 and M_2 are quasi-nonexpansive mappings. Thus, for any $p^\dagger \in F(M_1) \cap F(M_2)$ and by (1.5), we obtain

$$\begin{aligned}\rho(w_\gamma, p^\dagger) &= \rho((1-h_\gamma)u_\gamma \oplus h_\gamma M_2 u_\gamma, p^\dagger) \\ &\leq (1-h_\gamma)\rho(u_\gamma, p^\dagger) + h_\gamma \rho(M_2 u_\gamma, p^\dagger) \\ &\leq (1-h_\gamma)\rho(u_\gamma, p^\dagger) + h_\gamma \rho(u_\gamma, p^\dagger) \\ &= \rho(u_\gamma, p^\dagger). \end{aligned} \tag{4.1}$$

Using (1.5) and (4.1), we have

$$\begin{aligned}\rho(v_\gamma, p^\dagger) &= \rho((1-h_\gamma)M_2u_\gamma \oplus h_\gamma M_1 w_\gamma, p^\dagger)\\ &\le (1-h_\gamma)\rho(M_2u_\gamma, p^\dagger) + h_\gamma\rho(M_1w_\gamma, p^\dagger)\\ &\le (1-h_\gamma)\rho(u_\gamma, p^\dagger) + h_\gamma\rho(w_\gamma, p^\dagger)\\ &\le (1-h_\gamma)\rho(u_\gamma, p^\dagger) + h_\gamma\rho(u_\gamma, p^\dagger)\\ &= \rho(u_\gamma, p^\dagger). \end{aligned}\tag{4.2}$$

By (1.5) and (4.2), we have

$$\begin{aligned}\rho(u_{\gamma+1}, p^\dagger) &= \rho(M_1v_\gamma, p^\dagger)\\ &\le \rho(v_\gamma, p^\dagger)\\ &\le \rho(u_\gamma, p^\dagger).\end{aligned}\tag{4.3}$$

This implies that the sequence $\{\rho(u_\gamma, p^\dagger)\}$ is non-increasing and bounded below. Thus $\lim\limits_{\gamma\to+\infty} d(u_\gamma, p^\dagger)$ exists for each $p^\dagger \in F(M_1)\cap F(M_2)$.

Step b. Next, we show that

$$\lim_{\gamma\to+\infty}\rho(u_\gamma, M_1u_\gamma) = \lim_{\gamma\to+\infty}\rho(u_\gamma, M_2u_\gamma) = 0. \tag{4.4}$$

From **Step a**, it is shown that for all $p^\dagger \in F(M_1)\cap F(M_2)$, $\lim\limits_{\gamma\to+\infty}\rho(u_\gamma, p^\dagger)$ exists. Let

$$\lim_{\gamma\to+\infty} d(u_\gamma, p^\dagger) = z \ge 0. \tag{4.5}$$

If $z = 0$, then we get

$$\begin{aligned}\rho(u_\gamma, M_1u_\gamma) &\le \rho(u_\gamma, p^\dagger) + \rho(M_1u_\gamma, p^\dagger)\\ &\le \rho(u_\gamma, p^\dagger) + \rho(u_\gamma, p^\dagger)\\ &= 2\rho(u_\gamma, p^\dagger) \to 0 \text{ as } \gamma \to +\infty.\end{aligned}$$

Also,

$$\begin{aligned}\rho(u_\gamma, M_2u_\gamma) &\le \rho(u_\gamma, p^\dagger) + \rho(M_2u_\gamma, p^\dagger)\\ &\le \rho(u_\gamma, p^\dagger) + \rho(u_\gamma, p^\dagger)\\ &= 2\rho(u_\gamma, p^\dagger) \to 0 \text{ as } \gamma \to +\infty.\end{aligned}$$

Hence $\lim\limits_{\gamma\to+\infty}\rho(u_\gamma, M_1u_\gamma) = 0$ and $\lim\limits_{\gamma\to+\infty}\rho(u_\gamma, M_2u_\gamma) = 0$.

Now, suppose that $z > 0$. By (4.1), (4.2) and (4.5), we have

$$\limsup_{\gamma\to+\infty}\rho(w_\gamma, p^\dagger) \le \limsup_{\gamma\to+\infty}\rho(u_\gamma, p^\dagger) = z \tag{4.6}$$

and

$$\limsup_{\gamma\to+\infty}\rho(v_\gamma, p^\dagger) \le \limsup_{\gamma\to+\infty}\rho(u_\gamma, p^\dagger) = z. \tag{4.7}$$

Since $p^\dagger \in F(M_1)\cap F(M_2) \ne \emptyset$, we know that M_1 and M_2 are quasi-nonexpansive mappings. Thus we have

$$\limsup_{\gamma\to+\infty}\rho(M_1w_\gamma, p^\dagger) \le \limsup_{\gamma\to+\infty}\rho(w_\gamma, p^\dagger) \le z \tag{4.8}$$

and

$$\limsup_{\gamma\to+\infty}\rho(M_2u_\gamma,p^\dagger)\leq\limsup_{\gamma\to+\infty}\rho(u_\gamma,p^\dagger)=z. \tag{4.9}$$

By (1.5), we have

$$\begin{aligned}\rho(u_{\gamma+1},p^\dagger)&=\rho(M_1v_\gamma,p^\dagger)\\&\leq\rho(v_\gamma,p^\dagger).\end{aligned}$$

Therefore,

$$z\leq\liminf_{\gamma\to+\infty}\rho(v_\gamma,p^\dagger). \tag{4.10}$$

By (4.7) and (4.10), we have

$$z=\lim_{\gamma\to+\infty}\rho(v_\gamma,p^\dagger). \tag{4.11}$$

From (1.5), we have

$$z=\lim_{\gamma\to+\infty}\rho(v_\gamma,p^\dagger)=\lim_{\gamma\to+\infty}\rho((1-g_\gamma)M_2u_\gamma+g_\gamma M1w_\gamma,p^\dagger). \tag{4.12}$$

From Lemma 2.2, (4.8) and (4.9) and (4.12), we obtain

$$\lim_{\gamma\to+\infty}\rho(M_2u_\gamma,M_1w_\gamma)=0. \tag{4.13}$$

From (1.5), (4.11) and (4.13), we have

$$\begin{aligned}\rho(v_\gamma,p^\dagger)&=((1-g_\gamma)M_2u_\gamma+g_\gamma M_1w_\gamma,p^\dagger)\\&\leq\rho(M_2u_\gamma,p^\dagger)+g_\gamma\rho(M_1w_\gamma,M_2u_\gamma),\end{aligned}$$

which gives

$$z\leq\liminf_{\gamma\to+\infty}\rho(M_2u_\gamma,p^\dagger). \tag{4.14}$$

Using (4.9) and (4.14), we have

$$\lim_{k\to+\infty}z=\rho(M_2u_\gamma,p^\dagger). \tag{4.15}$$

Also,

$$\begin{aligned}\rho(M_2u_\gamma,p^\dagger)&\leq\rho(M_2u_\gamma,M_1w_\gamma)+\rho(M_1w_\gamma,p^\dagger)\\&\leq\rho(M_2u_\gamma,M_1w_\gamma)+\rho(w_\gamma,p^\dagger),\end{aligned}$$

which implies that

$$z\leq\liminf_{\gamma\to+\infty}\rho(w_\gamma,p^\dagger). \tag{4.16}$$

From (4.6) and (4.16), we obtain

$$z=\lim_{\gamma\to+\infty}\rho(w_\gamma,p^\dagger). \tag{4.17}$$

Finally, by (1.5), we have

$$\lim_{\gamma\to+\infty} z = \rho(w_\gamma, p^\dagger) = \lim_{\gamma\to+\infty} \rho((1-h_\gamma)u_\gamma \oplus h_\gamma M_2 u_\gamma, p^\dagger). \tag{4.18}$$

Now, due to (4.5), (4.9), (4.18) and Lemma 2.2, we have

$$\lim_{\gamma\to+\infty} \rho(u_\gamma, M_2 u_\gamma) = 0. \tag{4.19}$$

On the other hand, by (1.5) and (4.19), we have

$$\rho(w_\gamma, u_\gamma) = \rho((1-h_\gamma)u_\gamma \oplus h_\gamma M_2 u_\gamma, u_\gamma) \le h_\gamma \rho(u_\gamma, M_2 u_\gamma) \to 0 \text{ as } \gamma \to +\infty, \tag{4.20}$$

and

$$\begin{aligned}\rho(w_\gamma, M_1 w_\gamma) &= \rho((1-h_\gamma)u_\gamma \oplus h_\gamma M_2 u_\gamma, M_1 u_\gamma) \\ &\le (1-h_\gamma)\rho(u_\gamma, M_1 w_\gamma) + h_\gamma \rho(M_2 u_\gamma, M_1 u_\gamma) \\ &\le (1-h_\gamma)[\rho(u_\gamma, M_2 u_\gamma) + \rho(M_2 u_\gamma, M_1 w_\gamma)] + h_\gamma \rho(M_2 u_\gamma, M_1 w_\gamma).\end{aligned} \tag{4.21}$$

Now, using (4.13) and (4.19), we have

$$\lim_{\gamma\to+\infty} \rho(w_\gamma, M_1 w_\gamma) = 0. \tag{4.22}$$

Since M_1 satisfies condition (E), we obtain

$$\begin{aligned}\rho(u_\gamma, M_1 u_\gamma) &\le d(u_\gamma, w_\gamma) + \rho(w_\gamma, M_1 u_k) \\ &\le \rho(u_\gamma, w_\gamma) + \mu\rho(w_\gamma, M_1 w_\gamma) + \rho(w_\gamma, u_\gamma) \\ &\le 2\rho(u_\gamma, w_\gamma) + \mu\rho(w_\gamma, M_1 w_\gamma).\end{aligned}$$

By (4.19), (4.20) and (4.22), we have

$$\lim_{\gamma\to+\infty} \rho(u_\gamma, M_1 u_\gamma) = 0. \tag{4.23}$$

Hence $\lim_{\gamma\to+\infty} \rho(u_\gamma, M_1 u_\gamma) = \lim_{\gamma\to+\infty} \rho(u_\gamma, M_2 u_\gamma) = 0$.

Step c. Lastly, we will establish that the sequence $\{u_\gamma\}$ is $\triangle$-convergent to an element in Ω. Since from Theorem 4.1 Step (a), the sequence $\{u_\gamma\}$ is bounded. It follows that $\{u_\gamma\}$ has a $\triangle$-convergent subsequence. It is left to show that there exists a unique $\triangle$-limit to every $\triangle$-convergent subsequence of $\{u_\gamma\}$. Proving by contradiction, let $\{u_{\gamma_r}\}$ and $\{u_{\gamma_s}\}$ be two subsequences of $\{u_\gamma\}$ such that $\{u_{\gamma_r}\}$ and $\{u_{\gamma_s}\}$ are $\triangle$-convergent to u and v, respectively. Again, from Theorem 4.1 Step (b), we know that $\{u_{\gamma_r}\}$ is bounded and $\lim_{r\to+\infty} d(M_1 u_{\gamma_r}, u_{\gamma_r}) = 0$. So we can assume that $u \in F(M_1) \cap F(M_2)$. It follows that

$$r_a(\{u_{\gamma_r}\}, M_1 u) = \limsup_{r\to+\infty} \rho(u_{\gamma_r}, M_1 u).$$

Since M_1 satisfies condition (E), for some $\mu \ge 1$, we have

$$\begin{aligned}r_a(\{u_{\gamma_r}\}, M_1 u) &= \limsup_{r\to+\infty} d(u_{\gamma_r}, M_1 u) \\ &\le \mu \limsup_{r\to+\infty} \rho(M_1 u_{\gamma_r}, u_{\gamma_r}) + \limsup_{r\to+\infty} \rho(u_{\gamma_r}, u)\end{aligned}$$

$$= r_a(\{u_{\gamma_r}\}, u).$$

Since the asymptotic centre of $\{u_{\gamma_r}\}$ has a unique element, $M_1 u = u$.

Similarly, we can obtain $M_1 v = v$. Following the same approach, we can show that $M_2 = u$ and $M_2 v = v$, respectively. The uniqueness of asymptotic of a sequence ensures that

$$\begin{aligned}\limsup_{\gamma\to+\infty} \rho(u_\gamma, u) &= \limsup_{r\to+\infty} \rho(u_{\gamma_r}, u) < \limsup_{r\to+\infty} \rho(u_{\gamma_r}, v)\\ &= \limsup_{\gamma\to+\infty} \rho(u_\gamma, v) = \limsup_{s\to+\infty} \rho(u_{\gamma_s}, v)\\ &< \limsup_{s\to\infty} \rho(u_{\gamma_s}, u) = \limsup_{\gamma\to+\infty} \rho(u_\gamma, u),\end{aligned}$$

which is a contradiction, except $u = v$. This completes the proof. □

Next, we establish the following strong convergence theorems.

Theorem 4.2. *Let $\mathcal{G}$ be a nonempty closed convex subset of $\mathcal{S}$ and $M_1, M_2 : \mathcal{G} \to \mathcal{G}$ be two mappings enriched with the condition (E). If $\Omega = F(M_1) \cap F(M_2) \neq \emptyset$ and $\{u_\gamma\}$ is the sequence generated by* (3.12). *Then $\{u_\gamma\}$ converges strongly to a common fixed point of M_1 and M_2 if and only if $\liminf_{\gamma\to+\infty} D(u_\gamma, F(M_1) \cap F(M_2)) = 0$, where $D(u_\gamma, F(M_1) \cap F(M_2)) = \inf\{\rho(u_\gamma, p^\dagger) : p^\dagger \in F(M_1) \cap F(M_2)\}$.*

Proof. Suppose that $\liminf_{\gamma\to+\infty} D(u_\gamma, F(M_1) \cap F(M_2)) = 0$. By Theorem 4.1 step (a), we have $\liminf_{\gamma\to+\infty} D(u_\gamma, F(M_1) \cap F(M_2))$ exists and so

$$\lim_{\gamma\to+\infty} \rho(u_\gamma, F(M_1) \cap F(M_2)) = 0. \tag{4.24}$$

From (4.24), a subsequence $\{u_{\gamma_r}\}$ of the sequence $\{u_\gamma\}$ exists such that $\rho(u_{\gamma_r}, t_r) \leq \frac{1}{2^r}$ for all $r \geq 1$, where $\{t_r\}$ is a sequence in $F(M_1) \cap F(M_2)$. By Theorem 4.1 Step (a), we obtain

$$\rho(u_{\gamma_{r+1}}, t_r) \leq \rho(u_{\gamma_r}, t_r) \leq \frac{1}{2^r}. \tag{4.25}$$

Using (4.25), we get

$$\rho(t_{r+1}, t_r) \leq \rho(t_{r+1}, u_{\gamma_{r+1}}) + \rho(u_{\gamma_{r+1}}, t_r) \leq \frac{1}{2^{r+1}} + \frac{1}{2^r} < \frac{1}{2^{r-1}}.$$

This implies that $\{t_\gamma\}$ is a Cauchy sequence in $\mathcal{G}$. We know that $F(M_1) \cap F(M_2)$ is closed and $\{u_\gamma\}$ converges to some $t \in F(M_1) \cap F(M_2)$. Now,

$$\rho(u_{\gamma_r}, t) \leq \rho(u_{\gamma_r}, t_\gamma) + \rho(t_\gamma, t).$$

Letting $\gamma \to +\infty$, we obtain that $\{u_{\gamma_r}\}$ converges strongly to t. By Theorem 4.1 Step (a), $\lim_{\gamma\to+\infty} \rho(u_\gamma, t)$ exists. Thus $\{u_\gamma\}$ converges strongly to t. □

Two mappings $M_1, M_2 : \mathcal{S} \to \mathcal{S}$ are said to satisfy the condition (A) [50] if there exists a nondecreasing function $\tau : [0, \infty) \to [0, \infty)$ satisfying $\tau(0) = 0$ and $\tau(r) > 0$ for all $r \in (0, \infty)$ such that $\rho(u, M_1 u) \geq \tau(D(u, F(M_1) \cap F(M_2)))$ or $\rho(u, M_2 u) \geq \tau(D(u, F(M_1) \cap F(M_2)))$ for all $u \in \mathcal{S}$, where $D(u, F(M_1) \cap F(M_2))$ stands for the distance of u from $F(M_1) \cap F(M_2)$.

Theorem 4.3. *Let $\mathcal{G}$ be a nonempty closed convex subset of $\mathcal{S}$ and $M_1, M_2 : \mathcal{G} \to \mathcal{G}$ be two mappings enriched with the condition (E). Let $\Omega = F(M_1) \cap F(M_2) \neq \emptyset$ and $\{u_\gamma\}$ be the sequence generated by (3.12). Suppose that M_1 and M_2 satisfy condition (A). Then $\{u_\gamma\}$ converges strongly to a common fixed point of M_1 and M_2.*

Proof. By Theorem 4.1 Step (b), it follows that

$$\liminf_{\gamma\to+\infty} \rho(M_1u_\gamma, u_\gamma) = \liminf_{\gamma\to+\infty} \rho(M_2u_\gamma, u_\gamma) = 0. \tag{4.26}$$

Since M_1 and M_2 fulfill condition (A), we get $\rho(M_1u_\gamma, u_\gamma) \geq \tau(D(u_\gamma, F(M_1) \cap F(M_2))$ or $\rho(M_2u_\gamma, u_\gamma) \geq \tau(D(u_\gamma, F(M_1) \cap F(M_2))$. From (4.26), we obtain

$$\liminf_{\gamma\to+\infty} \tau(D(u_\gamma, \rho(M_1u_\gamma, u_\gamma) \geq \tau(D(u_\gamma, F(M_1) \cap F(M_2)))) = 0$$

or

$$\liminf_{\gamma\to+\infty} \tau(D(u_\gamma, \rho(M_2u_\gamma, u_\gamma) \geq \tau(D(u_\gamma, F(M_1) \cap F(M_2)))) = 0.$$

Again, since the function $\tau : [0, +\infty) \to [0, +\infty)$ is nondecreasing such that $\varrho(0) = 0$ and $\tau(r) > 0$ for all $r \in (0, +\infty)$, we have

$$\liminf_{\gamma\to+\infty} D(u_\gamma, \rho(M_1u_\gamma, u_\gamma) \geq \tau(D(u_\gamma, F(M_1) \cap F(M_2))) = 0$$

or

$$\liminf_{\gamma\to+\infty} D(u_\gamma, \rho(M_2u_\gamma, u_\gamma) \geq \tau(D(u_\gamma, F(M_1) \cap F(M_2))) = 0.$$

Hence all the assumptions of Theorem 4.2 are fulfilled. Therefore, $\{u_\gamma\}$ converges strongly to a common fixed point of $F(M_1)$ and $F(M_2)$. □

Now, we present some nontrivial example with nonlinear convex structure.

Example 4.1. Let $\mathcal{S} = \{(u_1, u_2, u_3) \in \mathbb{R}^3 : u_1 > 0, u_2 > 0, u_3 > 0\}$. If $u = (u_1, u_2, u_3)$, $v = (v_1, v_2, v_3)$ and $w = (w_1, w_2, w_3) \in \mathcal{S}$ and $\delta, \xi, \mu \in [0, 1]$ such that $\delta + \xi + \mu = 1$, we define a mapping $\mathcal{V} : \mathcal{S} \times \mathcal{S} \times \mathcal{S} \times [0, 1] \times [0, 1] \times [0, 1] \to \mathcal{S}$ by

$$\mathcal{V}(u, v, w, \delta, \xi, \mu) = (\delta u_1 + \xi u_2 + \mu u_3, \delta v_1 + \xi v_2 + \mu v_3, \delta w_1 + \xi w_2 + \mu w_3).$$

Also, let $\rho : \mathcal{S} \times \mathcal{S} \to [0, +\infty)$ be defined by

$$\rho(u, v) = |u_1v_1 + u_2v_2 + u_3v_3|.$$

Then $(\mathcal{S}, \rho, \mathcal{V})$ is not a normed space, but it is a convex metric space [49].

Now, let $\mathcal{G} = [\frac{1}{4}, 3] \times [\frac{1}{4}, 3] \times [\frac{1}{4}, 3] \in \mathcal{S}$ and $M : \mathcal{G} \to \mathcal{G}$ be a mapping defined by

$$M(u_1, u_2) = \begin{cases} (1, 1, 1), & \text{if } u \neq (\frac{8}{5}, \frac{8}{5}, \frac{8}{5}), \\ \left(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}\right), & \text{if } u = (\frac{8}{5}, \frac{8}{5}, \frac{8}{5}). \end{cases}$$

Then M satisfies condition (E), but it does not satisfy condition (C).

Firstly, we show that M does not satisfy condition (C) for $u = (u_1, u_2, u_3) = (\frac{8}{5}, \frac{8}{5}, \frac{8}{5})$ and $v = (v_1, v_2, v_3) = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$.

$$\frac{1}{2}\rho(u, Mu) = \frac{1}{2}\left|\frac{16}{15} + \frac{16}{15} + \frac{16}{15}\right| = \frac{48}{30} \leq \frac{24}{15} = \left|\frac{8}{15} + \frac{8}{15} + \frac{8}{15}\right| = \rho(u, v).$$

But

$$\rho(Mu, Mv) = \left|\frac{2}{3} + \frac{2}{3} + \frac{2}{3}\right| = \frac{6}{2} > \frac{24}{15} = \rho(u, v). \tag{4.27}$$

Therefore, M does not satisfy condition (C).

Next, we show that M satisfies condition (E) for $\mu = 3$. We consider the following cases:

Case I. If $u = (u_1, u_2, u_3) \neq (\frac{8}{5}, \frac{8}{5}, \frac{8}{5})$ and $v = (v_1, v_2, v_3) = (\frac{8}{5}, \frac{8}{5}, \frac{8}{5})$, then we have

$$\begin{aligned}3\rho(u, Mu) + \rho(u, v) &= 3|u_1 + u_2 + u_3| + \left|\frac{8}{5}u_1 + \frac{8}{5}u_2 + \frac{8}{5}u_3\right| \\ &> \left|\frac{2}{3}u_1 + \frac{2}{3}u_2 + \frac{2}{3}u_3\right| = \rho(u, Mv).\end{aligned}$$

Case II. If $u = (u_1, u_2, u_3) = (\frac{8}{5}, \frac{8}{5}, \frac{8}{5})$ and $v = (v_1, v_2, v_3) \neq (\frac{8}{5}, \frac{8}{5}, \frac{8}{5})$, then we have

$$\begin{aligned}3\rho(u, Mu) + \rho(u, v) &= 3\left|\frac{16}{15} + \frac{16}{15} + \frac{16}{15}\right| + \left|\frac{8}{5}v_1 + \frac{8}{5}v_2 + \frac{8}{5}v_3\right| \\ &= \frac{144}{15} + \left|\frac{8}{5}v_1 + \frac{8}{5}v_2 + \frac{8}{5}v_3\right| \\ &> \frac{24}{5} = \left|\frac{8}{5} + \frac{8}{5} + \frac{8}{5}\right| = \rho(u, Mv).\end{aligned}$$

Case III. If $u = (u_1, u_2, u_3) = (\frac{8}{5}, \frac{8}{5}, \frac{8}{5})$ and $v = (v_1, v_2, v_3) = (\frac{8}{5}, \frac{8}{5}, \frac{8}{5})$, then we have

$$\begin{aligned}3\rho(u, Mu) + \rho(u, v) &= 3\left|\frac{16}{15} + \frac{16}{15} + \frac{16}{15}\right| + \rho(u, v) = \frac{144}{15} + \rho(u, v) \\ &> \left|\frac{16}{15} + \frac{16}{15} + \frac{16}{15}\right| = \frac{48}{15} = \rho(u, Mv).\end{aligned}$$

Case IV. If $u = (u_1, u_2, u_3) \neq (\frac{7}{5}, \frac{7}{5}, \frac{7}{5})$ and $v = (v_1, v_2, v_3) \neq (\frac{7}{5}, \frac{7}{5}, \frac{7}{5})$, then we have

$$\begin{aligned}3\rho(u, Mu) + \rho(u, v) &= 3|u_1 + u_2 + u_3| + |u_1v_1 + u_2v_2 + u_3v_3| \\ &> |u_1 + u_2 + u_3| = \rho(u, Mv).\end{aligned}$$

Therefore, M satisfies condition (E) for $\mu = 3$. The fixed point of M is $(1, 1, 1)$.

5. A numerical result

In this section, we compare the applicability and efficiency of the new iterative method (1.5) with some well known iterative processes in the current literature. We will present some nontrivial numerical examples which will be used to show that our new method converges faster to the common fixed point of two mappings enriched with condition (E) than several other iterative methods.

Example 5.1. Let $\mathcal{S} = \mathbb{R}^2$ and $\mathcal{G} = \{g = (g_1, g_2) : (g_1, g_2) \in [0,1] \times [0,1]\}$ be a subset of $\mathcal{S}$ with the taxi-cap metric

$$\rho((u_1, u_2), (v_1, v_2)) = |u_1 - v_1| + |u_2 - v_2|,$$

for all (u_1, u_2) and (v_1, v_2) in $\mathcal{G}$. Let $M_1, M_2 : \mathcal{G} \to \mathcal{G}$ be defined by

$$M_1(u_1, u_2) = \begin{cases} (0,0), & \text{if } (u_1, u_2) \in [0, \frac{1}{26}) \times [0, \frac{1}{6}), \\ \left(\frac{7u_1}{8}, \frac{7u_2}{8}\right), & \text{if } (u_1, u_2) \in [\frac{1}{26}, 1] \times [\frac{1}{6}, 1], \end{cases}$$

$$M_2(u_1, u_2) = \begin{cases} (0,0), & \text{if } (u_1, u_2) \in [0, \frac{1}{26}) \times [0, \frac{1}{6}), \\ \left(\frac{4u_1}{5}, \frac{4u_2}{5}\right), & \text{if } (u_1, u_2) \in [\frac{1}{26}, 1] \times [\frac{1}{6}, 1]. \end{cases}$$

Now, if $u = (\frac{1}{26}, \frac{1}{6})$ and $v = (\frac{1}{50}, \frac{1}{23})$, then we have $M_1 u = (\frac{7}{208}, \frac{7}{48})$ and $M_1 v = (0,0)$. Thus

$$\begin{aligned} \frac{1}{2}\rho(u, M_1 u) &= \frac{1}{2}\left(\left|\frac{1}{26} - \frac{7}{208}\right| + \left|\frac{1}{6} - \frac{7}{48}\right|\right) = \frac{1}{78} \\ &< \frac{6353}{44850} = \left|\frac{1}{26} - \frac{1}{50}\right| + \left|\frac{1}{6} - \frac{1}{23}\right| = \rho(u, v), \end{aligned}$$

but

$$\rho(M_1 u, M_1 v) = \frac{1792}{9984} > \frac{6353}{44850} = \rho(u, v). \tag{5.1}$$

Thus M_1 does not satisfy condition (C).

We now show that M_1 satisfies condition (E) for different considered cases as follows:

Case 1. When $u = (u_1, u_2)$, $v = (v_1, v_2) \in [0, \frac{1}{26}) \times [0, \frac{1}{6})$, we have

$$\begin{aligned} \rho(u, M_1 v) &= |u_1| + |u_2| \leq 8(|u_1| + |u_2|) \\ &\leq 8(|u_1| + |u_2|) + |u_1 - v_1| + |u_2 - v_2| \\ &= 8\rho(u, M_1 u) + \rho(u, v). \end{aligned}$$

Case 2. When $u = (u_1, u_2)$, $v = (v_1, v_2) \in [\frac{1}{26}, 1] \times [\frac{1}{6}, 1]$, we have

$$\begin{aligned} \rho(u, M_1 v) &= |u - M_1 u| + |M_1 u - M_1 v| \\ &= |u - M_1 u| + \frac{7}{8}(|u_1 - v_1| + |u_2 - v_2|) \\ &\leq |u - M_1 u| + (|u_1 - v_1| + |u_2 - v_2|) \\ &\leq 8|u - M_1 u| + (|u_1 - v_1| + |u_2 - v_2|) \\ &= 8\rho(u, M_1 u) + \rho(u, v). \end{aligned}$$

Case 3. When $u = (u_1, u_2) \in [0, \frac{1}{26}) \times [0, \frac{1}{6})$ and $v = (v_1, v_2) \in [\frac{1}{26}, 1] \times [\frac{1}{6}, 1]$, we have

$$\rho(u, M_1 v) = \left|u_1 - \frac{7v_1}{8}\right| + \left|u_2 - \frac{7v_2}{8}\right|$$

$$
\begin{aligned}
&= \left|\frac{8u_1 - 7v_1}{8}\right| + \left|\frac{8u_2 - 7v_2}{8}\right| \\
&= \left|\frac{u_1 + 7u_1 - 7v_1}{8}\right| + \left|\frac{u_2 + 7u_2 - 7v_2}{8}\right| \\
&\leq \frac{1}{8}(|u_1| + |u_2|) + \frac{7}{8}(|u_1 - v_1| + |u_2 - v_2|) \\
&\leq |u_1| + |u_2| + (|u_1 - v_1| + |u_2 - v_2|) \\
&\leq 8(|u_1| + |u_2|) + (|u_1 - v_1| + |u_2 - v_2|) \\
&= 8\rho(u, M_1 u) + \rho(u, v).
\end{aligned}
$$

Case 4. When $u = (u_1, u_2) \in [\frac{1}{26}, 1] \times [\frac{1}{6}, 1]$ and $v = (v_1, v_2) \in [0, \frac{1}{26}) \times [0, \frac{1}{6})$, we have

$$
\begin{aligned}
\rho(u, M_1 v) = |u_1| + |u_2| &= 8\left(\left|\frac{u_1}{8}\right| + \left|\frac{u_2}{8}\right|\right) \\
&= 8\left(\left|u_1 - \frac{7u_1}{8}\right| + \left|u_2 - \frac{7u_2}{8}\right|\right) \\
&\leq 8\left(\left|u_1 - \frac{7u_1}{8}\right| + \left|u_2 - \frac{7u_2}{8}\right|\right) + |u_1 - v_1| + |u_2 - v_2| \\
&= 8\rho(u, M_1 u) + \rho(u, v).
\end{aligned}
$$

From the above cases, it is clear that M_1 satisfies (1.2) with $\mu = 8$. Hence, M_1 is a mapping enriched with the condition (E) for $\mu = 8$. The fixed point of M_1 is $(0, 0)$.

Following a similar approach, it can be proved that M_2 is a mapping enriched with condition (E) for $\mu = 5$, but it does not satisfy condition (C) for $u = (\frac{1}{26}, \frac{1}{6})$ and $v = (\frac{1}{50}, \frac{1}{23})$. The fixed point of M_2 is $(0, 0)$.

Observe that $F(M_1) \cap F(M_2) = \{(0, 0)\}$. Let $h_\gamma = h_\gamma = t_\gamma = \frac{1}{2}$, for all $\gamma \geq 1$ be the control parameters and $(0.3, 0.7)$ be the starting point. We use MATLAB R2015a to obtain Table 3, Table 4, Figure 3 and Figure 4. Clearly, the mixed-type Picard-S iterative converges faster to the common fixed point (0,0) than Man, Ishikawa, S, Noor, Abbas and Picard-Man iterative schemes.

Table 3. Convergence comparison of different iterative algorithms for contraction-like mappings.

u_γ	Mann	Ishikawa	S	Mixed-Type Picard-S
u_1	(0.300000, 0.700000)	(0.300000, 0.700000)	(0.300000, 0.700000)	(0.300000, 0.700000)
u_2	(0.281250, 0.656250)	(0.273047, 0.637109)	(0.254297, 0.593359)	(0.195000, 0.455000)
u_3	(0.263672, 0.615234)	(0.248515, 0.579869)	(0.215556, 0.502965)	(0.126750, 0.295750)
u_4	(0.247192, 0.576782)	(0.226188, 0.527771)	(0.182718, 0.426341)	(0.082388, 0.192237)
u_5	(0.231743, 0.540733)	(0.205866, 0.480354)	(0.154882, 0.361391)	(0.053552, 0.124954)
u_6	(0.217259, 0.506938)	(0.187370, 0.437198)	(0.131287, 0.306335)	(0.034809, 0.081220)
u_7	(0.203680, 0.475254)	(0.170536, 0.397918)	(0.111286, 0.259667)	(0.022626, 0.052793)
u_8	(0.190950, 0.445551)	(0.155215, 0.362168)	(0.094332, 0.220108)	(0.014707, 0.034316)
u_9	(0.179016, 0.417704)	(0.141270, 0.329629)	(0.079961, 0.186576)	(0.009559, 0.022305)
u_{10}	(0.167827, 0.391597)	(0.128577, 0.300014)	(0.067780, 0.158152)	(0.006214, 0.014498)
u_{11}	(0.157338, 0.367122)	(0.117026, 0.273060)	(0.057454, 0.134059)	(0.004039, 0.009424)
u_{12}	(0.147505, 0.344177)	(0.106512, 0.248527)	(0.048701, 0.113636)	(0.002625, 0.006126)
u_{13}	(0.138285, 0.322666)	(0.096942, 0.226198)	(0.041282, 0.096324)	(0.001706, 0.003982)
u_{14}	(0.129643, 0.302499)	(0.088233, 0.205876)	(0.034993, 0.081650)	(0.001109, 0.002588)
u_{15}	(0.121540, 0.283593)	(0.080305, 0.187379)	(0.029662, 0.069211)	(0.000721, 0.001682)
u_{16}	(0.113944, 0.265869)	(0.073090, 0.170544)	(0.025143, 0.058667)	(0.000469, 0.001093)

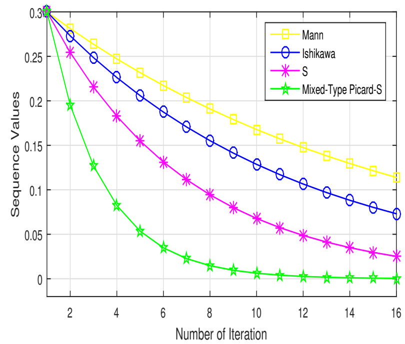

Figure 3. Graph corresponding to Table 3.

Table 4. Convergence comparison of different iterative algorithms for contraction-like mappings.

u_γ	Noor	Abbas	Picard-Man	Mixed-Type Picard-S
u_1	(0.300000, 0.700000)	(0.300000, 0.700000)	(0.300000, 0.700000)	(0.300000, 0.700000)
u_2	(0.269458, 0.628735)	(0.234302, 0.546704)	(0.246094, 0.574219)	(0.195000, 0.455000)
u_3	(0.242025, 0.564726)	(0.182991, 0.426979)	(0.201874, 0.471039)	(0.126750, 0.295750)
u_4	(0.217386, 0.507233)	(0.142917, 0.333473)	(0.165600, 0.386399)	(0.082388, 0.192237)
u_5	(0.195254, 0.455593)	(0.111619, 0.260445)	(0.135843, 0.316968)	(0.053552, 0.124954)
u_6	(0.175376, 0.409211)	(0.087175, 0.203409)	(0.111434, 0.260013)	(0.034809, 0.081220)
u_7	(0.157522, 0.367551)	(0.068084, 0.158863)	(0.091411, 0.213292)	(0.022626, 0.052793)
u_8	(0.141485, 0.330131)	(0.053174, 0.124073)	(0.074985, 0.174966)	(0.014707, 0.034316)
u_9	(0.127081, 0.296522)	(0.041529, 0.096902)	(0.061511, 0.143527)	(0.009559, 0.022305)
u_{10}	(0.114143, 0.266334)	(0.032435, 0.075681)	(0.050459, 0.117737)	(0.006214, 0.014498)
u_{11}	(0.102523, 0.239219)	(0.025332, 0.059107)	(0.041392, 0.096581)	(0.004039, 0.009424)
u_{12}	(0.092085, 0.214865)	(0.019784, 0.046163)	(0.033954, 0.079227)	(0.002625, 0.006126)
u_{13}	(0.082710, 0.192991)	(0.015452, 0.036054)	(0.027853, 0.064991)	(0.001706, 0.003982)
u_{14}	(0.074290, 0.173343)	(0.012068, 0.028158)	(0.022848, 0.053313)	(0.001109, 0.002588)
u_{15}	(0.066727, 0.155695)	(0.009425, 0.021992)	(0.018743, 0.043733)	(0.000721, 0.001682)
u_{16}	(0.059933, 0.139845)	(0.007361, 0.017176)	(0.015375, 0.035875)	(0.000469, 0.001093)

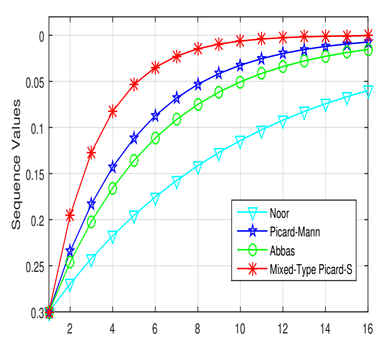

Figure 4. Graph corresponding to Table 4.

6. Conclusion

(i) In this article, we have introduced the mixed-type Picard-S iterative method (1.5) in hyperbolic spaces.

(ii) We have proved that our new iterative algorithm converges to the common fixed points of two contractive-like mappings. The analytical convergence results are supported with numerical examples. These examples are used to show that our new method converges faster than many existing iterative methods.

(iii) We have initiated and studied new notions of data dependence and weak w^2-stability results of iterative algorithm with two mappings.

(iv) Several strong and $\triangle$-convergence theorems have proved for common fixed points of mappings enriched with condition (E).

(v) We provided several novel and nontrivial examples of mappings enriched with condition (E). Further, we tested the competence of new iterative method with several existing methods for common fixed pints of mappings satisfying condition (E).

(vi) Our results are also valid in CAT(0) and linear spaces.

Declarations

Availablity of data and materials

Not applicable.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

The authors equally conceived of the study, participated in its design and coordination, drafted the manuscript, participated in the sequence alignment, and read and approved the final manuscript.

Acknowledgements

We would like to express our sincere gratitude to the anonymous referee for his/her helpful comments that will help to improve the quality of the manuscript.

References

[1] M. Abbas and T. Nazir, *A new faster iteration process applied to constrained minimization and feasibility problems*, Mat. Vesnik, 2014, 66(2), 223–234.

[2] R. P. Agarwal, D. O'Regan, D. R. Sahu, *Iterative construction of fixed points of nearly asymptotically nonexpansive mappings*, J. Nonlinear Convex Anal., 2007, 8(1), 61–79.

[3] V. Berinde, *On the approximation of fixed points of weak contractive mappings*, Carpath. J. Math., 2003, 19, 7–22.

[4] V. Berinde, *Approximating fixed points of weak contractions using the Picard iteration*, Nonlinear Anal. Forum, 2004, 9, 43–53.

[5] V. Berinde, *Picard iteration converges faster than Mann iteration for a class of quasicontractive operators*, Fixed Point Theory Appl., 2004, 2, 97–105.

[6] T. Cardinali and P. Rubbioni, *A generalization of the Caristi fixed point theorem in metric spaces*, Fixed Point Theory, 2010, 11(1), 3–10.

[7] S. Chang, G. Wanga, L. Wanga, Y. K. Tang and Z. L. Mab, $\triangle$-*Convergence theorems for multivalued nonexpansive*, Appl. Math. Comput., 2014, 249, 535–540.

[8] G. Das and J. P. Debata, *Fixed points of quasinonexpansive mappings*, Indian J. Pure Appl. Math., 1986, 17, 1263–1269.

[9] I. M. Esuabana, U. A. Abasiekwere, J. A. Ugboh and Z. Lipcsey, *Equivalent construction of ordinary differential equations from impulsive system*, Acad. J. Appl. Math. Sci., 2018, 4(8), 77–89.

[10] I. M. Esuabana and J. A. Ugboh, *Marching method: A new numerical method for finding roots of algebraic and transcendental equations*, Am. J. Comput. Appl. Math., 2019, 9(1), 6–11.

[11] I. M. Esuabana and J. A. Ugboh, *Survey of impulsive differential equations with continuous delay*, Int. J. Math. Trends Tech., 2018, 60(1), 22–28.

[12] M. Gabeleh, P. R. Patle and M. De La Sen, *Noncyclic φ-contractions in hyperbolic uniformly convex metric spaces*, J. Nonlinear Var. Anal., 2023, 7, 251–265.

[13] C. Garodia and I. Uddin, *New iterative method for solving split feasibility problem*, J. Appl. Anal. Comput., 2020, 10(3), 986–1004. DOI: 10.11948/20190179.

[14] K. Goebel and W. A. Kirk, *Iteration processes for nonexpansive mappings*, in: Topological Methods in Nonlinear Functional Analysis, S. P. Singh, S. Thomeier and B. Watson (eds.), Contemp. Math. Am. Math. Soc. AMS, Providence, RI., 1983, 21, 115–123.

[15] K. Goebel and S. Reich, *Uniform Convexity, Hyperbolic Geometry, and Nonexpansive Mappings*, Marcel Dekker, New York. 1984.

[16] F. Gürsoy and V. Karakaya, *A Picard-S hybrid type iteration method for solving a differential equation with retarded argument*, arXiv preprint, arXiv:1403.2546.

[17] F. Güsoy, *A Picard-S iterative scheme for approximating fixed point of weak-contraction mappings*, Filomat, 2016, 30(10), 2829–2845. DOI: 10.2298/FIL1610829G.

[18] A. M. Harder, *Fixed Point Theory and Stability Results for Fixed Point Iteration Procedures*, Ph.D. thesis, University of Missouri-Rolla, Missouri, 1987.

[19] A. M. Harder and T. L. Hicks, *A stable iteration procedure for nonexpansive mappings*, Math. Japan, 1988, 33(5), 687–692.

[20] M. Imdad and S. Dashputre, *Fixed point approximation of Picard normal S-iteration process for generalized nonexpansive mappings in hyperbolic spaces*, Math. Sci., 2016, 10, 131–138. DOI: 10.1007/s40096-016-0187-8.

[21] C. O. Imoru and M. O. Olantiwo, *On the stability of Picard and Mann iteration processes*, Carpath. J. Math., 2003, 19(2), 155–160.

[22] S. Ishikawa, *Fixed points and iteration of a nonexpansive mapping in a Banach space*, Proc. Am. Math. Soc., 1976, 59(1), 65–71.

[23] M. Jubair, F. Ali and J. Ali, *Convergence and stability of an iteration process and solution of a fractional differential equation*, J. Inequal. Appl., 2021, 2021, Paper No. 144. DOI: 10.1186/s13660-021-02677-w.

[24] A. R. Khan, H. Fukhar-ud-din, M. A. Khan, *An implicit algorithm for two finite families of nonexpansive maps in hyperbolic spaces*, Fixed Point Theory Appl., 2012, 2012, Paper No. 54.

[25] H. S. Khan, *A Picard-Mann hybrid iterative process*, Fixed Point Theory Appl., 2013, 2013, Paper No. 69.

[26] S. H. Khan and J. K. Kim, *Common fixed points of two nonexpansive mappings by a modified faster iteration scheme*, Bull. Korean Math. Soc., 2010, 47, 973–985.

[27] W. A. Kirk and B. Panyanak, *A concept of convergence in geodesic spaces*, Nonlinear Anal., 2008, 68(12), 3689–3696.

[28] U. Kohlenbach, *Some logical metatheorems with applications in functional analysis*, Trans. Am. Math. Soc., 2005, 357(1), 89–128.

[29] L. Leuştean, *A quadratic rate of asymptotic regularity for CAT(0) space*, J. Math. Anal. Appl., 2007, 25(1), 386–399.

[30] Z. Lipcsey, I. M. Esuabana, J. A. Ugboh and I. O. Isaac, *Integral representation of functions of bounded variation*, J. Math., 2019, 2019, Article ID 1065946. DOI: 10.1155/2019/1065946.

[31] W. R. Mann, *Mean value methods in iteration*, Proc. Am. Math. Soc., 1953, 4, 506–510.

[32] M. A. Noor, *New approximation schemes for general variational inequalities*, J. Math. Anal. Appl., 2000, 251(1), 217–229.

[33] A. E. Ofem, J. A. Abuchu, R. George, G. C. Ugwunnadi and O. K. Narain, *Some new results on convergence, weak w2-stability and data dependence of two multivalued almost contractive mappings in hyperbolic spaces*, Math., 2022, 10(20), Paper No. 3720.

[34] A. E. Ofem, J. A. Abuchu, G. C. Ugwunnadi, H. Işik and O. K. Narain, *On a four-step iterative algorithm and its application to delay integral equations in hyperbolic spaces*, Rend. Circ. Mat. Palermo Ser. 2 (in press). DOI: 10.1007/s12215-023-00908-1.

[35] A. E. Ofem and D. I. Igbokwe, *A new faster four step iterative algorithm for Suzuki generalized nonexpansive mappings with an application*, Adv. Theory Nonlinear Anal. Appl., 2021, 5(3), 482–506. DOI: 10.31197/atnaa.869046.a.

[36] A. E. Ofem, H. Işik, F. Ali and J. Ahmad, *A new iterative approximation scheme for Reich-Suzuki type nonexpansive operators with an application*, J. Inequal. Appl., 2022, 2022, Paper No. 28. DOI: 101186/s13660-022-02762-8.

[37] A. E. Ofem, H. Isik, G. C. Ugwunnadi, R. George and O. K. Narain, *Approximating the solution of a nonlinear delay integral equation by an efficient iterative algorithm in hyperbolic spaces*, AIMS Math., 2023, 8(7), 14919–14950.

[38] A. E. Ofem, U. E. Udofia and D. I. Igbokwe, *A robust iterative approach for solving nonlinear Volterra delay integro-differential equations*, Ural Math. J., 2021, 7(2), 59–85.

[39] A. E. Ofem, G. C. Ugwunnadi, O. K. Narain and J. K. Kim, *Approximating common fixed point of three multivalued mappings satisfying condition* (E) *in hyperbolic spaces*, Nonlinear Funct. Anal. Appl., 2023, 28(3), 623–646.

[40] M. O. Osilike, *Stability results for the Ishikawa fixed point iteration procedure*, Indian J. Pure Appl. Math., 1995, 26(10), 937–945.

[41] M. O. Osilike, *A stable iteration procedure for quasi-contractive maps*, Indian J. Pure Appl. Math., 1996, 27(1), 25–34.

[42] M. O. Osilike, *Stability of the Ishikawa iteration method for quasi-contractive maps*, Indian J. Pure Appl. Math., 1997, 28(9), 1251–265.

[43] M. O. Osilike, *Stability of the Mann and Ishikawa iteration procedures for* ϕ*-strong pseudocontractions and nonlinear equations of the* ϕ*-strongly accretive type*, J. Math. Anal. Appl., 1998, 227, 319–334.

[44] M. O. Osilike, *A note on the stability of iteration procedures for strong pseudocontractions and strongly accretive type equations*, J. Math. Anal. Appl., 2000, 250(2), 726–730.

[45] A. M. Ostrowski, *The round-off stability of iterations*, Z. Angew. Math. Mech., 1967, 47(2), 77–81.

[46] R. Pant and R. Pandey, *Existence and convergence results for a class of nonexpansive type mappings in hyperbolic spaces*, Appl. Gen. Topol., 2019, 20(1), 281–295. DOI: 10.4995/agt.2019.11057.

[47] S. Reich and I. Shafrir, *Nonexpansive iterations in hyperbolic spaces*, Nonlinear Anal., 1990, 15, 537–558.

[48] B. E. Rhoades, *Fixed point theorems and stability results for fixed point iteration procedures*, Indian J. Pure Appl. Math., 1990, 21(1), 1–9.

[49] G. S. Saluja and H. K. Nashine, *Convergence of an implicit iteration process for a finite family of asymptotically quasi-nonexpansive mappings in convex metric spaces*, Opuscula Math., 2010, 30(3), 331–340.

[50] H. F. Senter and W. G. Dotson, *Approximating fixed points of nonexpansive mappings*, Proc. Am. Math. Soc., 1974, 44, 375–380.

[51] S. M. Soltuz and T. Grosan, *Data dependence for Ishikawa iteration when dealing with contractive like operators*, Fixed Point Theory Appl., 2008, 2008, Article ID 242916.

[52] T. Suzuki, *Fixed point theorems and convergence theorems for some generalized nonexpansive mappings*, J. Math. Anal. Appl., 2008, 340, 1088–590.

[53] B. S. Thakurr, D. Thakur and M. Postolache, *A new iterative scheme for numerical reckoning of fixed points of Suzuki's generalized nonexpansive mappings*, Appl. Math. Comput., 2016, 275, 1088–1095.

[54] I. Timis, *On the weak stability of Picard iteration for some contractive type mappings*, Ann. Univ. Craiova, Math. Comput. Sci. Ser., 2010, 37(2), 106–114.

[55] K. Ullah, J. Ahmad, M. De la Sen and M. N. Khan, *Approximating fixed points of Reich–Suzuki type nonexpansive mappings in hyperbolic spaces*, J. Math., 2020, 2020, Article ID 2169652. DOI: 10.1155/2020/2169652.