

EXISTENCE FOR NONLINEAR FRACTIONAL EVOLUTIONARY EQUATIONS INVOLVING ψ -CAPUTO FRACTIONAL DERIVATIVE*

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Abstract The aim of this paper to deal with a new class of fractional evolutionary equations involving ψ -Caputo fractional derivative and nonlinear weakly continuous operators. Exploiting the Rothe method and using a surjectivity result for weakly continuous operators, the solvability for the problem is established. The result is applied to a quasistatic frictional contact problem.

Keywords Fractional evolutionary equation, ψ -Caputo fractional derivative, weakly continuous operator, quasistatic frictional contact problem.

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1. Introduction

We observe that recently there are many models in mechanics, engineering, chemistry and fluid flow which are successfully modeled by fractional differential equations. See [14, 16, 17, 21, 22, 29, 41] for instance. In [14], Han, Migórski and Zeng studied a quasistatic frictionless contact problem for a viscoelastic body in which the constitutive equation is modeled with the fractional Kelvin-Voigt law and the contact condition is described by the Clarke subdifferential of a nonconvex and nonsmooth functional. The variational formulation of this problem is provided in the form of a fractional hemivariational inequality. They transferred this inequality to an Volterra inclusion and apply the Rothe method to prove the existence of solutions to the inclusion. In [41], Zeng and Migórski generalized the problem in [14] and investigated a class of elliptic hemivariational inequalities involving the time-fractional order integral operator. Exploiting the Rothe method and using the surjectivity of multivalued pseudomonotone operators, a result on existence of solution to the problem is established. Then, this abstract result is applied to provide a theorem on the weak solvability of a fractional viscoelastic contact problem involving

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the fractional Kelvin-Voigt law. In [39], Zeng dealt with a new class of fractional evolutionary equations involving the time-fractional order integral and nonlinear weakly continuous operators. Exploiting the Rothe method and using a surjectivity result for weakly continuous operators, the solvability for the problem is established. The result is applied to prove existence of solutions to time-fractional nonstationary incompressible Navier-Stokes equation and Navier-Stokes-Voigt equation.

Very recently, many scholars pay attention to the generalization for fractional derivative, especially the ψ -Caputo fractional derivative (see [10, 23–25, 31]). In [13], Han, Li and Zeng introduced fractional viscoelastic constitutive equations modeled by fractional Kelvin-Voigt law with ψ -Caputo derivative. Motivated by above works, we wish to investigate in this paper the existence of solutions to the Cauchy problem for a class of time-fractional evolutionary equations involving ψ -Caputo fractional derivative and nonlinear weakly continuous operators.

Consider a separable reflexive Banach space V with its dual space V^* and $0 < T < \infty$. For $\alpha \in (0, 1)$ and $p > \frac{1}{\alpha}$, let

$$\mathcal{V} = L^p(0, T; V), \quad \mathcal{V}^* = L^{\frac{p}{p-1}}(0, T; V^*) \quad \text{and} \quad \mathcal{W} = \{v \in \mathcal{V} \mid {}^c D_{a+}^{\alpha, \psi} v \in \mathcal{V}\}.$$

Given nonlinear operators $A, B: V \rightarrow V^*$ and elements $f \in \mathcal{V}^*, u_0 \in V$, consider the Cauchy problem for fractional evolutionary equation with the ψ -Caputo fractional derivative.

Problem 1.1. Find $u \in \mathcal{W}$ such that

$$\begin{cases} B({}^c D_{a+}^{\alpha, \psi} u(t)) + Au(t) = f(t) & \text{a.e. } t \in (0, T), \\ u(0) = u_0. \end{cases}$$

The main novelties of the paper are following. First, we consider here the ψ -fractional calculus of order α . If we restrict the fractional derivative to $\psi(t) = t$, then the problems reduce to the classical Caputo fractional problem, which has been studied in [14, 39, 41]. If we restrict the fractional derivative and kernel function to $\alpha = 1$ and $\psi(t) = t$, respectively, then the problems reduce to the first-order evolutionary equations, which can be found in [38]. Second, instead of linearity, monotonicity and pseudomonotonicity widely used in the literature, we deal with time-fractional evolutionary equations with nonlinear weakly continuous operators. We emphasize that such kinds of time-fractional evolutionary equations with weakly continuous operators have not been studied so far which are the generalization of the equations in [13, 36–40]. Third, we consider less restrictive hypotheses for the data of Problem 1.1 than in other contributions. For instance, it is worth to pointing out that the coercivity hypothesis on the operator A can be moved. Fourth, we exploit the Rothe method with a sufficiently small time step to establish existence of solutions to time-fractional evolutionary equations which allows to omit smallness conditions which are met in some earlier contributions on evolutionary equations. Fifth, we provide new applications of the abstract existence result to study the quasistatic frictional contact problem.

In the rest of this paper, some preliminary materials are recalled in Section 2. In Section 3, we show the existence of solutions to the fractional evolutionary equation, which is based on the Rothe method combined with a surjectivity result for stationary equations involving weakly continuous operator. In Section 4, we deliver an existence result for a class of impulsive evolutionary equations by applying the

previous result in Section 3. In Section 5 we deliver a result on weak solvability for a quasistatic frictional contact problem.

2. Preliminaries

Let $(X, \|\cdot\|_X)$ be a Banach space. We denote by X^* its dual space and by $\langle \cdot, \cdot \rangle_X$ the duality pairing between X^* and X . For $0 < T < \infty$, $1 \leq p < \infty$, let $L^p(0, T; X)$ be the space of p -th integrable functions from $(0, T)$ into X , and $C([0, T]; X)$ be the space of all continuous functions from $[0, T]$ into X , endowed with usual norms.

At first, we introduce some notations, definitions, lemmas and results of ψ -Caputo fractional derivative (see [25, 31]). Let $\alpha > 0$, $h : [a, b] \rightarrow \mathbb{R}$ and $\psi \in C^n[a, b]$ be an increasing differentiable function such that $\psi'(t) \neq 0$ for all $t \in [a, b]$.

Definition 2.1. The left-sided ψ -Riemann-Liouville fractional of order α of a function h is given by

$$I_{a+}^{\alpha, \psi} h(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \psi'(s) (\psi(t) - \psi(s))^{\alpha-1} h(s) ds,$$

where $\Gamma(\cdot)$ is a gamma function.

Definition 2.2. Let $n-1 < \alpha < n$. The left-sided ψ -Riemann-Liouville fractional derivative of order α of a function h is defined by

$$D_{a+}^{\alpha, \psi} h(t) = \left[\frac{1}{\psi'(t)} \frac{d}{dt} \right]^n I_{a+}^{n-\alpha, \psi} h(t).$$

Definition 2.3. Let $n-1 < \alpha < n$ and $h \in C^{n-1}[a, b]$. The left-sided ψ -Caputo fractional derivative of function h of order α is determined as

$${}^c D_{a+}^{\alpha, \psi} h(t) = D_{a+}^{\alpha, \psi} \left[h(t) - \sum_{k=0}^{n-1} \frac{h_{\psi}^{[k]}(a)}{k!} (\psi(t) - \psi(a))^k \right],$$

where $h_{\psi}^{[k]}(t) = \left[\frac{1}{\psi'(t)} \frac{d}{dt} \right]^k h(t)$ and $n = [\alpha] + 1$ for $\alpha \notin \mathbb{N}$, $n = \alpha$ for $\alpha \in \mathbb{N}$. Further, if $h \in C^n[a, b]$ and $\alpha \notin \mathbb{N}$, then

$$\begin{aligned} {}^c D_{a+}^{\alpha, \psi} h(t) &= I_{a+}^{n-\alpha, \psi} \left[\frac{1}{\psi'(t)} \frac{d}{dt} \right]^n h(t) \\ &= \frac{1}{\Gamma(n-\alpha)} \int_a^t \psi'(s) (\psi(t) - \psi(s))^{n-\alpha-1} h_{\psi}^{[n]}(s) ds. \end{aligned}$$

Thus, if $\alpha = n \in \mathbb{N}$, one has

$${}^c D_{a+}^{\alpha, \psi} h(t) = h_{\psi}^{[n]}(t).$$

Lemma 2.1. Let $\alpha > 0$ and $h : [a, b] \rightarrow \mathbb{R}$.

(1) If $h \in C[a, b]$, then ${}^c D_{a+}^{\alpha, \psi} I_{a+}^{\alpha, \psi} h(t) = h(t)$.

(2) If $h \in C^{n-1}[a, b]$, then $I_{a+}^{\alpha, \psi} {}^c D_{a+}^{\alpha, \psi} h(t) = h(t) - \sum_{k=0}^{n-1} c_k [h(t) - h(a)]^k$, where

$$c_k = \frac{h_{\psi}^{[k]}(a)}{k!}.$$

Lemma 2.2. Let $\alpha > 0$ and $h, \psi \in C[a, b]$.

- (i) $I_{a+}^{\alpha; \psi}(\cdot)$ is linear and bounded from $C[a, b]$ to $C[a, b]$.
- (ii) $I_{a+}^{\alpha; \psi} h(a) = \lim_{t \rightarrow a+} I_{a+}^{\alpha; \psi} h(t) = 0$.

Lemma 2.3. Let $\alpha, \beta > 0$ and $h : [a, b] \rightarrow \mathbb{R}$. Then

- (1) $I_{a+}^{\alpha; \psi} [h(t) - h(a)]^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)} [h(t) - h(a)]^{\alpha+\beta-1}$.
- (2) ${}^c D_{a+}^{\alpha; \psi} [h(t) - h(a)]^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)} [h(t) - h(a)]^{\beta-\alpha-1}$.
- (3) ${}^c D_{a+}^{\alpha; \psi} [h(t) - h(a)]^k = 0, \forall k \in \{0, 1, \dots, n-1\}, n \in \mathbb{N}$.
- (4) $I_{a+}^{\alpha; \psi} I_{a+}^{\beta; \psi} h(t) = I_{a+}^{\alpha+\beta; \psi} h(t)$.

At the end of this section, we also recall some useful notations and results which will be used in the sequel.

Definition 2.4 ([11, 30]). Let X be a Banach space. An operator $F : X \rightarrow X^*$ is said to be weakly continuous, if for any sequence $\{u_n\}_{n \geq 1} \subset X$ with $u_n \rightarrow u$ weakly in X , we have $Fu_n \rightarrow Fu$ weakly in X^* .

Definition 2.5 ([11]). Let X be a Banach space. An operator $F : X \rightarrow X^*$ is said to be coercive, if

$$\lim_{\|u\|_X \rightarrow \infty} \frac{\langle Fu, u \rangle_X}{\|u\|_X} = \infty.$$

Theorem 2.1. Let X be a separable reflexive Banach space and $F : X \rightarrow X^*$ be weakly continuous and coercive. Then F is surjective, i.e., for every $f \in X^*$, there is $u \in X$ such that $Fu = f$.

Lemma 2.4 (Lemma 2, [32]). Let $\{u_n\}$ and $\{w_n\}$ be nonnegative sequences satisfying

$$u_n \leq C + \sum_{i=1}^{n-1} w_i u_i \quad \text{for } n \geq 1$$

with a constant $C > 0$. Then

$$u_n \leq C \exp \left(\sum_{i=1}^{n-1} w_i \right) \quad \text{for } n \geq 1.$$

3. Existence of solutions

This section is devoted to present the existence of solutions to Problem 1.1. We will consider the following hypotheses on the data to prove the main result.

$\underline{H(B)}$: $B : V \rightarrow V^*$ and its Nemytski operator $\mathcal{B} : \mathcal{V} \rightarrow \mathcal{V}^*$ are weakly continuous, and there exists $\eta_B > 0$ such that

$$\langle Bv, v \rangle_V \geq \eta_B \|v\|_V^2, \quad \forall v \in V.$$

$\underline{H(A)_1}$: $A : V \rightarrow V^*$ is weakly continuous, and there exist $a_0, a_1 \geq 0$ such that

$$\|Av\|_{V^*} \leq a_0 + a_1 \|v\|_V, \quad \forall v \in V.$$

$\underline{H(A)_2}$: $A : V \rightarrow V^*$ is weakly continuous, and there exist $\bar{a}_0, \bar{a}_1, \bar{a}_2 \geq 0, \eta_A > 0$ such that

$$\|Av\|_{V^*} \leq \bar{a}_0 + \bar{a}_1 \|v\|_V + \bar{a}_2 \|v\|_V^2, \quad \forall v \in V$$

and

$$\langle Av, v \rangle_V \geq \eta_A \|v\|_V^2, \quad \forall v \in V.$$

$\underline{H}(\psi)$: $\psi(0) = 0$ and $k_1 \leq \psi'(t) \leq k_2$ for all $t \in [0, T]$ with some constants $k_1, k_2 > 0$.

$\underline{H}(f)$: $f \in L^\infty(0, T; V^*)$.

Theorem 3.1. Assume that $H(B)$, $H(f)$ hold, and either $H(A)_1$ or $H(A)_2$ is satisfied. Then, Problem 1.1 has a solution $u \in \mathcal{W}$.

Remark 3.1. (i) If $\psi(t) = t$ for all $t \in [0, T]$, then Theorem 3.1 reduces to Theorem 2 in [39].

(ii) If B is linear and bounded, then $\mathcal{B}$ is also linear and bounded, and hence they are both weakly continuous.

(iii) If A, B are linear and bounded, then Theorem 3.1 reduces to Theorem 3.1 in [13] when $j \equiv 0$.

The proof of Theorem 3.1 will be divided into a sequence of lemmas by applying the Rothe method (see [18, 30]). We begin the proof of Theorem 3.1 with the following auxiliary problem associated with Problem 1.1.

Problem 3.1. Find $w \in \mathcal{V}$ such that

$$Bw(t) + A(u_0 + I_{a+}^{\alpha, \psi} w(t)) = f(t) \quad \text{for a.e. } t \in (0, T).$$

Let $w(t) = {}^c D_{a+}^{\alpha, \psi} u(t)$ for a.e. $t \in (0, T)$. Then

$$u(t) = u_0 + I_{a+}^{\alpha, \psi} w(t), \quad \forall t \in [0, T]. \quad (3.1)$$

Using the above notation, it is easy to observe that $w \in \mathcal{V}$ solves Problem 3.1 if and only if $u \in \mathcal{W}$ defined by (3.1) is a solution to Problem 1.1 (see Lemma 3.1 in [13]). Therefore, in what follows, to prove Theorem 3.1, it is enough to establish existence of solutions to Problem 3.1.

Consider the following notations ([13, 14, 18, 30, 41]). Let $N \in \mathbb{N}$ be fixed, $\tau = \frac{T}{N}$ denote the time step, and $t_k = k\tau$ for $k = 1, 2, \dots, N$. For any $g \in \mathcal{V}$, consider the following approximations

$$\begin{aligned} g_\tau^k &= \frac{1}{\tau} \int_{t_{k-1}}^{t_k} g(s) ds \quad \text{for } k = 1, \dots, N, \\ I_{t_n}^{\alpha, \psi} g(t_n) &= \frac{1}{\Gamma(\alpha)} \sum_{j=1}^n \int_{t_{j-1}}^{t_j} \psi'(s) (\psi(t_n - s))^{\alpha-1} g(t_j) ds \\ &= \frac{1}{\Gamma(\alpha+1)} \sum_{j=1}^n g(t_j) \left((\psi(t_n) - \psi(t_{j-1}))^\alpha - (\psi(t_n) - \psi(t_j))^\alpha \right) \\ &= \frac{\tau^\alpha}{\Gamma(\alpha+1)} \sum_{j=1}^n g(t_j) \left((\psi'(\xi_{j-1}))^\alpha (n-j+1)^\alpha - (\psi'(\xi_j))^\alpha (n-j)^\alpha \right), \end{aligned}$$

where $\xi_j \in (t_j, t_n)$ ($j = 0, 1, \dots, n-1$).

Using the above notations, consider the following Rothe problem.

Problem 3.2. Find a sequence $\{w_\tau^k\}_{k=1}^N \subset V$ such that

$$Bw_\tau^k + Av_\tau^k = f_\tau^k \quad (3.2)$$

for $k = 1, \dots, N$, where

$$v_\tau^k = u_0 + \frac{\tau^\alpha}{\Gamma(\alpha+1)} \sum_{j=1}^k w_\tau^j \left((\psi'(\xi_{j-1}))^\alpha (k-j+1)^\alpha - (\psi'(\xi_j))^\alpha (k-j)^\alpha \right).$$

Lemma 3.1. Under the hypotheses of Theorem 3.1, there exists $\tau_0 > 0$ such that for all $\tau \in (0, \tau_0)$, Problem 3.2 has a solution $\{w_\tau^k\}_{k=1}^N \subset V$.

Proof. First, we note that Problem 3.2 can be equivalently formulated as follows: find $w_\tau^k \in V$ such that

$$F(w_\tau^k) = f_\tau^k \quad \text{for all } k = 1, \dots, N, \quad (3.3)$$

where $F: V \rightarrow V^*$ is defined by

$$F(v) = Bv + A \left(u_0 + \frac{\tau^\alpha}{\Gamma(\alpha+1)} \sum_{j=1}^{k-1} w_\tau^j \left((\psi'(\xi_{j-1}))^\alpha (k-j+1)^\alpha - (\psi'(\xi_j))^\alpha (k-j)^\alpha \right) + \frac{\tau^\alpha (\psi'(\xi_{n-1}))^\alpha}{\Gamma(\alpha+1)} v \right).$$

We will prove now that equation (3.3) is solvable for small values of τ . We show that F is surjective. By Theorem 2.1, it is sufficient to show that F is weakly continuous and coercive.

The weak continuity of F is obvious. It remains to prove that F is coercive. We show it separately under $H(A)_1$ and $H(A)_2$.

First, we suppose $H(A)_1$. For all $v \in V$, one has

$$\begin{aligned} & \langle Fv, v \rangle_V \\ & \geq \eta_B \|v\|_V^2 - \left(a_0 + a_1 (K_\tau + \frac{\tau^\alpha (\psi'(\xi_{n-1}))^\alpha}{\Gamma(\alpha+1)} \|v\|_V) \right) \|v\|_V \\ & = \left(\eta_B - \frac{a_1 \tau^\alpha (\psi'(\xi_{n-1}))^\alpha}{\Gamma(\alpha+1)} \right) \|v\|_V^2 - (a_0 + a_1 K_\tau) \|v\|_V \\ & \geq \left(\eta_B - \frac{a_1 \tau^\alpha k_2^\alpha}{\Gamma(\alpha+1)} \right) \|v\|_V^2 - (a_0 + a_1 K_\tau) \|v\|_V, \end{aligned}$$

where

$$K_\tau = \|u_0\|_V + \frac{\tau^\alpha}{\Gamma(\alpha+1)} \sum_{j=1}^{k-1} \|w_\tau^j\|_V \left((\psi'(\xi_{j-1}))^\alpha (k-j+1)^\alpha - (\psi'(\xi_j))^\alpha (k-j)^\alpha \right). \quad (3.4)$$

Then, F is coercive for all $\tau \in (0, \tau_0)$ with $\tau_0 = \left(\frac{\eta_B \Gamma(\alpha+1)}{a_1 k_2^\alpha} \right)^{1/\alpha}$.

Second, we prove the coercivity of F under $H(A)_2$. Denote

$$z_\tau = u_0 + \frac{\tau^\alpha}{\Gamma(\alpha+1)} \sum_{j=1}^{k-1} w_\tau^j \left((\psi'(\xi_{j-1}))^\alpha (k-j+1)^\alpha - (\psi'(\xi_j))^\alpha (k-j)^\alpha \right),$$

$$C_\tau = \frac{\tau^\alpha (\psi'(\xi_{n-1}))^\alpha}{\Gamma(\alpha + 1)}$$

and $v_\tau = z_\tau + C_\tau v$. Then $v = \frac{1}{C_\tau}(v_\tau - z_\tau)$, and hence

$$\begin{aligned} & \langle Av_\tau, v \rangle_V \\ &= \langle Av_\tau, \frac{1}{C_\tau}(v_\tau - z_\tau) \rangle_V \\ &= \frac{1}{C_\tau} \langle Av_\tau, v_\tau \rangle_V - \frac{1}{C_\tau} \langle Av_\tau, z_\tau \rangle_V \\ &\geq \frac{\eta_A}{C_\tau} \|v_\tau\|_V^2 - \frac{1}{C_\tau} \langle Av_\tau, z_\tau \rangle_V \\ &\geq \frac{\eta_A}{C_\tau} \|v_\tau\|_V^2 - \frac{1}{C_\tau} \|z_\tau\|_V \|Av_\tau\|_{V^*} \\ &\geq \frac{\eta_A}{C_\tau} \|v_\tau\|_V^2 - \frac{1}{C_\tau} \|z_\tau\|_V (\bar{a}_0 + \bar{a}_1 \|v_\tau\|_V + \bar{a}_2 \|v_\tau\|_V^2) \\ &= \frac{\eta_A}{C_\tau} \|v_\tau\|_V^2 - \frac{1}{C_\tau} \|z_\tau\|_V (\bar{a}_0 + \bar{a}_1 \|z_\tau + C_\tau v\|_V + \bar{a}_2 \|z_\tau + C_\tau v\|_V^2) \\ &\geq \frac{\eta_A}{C_\tau} \|v_\tau\|_V^2 - \frac{1}{C_\tau} \|z_\tau\|_V (\bar{a}_0 + \bar{a}_1 \|z_\tau\|_V + \bar{a}_1 C_\tau \|v\|_V + \bar{a}_2 \|z_\tau\|_V^2 \\ &\quad + 2\bar{a}_2 C_\tau \|z_\tau\|_V \|v\|_V + \bar{a}_2 C_\tau^2 \|v\|_V^2) \\ &\geq \frac{\eta_A}{C_\tau} \|v_\tau\|_V^2 - \bar{a}_2 C_\tau \|z_\tau\|_V \|v\|_V^2 - \|z_\tau\|_V (\bar{a}_1 + 2\bar{a}_2 \|z_\tau\|_V) \|v\|_V \\ &\quad - \frac{1}{C_\tau} \|z_\tau\|_V (\bar{a}_0 + \bar{a}_1 \|z_\tau\|_V + \bar{a}_2 \|z_\tau\|_V^2). \end{aligned}$$

This gives

$$\begin{aligned} & \langle Fv, v \rangle_V \\ &\geq \eta_B \|v\|_V^2 + \frac{\eta_A}{C_\tau} \|v_\tau\|_V^2 - \bar{a}_2 C_\tau \|z_\tau\|_V \|v\|_V^2 - \|z_\tau\|_V (\bar{a}_1 + 2\bar{a}_2 \|z_\tau\|_V) \|v\|_V \\ &\quad - \frac{1}{C_\tau} \|z_\tau\|_V (\bar{a}_0 + \bar{a}_1 \|z_\tau\|_V + \bar{a}_2 \|z_\tau\|_V^2) \\ &\geq \eta_B \|v\|_V^2 - \frac{\bar{a}_2 \tau^\alpha (\psi'(\xi_{n-1}))^\alpha}{\Gamma(\alpha + 1)} K_\tau \|v\|_V^2 - K_\tau (\bar{a}_1 + 2\bar{a}_2 K_\tau) \|v\|_V \\ &\quad - \frac{\Gamma(\alpha + 1)}{\tau^\alpha (\psi'(\xi_{n-1}))^\alpha} K_\tau (\bar{a}_0 + \bar{a}_1 K_\tau + \bar{a}_2 K_\tau^2) \\ &\geq \left(\eta_B - \frac{\bar{a}_2 \tau^\alpha k_2^\alpha K_\tau}{\Gamma(\alpha + 1)} \right) \|v\|_V^2 - K_\tau (\bar{a}_1 + 2\bar{a}_2 K_\tau) \|v\|_V \\ &\quad - \frac{\Gamma(\alpha + 1)}{\tau^\alpha k_2^\alpha} K_\tau (\bar{a}_0 + \bar{a}_1 K_\tau + \bar{a}_2 K_\tau^2). \end{aligned}$$

It is easy to verify that we can choose τ_0 sufficiently small such that for all $\tau \in (0, \tau_0)$, $\eta_B - \frac{\bar{a}_2 \tau^\alpha k_2^\alpha K_\tau}{\Gamma(\alpha + 1)} > 0$, and hence F is coercive for all $\tau \in (0, \tau_0)$.

Therefore, we conclude from Theorem 2.1 that Problem 3.2 is solvable for all $\tau \in (0, \tau_0)$. This completes the proof. $\square$

Lemma 3.2. *Under the hypotheses of Theorem 3.1, there exists $\tau_0 > 0$ such that for all $\tau \in (0, \tau_0)$, the following estimates are satisfied*

$$\max_{k=1,\dots,N} \|w_\tau^k\|_V \leq C, \quad (3.5)$$

$$\max_{k=1,\dots,N} \|v_\tau^k\|_V \leq C, \quad (3.6)$$

where $C > 0$ is independent of τ .

Proof. We start with the proof of (3.5).

First, we assume that $H(A)_1$ holds. For all $1 \leq k \leq N$, we have

$$Bw_\tau^k + Av_\tau^k = f_\tau^k. \quad (3.7)$$

Multiplying equation (3.7) by w_τ^k , similarly as in the first part of the proof of Lemma 3.1, we obtain

$$\begin{aligned} & \langle f_\tau^k, w_\tau^k \rangle_V \\ &= \langle Bw_\tau^k, w_\tau^k \rangle_V + \langle Av_\tau^k, w_\tau^k \rangle_V \\ &\geq \left(\eta_B - \frac{a_1 \tau^\alpha k_2^\alpha}{\Gamma(\alpha+1)} \right) \|w_\tau^k\|_V^2 - (a_0 + a_1 K_\tau) \|w_\tau^k\|_V. \end{aligned}$$

Then

$$\|f_\tau^k\|_{V^*} + a_0 + a_1 K_\tau \geq \left(\eta_B - \frac{a_1 \tau^\alpha k_2^\alpha}{\Gamma(\alpha+1)} \right) \|w_\tau^k\|_V,$$

where K_τ is defined in (3.4). Now, we choose $\tau_0 = \left(\frac{\eta_B \Gamma(\alpha+1)^{1/\alpha}}{2a_1 k_2^\alpha} \right)$ to get $\eta_B - \frac{a_1 \tau^\alpha k_2^\alpha}{\Gamma(\alpha+1)} > \frac{\eta_B}{2}$ for all $\tau \in (0, \tau_0)$. Therefore, we deduce

$$\begin{aligned} & \|w_\tau^k\|_V \\ &\leq \frac{2}{\eta_B} (\|f_\tau^k\|_{V^*} + a_0) + \frac{2}{\eta_B} a_1 K_\tau \\ &= \frac{2}{\eta_B} (\|f_\tau^k\|_{V^*} + a_0) + \frac{2}{\eta_B} a_1 \|u_0\|_V \\ &\quad + \frac{2a_1 \tau^\alpha}{\eta_B \Gamma(\alpha+1)} \sum_{j=1}^{k-1} \|w_\tau^j\|_V \left((\psi'(\xi_{j-1}))^\alpha (k-j+1)^\alpha - (\psi'(\xi_j))^\alpha (k-j)^\alpha \right). \end{aligned}$$

By denoting

$$m_0 = \frac{2}{\eta_B} (\|f\|_{L^\infty} + a_0 + a_1 \|u_0\|_V),$$

it follows from Lemma 2.4 that

$$\begin{aligned} \|w_\tau^k\|_V &\leq m_0 \exp \left(\frac{2a_1}{\eta_B \Gamma(\alpha)} \sum_{j=1}^{k-1} \int_{t_{j-1}}^{t_j} \psi'(s) (\psi(t_n) - \psi(s))^{\alpha-1} ds \right) \\ &\leq m_0 \exp \left(\frac{2a_1}{\eta_B \Gamma(\alpha)} \int_0^{t_n} \psi'(s) (\psi(t_n) - \psi(s))^{\alpha-1} ds \right) \end{aligned}$$

$$\leq m_0 \exp\left(\frac{2a_1 T^\alpha}{\eta_B \Gamma(\alpha+1)}\right).$$

This proves inequality (3.5) in the case of $H(A)_1$.

Second, we assume hypothesis $H(A)_2$. Multiplying equation (3.7) by w_τ^k , analogously as in the second part of the proof of Lemma 3.1, we obtain

$$\begin{aligned} & \langle f_\tau^k, w_\tau^k \rangle_V \\ &= \langle Bw_\tau^k, w_\tau^k \rangle_V + \langle Aw_\tau^k, w_\tau^k \rangle_V \\ &\geq \left(\eta_B - \frac{\bar{a}_2 \tau^\alpha k_2^\alpha K_\tau}{\Gamma(\alpha+1)}\right) \|w_\tau^k\|_V^2 + \frac{\eta_A \Gamma(\alpha+1)}{\tau^\alpha k_2^\alpha} \|v_\tau^k\|_V^2 - (K_\tau(\bar{a}_1 + 2\bar{a}_2 K_\tau)) \|w_\tau^k\|_V \\ &\quad - \frac{\Gamma(\alpha+1)}{\tau^\alpha k_2^\alpha} K_\tau(\bar{a}_0 + \bar{a}_1 K_\tau + \bar{a}_2 K_\tau^2). \end{aligned}$$

Hence, we get

$$\begin{aligned} \left(\eta_B - \frac{\bar{a}_2 \tau^\alpha k_2^\alpha K_\tau}{\Gamma(\alpha+1)}\right) \|w_\tau^k\|_V^2 &\leq (\|f\|_{L^\infty} + (K_\tau(\bar{a}_1 + 2\bar{a}_2 K_\tau))) \|w_\tau^k\|_V \\ &\quad + \frac{\Gamma(\alpha+1)}{\tau^\alpha k_2^\alpha} (K_\tau(\bar{a}_0 + \bar{a}_1 K_\tau + \bar{a}_2 K_\tau^2) - \eta_A \|v_\tau^k\|_V^2). \end{aligned}$$

We can choose $\tau_0 > 0$ such that $\eta_B - \frac{\bar{a}_2 \tau^\alpha k_2^\alpha K_\tau}{\Gamma(\alpha+1)} > \frac{\eta_B}{2}$ for all $\tau \in (0, \tau_0)$. With this choice of τ_0 , we consider the following two cases:

- (i) $\bar{a}_0 \|u_0\|_V + \bar{a}_1 \|u_0\|_V^2 + \bar{a}_2 \|u_0\|_V^3 \geq \eta_A \|v_\tau^k\|_V^2$;
- (ii) $\bar{a}_0 \|u_0\|_V + \bar{a}_1 \|u_0\|_V^2 + \bar{a}_2 \|u_0\|_V^3 < \eta_A \|v_\tau^k\|_V^2$.

If (i) holds, then $\{v_\tau^k\}$ is bounded. We have

$$\begin{aligned} & \eta_B \|w_\tau^k\|_V^2 \\ &\leq \langle Bw_\tau^k, w_\tau^k \rangle \\ &\leq \|f_\tau^k\|_{V^*} \|w_\tau^k\|_V + \|Aw_\tau^k\|_{V^*} \|w_\tau^k\|_V \\ &\leq \|f\|_{L^\infty} \|w_\tau^k\|_V + (\bar{a}_0 + \bar{a}_1 \|v_\tau^k\|_V + \bar{a}_2 \|v_\tau^k\|_V^2) \|w_\tau^k\|_V, \end{aligned}$$

and hence

$$\eta_B \|w_\tau^k\|_V \leq \|f\|_{L^\infty} + \bar{a}_0 + \bar{a}_1 \|v_\tau^k\|_V + \bar{a}_2 \|v_\tau^k\|_V^2,$$

which implies that $\{w_\tau^k\}$ is bounded.

If (ii) holds, then

$$\begin{aligned} & \|w_\tau^k\|_V^2 \\ &\leq \frac{2}{\eta_B} (\|f\|_{L^\infty} + K_\tau(\bar{a}_1 + 2\bar{a}_2 K_\tau)) \|w_\tau^k\|_V \\ &\quad + \frac{2\Gamma(\alpha+1)}{\eta_B \tau^\alpha k_2^\alpha} (K_\tau(\bar{a}_0 + \bar{a}_1 K_\tau + \bar{a}_2 K_\tau^2) - \eta_A \|v_\tau^k\|_V^2) \\ &= \frac{2}{\eta_B} (\|f\|_{L^\infty} + K_\tau(\bar{a}_1 + 2\bar{a}_2 K_\tau)) \|w_\tau^k\|_V \\ &\quad + \frac{2\Gamma(\alpha+1)}{\eta_B \tau^\alpha k_2^\alpha} \left(\|u_0\|_V + \frac{\tau^\alpha}{\Gamma(\alpha+1)} \sum_{j=1}^{k-1} \|w_\tau^j\|_V ((\psi'(\xi_{j-1}))^\alpha (k-j+1)^\alpha \right. \end{aligned}$$

$$\begin{aligned}
& -(\psi'(\xi_j))^\alpha(k-j)^\alpha) \times \left(\bar{a}_0 + \bar{a}_1 \left(\|u_0\|_V \right. \right. \\
& + \frac{\tau^\alpha}{\Gamma(\alpha+1)} \sum_{j=1}^{k-1} \|w_\tau^j\|_V \left((\psi'(\xi_{j-1}))^\alpha(k-j+1)^\alpha - (\psi'(\xi_j))^\alpha(k-j)^\alpha \right) \\
& + \bar{a}_2 \left(\|u_0\|_V + \frac{\tau^\alpha}{\Gamma(\alpha+1)} \sum_{j=1}^{k-1} \|w_\tau^j\|_V \left((\psi'(\xi_{j-1}))^\alpha(k-j+1)^\alpha \right. \right. \\
& \left. \left. - (\psi'(\xi_j))^\alpha(k-j)^\alpha \right) \right)^2 \left. \right) - \eta_A \|v_\tau^k\|_V^2 \Big) \\
& = \frac{2}{\eta_B} \left(\|f\|_{L^\infty + K_\tau(\bar{a}_1 + 2\bar{a}_2 K_\tau)} \right) \|w_\tau^k\|_V \\
& + \frac{2\Gamma(\alpha+1)}{\eta_B \tau^\alpha k_2^\alpha} \left(\bar{a}_0 \|u_0\|_V + \bar{a}_1 \|u_0\|_V^2 + \bar{a}_2 \|u_0\|_V^3 - \eta_A \|v_\tau^k\|_V^2 \right. \\
& + \frac{\tau^\alpha}{\Gamma(\alpha+1)} \left(m_1 + \frac{\tau^\alpha}{\Gamma(\alpha+1)} (\bar{a}_1 + 3\bar{a}_2 \|u_0\|_V) \right. \\
& \times \sum_{j=1}^{k-1} \|w_\tau^j\|_V \left((\psi'(\xi_{j-1}))^\alpha(k-j+1)^\alpha - (\psi'(\xi_j))^\alpha(k-j)^\alpha \right) \\
& + \bar{a}_2 \frac{\tau^{2\alpha}}{\Gamma^2(\alpha+1)} \left(\sum_{j=1}^{k-1} \|w_\tau^j\|_V \left((\psi'(\xi_{j-1}))^\alpha(k-j+1)^\alpha - (\psi'(\xi_j))^\alpha(k-j)^\alpha \right) \right)^2 \Big) \\
& \times \sum_{j=1}^{k-1} \|w_\tau^j\|_V \left((\psi'(\xi_{j-1}))^\alpha(k-j+1)^\alpha - (\psi'(\xi_j))^\alpha(k-j)^\alpha \right) \\
& \leq \frac{2}{\eta_B} \left(\|f\|_{L^\infty + K_\tau(\bar{a}_1 + 2\bar{a}_2 K_\tau)} \right) \|w_\tau^k\|_V + \frac{2}{\eta_B k_2^\alpha} \left(m_1 + \frac{\tau^\alpha}{\Gamma(\alpha+1)} (\bar{a}_1 + 3\bar{a}_2 \|u_0\|_V) \right. \\
& \times \sum_{j=1}^{k-1} \|w_\tau^j\|_V \left((\psi'(\xi_{j-1}))^\alpha(k-j+1)^\alpha - (\psi'(\xi_j))^\alpha(k-j)^\alpha \right) \\
& + \bar{a}_2 \frac{\tau^{2\alpha}}{\Gamma^2(\alpha+1)} \left(\sum_{j=1}^{k-1} \|w_\tau^j\|_V \left((\psi'(\xi_{j-1}))^\alpha(k-j+1)^\alpha - (\psi'(\xi_j))^\alpha(k-j)^\alpha \right) \right)^2 \Big) \\
& \times \sum_{j=1}^{k-1} \|w_\tau^j\|_V \left((\psi'(\xi_{j-1}))^\alpha(k-j+1)^\alpha - (\psi'(\xi_j))^\alpha(k-j)^\alpha \right),
\end{aligned}$$

where $m_1 = \bar{a}_0 + 2\bar{a}_1 \|u_0\|_V + 3\bar{a}_2 \|u_0\|_V^2$. It follows that

$$\begin{aligned}
& \|w_\tau^k\|_V \\
& \leq \frac{2}{\eta_B} \left(\|f\|_{L^\infty + K_\tau(\bar{a}_1 + 2\bar{a}_2 K_\tau)} \right) + \left(\frac{2}{\eta_B k_2^\alpha} \left(m_1 + \frac{\tau^\alpha}{\Gamma(\alpha+1)} (\bar{a}_1 + 3\bar{a}_2 \|u_0\|_V) \right. \right. \\
& \times \sum_{j=1}^{k-1} \|w_\tau^j\|_V \left((\psi'(\xi_{j-1}))^\alpha(k-j+1)^\alpha - (\psi'(\xi_j))^\alpha(k-j)^\alpha \right) + \bar{a}_2 \frac{\tau^{2\alpha}}{\Gamma^2(\alpha+1)} \\
& \times \left(\sum_{j=1}^{k-1} \|w_\tau^j\|_V \left((\psi'(\xi_{j-1}))^\alpha(k-j+1)^\alpha - (\psi'(\xi_j))^\alpha(k-j)^\alpha \right) \right)^2 \Big)
\end{aligned}$$

$$\begin{aligned}
& \times \sum_{j=1}^{k-1} \|w_\tau^j\|_V \left((\psi'(\xi_{j-1}))^\alpha (k-j+1)^\alpha - (\psi'(\xi_j))^\alpha (k-j)^\alpha \right)^{\frac{1}{2}} \\
& \leq \frac{2}{\eta_B} (\|f\|_{L^\infty + K_\tau(\bar{a}_1 + 2\bar{a}_2 K_\tau)}) + \frac{1}{\eta_B k_2^\alpha} \sum_{j=1}^{k-1} \|w_\tau^j\|_V \left((\psi'(\xi_{j-1}))^\alpha (k-j+1)^\alpha \right. \\
& \quad \left. - (\psi'(\xi_j))^\alpha (k-j)^\alpha \right) \\
& \quad + \frac{1}{2} \left(m_1 + \frac{\tau^\alpha}{\Gamma(\alpha+1)} (\bar{a}_1 + 3\bar{a}_2 \|u_0\|_V) \sum_{j=1}^{k-1} \|w_\tau^j\|_V \left((\psi'(\xi_{j-1}))^\alpha (k-j+1)^\alpha \right. \right. \\
& \quad \left. \left. - (\psi'(\xi_j))^\alpha (k-j)^\alpha \right) \right. \\
& \quad \left. + \bar{a}_2 \frac{\tau^{2\alpha}}{\Gamma^2(\alpha+1)} \left(\sum_{j=1}^{k-1} \|w_\tau^j\|_V \left((\psi'(\xi_{j-1}))^\alpha (k-j+1)^\alpha - (\psi'(\xi_j))^\alpha (k-j)^\alpha \right) \right)^2 \right) \\
& \leq m_2 + m_3 \sum_{j=1}^{k-1} \|w_\tau^j\|_V \left((\psi'(\xi_{j-1}))^\alpha (k-j+1)^\alpha - (\psi'(\xi_j))^\alpha (k-j)^\alpha \right) \\
& \quad + m_4 \left(\sum_{j=1}^{k-1} \|w_\tau^j\|_V \left((\psi'(\xi_{j-1}))^\alpha (k-j+1)^\alpha - (\psi'(\xi_j))^\alpha (k-j)^\alpha \right) \right)^2,
\end{aligned}$$

where

$$\begin{aligned}
m_2 &= \frac{2}{\eta_B} (\|f\|_{L^\infty} + \bar{a}_1 \|u_0\|_V + 2\bar{a}_2 \|u_0\|_V^2) + \frac{1}{2} m_1, \\
m_3 &= \frac{2T^\alpha \eta_B \Gamma(\alpha+1) (\bar{a}_1 + 4\bar{a}_2 \|u_0\|_V)}{+} \frac{T^\alpha}{2\Gamma(\alpha+1)} (\bar{a}_1 + 3\bar{a}_2 \|u_0\|_V) + \frac{1}{\eta_B k_2^\alpha}, \\
m_4 &= \left(\frac{4}{\eta_B} + \frac{1}{2} \right) \frac{\bar{a}_2 T^{2\alpha}}{\Gamma^2(\alpha+1)}.
\end{aligned}$$

Here we use the elementary fact: if $x^2 \leq ax + b$ for $a, b, x \geq 0$, then $x \leq a + \sqrt{b}$. Moreover, for w_τ^1 , we have

$$\begin{aligned}
& \|f\|_{L^\infty} \|w_\tau^1\|_V \\
& \geq \langle f_\tau^1, w_\tau^1 \rangle_V \\
& = \langle Bw_\tau^1, w_\tau^1 \rangle_V + \langle Av_\tau^1, w_\tau^1 \rangle_V \\
& \geq \eta_B \|w_\tau^1\|_V^2 + \frac{\Gamma(\alpha+1)}{\tau^\alpha (\psi'(\xi_0))^\alpha} \langle Av_\tau^1, v_\tau^1 \rangle_V - \frac{\Gamma(\alpha+1)}{\tau^\alpha (\psi'(\xi_0))^\alpha} \langle Av_\tau^1, u_0 \rangle_V \\
& \geq \eta_B \|w_\tau^1\|_V^2 + \frac{\eta_A \Gamma(\alpha+1)}{\tau^\alpha (\psi'(\xi_0))^\alpha} \|v_\tau^1\|_V^2 \\
& \quad - \frac{\Gamma(\alpha+1)}{\tau^\alpha (\psi'(\xi_0))^\alpha} \|u_0\|_V \left(\bar{a}_0 + \bar{a}_1 \|u_0\|_V + \frac{\tau^\alpha (\psi'(\xi_0))^\alpha}{\Gamma(\alpha+1)} w_\tau^1\|_V + \bar{a}_2 \|u_0\|_V \right. \\
& \quad \left. + \frac{\tau^\alpha (\psi'(\xi_0))^\alpha}{\Gamma(\alpha+1)} w_\tau^1\|_V^2 \right) \\
& \geq \left(\eta_B - \frac{\bar{a}_2 k_2^\alpha \tau^\alpha \|u_0\|_V}{\Gamma(\alpha+1)} \right) \|w_\tau^1\|_V^2 - (\bar{a}_1 \|u_0\|_V + 2\bar{a}_2 \|u_0\|_V^2) \|w_\tau^1\|_V
\end{aligned}$$

$$-\frac{\Gamma(\alpha+1)}{\tau^\alpha k_2^\alpha} \|u_0\|_V (\bar{a}_0 + \bar{a}_1 \|u_0\|_V + \bar{a}_2 \|u_0\|_V^2) + \frac{\eta_A \Gamma(\alpha+1)}{\tau^\alpha k_2^\alpha} \|v_\tau^1\|_V^2,$$

and hence

$$\begin{aligned} & \left(\eta_B - \frac{\bar{a}_2 k_2^\alpha \tau^\alpha \|u_0\|_V}{\Gamma(\alpha+1)} \right) \|w_\tau^1\|_V^2 \\ & \leq (\|f\|_{L^\infty} + \bar{a}_1 \|u_0\|_V + 2\bar{a}_2 \|u_0\|_V^2) \|w_\tau^1\|_V \\ & \quad + \frac{\Gamma(\alpha+1)}{\tau^\alpha k_2^\alpha} (\bar{a}_0 \|u_0\|_V + \bar{a}_1 \|u_0\|_V^2 + \bar{a}_2 \|u_0\|_V^3 - \eta_A \|v_\tau^1\|_V^2) \\ & \leq (\|f\|_{L^\infty} + \bar{a}_1 \|u_0\|_V + 2\bar{a}_2 \|u_0\|_V^2) \|w_\tau^1\|_V. \end{aligned}$$

Obviously, $\eta_B - \frac{\bar{a}_2 k_2^\alpha \tau^\alpha \|u_0\|_V}{\Gamma(\alpha+1)} > \frac{\eta_B}{2}$ for all $\tau \in (0, \tau_0)$. Then

$$\|w_\tau^1\|_V \leq \frac{2}{\eta_B} (\|f\|_{L^\infty} + \bar{a}_1 \|u_0\|_V + 2\bar{a}_2 \|u_0\|_V^2),$$

which implies that w_τ^1 is bounded. We conclude that (3.5) holds by induction.

Moreover, condition (3.6) is a consequence of (3.5). This completes the proof of the lemma. $\square$

In the final step of the Rothe method, we prove the convergence of suitably defined approximate sequences of functions towards a solution to Problem 3.1.

We consider the piecewise constant interpolants $\bar{w}_\tau, \bar{v}_\tau \in \mathcal{V}$, $\bar{f}_\tau \in \mathcal{V}^*$ defined by

$$\bar{w}_\tau(t) = w_\tau^k, \quad \bar{v}_\tau(t) = v_\tau^k, \quad \bar{f}_\tau(t) = f_\tau^k,$$

respectively, for a.e. $t \in (t_{k-1}, t_k]$ with $k = 1, \dots, N$.

Lemma 3.3. *Under the hypotheses of Theorem 3.1, for a subsequence, we have*

$$\bar{w}_\tau \rightarrow w \text{ weakly in } \mathcal{V},$$

where $w \in \mathcal{V}$ is a solution to Problem 3.1.

Proof. By denoting $\mathcal{A}: \mathcal{V} \rightarrow \mathcal{V}^*$ by the Nemytski operator of A , the equation (3.7) turns out to be

$$\mathcal{B}\bar{w}_\tau + \mathcal{A}\bar{v}_\tau = \bar{f}_\tau. \quad (3.8)$$

By (3.5), the sequence $\{\bar{w}_\tau\}$ remains bounded in $\mathcal{V}$. Exploiting the reflexivity of $\mathcal{V}$, for a subsequence, we get

$$\bar{w}_\tau \rightarrow w \text{ weakly in } \mathcal{V}. \quad (3.9)$$

Subsequently, we will show that $w \in \mathcal{V}$ is a solution to Problem 3.1. Since the fractional integral operator $I_{a+}^{\alpha, \psi}: \mathcal{V} \rightarrow C([0, T]; V)$ is linear and continuous, it is also weakly continuous. Therefore, from (3.9), we deduce

$$I_{a+}^{\alpha, \psi} \bar{w}_\tau \rightarrow I_{a+}^{\alpha, \psi} w \text{ weakly in } C([0, T]; V), \text{ as } \tau \rightarrow 0.$$

This also gives

$$I_{a+}^{\alpha, \psi} \bar{w}_\tau(t) \rightarrow I_{a+}^{\alpha, \psi} w(t) \text{ weakly in } V, \text{ as } \tau \rightarrow 0 \quad (3.10)$$

for all $t \in [0, T]$. Indeed, let $v^* \in V^*$ and $t \in [0, T]$. We choose $\varphi \in (C([0, T]; V))^*$ defined by $\varphi(y) = \langle v^*, y(t) \rangle_V$ for $y \in C([0, T]; V)$. We have

$$\langle v^*, I_{a+}^{\alpha, \psi} \bar{w}_\tau(t) \rangle_V = \varphi(I_{a+}^{\alpha, \psi} \bar{w}_\tau) \rightarrow \varphi(I_{a+}^{\alpha, \psi} w) = \langle v^*, I_{a+}^{\alpha, \psi} w(t) \rangle_V, \text{ as } \tau \rightarrow 0.$$

Since $v^* \in V^*$ is arbitrary, (3.10) follows.

On the other hand, from (3.5), there exists a constant $C > 0$ such that $\|\bar{w}_\tau(s)\| \leq C$ for a.e. $s \in (0, T)$. Then, for $t \in (t_{k-1}, t_k]$, we have

$$\begin{aligned}
 & \|\bar{v}_\tau(t) - u_0 - I_{a+}^{\alpha, \psi} \bar{w}_\tau(t)\|_V \\
 &= \frac{1}{\Gamma(\alpha)} \left\| \int_0^{t_k} \psi'(s)(\psi(t_k) - \psi(s))^{\alpha-1} \bar{w}_\tau(s) ds \right. \\
 & \quad \left. - \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} \bar{w}_\tau(s) ds \right\|_V \\
 &\leq \frac{1}{\Gamma(\alpha)} \left\| \int_0^t \psi'(s)((\psi(t_k) - \psi(s))^{\alpha-1} - (\psi(t) - \psi(s))^{\alpha-1}) \bar{w}_\tau(s) ds \right\|_V \\
 & \quad + \frac{1}{\Gamma(\alpha)} \left\| \int_t^{t_k} \psi'(s)(\psi(t_k) - \psi(s))^{\alpha-1} \bar{w}_\tau(s) ds \right\|_V \\
 &\leq \frac{C}{\Gamma(\alpha)} \int_0^t |\psi'(s)((\psi(t_k) - \psi(s))^{\alpha-1} - (\psi(t) - \psi(s))^{\alpha-1})| ds \\
 & \quad + \frac{C}{\Gamma(\alpha)} \int_t^{t_k} \psi'(s)(\psi(t_k) - \psi(s))^{\alpha-1} ds \\
 &\leq \frac{C}{\Gamma(\alpha+1)} |\psi^\alpha(t_k) - (\psi(t_k) - \psi(t))^\alpha - \psi^\alpha(t)| \\
 & \quad + \frac{C}{\Gamma(\alpha+1)} (\psi(t_k) - \psi(t))^\alpha. \tag{3.11}
 \end{aligned}$$

Since $t \rightarrow t_k$, as $\tau \rightarrow 0$, we deduce that $\bar{v}_\tau(t) - u_0 - I_{a+}^{\alpha, \psi} \bar{w}_\tau(t) \rightarrow 0$, as $\tau \rightarrow 0$ for all $t \in [0, T]$.

Combining (3.10) and (3.11), we have

$$\bar{v}_\tau(t) \rightarrow u_0 + I_{a+}^{\alpha, \psi} w(t) \text{ weakly in } V, \text{ as } \tau \rightarrow 0 \tag{3.12}$$

for all $t \in [0, T]$.

Moreover, having in mind $H(A)_1$ or $H(A)_2$, and (3.12), by applying the Lebesgue dominated convergence theorem, for any $v \in \mathcal{V}$, we obtain

$$\begin{aligned}
 \lim_{\tau \rightarrow 0} \langle \mathcal{A} \bar{v}_\tau, v \rangle_{\mathcal{V}} &= \lim_{\tau \rightarrow 0} \int_0^T \langle \mathcal{A} \bar{v}_\tau(t), v(t) \rangle_V dt \\
 &= \int_0^T \lim_{\tau \rightarrow 0} \langle \mathcal{A} \bar{v}_\tau(t), v(t) \rangle_V dt \\
 &= \int_0^T \langle \mathcal{A}(u_0 + I_{a+}^{\alpha, \psi} w(t)), v(t) \rangle_V dt \\
 &= \langle \mathcal{A}(u_0 + I_{a+}^{\alpha, \psi} w), v \rangle_{\mathcal{V}}.
 \end{aligned}$$

Then, we infer

$$\mathcal{A} \bar{v}_\tau \rightarrow \mathcal{A}(u_0 + I_{a+}^{\alpha, \psi} w) \text{ weakly in } \mathcal{V}^*, \text{ as } \tau \rightarrow 0. \tag{3.13}$$

Since $\mathcal{B}$ is weakly continuous,

$$\mathcal{B} w_\tau \rightarrow \mathcal{B} w \text{ weakly in } \mathcal{V}^*, \text{ as } \tau \rightarrow 0. \tag{3.14}$$

As concerns the approximation of the right-hand side, [5, Lemma 3.3] implies that $\bar{f}_\tau \rightarrow f$ in $\mathcal{V}^*$, as $\tau \rightarrow 0$. Hence and from (3.9), (3.13) and (3.14), we pass to the limit in (3.8) and obtain

$$\mathcal{B}w + \mathcal{A}(u_0 + I_{a+}^{\alpha, \psi} w) = f.$$

The latter implies that $w \in \mathcal{V}$ is a solution to Problem 3.1. This completes the proof of the lemma. $\square$

Finally, taking into account Lemmas 3.1 and 3.3, we conclude that the assertion of Theorem 3.1 is satisfied.

4. Impulsive evolutionary equations

In this section, we apply our previous results to the existence and uniqueness of solution to an impulsive evolutionary equation. Given $0 < T < +\infty$ and a finite set of points

$$E = \{t_1, \dots, t_m\}, \quad 0 = t_0 < t_1 < t_2 < \dots < t_m < t_{m+1} = T.$$

The elements of E are called impulsive points. Consider the following impulsive evolutionary equation.

$$\begin{cases} B({}^c D_{a+}^{\alpha, \psi} u(t)) + Au(t) = f(t), & t \in [0, T] - E, \\ \Delta u(t_k) = G(t_k, u(t_k)), & t_k \in E, \\ u(0) = u_0, \end{cases} \quad (4.1)$$

where $G : [0, T] \times V \rightarrow V$ is a continuous function. $\Delta u(t_k) = u(t_k^+) - u(t_k^-)$ where $u(t_k^+)$ and $u(t_k^-)$ denote the right and the left limits of $u(t)$ at $t = t_k \in D$, respectively.

For $k \in \{0, 1, \dots, m\}$, let

$$\mathcal{V}_k = L^2((t_k, t_{k+1}); V), \quad \mathcal{V}_k^* = L^2((t_k, t_{k+1}); V^*), \quad \mathcal{W}_k = W^{1,2}((t_k, t_{k+1}); V)$$

and

$$\mathcal{W}' = \bigcap_{k=0}^m \mathcal{W}_k.$$

We can obtain the following result for (4.1).

Theorem 4.1. *Under the hypotheses of one of Theorem 3.1, (4.1) has a unique solution in $\mathcal{W}'$.*

Proof. By the impulsive points, we divide the interval $(0, T)$ into $m + 1$ subintervals $\sigma_0 = (0, t_1)$, $\sigma_1 = (t_1, t_2)$, $\dots$, $\sigma_m = (t_m, T)$. We solve the problem first in the interval σ_0 , then in the interval σ_1 and so on until the final interval σ_m . The specific steps are as follows.

Step 1. We consider the following evolutionary problem

$$\begin{cases} B({}^c D_{a+}^{\alpha, \psi} u(t)) + Au(t) = f(t) \quad \text{a.e. } t \in \sigma_0, \\ u(0) = x_0. \end{cases} \quad (4.2)$$

According to Theorem 3.1, we deduce that problem (4.2) admits a solution $u^0 \in \mathcal{W}_0$. Hence $u^0 \in C(\sigma_0; V)$. Therefore, the left limit $u^0(t_1^-) \in V$ exists and we define $u^0(t_1) = u^0(t_1^-)$. Since G is continuous, $u^0(t_1^+)$ is well defined and it is given by

$$\theta_1 := u^0(t_1^+) = u^0(t_1) + G(t_1, u^0(t_1)) \in V.$$

Step 2. We consider the following evolutionary problem

$$\begin{cases} B({}^c D_{a^+}^{\alpha, \psi} u(t)) + Au(t) = f(t) & \text{a.e. } t \in \sigma_1, \\ u(t_1) = \theta_1. \end{cases} \quad (4.3)$$

By Theorem 3.1, we can get a solution $u^1 \in \mathcal{W}_1$ of problem (4.3). Hence $u^1 \in C(\sigma_1; V)$. Similar to Step 1, let

$$\theta_2 := u^1(t_2^+) = u^1(t_2) + G(t_2, u^1(t_2)) \in V.$$

Step 3. For $k = 0, 1, \dots, m$, we continue the above process and get that u^k solves

$$\begin{cases} B({}^c D_{a^+}^{\alpha, \psi} u(t)) + Au(t) = f(t) & \text{a.e. } t \in \sigma_k, \\ u(t_k) = \theta_k, \end{cases}$$

where

$$\theta_k = u^{k-1}(t_k) + G(t_k, u^{k-1}(t_k)) \in V.$$

Let $\bar{u}: [0, T] \rightarrow V$ be

$$\bar{u}(t) = \begin{cases} u^0(t), & t \in [0, t_1], \\ u^1(t), & t \in (t_1, t_2], \\ \dots, & \dots \\ u^m(t), & t \in (t_m, T]. \end{cases}$$

It is clear that $u \in \mathcal{W}'$ is a solution to problem (4.1). The proof is complete. $\square$

5. Quasistatic frictional contact problem

In this part, we consider a quasistatic frictional contact problem. For more quasistatic frictional contact problems, we refer to [15, 33].

An elastic body occupies an open, bounded and connected set $\Omega \subset \mathbb{R}^d$, $d = 1, 2, 3$. The boundary of Ω is denoted by $\Gamma = \partial\Omega$ and it is assumed to be Lipschitz continuous. We also suppose that Γ consists of three mutually disjoint and measurable parts $\bar{\Gamma}_1, \bar{\Gamma}_2$ such that $\text{meas}(\Gamma_1) > 0$. Volume forces of density f_0 act in Ω and surface traction of density f_N are applied on Γ_2 . The body is in frictional contact with a foundation over the potential contact surface Γ_3 . Let $\nu = (\nu_i)$ be the outward unit normal at Γ and let $\mathbb{S}^d$ be the space of second order symmetric tensors on $\mathbb{R}^d$. The inner products and the corresponding norms on $\mathbb{R}^d$ and $\mathbb{S}^d$ are

$$u \cdot v = x_i y_i, \quad \|v\|_{\mathbb{R}^d} = (v \cdot v)^{\frac{1}{2}} \quad \text{for all } u, v \in \mathbb{R}^d,$$

$$\sigma : \tau = \sigma_{ij} \tau_{ij}, \quad \|\tau\|_{\mathbb{S}^d} = (\tau : \tau)^{\frac{1}{2}} \text{ for all } \sigma, \tau \in \mathbb{S}^d,$$

where the indices i and j run between 1 and d , and the summation convention over repeated indices is used. We will use the spaces V , H and Y defined by

$$V = \{v \in H^1(\Omega; \mathbb{R}^d) \mid v = \mathbf{0} \text{ on } \Gamma_1\}, \quad H = L^2(\Omega; \mathbb{R}^d), \quad Y = L^2(\Omega; \mathbb{S}^d).$$

The space H and Y will be endowed with the Hilbertian structure given by the inner product

$$\langle u, v \rangle_H = \int_{\Omega} u \cdot v \, dx, \quad \langle \sigma, \tau \rangle_Y = \int_{\Omega} \sigma : \tau \, dx.$$

On the space V we consider the inner product and the corresponding norm given by

$$\langle u, v \rangle_V = \langle \varepsilon(u), \varepsilon(v) \rangle_Y, \quad \|v\|_V = \|\varepsilon(v)\|_Y \text{ for all } u, v \in V,$$

where $\varepsilon : H^1(\Omega; \mathbb{R}^d) \rightarrow Y$ denotes the deformation operator, given by

$$\varepsilon(u) = (\varepsilon_{ij}(u)), \quad \varepsilon_{ij}(u) = \frac{1}{2}(x_{i,j} + x_{j,i}).$$

Let $\gamma : V \rightarrow L^2(\Gamma; \mathbb{R}^d)$ be the trace operator. By the Sobolev trace theorem, we have

$$\|v\|_{L^2(\Gamma; \mathbb{R}^d)} \leq \|\gamma\| \|v\|_V \text{ for all } v \in V.$$

For a vector field, notation x_ν and u_τ represent the normal and tangential components of u on Γ given by $x_\nu = u \cdot \nu$ and $u_\tau = u - x_\nu \nu$. Also, σ_ν and σ_τ represent the normal and tangential components of the stress field σ on the boundary, i.e., $\sigma_\nu = (\sigma \nu) \cdot \nu$ and $\sigma_\tau = \sigma \nu - \sigma_\nu \nu$.

Consider the following quasistatic frictional contact problem.

Problem 5.1. Find a displacement field $u : \Omega \times [0, T] \rightarrow \mathbb{R}^d$ and a stress field $\sigma : \Omega \times [0, T] \rightarrow \mathbb{S}^d$ such that

$$\operatorname{Div} \sigma + f_0 = \mathbf{0} \text{ in } \Omega \times \mathbf{I}, \quad (5.1)$$

$$\sigma = \mathbf{B}(\varepsilon({}^c D_{a+}^{\alpha, \psi} u(t))) + \mathbf{A}(\varepsilon(u)) \text{ in } \Omega \times \mathbf{I}, \quad (5.2)$$

$$u = \mathbf{0} \text{ on } \Gamma_1 \times \mathbf{I}, \quad (5.3)$$

$$\sigma \nu = f_N \text{ on } \Gamma_2 \times \mathbf{I}, \quad (5.4)$$

$$u(0) = u_0 \text{ in } \Omega. \quad (5.5)$$

The description of Problem 5.1 can be found in [13, 33]. We will make the following hypotheses for Problem 5.1.

$$\left\{ \begin{array}{l} \mathbf{B} : \Omega \times \mathbb{S}^d \rightarrow \mathbb{S}^d \text{ is such that,} \\ \text{(a) } \mathbf{B}(\cdot, \varepsilon) \text{ is measurable on } \Omega \text{ for all } \varepsilon \in \mathbb{S}^d \text{ and,} \\ \mathbf{B}(w, \cdot) \text{ is linear and bounded on } \mathbb{S}^d \text{ for a.e. } w \in \Omega, \\ \text{(b) there exists } \eta_{\mathbf{B}} > 0 \text{ such that } \mathcal{B}(w, \varepsilon) : \varepsilon \geq \eta_{\mathbf{B}} \|\varepsilon\|^2, \\ \text{for all } \varepsilon \in \mathbb{S}^d, \text{ a.e. } w \in \Omega, \end{array} \right. \quad (5.6)$$

$$\left\{ \begin{array}{l} \mathbf{A}: \Omega \times \mathbb{S}^d \rightarrow \mathbb{S}^d \text{ is such that,} \\ \text{(a) } \mathbf{A}(\cdot, \varepsilon) \text{ is measurable on } \Omega \text{ for all } \varepsilon \in \mathbb{S}^d \text{ and,} \\ \mathbf{A}(u, \cdot) \text{ is weakly continuous on } \mathbb{S}^d \text{ for a.e. } u \in \Omega, \\ \text{(b) there exists } \bar{a}_0, \bar{a}_1 > 0 \text{ such that } \|\mathcal{A}(w, \varepsilon)\| \leq \bar{a}_0 + \bar{a}_1 \|\varepsilon\|, \\ \text{for all } \varepsilon \in \mathbb{S}^d, \text{ a.e. } w \in \Omega. \end{array} \right. \quad (5.7)$$

The densities of body forces and surface tractions are assumed to satisfy

$$f_0 \in L^\infty(0, T; H) \quad \text{and} \quad f_N \in L^\infty(0, T; L^2(\Gamma_2; \mathbb{R}^d)).$$

The function $f: [0, T] \rightarrow V^*$ is given by

$$\langle f(t), v \rangle_V = \langle f_0(t), v \rangle_H + \langle f_N(t), v \rangle_{L^2(\Gamma_2; \mathbb{R}^d)}$$

for all $v \in V$, $t \in [0, T]$.

The variational formulation of Problem 5.1 has the following form.

Problem 5.2. Find $u \in \mathcal{W}$ such that

$$\left\{ \begin{array}{l} \langle \mathbf{B}(\varepsilon({}^c D_{a+}^{\alpha, \psi} u(t))), \varepsilon(v) \rangle_{\mathcal{H}} + \langle \mathbf{A}(\varepsilon(u(t))), \varepsilon(v) \rangle_{\mathcal{H}} = \langle f(t), v \rangle_V, \\ \forall v \in V, \quad \text{a.e. } t \in [0, T], \\ u(0) = u_0. \end{array} \right.$$

Set

$$B: V \rightarrow V^*, \quad \langle B(u), v \rangle = \langle \mathbf{B}(\varepsilon(u)), \varepsilon(v) \rangle_{\mathcal{H}} \quad \text{for } u, v \in V, \quad (5.8)$$

$$A: V \rightarrow V^*, \quad \langle A(u), v \rangle = \langle \mathbf{A}(\varepsilon(u)), \varepsilon(v) \rangle_{\mathcal{H}} \quad \text{for } u, v \in V. \quad (5.9)$$

It is easy to verify the following lemma.

Lemma 5.1. Under the assumption (5.6) the operator B defined by (5.8) satisfies $H(B)$. Under the assumption (5.7) the operator A defined by (5.9) satisfies $H(A)$.

Proof. Since $\mathbf{B}(w, \cdot)$ is linear and bounded on $\mathbb{S}^d$ for a.e. $w \in \Omega$, B is linear and bounded on V , and hence $\mathcal{B}$ is linear and bounded on $\mathcal{V}$, which shows that $H(B)$ holds. $H(A)$ is easy to verify. $\square$

Let us consider the following evolution inclusion.

$$\left\{ \begin{array}{l} B({}^c D_{a+}^{\alpha, \psi} u(t)) + Au(t) = f(t), \quad t \in I, \\ u(0) = u_0. \end{array} \right. \quad (5.10)$$

Theorem 5.1. Under hypotheses (5.6) and (5.7), Problem 5.1 has a solution.

Proof. By Lemma 5.1, all the hypotheses of Theorem 3.1 are satisfied, and hence (5.10) has a solution. Therefore, Problem 5.2 has a solution. $\square$

6. Conclusions

This work presents the result on the existence of solutions to a new class of fractional evolutionary equations involving ψ -Caputo fractional derivative and nonlinear weakly continuous operators. As a result, the main conclusions obtained can be applied in more general practical problems. At the same time, we are able to generalize to more complex equations and mixed fractional order evolutionary inclusion problems by using similar methods. Moreover, in the future, we will consider more general assumptions that are interesting and worthwhile.

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