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GALERKIN-FEM FOR OBTAINING THE NUMERICAL SOLUTION OF THE LINEAR FRACTIONAL KLEIN-GORDON EQUATION

M. M. Khader[1,2,†] and Khadijah M. Abualnaja[3]

Abstract In this paper, an efficient numerical method for solving the linear fractional Klein-Gordon equation (LFKGE) is introduced. The proposed method depends on the Galerkin finite element method (GFEM) using quadratic B-spline base functions and replaces the Caputo fractional derivative using $L2$ discretization formula. The introduced technique reduces LFKGE to a system of algebraic equations, which solved using conjugate gradient method. The study the stability analysis to the approximation obtained by the proposed scheme is given. To test the accuracy of the proposed method we evaluated the error norm L_2. It is shown that the presented scheme is unconditionally stable. Numerical example is given to show the validity and the accuracy of the introduced algorithm.

Keywords Fractional Klein-Gordon equation, Galerkin finite element method, quadratic B-spline functions, stability analysis.

MSC(2010) 41A30, 41A55, 65N12, 65N30.

1. Introduction

Fractional differential equations (FDEs) have been the focus of many studies due to their frequent appearance in various applications in fluid mechanics, viscoelasticity, biology, physics and engineering [15]. Consequently, considerable attention has been given to the solutions of FDEs of physical interest. Most FDEs do not have exact solutions, so approximate and numerical techniques must be used ([1,2,6–11,17]).

Definition 1.1. The Caputo fractional derivative operator D^α of order α is defined as follows:

$$D^\alpha f(x) = \begin{cases} \frac{1}{\Gamma(m-\alpha)} \int_0^x \frac{f^{(m)}(t)}{(x-t)^{\alpha-m+1}} dt, & m-1 < \alpha < m, \quad x > 0; \\ f^{(m)}(x), & \alpha = m. \end{cases}$$

For more details on fractional derivatives definitions and its properties see ([3,18]).

†the corresponding author.
Email address:mohamedmbd@yahoo.com(M. M. Khader)
[1]Department of Mathematics and Statistics, College of Science, Al Imam Mohammad Ibn Saud Islamic University (IMSIU), Riyadh, Saudi Arabia
[2]Department of Mathematics, College of Science, Benha University, Benha, Egypt
[3]Department of Mathematics and Statistics, College of Science, Taif University, Taif, KSA

The Klein-Gordon equation plays a significant role in mathematical physics and many scientific applications such as, solid-state physics, nonlinear optics, and quantum field theory [20]. The equation has attracted much attention for studying solitons [19] and condensed matter physics, in investigating the interaction of solitons in a collisionless plasma, the recurrence of initial states, and in examining the nonlinear wave equations [4]. Wazwaz has obtained the various exact traveling wave solutions such as compactons, solitons and periodic solutions by using the tanh method [20]. The study of numerical solutions of the Klein-Gordon equation has been investigated considerably in the last few years. In the previous studies, the most papers have carried out different spatial discretization of the equation [21].

The main goal in this paper is concerned to obtain the numerical solution of the linear fractional Klein-Gordon equation (LFKGE) of the form

$$D_t^{\alpha} u(x,t) - u_{xx}(x,t) = f(x,t), \quad 0 < x < 1, \quad 0 \le t \le T, \tag{1.1}$$

the parameter α refers to the Caputo fractional derivative with respect to time, $1 < \alpha \le 2$, and the function $f(x,t)$ is a source term. We assume the boundary and initial conditions

$$u(0,t) = 0, \quad u(1,t) = 0. \tag{1.2}$$

$$u(x,0) = g_1(x), \quad u_t(x,0) = g_2(x), \qquad 0 < x < 1, \tag{1.3}$$

here g_1 and g_2 are given functions. Note that, at $\alpha = 2$, Eq.(1.1) is the classical LKGE.

Many of the numerical methods for solving fractional PDEs that have been proposed differ essentially in the way in which the normal and fractional derivatives are discretized. In this paper, the Galerkin FEM is applied to solve numerically the (LFKGE). The main idea behind FEM is to divide the whole region of the given problem into an equivalent system of finite elements with associated nodes and to choose the most appropriate element type to model most closely the actual physical behavior. Thus, by means of the FEM, a huge problem is converted into many solvable small problem. So, the elements must be made small enough to give usable results and yet large enough to reduce computational effort [12].

2. Approximate formula of the Caputo fractional derivative

In this section, we introduce a discrete approximation of the fractional derivative $D_t^{\alpha} u(x,t)$. Here for some positive integers N and M, the grid sizes in space and time for the finite difference algorithm are defined by $h = \frac{1}{N}$ and $\tau = \frac{T}{M}$, respectively. The grid points in the space interval $[0,1]$ are the numbers $x_i = ih,\ i = 0,1,...,N$ and the grid points in the time interval $[0,T]$ are labeled $t_n = n\tau,\ n = 0,1,\dots$. Now, the discrete approximation of the fractional derivative $D_t^{\alpha} u(x,t)$ can be obtained by a simple quadrature formula as follows [14]

$$\begin{aligned} & D_t^{\alpha} u(x_i,t_n) \\ =& \frac{1}{\Gamma(2-\alpha)} \int_0^{t_n} \frac{\partial^2}{\partial s^2} u(x_i,s)(t_n - s)^{1-\alpha} ds \\ =& \frac{1}{\Gamma(2-\alpha)} \sum_{j=1}^{n} \int_{(j-1)\tau}^{j\tau} \left[\frac{u_i^{j+1} - 2u_i^j + u_i^{j-1}}{\tau^2} + O(\tau^2) \right] (n\tau - s)^{1-\alpha} ds \end{aligned}$$

$$=\frac{1}{\Gamma(2-\alpha)(2-\alpha)}\sum_{j=1}^{n}\left[\frac{u_i^{j+1}-2u_i^j+u_i^{j-1}}{\tau^2}+O(\tau^2)\right][(n-j+1)^{2-\alpha}-(n-j)^{2-\alpha}][\tau^{2-\alpha}]$$

$$=\frac{1}{\Gamma(3-\alpha)\tau^{\alpha}}\sum_{j=1}^{n}(u_i^{j+1}-2u_i^j+u_i^{j-1})\left[(n-j+1)^{2-\alpha}-(n-j)^{2-\alpha}\right]$$

$$+\frac{1}{\Gamma(3-\alpha)}\sum_{j=1}^{n}\left[(n-j+1)^{2-\alpha}-(n-j)^{2-\alpha}\right]O(\tau^{4-\alpha}).$$

Setting and shifting indices, we have

$$r_{\alpha,\tau}=\frac{1}{\Gamma(3-\alpha)\tau^{\alpha}}\quad\text{and}\quad\omega_k^{(\alpha)}=(k+1)^{2-\alpha}-k^{2-\alpha},\quad k=0,1,\cdots,\tag{2.1}$$

and

$$D_t^{\alpha}u(x_i,t_n)=r_{\alpha,\tau}\sum_{k=1}^{n}\omega_k^{(\alpha)}(u_i^{n-k+2}-2u_i^{n-k+1}+u_i^{n-k})+\frac{1}{\Gamma(3-\alpha)}n^{2-\alpha}O(\tau^{4-\alpha})$$

$$=r_{\alpha,\tau}\sum_{k=1}^{n}\omega_k^{(\alpha)}(u_i^{n-k+2}-2u_i^{n-k+1}+u_i^{n-k})+O(\tau^2).$$

The second-order approximation formula for the computation of Caputo's fractional derivative is given by the expression

$$D_t^{\alpha}u_i^n\cong r_{\alpha,\tau}\sum_{k=1}^{n}\omega_k^{(\alpha)}(u_i^{n-k+2}-2u_i^{n-k+1}+u_i^{n-k}).\tag{2.2}$$

3. Procedure solution using Galerkin FEM with quadratic B-spline functions

In order to solve Eq.(1.1) with the boundary and initial conditions (1.2)-(1.3) by using Galerkin FEM, we will define quadratic B-spline base functions [13]. Let us consider the interval $[0,1]$ is partitioned into N finite elements of uniformly equal length by the knots $x_m, m=0,1,...,N$ such that $0=x_0<x_1<...<x_N=1$ and $h=x_{m+1}-x_m$. The quadratic B-splines $\{\phi_m(x)\}_{m=-1}^{N}$ at the knots x_m can be defined on $[0,1]$ as follows ([5,16]):

$$\phi_m(x)=\frac{1}{h^2}\begin{cases}(x-x_{m-1})^2, & [x_{m-1},x_m],\\ (x-x_{m-1})^2-3(x-x_m)^2, & [x_m,x_{m+1}],\\ (x-x_{m-1})^2-3(x-x_m)^2+3(x-x_{m+1})^2, & [x_{m+1},x_{m+2}],\\ 0, & \text{otherwise.}\end{cases}\tag{3.1}$$

The set of splines $\{\phi_{-1}(x),\phi_0(x),...,\phi_N(x)\}$ forms a basis for the function defined over $[0,1]$. Therefore, any approximate solution $U_N(x,t)$ can be written in terms of the quadratic B-splines trial functions as follows

$$U_N(x,t)=\sum_{m=-1}^{N}c_m(t)\phi_m(x),\tag{3.2}$$

where $c_m(t)$'s are unknowns, time dependent parameters to be determined from the boundary and weighted residual conditions. Each quadratic B-spline function covers three elements, so that each element $[x_m, x_{m+1}]$ is covered by three quadratic B-spline functions. For this problem, the finite elements are identified with the interval $[x_m, x_{m+1}]$ and the elements knots x_m, x_{m+1}. Using the nodal values, U_m and U'_m are given in terms of the parameter $c_m(t)$ as follows:

$$\begin{aligned} U_N(x_m) &= U_m = c_{m-1} + c_m, \\ U'_N(x_m) &= U'_m = (-2c_{m-1} + 2c_m)/h, \end{aligned} \tag{3.3}$$

the variation of $U_N(x,t)$ over the typical element $[x_m, x_{m+1}]$ can be defined in the form

$$U_N = \sum_{j=m-1}^{m+1} c_j \phi_j. \tag{3.4}$$

Now, the first step of the implementation of the Galerkin method is to construct the weak form of Eq.(1.1) as follows.

Multiply Eq.(1.1) by the weight function $\Psi(x)$ and integrate w.r.t., x from 0 to 1 to obtain

$$\int_0^1 (D^\alpha U - U_{xx})\Psi dx = \int_0^1 f\Psi dx, \tag{3.5}$$

where $\Psi(x)$ is the weighted function taken as quadratic B-spline function. In the previous equation we use the integration by parts to distribute the degree of dependent variable between itself and weight function, so we obtain

$$\int_0^1 (D^\alpha(U)\Psi + U_x\Psi_x)dx = [\Psi\, U_x]_0^1 + \int_0^1 f\Psi dx. \tag{3.6}$$

Since the weak formulation (3.6) is valid on the whole region, particularly it is satisfied over the typical element $[x_m, x_{m+1}]$, thus Eq.(3.6) can be rewritten as follows

$$\int_{x_m}^{x_{m+1}} (D^\alpha(U)\Psi + U_x\Psi_x)dx = [\Psi\, U_x]_{x_m}^{x_{m+1}} + \int_{x_m}^{x_{m+1}} f\Psi dx. \tag{3.7}$$

To change from the global coordinate system into the local one, we use the transformation $\xi = x - x_m$. Thus, the weak form (3.7) transforms into the form

$$\int_0^h (D^\alpha(U)\Psi + U_\xi\Psi_\xi)d\xi = [\Psi\, U_\xi]_0^h + \int_0^h f\Psi d\xi. \tag{3.8}$$

This equation represents the element equation for a typical element e. Also, the basis functions defined in formula (3.1) can be rewritten in the following form

$$\begin{aligned} \phi_{m-1} &= (1 - \frac{\xi}{h})^2, \\ \phi_m &= (1 + \frac{2\xi}{h} - 2(\frac{\xi}{h})^2), \\ \phi_{m+1} &= (\frac{\xi}{h})^2. \end{aligned} \tag{3.9}$$

Substituting from Eqs.(3.9) into Eq.(3.8), we obtain:

$$R^e\dot{C}(t) + S^eC(t) = P^eC(t) + F(t), \tag{3.10}$$

where the dot refers α^{th} fractional derivative with respect to time and R^e_{ij}, S^e_{ij} and P^e_{ij} are element matrices and defined in the following form

$$R^e_{ij} = \int_0^h \phi_i \phi_j d\xi, \quad S^e_{ij} = \int_0^h \phi'_i \phi'_j d\xi, \quad P^e_{ij} = [\phi_i \phi'_j]_0^h, \quad F(t) = \int_0^h f\Psi d\xi, \quad (3.11)$$

where $i, j = m-1, m, m+1$. The element matrices are computed as follows

$$R^e_{ij} = \frac{h}{30}\begin{pmatrix} 6 & 13 & 1 \\ 13 & 54 & 13 \\ 1 & 13 & 6 \end{pmatrix}, \quad S^e_{ij} = \frac{2}{3h}\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}, \quad P^e_{ij} = \frac{2}{h}\begin{pmatrix} 1 & -1 & 0 \\ 1 & -2 & 1 \\ 0 & -1 & 1 \end{pmatrix}.$$

Assembling all contributions coming from all the elements, Eq.(3.10) yields the system

$$R\dot{C}(t) + SC(t) = PC(t) + F(t), \quad (3.12)$$

where $C(t)$ is the vector of unknowns, R, S and P are $(N+2) \times (N+2)$ global matrices with generalized m^{th} row, and defined as follows, respectively,

$$\begin{aligned} R &: \frac{1}{30}(1 \quad 26 \quad 66 \quad 26 \quad 1), \\ S &: \frac{2}{3}(-1 \quad -2 \quad 6 \quad -2 \quad -1), \\ P &: (0 \quad 0 \quad 0 \quad 0 \quad 0). \end{aligned}$$

If time parameter $C(t)$'s and its fractional time derivative $\dot{C}(t)$'s in Eq.(3.12) are discretized by the Crank-Nicolson formula and $L2$ formula, respectively

$$c_m = \frac{1}{2}(c^n_m + c^{n+1}_m), \quad (3.13)$$

and

$$\dot{C} = D^\alpha C = \frac{(\tau)^{-\alpha}}{\Gamma(3-\alpha)} \sum_{k=0}^{n-1} w^{(\alpha)}_k [c^{n-k}_m - 2c^{n-k-1}_m + c^{n-k-2}_m], \quad (3.14)$$

we obtain a recurrence relationship between successive time levels relating unknown parameters c^{n+1}_m as follows

$$\begin{aligned} &(3-\lambda)c^{n+1}_{m-2} + (78-2\lambda)c^{n+1}_{m-1} + (198+6\lambda)c^{n+1}_m + (78-2\lambda)c^{n+1}_{m+1} + (3-\lambda)c^{n+1}_{m+2} \\ &= (6+\lambda)c^n_{m-2} + (156+2\lambda)c^n_{m-1} + (396-6\lambda)c^n_m + (156+2\lambda)c^n_{m+1} + (6+\lambda)c^n_{m+2} \\ &- 3(c^{n-1}_{m-2} + 26c^{n-1}_{m-1} + 66c^{n-1}_m + 26c^{n-1}_{m+1} + c^{n-1}_{m+2}) - 3\sum_{k=1}^{n} w^{(\alpha)}_k [(c^{n-k+1}_{m-2} - 2c^{n-k}_{m-2} + c^{n-k-1}_{m-2}) \\ &+ 26(c^{n-k+1}_{m-1} - 2c^{n-k}_{m-1} + c^{n-k-1}_{m-1}) + 66(c^{n-k+1}_m - 2c^{n-k}_m + c^{n-k-1}_m) \\ &+ 26(c^{n-k+1}_{m+1} - 2c^{n-k}_{m+1} + c^{n-k-1}_{m+1}) + (c^{n-k+1}_{m+2} - 2c^{n-k}_{m+2} + c^{n-k-1}_{m+2})] + f^{n+1}_m, \end{aligned} \quad (3.15)$$

where

$$\lambda = \frac{30(\tau)^\alpha \Gamma(3-\alpha)}{h^2}.$$

The system (3.15) consists of $N+2$ linear equations including $N+2$ unknown parameters $(c_{-1}, c_0, ..., c_N)^T$. Adapting boundary conditions (1.2) to the system (3.12), we obtain $N \times N$ penta-diagonal matrix system.

Initial state

The starting values $d^0 = (c^0_{-1}, c^0_0, ..., c^0_{N-1}, c^0_N)^T$ and $d^1 = (c^1_{-1}, c^1_0, ..., c^1_{N-1}, c^1_N)^T$ are determined from the initial conditions $U(x,0)$ and $U_t(x,0)$ by interpolating using quadratic splines. Using relations at the knots $U_N(x_k,0) = U(x_k,0)$, $(k = 0,1,...,N)$ together with $U'_N(x_N,0) = U'(x_N,0)$, initial vectors d^0 and d^1 can be found from the following matrix equations

$$\begin{bmatrix} 1 & 1 & & & & \\ 1 & 1 & & & & \\ & 1 & 1 & & & \\ & & \ddots & & & \\ & & & 1 & 1 & \\ & & & & \frac{-2}{h} & \frac{2}{h} \end{bmatrix} \begin{bmatrix} c^0_{-1} \\ c^0_0 \\ c^0_1 \\ \vdots \\ c^0_{N-1} \\ c^0_N \end{bmatrix} = \begin{bmatrix} U^0_0 \\ U^0_1 \\ U^0_2 \\ \vdots \\ U^0_N \\ U'_N \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & & & & \\ 1 & 1 & & & & \\ & 1 & 1 & & & \\ & & \ddots & & & \\ & & & 1 & 1 & \\ & & & & \frac{-2}{h} & \frac{2}{h} \end{bmatrix} \begin{bmatrix} c^1_{-1} \\ c^1_0 \\ c^1_1 \\ \vdots \\ c^1_{N-1} \\ c^1_N \end{bmatrix} = \begin{bmatrix} U^1_0 \\ U^1_1 \\ U^1_2 \\ \vdots \\ U^1_N \\ U'_N \end{bmatrix}.$$

4. Stability analysis

In this section, we use the John von Neumann method to study the stability analysis of the finite element scheme (3.15). In our study we will consider the zero source term, i.e., $f(x,t) = 0$. In the fractional John von Neumann stability procedure, the stability of the introduced scheme is decided by putting

$$c^n_m = \xi^n e^{\mathrm{i}m\varphi}. \tag{4.1}$$

Inserting this expression into the finite element scheme (3.15), we get:

$$\begin{aligned} &\xi^{n+1}((3-\lambda)e^{-2\mathrm{i}\varphi} + (78-2\lambda)e^{-\mathrm{i}\varphi} + (198+6\lambda) + (78-2\lambda)e^{\mathrm{i}\varphi} + (3-\lambda)e^{2\mathrm{i}\varphi}) \\ &= \xi^n((6+\lambda)e^{-2\mathrm{i}\varphi} + (156+2\lambda)e^{-\mathrm{i}\varphi} + (396-6\lambda) + (156+2\lambda)e^{\mathrm{i}\varphi} + (6+\lambda)e^{2\mathrm{i}\varphi}) \\ &- 3\xi^{n-1}(e^{-2\mathrm{i}\varphi}+26e^{-\mathrm{i}\varphi}+66+26e^{\mathrm{i}\varphi}+e^{2\mathrm{i}\varphi}) - 3\sum_{k=1}^{n} w^{(\alpha)}_k[(\xi^{n-k+1}-2\xi^{n-k}+\xi^{n-k-1}) \\ &\times (e^{-2\mathrm{i}\varphi} + 26e^{-\mathrm{i}\varphi} + 66 + 26e^{\mathrm{i}\varphi} + e^{2\mathrm{i}\varphi})]. \end{aligned} \tag{4.2}$$

The stability of the scheme is determined by the behavior of ξ^n. In the John von Neumann method, the stability analysis is carried out using the amplification factor ζ defined by

$$\xi^{n+1} = \zeta\xi^n. \tag{4.3}$$

Of course, ζ depends on n. But, let us assume for the moment that, ζ is independent of time. Then, inserting this expression into Eq.(4.2), one gets the following expression for the amplification factor ζ of the LFKGE.

$$\begin{aligned} &\zeta((3-\lambda)e^{-2\mathrm{i}\varphi} + (78-2\lambda)e^{-\mathrm{i}\varphi} + (198+6\lambda) + (78-2\lambda)e^{\mathrm{i}\varphi} + (3-\lambda)e^{2\mathrm{i}\varphi}) \\ &= ((6+\lambda)e^{-2\mathrm{i}\varphi} + (156+2\lambda)e^{-\mathrm{i}\varphi} + (396-6\lambda) + (156+2\lambda)e^{\mathrm{i}\varphi} + (6+\lambda)e^{2\mathrm{i}\varphi}) \\ &- 3\zeta^{-1}(e^{-2\mathrm{i}\varphi}+26e^{-\mathrm{i}\varphi}+66+26e^{\mathrm{i}\varphi}+e^{2\mathrm{i}\varphi}) - 3\sum_{k=1}^{n} w^{(\alpha)}_k[(\zeta^{n-k+1}-2\zeta^{n-k}+\zeta^{n-k-1}) \\ &\times (e^{-2\mathrm{i}\varphi} + 26e^{-\mathrm{i}\varphi} + 66 + 26e^{\mathrm{i}\varphi} + e^{2\mathrm{i}\varphi})]. \end{aligned} \tag{4.4}$$

In view of the Fourier stability analysis, the scheme (4.4) will be stable as long as $|\zeta| < 1$. Considering the extreme value $\zeta = 1$, and from the Eqs.(4.2) and (4.4), we can obtain the following inequality

$$\lambda(2\sin^2(\varphi/2) + \sin^2(\varphi)) \geq 0. \tag{4.5}$$

Substituting the Fourier mode (4.1) into the recurrence relationship (4.4), and following a similar way, we again obtain (4.5). Since $\lambda \geq 0$, both of the schemes are unconditionally stable.

In the following section, we will present the numerical results of the LFKGE by using Galerkin FEM with quadratic B-spline base functions, and the accuracy of the method is measured by the error norm L_2

$$L_2 = ||U^{exact} - U_N||_2 \simeq \sqrt{h\sum_{j=0}^{N} |U_j^{exact} - (U_N)_j|^2}. \tag{4.6}$$

5. Numerical results

In this section, we implement the proposed method to solve LFKGE (1.1) of the form

$$D_t^{\alpha} u(x,t) - u_{xx}(x,t) = f(x,t), \quad 0 < x < 1, \quad t > 0,$$

with the source function $f(x,t) = \alpha![1-(\alpha+1)x]e^{-t} - x^{\alpha}(1-x)e^{-t}$, initial conditions $u(x,0) = x^{\alpha}(1-x), \quad u_t(x,0) = x^{\alpha}(x-1)$, and Dirichlet conditions $u(0,t) = u(1,t) = 0$. The exact solution to this problem is $u(x,t) = x^{\alpha}(1-x)e^{-t}$.

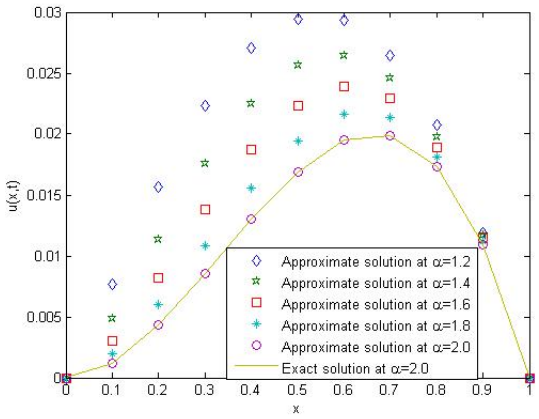

Figure 1. The approximate solution with different values of α at $T = 2$.

The obtained numerical results by means of the proposed method are shown in table 1 and figures 1-2. In table 1, the absolute error between the exact solution u_{ex} and the approximate solution u_{approx} at $N = 40$ and $N = 80$ with the final time $T = 1$ are given, where the L_2-norm in these cases are 2.19×10^{-3} and 4.46×10^{-5}, respectively. In figure 1, the behavior of the approximate solution with different values of α ($\alpha = 1.2, 1.4, 1.6, 1.8, 2.0$) at final time $T = 2$. and in figure 2, the behavior of the approximate solution with different values of N ($N = 30, 50, 70$) at final time $T = 1$ is given.

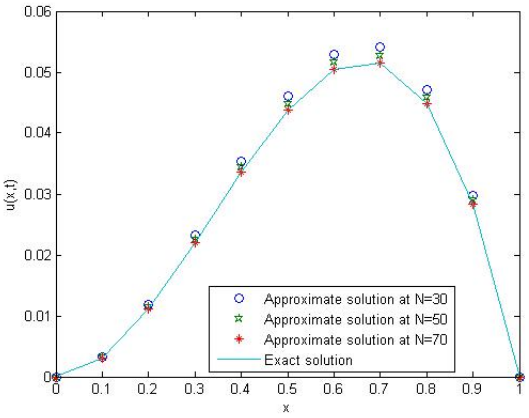

Figure 2. The approximate solution with different values of N at $T = 1$.

Table 1. The absolute error at $N = 40$, $N = 80$ and $T = 1$.

x	$\lvert u_{ex} - u_{approx}\rvert$ at $N = 40$	$\lvert u_{ex} - u_{approx}\rvert$ at $N = 80$
0.0	1.123456e-03	8.124578e-05
0.1	2.789456e-03	2.985621e-05
0.2	4.456123e-03	1.014752e-05
0.3	1.741852e-03	5.789123e-05
0.4	7.852741e-03	1.321456e-05
0.5	5.963456e-03	3.159789e-05
0.6	6.852159e-03	3.159753e-05
0.7	3.123789e-03	1.258741e-05
0.8	1.759510e-03	2.154236e-05
0.9	3.003045e-03	2.056478e-05
1.0	0.012980e-03	0.003456e-05

6. Conclusion and remarks

The properties of the Galerkin finite element method with quadratic B-spline base functions are used to solve numerically the linear fractional Klein-Gordon equation. The introduced technique reduces the proposed problem to the solution of a linear system of algebraic equations which solved by using conjugate gradient method. The Caputo fractional derivative is approximated using $L2$ formula. The solution obtained using the suggested method is in excellent agreement with the exact solution and shows that this approach can be solved the problem effectively and applicable to many other problems. It is evident that the overall errors can be made smaller by adding new terms from the series (3.2). Comparisons are made between approximate solution and exact solution to illustrate the validity and the great potential of the technique. All computations in this paper are done using Matlab 8.

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