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ON AN ITERATIVE METHOD FOR A CLASS OF 2 POINT & 3 POINT NONLINEAR SBVPS

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Abstract In this article, we propose a novel modification to Quasi-Newton method, which is now a days popularly known as variation iteration method (VIM) and use it to solve the following class of nonlinear singular differential equations which arises in chemistry $-y''(x) - \frac{\alpha}{x}y'(x) = f(x, y),\ x \in (0, 1)$, where $\alpha \geq 1$, subject to certain two point and three point boundary conditions. We compute the relaxation parameter as a function of Bessel and the modified Bessel functions. Since rate of convergence of solutions to the iterative scheme depends on the relaxation parameter, thus we can have faster convergence. We validate our results for two point and three point boundary conditions. We allow $\partial f/\partial y$ to take both positive and negative values.

Keywords Singular differential equation, quasi-Newton method, Bessel function, modified Bessel function, two point boundary condition, three point boundary condition.

MSC(2010) 34B16, 34B27, 34B60.

1. Introduction

In [39], we proposed that the appropriate equation for the thermal balance between the heat generated by the chemical reaction and that the heat conducted away can be written as

$$\nabla^2 u(P) = f(P, u(P), du(P)/dP),$$

which gives rise to singular differential equation given by

$$-(x^\alpha y')' = x^\alpha f(x, y, x^\alpha y'),\ x \in (0, 1),$$

whenever we are interested in radially symmetric solutions. Such nonlinear singular boundary value problems (SBVPs) pose great challenges to researchers, due to unpredictable behavior of their solutions in the neighborhood of the singular points. It is also not easy to obtain the closed form solutions of these nonlinear SBVPs. So, researchers try to develop efficient numerical schemes which can produce accurate and stable results.

The main objective of our work is to compute the relaxation parameter or Lagrange's multiplier for Quasi-Newton's method (VIM) (see [14–19, 31, 35–37, 40, 41]

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and find the approximate solutions of the following class of nonlinear SBVPs

$$-(x^\alpha y')' = x^\alpha f(x, y), \quad 0 < x < 1, \quad ' \equiv \frac{d}{dx}, \tag{1.1}$$

$$y'(0) = A, \quad a_1 y(1) + b_1 y'(1) = c_1, \text{ or} \tag{1.2}$$

$$y'(0) = 0, \quad y(1) = \delta y(\eta), \ \delta > 0, \ 0 < \eta < 1, \tag{1.3}$$

where α, A, a_1, b_1, c_1 are real constants and $\alpha \geq 1$. Here we assume that $f(x, y)$ is continuous and Lipschitz continuous in $D = \{(x, y) \in [0, 1] \times \mathbb{R}\}$.

Chandrashekhar [4] in Chapter IV (Polytropic and Isothermal Gas Spheres) derived Lane-Emden equation of index n ($\alpha = 2, \ f(x, y) = y^n$). SBVP (1.1) is also derived by several researchers (see [1,5,23,25]).

As far as analytical results are considered, enormous literature is available for two point SBVP (see [6,12,26,27,33]). Russell and Shampine [33] showed that the above class have unique solution for $\alpha = 1$ if $K < j_0^2$, where j_0 is the first positive zero of Bessel function $J_0(x)$, for $\alpha = 2$, the problem has unique solution if $K < \pi^2$, where K is Lipschitz constant. Chawla and Shivakumar [6] have shown that the SBVP (1.1)-(1.2) has unique solution for all α, if $K = \frac{\partial f}{\partial y} < K_1^2$, where K_1 is the first zero of $J_{\left(\frac{\alpha-1}{2}\right)}(\sqrt{K})$. El-Gebeily and Boumenir [12] have shown that the problem has a unique solution for certain boundary conditions under the assumption that the range of $\frac{\partial f}{\partial y}$ has empty intersection with the closure of the spectrum of the singular differential operator, where f denotes the nonlinearity. Pandey and Verma [26,27] generalized some of these results for a general class of SBVPs.

The numerical solutions of these SBVP have been discussed by several methods such as Cubic spline and B-spline methods ([7, 32, 34]), mixed decomposition-spline method (MDSM) [20], finite difference method ([8,28–30]), two fold spline Chebyshev linearization apporach [21], patching approach [22]. These methods are very popular and have several advantages, but they need a lot of computational work.

Iterative methods are preferred over other numerical methods as they take less computational work and provide highly accurate approximations or even exact solutions. Recently, researchers have used Adomian decomposition method (ADM), modified Adomian decomposition method (MDM) and Homotopy analysis method (HAM) ([2,3,9,11]) for non-linear SBVPs.

Literature survey shows that Quasi-Newton method also referred as varitional iteration methods (VIMs) (see [14–19,31,35–37,40,41]) are very efficient for solving nonlinear differential equations. Ravikanth et al. [31], Wazwaz [40] and Singh et al. [37] considered class of SBVPs (1.1)-(1.2) and discussed certain aspects of iterative scheme referred as variational method. Variation iteration methods are still under investigation, recently, e.g., Daliri et al. [10] used it on a class of nonlinear Fredholm integral equations, Zhang et al. [42] discussed it on a family of fifth-order convergent methods for solving nonlinear equations, Ghorbani et al. [13] used it on nonlinear two-point boundary value problems, Zellal et al. [43] used it on biological population model, Noeiaghdam [24] used it on epidemiological model of computer viruses.

As far as analytical results of three point singular BVPs are considered, the reader is suggested to read ([38,39] and the references there in). But there are very few papers which talk about numerical solutions for a class of nonlinear singular three point BVPs. So, the result of this paper fills the gap existing in literature on numerical solutions for class of singular three point BVPs.

In this paper, we propose a modification to Quasi-Newton method (or VIM) and use it to solve a class of two/three point nonlinear SBVPs (1.1)-(1.3). We generalize the relaxation parameter (or Lagranges multiplier) (λ) and compute it as a function of the variable (ω). The relaxation parameter (λ) is expressed in terms of Bessel and modified Bessel functions.

For two point SBVPs, when $\omega = 0$ our results coincide with the results in [31,40]. For positive values of ω our scheme converges faster. We allow $\frac{\partial f}{\partial y}$ to take both positive and negative values.

For three point SBVPs, we have given two examples whose results do not exist in literature.

We have organized this paper into the following sections. In section 2, we discuss the basic ideas of Quasi-Newton iteration method (or VIM) and its convergence. In section 3, we verify our results with suitable test examples.

2. The basic idea of Quasi-Newton iteration method (or VIM)

Roots of nonlinear equation $\phi(x) = 0$ can be computed by Newton's method, which we can write as

$$x_{n+1} = x_n - \frac{\phi(x_n)}{\phi'(x_n)}.$$

If we replace $\frac{1}{\phi'(x_n)}$ by an approximation (say λ), the resulting method

$$x_{n+1} = x_n - \lambda\phi(x_n)$$

is then referred as Quasi-Newton iteration method. This approximation λ is referred as relaxation parameter. Since x_n is an approximated root of $\phi(x_n) \neq 0$, so, we look for an optimal value of λ such that the difference $x_{n+1} - x_n$ which is equal to $\lambda\phi(x_n)$ is minimized. Using this optimal value of λ, we generate a sequence $\{x_n\}$ which converges to a root of the nonlinear equation.

The solution of the differential equation (1.1) is a zero of (1.1). So, the above ideas can be used to compute the solution of (1.1). In next paragraph we will discuss some of the preliminary results which will use the above concepts.

Schunk [35] used these concepts, to calculate the bending of cylindrical panels while Zhukov [41] used this method for thin rectangular slabs. The method was strengthened by Kirichenko and Krys'ko [18]. They considered a class of equations which was described by positive definite operators. Inokuti et al. [17] referred the relaxation parameter as Lagrange's parameter and solved the nonlinear equations, which may involve algebraic, differential, integral, or finite difference operators. He ([14–16]) popularized this method and after that several authors started referring this method as He's variation iteration method.

In variational iteration method [14], the following non-linear differential equation is considered

$$L(y) + N(y) = g(x), \tag{2.1}$$

where L is a linear operator, N is a non-linear operator and $g(x)$ is the source term. We can write correction functional as

$$y_{n+1}(x) = y_n + \int_0^x \lambda \left[L(y_n(t)) + N(\tilde{y}_n(t)) - g(t)\right] dt, \qquad n \geq 0, \tag{2.2}$$

where $\tilde{y}_n$ is treated as restricted variation, i.e., $\delta\tilde{y}_n = 0$ and λ is the relaxation parameter which is identified optimally with the help of variational theory.

2.1. Relaxation parameter in terms of Bessel functions

For the nonlinear SBVPs (1.1), we define the following iterative scheme

$$y_{n+1}(x) = y_n + \int_0^x \lambda \left(-\ddot{y}_n(t) - \frac{\alpha}{t}\dot{y}_n(t) - \tilde{f}(t, y_n)\right) dt, \qquad n \geq 0. \tag{2.3}$$

To find an optimal value of relaxation parameter "λ", such that the second term (right hand side) of the above equation will be minimized, we write the iterative scheme (correction functional) as suggested by Soltani and Shirzadi (see [36]))

$$y_{n+1}(x) = y_n(x) + \int_0^x \lambda \left(-\ddot{y}_n(t) - \frac{\alpha}{t}\dot{y}_n(t) - \omega y_n(t) - \tilde{f}(t, y_n) + \omega\tilde{y}_n(t)\right) dt, \tag{2.4}$$

where $\dot{} \equiv \frac{d}{dt}$. When $\omega = 0$, this scheme is same as considered in [31, 40].

By taking the variation on both sides of (2.3), we get

$$\delta y_{n+1}(x) = \delta y_n(x) + \delta \int_0^x \lambda \left(-\ddot{y}_n(t) - \frac{\alpha}{t}\dot{y}_n(t) - \omega y_n(t) - \tilde{f}(t, y_n) + \omega\tilde{y}_n(t)\right) dt.$$

Hence

$$\delta y_{n+1}(x) = \delta y_n(x) + \delta \int_0^x \lambda \left(-\ddot{y}_n(t) - \frac{\alpha}{t}\dot{y}_n(t) - \omega y_n(t)\right) dt, \text{ where } \delta\tilde{y}_n = 0.$$

Integrating by parts, we get

$$\begin{aligned}\delta y_{n+1}(x) &= \left(1 + \lambda_x(x) - \frac{\alpha\lambda(x)}{x}\right)\delta y_n(x) - \delta\lambda(x) y_n'(x) \\ &\quad - \int_0^x \left(\lambda_{tt} - \alpha\frac{(t\lambda_t - \lambda)}{t^2} + \omega\lambda\right)\delta y_n(t) dt \\ &= 0.\end{aligned}$$

Hence, we get

$$1 + \lambda_x(x) - \frac{\alpha\lambda(x)}{x} = 0, \tag{2.5}$$

$$\lambda(x) = 0, \tag{2.6}$$

$$-\lambda_{tt}(t) + \alpha\frac{(t\lambda_t(t) - \lambda(t))}{t^2} - \omega\lambda(t) = 0. \tag{2.7}$$

We can write (2.7) as follows

$$t^2\lambda_{tt} - t\alpha\lambda_t + (\alpha + t^2\omega)\lambda = 0. \tag{2.8}$$

The Standard Bessel's equation given by

$$z^2\frac{d^2w}{dz^2} + z\frac{dw}{dz} + (z^2 - \nu^2)w = 0, \tag{2.9}$$

is transformed (Lommel's transformations $z = \beta\zeta^\gamma$, $w = \zeta^{-a}v(\zeta)$) into (2.10)

$$\zeta^2 \frac{d^2v}{d\zeta^2} + \zeta(1-2a)\frac{dv}{d\zeta} + \left[(\beta\gamma\zeta^\gamma)^2 + (a^2 - \nu^2\gamma^2)\right] v = 0. \tag{2.10}$$

Now, if we set $\nu = \frac{(1-\alpha)}{2}$, $a = \frac{(1+\alpha)}{2}$, $\gamma = 1$, $\beta^2 = \omega$, $\zeta = t$ then (2.10) is reduced into (2.8). The transformed Bessel's equation (2.10) has two linearly independent solutions, which are defined as

$$v_1(\zeta) = \zeta^a w_1\left(\beta\zeta^\gamma\right), \quad v_2(\zeta) = \zeta^a w_2\left(\beta\zeta^\gamma\right), \tag{2.11}$$

where $w_1(z)$ and $w_2(z)$ are two linearly independent solutions of the Bessel's equation (2.9). Hence, we obtain two linearly independent solutions of (2.8) in terms of $w_1(z)$ and $w_2(z)$. The bounded solution of (2.8) is given by $t^{\nu+\alpha} J_{-\nu}\left(t\sqrt{\omega}\right)$, if $\omega > 0$ or $t^{\nu+\alpha} I_{-\nu}\left(t\sqrt{\omega}\right)$, if $\omega < 0$, where $J_{-\nu}$ and $Y_{-\nu}$ are Bessel functions of first and second kind, respectively and $I_{-\nu}$ and K_ν are modified Bessel functions of first and second kind, respectively.

By using the conditions (2.5) and (2.6), we obtain the optimal values of the relaxation parameter. For $\omega > 0$, we get

$$\lambda(t) = \frac{\pi x t^\nu t^\alpha}{2x^\nu x^\alpha}\left[J_{-\nu}\left(t\sqrt{\omega}\right) Y_{-\nu}\left(x\sqrt{\omega}\right) - J_{-\nu}\left(x\sqrt{\omega}\right) Y_{-\nu}\left(t\sqrt{\omega}\right)\right], \tag{2.12}$$

and similarly for $\omega < 0$, we get

$$\lambda(t) = \frac{t^\alpha t^\nu\ x}{x^\nu\ x^\alpha}\left[I_{-\nu}\left(x\sqrt{|\omega|}\right) K_\nu\left(t\sqrt{|\omega|}\right) - I_{-\nu}\left(t\sqrt{|\omega|}\right) K_\nu\left(x\sqrt{|\omega|}\right)\right]. \tag{2.13}$$

The successive approximation y_{n+1}, $n \geq 0$ can be computed from the correctional functional (2.3), where λ is given by equation ((2.12) or (2.13)), and the sequence $\{y_n(x)\}$ converges uniformly (will be proved in the next section) to the exact solution (say $y(x)$) of the nonlinear SBVP (1.1), where the initial approximation y_0 may be chosen, so that it satisfies at least the initial or boundary conditions.

2.2. Convergence Analysis

To prove that, the limit of the sequence $\{y_n(x)\}$ obtained from (2.3) will converge to the solutions of (1.1)-(1.2), we have to prove that the sequence is convergent.

It is clear that

$$y_0(x) + \sum_{i=1}^{n}(y_i - y_{i-1}) = y_n(x) \tag{2.14}$$

is n^{th} partial sum of the infinite series

$$y_0(x) + \sum_{i=1}^{\infty}(y_i - y_{i-1}). \tag{2.15}$$

Therefore, to prove that the sequence $\{y_n(x)\}$ converges (uniformly), it is enough to prove that (2.15) converges (uniformly).

Theorem 2.1. *Let $\omega > 0$, $y_n(x) \in C^2[0,1]$, for $n = 0, 1, 2, \cdots$, and there exist $N \geq 0$ such that for all $f(x,u)$, $f(x,v) \in D$*

$$|f(x,u) - f(x,v)| \leq N|u - v|, \tag{2.16}$$

where $D = \{(x, y) \in [0, 1] \times R\}$*, then the sequences defined by (2.14), where* y_{n+1} *is given by* (2.3)*, will converge uniformly to the exact solutions of nonlinear SBVP (1.1)-(1.2).*

Proof. For $n = 0$ from (2.3) we get,

$$y_1(x) = y_0(x) - \int_0^x \lambda \left(\ddot{y}_0(t) + \frac{\alpha}{t}\dot{y}_0(t) + \tilde{f}(t, y_0)\right) dt.$$

Integrating by parts and using equations (2.5)–(2.7) on the right hand side of above equation, we get

$$|y_1(x) - y_0(x)| = \left|-\int_0^x \left(\left(-\lambda_t(t) + \frac{\alpha\lambda(t)}{t}\right)\dot{y}_0(t) + \lambda\tilde{f}(t, y_0)\right) dt\right| \tag{2.17}$$

$$\leq \int_0^x \left(\left|-\lambda_t(t) + \frac{\alpha\lambda(t)}{t}\right| |\dot{y}_0(t)| + \left|\lambda\tilde{f}(t, y_0)\right|\right) dt. \tag{2.18}$$

For $n = 1$ from (2.3) by similar analysis, we get

$$|y_2(x) - y_1(x)| = \Bigg| \int_0^x \Bigg(\left(-\lambda_{tt}(t) + \alpha\frac{(t\lambda_t(t) - \lambda)}{t^2}\right)(y_1(t) - y_0(t))$$

$$+ \lambda\left(\tilde{f}(t, y_1) - \tilde{f}(t, y_0)\right)\Bigg)dt\Bigg| \tag{2.19}$$

$$\leq \left|\int_0^x \lambda\left(\omega(y_1(t) - y_0(t)) - \left(\tilde{f}(t, y_1) - \tilde{f}(t, y_0)\right)\right) dt\right| \tag{2.20}$$

$$\leq \int_0^x |\lambda| \left(|\omega| \, |(y_1(t) - y_0(t))| + \left|\left(\tilde{f}(t, y_1) - \tilde{f}(t, y_0)\right)\right|\right) dt. \tag{2.21}$$

Further, by using Lipschitz condition, we get

$$|y_2(x) - y_1(x)| \leq \int_0^x |\lambda| \left(|(\omega + N)| \, |(y_1(t) - y_0(t))|\right) dt, \tag{2.22}$$

where N is Lipschitz constant. In general, we have

$$|y_{n+1}(x) - y_n(x)| \leq \int_0^x |\lambda| \left(|(\omega + N)| \, |(y_n(t) - y_{n-1}(t))|\right) dt. \tag{2.23}$$

Using series expansion of $J_{-\nu}$, $Y_{-\nu}$, $I_{-\nu}$ and K_ν (Appendix 5), we can easily conclude

$$\left|\frac{\lambda}{t}\right| \quad \& \quad \lambda_t$$

are bounded for all $t \leq x \leq 1$ and $\alpha \geq 1$. So, we define

$$(M_1)_\infty = \sup\left\{\left|-\lambda_t(t) + \frac{\alpha\lambda(t)}{t}\right| |\dot{y}_0(t)| + \left|\lambda\tilde{f}(t, y_0)\right|\right\}, \tag{2.24}$$

$$(M_2)_\infty = \sup\left\{|\lambda| \, |(\omega + N)|\right\}. \tag{2.25}$$

Consider

$$M = \max\left\{(M_1)_\infty, (M_2)_\infty\right\}. \tag{2.26}$$

From equations (2.18), (2.24) and (2.26), we get

$$|y_1(x) - y_0(x)| \leq \int_0^x (M_1)_\infty dt \leq \int_0^x M dt = Mx. \tag{2.27}$$

Similarly from equation (2.22),(2.25) and (2.26) we get

$$|y_2(x) - y_1(x)| \leq \int_0^x (M_2)_\infty |y_1(t) - y_0(t)| dt \leq \int_0^x M \times Mt dt = \frac{M^2 x^2}{2!}. \tag{2.28}$$

In general,

$$\begin{aligned} |y_{n+1}(x) - y_n(x)| &\leq \int_0^x (M_2)_\infty |y_n(t) - y_{n-1}(t)| dt \leq \int_0^x M \times \frac{M^n t^n}{n!} dt \\ &= \frac{M^{n+1} x^{n+1}}{(n+1)!}, \quad \forall x \in [0,1]. \end{aligned} \tag{2.29}$$

As the series $\sum_{n=0}^{\infty} \frac{M^{n+1}x^{n+1}}{(n+1)!}$ is convergent, $\forall x \in [0,1]$, therefore the series defined by (2.15)

$$|y_0(x)| + \sum_{i=1}^{\infty} |(y_i(x) - y_{i-1}(x))| \leq |y_0(x)| + \sum_{n=0}^{\infty} \frac{M^n x^n}{(n)!}, \tag{2.30}$$

is absolutely convergent, i.e., the sequence of partial sums is convergent for $x \in [0,1]$. Hence, by the Weierstrass M-Test

$$|y_0(x)| + \sum_{i=1}^{\infty} |(y_i(x) - y_{i-1}(x))|$$

converges uniformly $\forall x \in [0,1]$.

Similarly, from the convergence analysis for $\omega < 0$, we arrive at the following theorem.

Theorem 2.2. *Let $\omega < 0$, $y_n(x) \in C^2[0,1]$ for $n \in 0,1,2,\cdots$ and that there exist $N \geq 0$ such that for all $f(x,u)$, $f(x,v) \in D$,*

$$|f(x,u) - f(x,v)| \leq N|u - v|, \tag{2.31}$$

then the sequence defined by (2.14) where y_{n+1} is given by (2.3)*, will converge uniformly to the exact solutions of nonlinear SBVP (1.1)-(1.2).*

3. Numerical Illustrations

In this section, we consider four examples of two point and two examples of three point SBVPs. We illustrate that our modified version of Quasi-Newton's method gives very good results. In the limiting case $\omega \to 0$, the numerical and analytical results are exactly the same in result ([31, 40]). We use (2.3) to compute the solution. We introduce ω to have a better control over convergence. In the limiting case, since ωy_n and $\omega \tilde{y_n}$ will be canceled, so it does not affect our original iterative scheme. Thus, we use (2.3), but as λ is the new relaxation parameter which involves ω, so this method is not same as proposed in [31].

Example 3.1. Consider the linear singular two point boundary value problem

$$-y''(x) - \frac{1}{x}y'(x) = y(x) - \frac{5}{4} - \frac{x^2}{16}, \qquad 0 < x < 1, \tag{3.1}$$

$$y'(0) = 0, \quad y(1) = \frac{17}{16}. \tag{3.2}$$

Solution:
The exact solution of this problem is $y(x) = 1 + \frac{x^2}{16}$. Here $\frac{\partial f}{\partial y} > 0$. Now by using the equation (2.3), we get

$$y_{n+1}(x) = y_n(x) - \int_0^x \lambda \left(\ddot{y}_n(t) + \frac{1}{t}\dot{y}_n(t) + y_n(t) - \frac{5}{4} - \frac{t^2}{16} \right) dt. \tag{3.3}$$

Here, λ is given by (2.12). Using the equation (3.3) with initial approximation $y_0(x) = a$, we get the following successive approximations

$$\begin{aligned} y_0(x) =& a, \\ y_1(x) =& \frac{((4a-5)\omega+1)J_0\left(\sqrt{\omega}x\right)}{4\omega^2} - \frac{16a - x^2 - 20}{16\omega} + a - \frac{1}{4\omega^2}, \\ &\vdots \end{aligned}$$

Now, we find the values of a by imposing the boundary condition '$y(1) = \frac{17}{16}$' on the above approximations for different values of ω. The solutions at different space points are displayed in tables 1 and 2.

Table 1. Solution (y_1) of Example 3.1 for different values of ω.

x/y_1	$\omega = 0$ ([31])	$\omega = 0.1$	$\omega = 0.2$	$\omega = 0.3$	$\omega = 0.6$	$\omega = 0.72$	$\omega = 0.9$	Exact
0	0.994792	0.995335	0.995874	0.996407	0.997976	0.998591	0.999501	1
0.1	0.99543	0.995972	0.996509	0.997041	0.998606	0.99922	1.00013	1.000625
0.2	0.99735	0.997887	0.99842	0.998947	1.0005	1.00111	1.00201	1.0025
0.3	1.00057	1.00109	1.00162	1.00213	1.00366	1.00426	1.00514	1.005625
0.4	1.0051	1.00561	1.00612	1.00662	1.0081	1.00868	1.00953	1.01
0.5	1.01099	1.01147	1.01195	1.01243	1.01382	1.01437	1.01518	1.015625
0.6	1.01827	1.01871	1.01915	1.01958	1.02086	1.02136	1.0221	1.0225
0.7	1.02699	1.02737	1.02775	1.02812	1.02922	1.02965	1.03028	1.030625
0.8	1.03723	1.03752	1.0378	1.03809	1.03893	1.03925	1.03974	1.04
0.9	1.04903	1.0492	1.04937	1.04953	1.05001	1.0502	1.05047	1.050625
1	1.0625	1.0625	1.0625	1.0625	1.0625	1.0625	1.0625	1.0625

Table 2. Solution (y_2) of Example 3.1 for different values of ω.

x/y_2	$\omega = 0$ ([31])	$\omega = 0.1$	$\omega = 0.2$	$\omega = 0.3$	$\omega = 0.6$	$\omega = 0.72$	$\omega = 0.9$	Exact
0	1.00014	1.00011	1.00009	1.00007	1.00002	1.00001	1	1
0.1	1.00077	1.00074	1.00071	1.00069	1.00065	1.00064	1.00063	1.000625
0.2	1.00264	1.00261	1.00259	1.00257	1.00252	1.00251	1.0025	1.0025
0.3	1.00576	1.00574	1.00571	1.00569	1.00565	1.00564	1.00563	1.005625
0.4	1.01014	1.01011	1.01009	1.01007	1.01002	1.01001	1.01	1.01
0.5	1.01576	1.01573	1.01571	1.01569	1.01565	1.01564	1.01563	1.015625
0.6	1.02262	1.0226	1.02258	1.02256	1.02252	1.02251	1.0225	1.0225
0.7	1.03074	1.03072	1.0307	1.03068	1.03064	1.03063	1.03063	1.030625
0.8	1.04009	1.04007	1.04006	1.04004	1.04001	1.04001	1.04	1.04
0.9	1.05068	1.05067	1.05066	1.05065	1.05063	1.05063	1.05063	1.050625
1	1.0625	1.0625	1.0625	1.0625	1.0625	1.0625	1.0625	1.0625

Example 3.2. Consider the following nonlinear two-point SBVP (α=2 and $f(x,y)$= y^γ), derived by Chandrasekhar ([4]) where γ is a physical constant, in connection with the equilibrium of iso thermal gas spheres. We consider the case of $\gamma = 5$.

$$-y''(x) - \frac{2}{x}y'(x) = y^5, \qquad 0 < x < 1, \tag{3.4}$$

$$y'(0) = 0, \quad y(1) = \sqrt{\frac{3}{4}}. \tag{3.5}$$

Solution:
The exact solution of this problem is $y(x) = (1 + \frac{x^2}{3})^{-\frac{1}{2}}$. Here $\frac{\partial f}{\partial y} > 0$. Now by using the equation (2.3), we get

$$y_{n+1}(x) = y_n(x) - \int_0^x \lambda \left(\ddot{y}_n(t) + \frac{1}{t}\dot{y}_n(t) + (y_n(t))^5 \right) dt, \tag{3.6}$$

where λ is defined by (2.12). Using the equation (3.6) with initial approximation $y_0(x) = a$, we get the following successive approximations (to save some space we do not write $y_2(x)$)

$$\begin{aligned} y_0(x) =& a, \\ y_1(x) =& a - \frac{a^5\left(\sqrt{\omega}x - \sin\left(\sqrt{\omega}x\right)\right)}{\omega^{3/2}x}, \\ & \vdots \end{aligned}$$

Now we find the values of a by imposing the boundary condition $y(1) = \sqrt{\frac{3}{4}}$ on the above approximations for different values of ω. The solutions at different space points are displayed in table 3.

Table 3. Solution (y_2) of Example 3.2 for different values of ω.

x/y_2	$\omega = 0$ ([31])	$\omega = 0.1$	$\omega = 0.2$	$\omega = 1$	$\omega = 2$	$\omega = 2.3$	Exact
0	0.993678	0.993989	0.994293	0.996453	0.998454	0.9989	1
0.1	0.992067	0.992376	0.992677	0.994819	0.996804	0.997247	0.998337488
0.2	0.987282	0.987583	0.987878	0.989967	0.991904	0.992336	0.993399268
0.3	0.979461	0.97975	0.980032	0.982038	0.983896	0.98431	0.985329278
0.4	0.968827	0.969099	0.969366	0.971256	0.973006	0.973397	0.974354704
0.5	0.955679	0.95593	0.956176	0.95792	0.959533	0.959892	0.960768923
0.6	0.940377	0.940602	0.940822	0.942382	0.94382	0.94414	0.944911183
0.7	0.923325	0.923517	0.923704	0.925027	0.926243	0.926512	0.927145541
0.8	0.904958	0.905104	0.905248	0.906258	0.90718	0.907382	0.907841299
0.9	0.885714	0.885799	0.885883	0.886468	0.886997	0.887112	0.887356509
1	0.866025	0.866025	0.866025	0.866025	0.866025	0.866025	0.866025404

Example 3.3. Consider the nonlinear two point SBVP ([20]),

$$-y''(x) - \frac{1}{x}y'(x) = e^y, \qquad 0 < x < 1, \tag{3.7}$$

$$y'(0) = 0, \quad y(1) = 0. \tag{3.8}$$

Solution:
The exact solution of this problem is $y(x) = 2\ \ln\left(\frac{C+1}{C\ x^2+1}\right)$, where $C = 3 - 2\sqrt{2}$. Here $\frac{\partial f}{\partial y} > 0$. Now by using the equations (2.3), we get

$$y_{n+1}(x) = y_n(x) - \int_0^x \lambda\left(\ddot{y}_n(t) + \frac{1}{t}\dot{y}_n(t) + e^{y_n(t)}\right) dt, \tag{3.9}$$

where λ is defined by (2.12). Using the equation (3.9) with initial approximation $y_0(x) = a$, we get the successive approximations. Since expressions are lengthy, we are not mentioning it here. We find the values of a by imposing the boundary condition $y(1) = 0$ on the above approximations for different values of ω. The solutions at different space points are displayed in table 4.

Table 4. Solution (y_1) of Example 3.3 for different values of ω.

x/y_1	$\omega = 0$	$\omega = 0.2$	$\omega = 0.4$	$\omega = 0.6$	$\omega = 0.7$	$\omega = 0.78$	Exact
0	0.357403	0.350549	0.343887	0.337407	0.334233	0.331724	0.316694
0.1	0.353829	0.347	0.340362	0.333905	0.330742	0.328242	0.313266
0.2	0.343107	0.336358	0.329797	0.323415	0.320288	0.317817	0.303015
0.3	0.325237	0.318639	0.312224	0.305984	0.302927	0.30051	0.286047
0.4	0.300218	0.293869	0.287696	0.28169	0.278748	0.276422	0.262531
0.5	0.268052	0.262086	0.256285	0.250643	0.247879	0.245694	0.232697
0.6	0.228738	0.223337	0.218087	0.212982	0.210481	0.208504	0.196827
0.7	0.182276	0.177681	0.173216	0.168875	0.16675	0.16507	0.155248
0.8	0.128665	0.125185	0.121805	0.118521	0.116914	0.115645	0.108323
0.9	0.0679066	0.0659284	0.0640089	0.0621457	0.0612344	0.0605148	0.0564386
1	-5.55112E-17	-1.11022E-16	1.11022E-16	-5.55112E-17	5.55112E-17	0	0

Example 3.4. * Consider the following nonlinear two-point SBVP which occurs in diffusion problems with Michaelis-Menten kinetics ([1]),

$$y''(x) + \frac{2}{x}y'(x) = \frac{ny}{y+k}, \quad 0 < x < 1, \tag{3.10}$$

$$y'(0) = 0, \quad 5y(1) + y'(1) = 5, \tag{3.11}$$

where $n = 0.76129$ and $k = 0.03119$.

Solution:
Here $\frac{\partial f}{\partial y} < 0$. Now by using the equations (2.3), we get

$$y_{n+1}(x) = y_n(x) - \int_0^x \lambda\left(\ddot{y}_n(t) + \frac{2}{t}\dot{y}_n(t) - \frac{0.76129\ y}{y + 0.03119}\right) dt, \tag{3.12}$$

where λ is defined by (2.13). Using the equation (3.12) with initial approximation $y_0(x) = a$, we get the following successive approximations

$$\begin{aligned} y_0(x) =& a, \\ y_1(x) =& a - \frac{\frac{0.76129ax}{\omega} - \frac{0.76129a\left(\sinh\left(\sqrt{\omega}x\right)\right)}{\omega^{3/2}}}{ax + 0.03119x} \\ &\vdots \end{aligned}$$

*Exact solution is not known.

Now, we compute the values of a by using the boundary condition '$y(1) = 0$' on the above approximations for different values of ω. The solutions at different space points are displayed in table 5. The exact solution of this problem is not available. So, making use of absolute residual error (table 6), we show the efficiency of our technique and show how well the approximate solution satisfies nonlinear SBVP (1.1)–(1.2). We plot the residual errors and figure 1 shows that residual error is more near $x = 1$, but near the point of singularity it is very very less. Residual error is defined as follows

$$R_\omega = \left| y''(x) + \frac{2}{x}y'(x) - \frac{ny}{y+k} \right|,$$

where $n = 0.76129$ and $k = 0.03119$.

Table 5. Solution (y_1) of Example 3.4 for different values of ω.

x/y_1	$\omega = -3$	$\omega = -1$	$\omega = -0.5$	$\omega = -0.1$	$\omega = 0$	$\omega = 0$ ([31])
0	0.793101	0.817568	0.823268	0.827713	0.828808	0.828808024
0.1	0.794324	0.818791	0.824491	0.828935	0.830031	0.830030824
0.2	0.798014	0.822467	0.828163	0.832605	0.833699	0.833699224
0.3	0.804238	0.828618	0.834295	0.838722	0.839813	0.839813223
0.4	0.813109	0.83728	0.842906	0.847292	0.848373	0.848372822
0.5	0.824787	0.848507	0.854022	0.858319	0.859378	0.85937802
0.6	0.839486	0.862366	0.867676	0.871811	0.872829	0.872828818
0.7	0.857475	0.87894	0.883909	0.887774	0.888725	0.888725216
0.8	0.879086	0.89833	0.902769	0.906219	0.907067	0.907067213
0.9	0.90472	0.920653	0.924315	0.927156	0.927855	0.92785481
1	0.934858	0.946046	0.948611	0.9506	0.951088	0.951088007

Table 6. Residual Errors of Example 3.4 for different values of ω.

x/R_ω	$R_{\omega=-3}$	$R_{\omega=-1}$	$R_{\omega=-0.5}$	$R_{\omega=-0.1}$	$R_{\omega=0}$
0	0	0	0	0	0
0.1	0.00362525	0.00118256	0.000571692	8.29781E-05	3.92021E-05
0.2	0.0145672	0.00473802	0.00228916	0.00033263	0.000156143
0.3	0.0330256	0.0106898	0.00515958	0.000751083	0.000348856
0.4	0.0593387	0.0190771	0.00919487	0.0013418	0.000614153
0.5	0.0939897	0.0299551	0.0144116	0.00210944	0.000947756
0.6	0.137618	0.0433951	0.020831	0.00305974	0.00134446
0.7	0.191033	0.0594853	0.0284788	0.0041993	0.00179834
0.8	0.255232	0.078331	0.0373853	0.00553538	0.00230293
0.9	0.331422	0.100055	0.0475849	0.00707574	0.00285147
1	0.421043	0.1248	0.0591165	0.00882843	0.00343707

3.1. Problems based on three point BVP

The exact solutions of the following three point BVPs are not available, so with the help of absolute residual error, we show the efficiency and accuracy of our proposed technique. We consider the following expression for residual error

$$R_\omega = \left| -y''(x) - \frac{\alpha}{x}y'(x) - f(x,y) \right|.$$

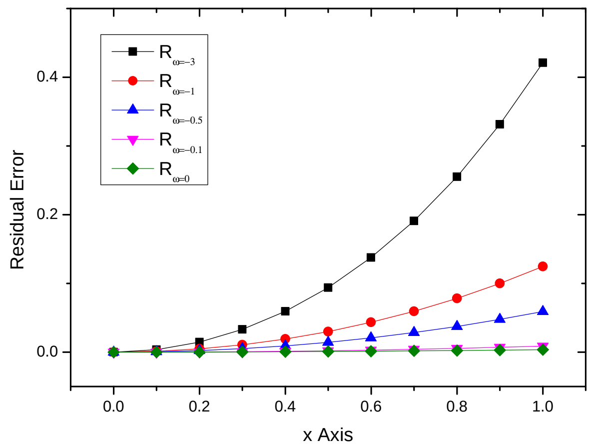

Figure 1. Residual Errors of Example 3.4 for different values of ω.

Example 3.5. Consider the nonlinear three point singular boundary value problem

$$-y''(x) - \frac{2}{x}y'(x) = \frac{3}{4}e^{y(x)}, \qquad 0 < x < 1, \tag{3.13}$$

$$y'(0) = 0, \quad y(1) = \frac{2}{5}y\left(\frac{1}{2}\right). \tag{3.14}$$

Solution:
Here $\frac{\partial f}{\partial y} > 0$, so from equation (2.3), we get

$$y_{n+1}(x) = y_n(x) - \int_0^x \lambda\left(\ddot{y}_n(t) + \frac{1}{t}\dot{y}_n(t) + \frac{3}{4}e^{y_n(t)}\right)dt, \tag{3.15}$$

were λ is given by (2.12). We choose the initial approximation $y_0(x) = a$, now from equation (3.15), we get that

$$\begin{aligned} y_0(x) =& a, \\ y_1(x) =& a - \frac{3e^a\left(\sqrt{\omega}\; x - \sin\left(\sqrt{\omega}\; x\right)\right)}{4\omega^{3/2}x}, \\ &\vdots \end{aligned}$$

Now we find the values of a by imposing the boundary condition '$y(1) = \frac{2}{5}y\left(\frac{1}{2}\right)$' on the above approximations for different values of ω. The solutions and residual error at different space points are displayed in tables 7, 8 and figure 2.

Table 7. Solution (y_1) of Example 3.5 for different values of ω.

x/y_1	$\omega = 0$	$\omega = 0.1$	$\omega = 0.3$	$\omega = 0.5$	$\omega = 0.7$	$\omega = 1$
0	0.237846	0.236163	0.232841	0.229576	0.226366	0.221653
0.1	0.236261	0.234581	0.231264	0.228004	0.224799	0.220093
0.2	0.231504	0.229833	0.226534	0.223292	0.220105	0.215425
0.3	0.223576	0.221923	0.218661	0.215454	0.212303	0.207674
0.4	0.212476	0.210856	0.207658	0.204515	0.201425	0.196889
0.5	0.198205	0.196638	0.193546	0.190506	0.187519	0.183133
0.6	0.180763	0.179279	0.176349	0.17347	0.170641	0.166489
0.7	0.16015	0.158787	0.156099	0.153458	0.150863	0.147056
0.8	0.136365	0.135177	0.132832	0.130529	0.128267	0.124949
0.9	0.109409	0.108461	0.106589	0.104751	0.102946	0.100299
1	0.0792821	0.0786554	0.0774182	0.0762025	0.0750076	0.0732533

Table 8. Residual Error of Example 3.5 for different values of ω.

x/R_ω	$R_{\omega=0}$	$R_{\omega=0.1}$	$R_{\omega=0.3}$	$R_{\omega=0.5}$	$R_{\omega=0.7}$	$R_{\omega=1}$
0	0	0	0	0	0	0
0.1	0.00150736	0.00134393	0.00101888	0.000696179	0.000375749	0.000100777
0.2	0.00601514	0.0053607	0.00405962	0.00276869	0.00148763	0.00041608
0.3	0.0134806	0.0120054	0.00907481	0.00616992	0.00329007	0.000984209
0.4	0.0238333	0.0212042	0.0159866	0.0108219	0.00570876	0.001867
0.5	0.0369762	0.0328556	0.0246885	0.0166185	0.00864317	0.00314692
0.6	0.0527871	0.0468318	0.0350474	0.023428	0.0119697	0.00492325
0.7	0.0711207	0.0629818	0.0469067	0.0310968	0.0155458	0.00730751
0.8	0.0918104	0.081133	0.0600896	0.0394532	0.0192143	0.0104184
0.9	0.114672	0.101095	0.0744023	0.0483116	0.0228082	0.0143763
1	0.139504	0.122662	0.0896384	0.0574772	0.0261558	0.0192983

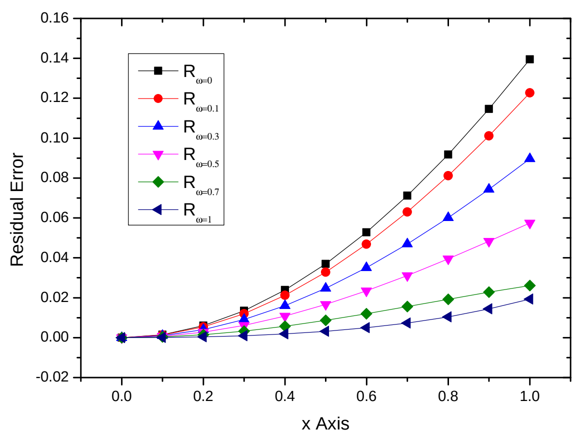

Figure 2. Residual Error of Example 3.5 for different values of ω.

Example 3.6. Consider the nonlinear three point singular boundary value problem

$$-y''(x) - \frac{2}{x}y'(x) = 1 - 2y^3(x), \qquad 0 < x < 1, \tag{3.16}$$

$$y'(0) = 0, \quad y(1) = \frac{1}{3}y\left(\frac{1}{4}\right). \tag{3.17}$$

Solution:
Here $\frac{\partial f}{\partial y} < 0$, so from equation (2.3), we get

$$y_{n+1}(x) = y_n(x) - \int_0^x \lambda \left(\ddot{y}_n(t) + \frac{1}{t}\dot{y}_n(t) + 1 - 2y^3(t)\right) dt, \tag{3.18}$$

where λ is given by (2.13) and we choose the initial approximation $y_0(x) = a$. Now from equation (3.18), we get that

$$\begin{aligned} y_0(x) =& a, \\ y_1(x) =& a - \frac{\left(2a^3 - 1\right)\left(x\sqrt{|k|} - \sinh\left(x\sqrt{|k|}\right)\right)}{x\,|k|^{3/2}}, \\ &\vdots \end{aligned}$$

Now we find the values of a by imposing the boundary condition '$y(1) = \frac{1}{3}y\left(\frac{1}{4}\right)$' on the above approximations for different values of ω. The solutions at different space points are displayed in tables 9, 10 and figure 3.

Table 9. Solution (y_1) of Example 3.6 for different values of ω.

x/y_1	$\omega = -2$	$\omega = -1.5$	$\omega = -1$	$\omega = -0.7$	$\omega = -0.5$	$\omega = -0.3$	$\omega = 0$
0	0.261313	0.255398	0.249569	0.246114	0.243828	0.241557	0.238177
0.1	0.259704	0.253786	0.247954	0.244497	0.242209	0.239937	0.236555
0.2	0.254858	0.248934	0.243097	0.239637	0.237348	0.235074	0.23169
0.3	0.246717	0.2408	0.23497	0.231515	0.22923	0.22696	0.223582
0.4	0.235183	0.229309	0.223524	0.220097	0.217831	0.21558	0.212231
0.5	0.220116	0.214358	0.20869	0.205334	0.203116	0.200913	0.197636
0.6	0.201335	0.195811	0.190379	0.187165	0.185041	0.182932	0.179798
0.7	0.17861	0.1735	0.16848	0.165512	0.163552	0.161607	0.158717
0.8	0.151665	0.147221	0.142861	0.140285	0.138584	0.136897	0.134392
0.9	0.12017	0.116735	0.113366	0.111375	0.110062	0.108759	0.106825
1	0.0837349	0.0817604	0.079815	0.0786619	0.0778993	0.0771414	0.0760139

Table 10. Residual Error of Example 3.6 for different values of ω.

$x/\ R_\omega$	$R_{\omega=-2}$	$R_{\omega=-1.5}$	$R_{\omega=-1}$	$R_{\omega=-0.7}$	$R_{\omega=-0.5}$	$R_{\omega=-0.3}$	$R_{\omega=0}$
0	0	0	0	0	0	0	4.51028 E-17
0.1	0.00256251	0.00179147	0.00101577	0.000548266	0.000235771	7.73604E-05	0.000548203
0.2	0.0103294	0.00722957	0.00411575	0.00224134	0.000989362	0.000264404	0.00214823
0.3	0.0235388	0.0165046	0.00945628	0.00522193	0.00239722	0.000428686	0.00466939
0.4	0.0425886	0.0299309	0.0172923	0.00972074	0.00467869	0.000358431	0.00790413
0.5	0.0680352	0.0479414	0.0279678	0.0160451	0.00812342	0.00022376	0.0115833
0.6	0.100591	0.071078	0.0419016	0.024562	0.013073	0.00164118	0.0153979
0.7	0.141114	0.0999744	0.0595652	0.0356741	0.0198956	0.00423675	0.0190262
0.8	0.190585	0.135329	0.0814511	0.0497868	0.0289529	0.00833947	0.022168
0.9	0.25007	0.177858	0.108029	0.0672648	0.0405574	0.0142227	0.0245846
1	0.320643	0.228231	0.139683	0.0883746	0.0549177	0.0220532	0.0261442

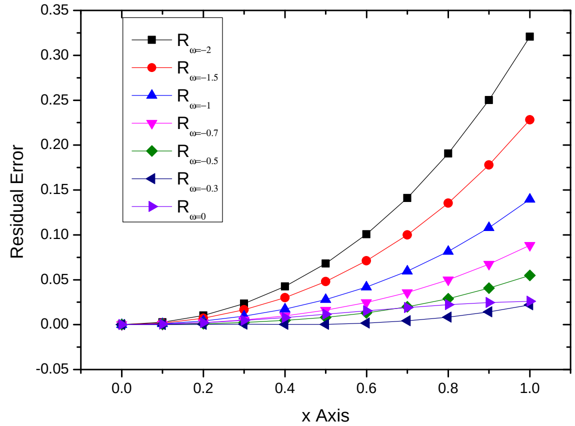

Figure 3. Residual Error of Example 3.6 for different values of ω.

4. Conclusion

In this paper, we have shown that introduction of parameter ω (correction term) in the iterative scheme greatly influence the convergence of the solution. In Tables 1, 2, 3 and 4 we have shown that when $\omega = 0$, our results are same as results in [32, 40]. For $\omega > 0$ the results are improved and are getting closer and closer to exact solutions. In tables 1 and 2, we have taken values of ω up to 0.72 which is less than square of first positive zeros of respective Bessel functions (see [33]) and we have also taken value of $\omega = 0.9$ which is greater that square of first positive zeros of respective Bessel functions (see [12]). In table 5, due to the absence of exact results, we compare the results with the results given in [32]. This table also shows that when value of ω is increasing the results are better (see Table 6). In table 7, 8, 9, 10 we depict the solutions of a class of three point SBVPs that does not exist in literature.

This techniques is extremely powerful and gives accurate solution in few iterations. The drawback of the method is that it is software dependent if we want to further increase the accuracy then software (Mathematica v11.3) may not be able to compute the solutions of integral in close form. But for few iterations this method should be preferred over other methods. To over come this we may have to use some interpolation technique for each iteration where we don't find close form solution of the integral but a discrete data which may be converted into a polynomial form by using interpolation. This method can also be extended easily for other class of IVPs or BVPs with singularity, the only issue that we need to take care of, is that integral in the iterative scheme must exist. Figures 1, 2, 3 depict that the residual errors are more towards the end point. But when $\omega < 0$ and close to zero then residual errors are uniform throughout the interval. When $\omega > 0$ and close to first eigenvalue of the corresponding BVPs then residual errors are uniform throughout the interval. Tables clarify that the results are quite encouraging for computing solutions of a class of three point nonlinear SBVPs.

5. Appendix

For $\omega > 0$, the Lagrange multipliers is

$$\lambda(t) = \frac{\pi x t^{\nu} t^{\alpha}}{2 x^{\nu} x^{\alpha}} \left[\left(J_{-\nu}\left(t\sqrt{\omega}\right) Y_{-\nu}\left(x\sqrt{\omega}\right) - J_{-\nu}\left(x\sqrt{\omega}\right) Y_{-\nu}\left(t\sqrt{\omega}\right)\right)\right],$$

where $J_{-\nu}\left(t\sqrt{\omega}\right) = \sum_{m=0}^{\infty} \frac{(-1)^m \left(\frac{t\sqrt{\omega}}{2}\right)^{-\nu+2m}}{m!\ \Gamma(m-\nu+1)}$, $\quad J_{\nu}\left(t\sqrt{\omega}\right) = \sum_{m=0}^{\infty} \frac{(-1)^m \left(\frac{t\sqrt{\omega}}{2}\right)^{\nu+2m}}{m!\ \Gamma(m+\nu+1)}$, and

$$Y_{-\nu}\left(t\sqrt{\omega}\right) = \frac{2}{\pi} J_{-\nu}\left(\ln \frac{t\sqrt{\omega}}{2} + \gamma\right) - \frac{1}{\pi} \sum_{-\nu-1}^{m=0} \frac{(-\nu-m-1)!}{m!} \left(\frac{t\sqrt{\omega}}{2}\right)^{2m+\nu} +$$
$$\frac{1}{\pi} \sum_{m=0}^{\infty} \frac{(-1)^{m-1}\left[\left(1+\frac{1}{2}+\cdots+\frac{1}{m}\right) + \left(1+\frac{1}{2}+\cdots+\frac{1}{m-\nu}\right)\right]}{m!\ (m-\nu)!} \left(\frac{t\sqrt{\omega}}{2}\right)^{2m-\nu}.$$

We also have a relation

$$Y_{-\nu}\left(t\sqrt{\omega}\right) = \frac{J_{\nu}\left(t\sqrt{\omega}\right) - \cos\nu\pi J_{-\nu}\left(t\sqrt{\omega}\right)}{\sin\nu\pi}.$$

i.e.,

$$\begin{aligned}
\lambda(t) = \frac{\pi x t^{\alpha}}{2\ x^{\alpha}} \Bigg[\Bigg(& \frac{2}{\pi} \left(\sum_{m=0}^{\infty} \frac{(-1)^m x^{-2\nu+2m} \left(\frac{\sqrt{\omega}}{2}\right)^{-\nu+2m}}{m!\ \Gamma(m-\nu+1)} \right) \left(\ln \frac{x\sqrt{\omega}}{2} + \gamma \right) \\
& - \frac{1}{\pi} \sum_{m=0}^{-\nu-1} \frac{(-\nu-m-1)!}{m!} (x)^{2m} \left(\frac{\sqrt{\omega}}{2}\right)^{2m+\nu} \\
+ \frac{1}{\pi} \sum_{m=0}^{\infty} & \frac{(-1)^{m-1}\left[\left(1+\frac{1}{2}+\cdots+\frac{1}{m}\right) + \left(1+\frac{1}{2}+\cdots+\frac{1}{m-\nu}\right)\right]}{m!\ (m-\nu)!} (x)^{2m-2\nu} \left(\frac{\sqrt{\omega}}{2}\right)^{2m-\nu} \\
+ & \left(\sum_{m=0}^{\infty} \frac{(-1)^m x^{-2\nu+2m} \left(\frac{\sqrt{\omega}}{2}\right)^{-\nu+2m}}{m!\ \Gamma(m-\nu+1)} \right) \cot\nu\pi \Bigg) \sum_{m=0}^{\infty} \frac{(-1)^m t^{2m} \left(\frac{\sqrt{\omega}}{2}\right)^{-\nu+2m}}{m!\ \Gamma(m-\nu+1)} \Bigg] \\
- \frac{\pi x}{2\ x^{\alpha}} & \left[\left(\sum_{m=0}^{\infty} \frac{(-1)^m x^{-2\nu+2m} \left(\frac{\sqrt{\omega}}{2}\right)^{-\nu+2m}}{m!\ \Gamma(m-\nu+1)} \right) \csc\nu\pi \sum_{m=0}^{\infty} \frac{(-1)^m t^{2m+1} \left(\frac{\sqrt{\omega}}{2}\right)^{\nu+2m}}{m!\ \Gamma(m+\nu+1)} \right].
\end{aligned} \tag{5.1}$$

Now for $\omega < 0$, the Lagrange multipliers is

$$\lambda(t) = \frac{t^{\alpha} t^{\nu}\ x}{x^{\nu}\ x^{\alpha}} \left[I_{-\nu}\left(x\sqrt{|\omega|}\right) K_{\nu}\left(t\sqrt{|\omega|}\right) - I_{-\nu}\left(t\sqrt{|\omega|}\right) K_{\nu}\left(x\sqrt{|\omega|}\right)\right],$$

where $I_{\nu}\left(x\sqrt{|\omega|}\right) = \sum_{m=0}^{\infty} \frac{\left(\frac{x\sqrt{|\omega|}}{2}\right)^{2m+\nu}}{m!\ \Gamma(m+\nu+1)}$, $\quad I_{-\nu}\left(x\sqrt{|\omega|}\right) = \sum_{m=0}^{\infty} \frac{\left(\frac{x\sqrt{|\omega|}}{2}\right)^{2m-\nu}}{m!\ \Gamma(m-\nu+1)}$, and

$$K_{\nu}\left(x\sqrt{|\omega|}\right) = \frac{\pi}{2\sin\nu\pi} \left[I_{-\nu}\left(x\sqrt{|\omega|}\right) - I_{\nu}\left(x\sqrt{|\omega|}\right)\right].$$

$$\lambda(t) = \frac{\pi\ t^{\alpha}}{2x^{\alpha}\ \sin\nu\pi}\left[\sum_{m=0}^{\infty}\frac{(t)^{2m}\left(\frac{\sqrt{|\omega|}}{2}\right)^{2m-\nu}}{m!\ \Gamma(m-\nu+1)}\left(\sum_{m=0}^{\infty}\frac{(x)^{2m+\alpha}\left(\frac{\sqrt{|\omega|}}{2}\right)^{2m-\nu}}{m!\ \Gamma(m-\nu+1)}\right.\right.$$

$$\left.\left.-\left(\sum_{m=0}^{\infty}\frac{(x)^{2m+\alpha}\left(\frac{\sqrt{|\omega|}}{2}\right)^{2m-\nu}}{m!\ \Gamma(m-\nu+1)}-\sum_{m=0}^{\infty}\frac{(x)^{2m+1}\left(\frac{\sqrt{|\omega|}}{2}\right)^{2m+\nu}}{m!\ \Gamma(m+\nu+1)}\right)\right)\right]$$

$$-\frac{\pi}{2\sin\nu\pi}\left[\sum_{m=0}^{\infty}\frac{(x)^{2m}\left(\frac{\sqrt{|\omega|}}{2}\right)^{2m-\nu}}{m!\ \Gamma(m-\nu+1)}\sum_{m=0}^{\infty}\frac{(t)^{2m+1}\left(\frac{\sqrt{|\omega|}}{2}\right)^{2m+\nu}}{m!\ \Gamma(m+\nu+1)}\right]. \qquad (5.2)$$

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