EXPLICIT PEAKON SOLUTIONS TO A FAMILY OF WAVE-BREAKING EQUATIONS

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Abstract The singular traveling wave solutions of a general 4-parameter family equation which unifies the Camass-Holm equation, the Degasperis-Procesi equation and the Novikov equation are investigated in this paper. At first, we obtain the explicit peakon solutions for one of its specific case that $a = (p+2)c$, $b = (p + 1)c$ and $c = 1$, which is referred to a generalized Camassa-Holm-Novikov (CHN) equation, by reducing it to a second-order ordinary differential equation (ODE) and solving its associated first-order integrable ODE. By observing the characteristics of peakon solutions to the CHN equation, we construct the peakon solutions for the general 4-parameter breaking wave equation. It reveals that singularities of the peakon solutions come up only when the solutions attain singular points of the equation, which might be a universal principal for all singular traveling wave solutions for wave breaking equations.

Keywords wave-breaking equations, singular wave solutions, peakon solutions, singular line.

MSC(2010) 34C23, 34C60, 35B65, 35Q35.

1. Introduction

There have been a considerable amount of nonlinear wave equations proposed in recent decades to describe the propagation of shallow water waves, especially the equation in form $u_t - u_{txx} = f(u, u_x, u_{xx}, u_{xxx})$ to model breaking waves, for instance the Camassa-Holm equation [3]

$$u_t - u_{txx} + 3uu_x - 2u_xu_{xx} - uu_{xxx} = 0, \tag{1.1}$$

the Degasperis-Procesi equation [5]

$$u_t - u_{txx} + 4uu_x - 3u_xu_{xx} - uu_{xxx} = 0, \tag{1.2}$$

the Novikov equation [21]

$$u_t - u_{txx} + 4u^2u_x - 3uu_xu_{xx} - u^2u_{xxx} = 0, \tag{1.3}$$

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the $b-$equation [9, 23]

$$u_t - u_{txx} + (b+1)uu_x - bu_xu_{xx} - uu_{xxx} = 0 \tag{1.4}$$

and the $\theta-$class equation [10]

$$u_t - u_{txx} + uu_x + (\theta - 1)u_xu_{xx} - \theta uu_{xxx} = 0. \tag{1.5}$$

Among these equations, the first three ones have been shown to be integrable in the sense of possessing a Lax pair, a bi-Hamiltonian structure, as well as local conservation laws, and all to admit peakon solutions [19] in the form

$$u(x,t) = \alpha e^{-|x-vt|}. \tag{1.6}$$

It is testified [10] that the $\theta-$class equation (1.5) admits peakon solutions for the case when $0 < \theta < \frac{1}{2}$, while every strong solution exists globally in time for the case when $\frac{1}{2} \le \theta \le 1$. Actually, by a time rescaling transformation $t \to t/\theta$, (1.5) is transformed to (1.4) with $b = 1/\theta - 1$, thus equations (1.4) and (1.5) are of same family.

Recently a more general 4-parameter family equation, which unifies the Camassa-Holm equation, the Degasperis-Procesi equation and the Novikov equation, is proposed [1] and given by

$$u_t - u_{txx} + au^pu_x - bu^{p-1}u_xu_{xx} - cu^pu_{xxx} = 0, \tag{1.7}$$

where a, b and c are not all zero parameters and $p \ne 0$. It has been proven that [1] (1.7), as well as the Camassa-Holm and the Novikov equations, admits single-solitary peakon solutions of the form (1.6) and multi-peakon solutions if and only if its parameters satisfy $b = (p+1)c$, $a = (p+2)c$ and $p > -1$. In this case, (1.7) reduces to a one-parameter family of equation which can be written as

$$m_t + (p+1)u^{p-1}u_xm + u^pm_x = 0, \quad m = u - u_{xx}, \tag{1.8}$$

via the scaling transformation $t \to t/c$. Equation (1.8) is referred to a generalized Camassa-Holm-Novikov equation since it unifies the Camassa-Holm equation and the Novikov equation. For two specific cases of equation (1.8) that $p = 2$ and $p = 3$ have been investigated in [26] recently. It is well known that the peakon solution is charactered by having discontinuities in the first derivative at its peak. The peakon solutions of the Camassa-Holm equation are orbital stable in the H^1 norm [4], which means that these wave patterns are physically recognizable.

As is well known, the traveling wave solutions are a class of invariant solutions of PDEs which usually [20, 24] admit the two Lie point symmetries $X_1 = \frac{\partial}{\partial x}$ and $X_2 = \frac{\partial}{\partial t}$, and thus $\xi = x - vt$ is an invariant which can be applied to reduce nonlinear PDEs to ODEs. Under the traveling wave frame, that is, by supposing $u(x,t) = \phi(\xi)$, equation (1.7) becomes the following third-order ODE

$$(v - c\phi^p)\phi''' - b\phi^{p-1}\phi'\phi'' + (a\phi^p - v)\phi' = 0, \tag{1.9}$$

and equation (1.8) leads to

$$(v - \phi^p)\phi''' - (p+1)\phi^{p-1}\phi'\phi'' + ((p+2)\phi^p - v)\phi' = 0, \tag{1.10}$$

where $'$ denotes the derivative with respect to ξ.

From the mathematical point of view, it is easy to interpret the solutions of the third-order ODEs (1.9) and (1.10) in $C^3(\mathbb{R})$ space. However, the "blow-up" phenomena has been seen often in application world which implies that the solutions of some nonlinear wave equations arising in the investigation of some fundamental physical problem may lose their smoothness, that is to say that the solutions may blow up some times [14, 15, 23, 26]. Therefore, it is necessary to extend the classical solutions to weak solutions which are usually named as singular wave solutions for nonlinear wave equations. Clearly, equation (1.9) has singularities when $v - c\phi^p = 0$. There have been various methods proposed [2, 6–8, 11, 12, 16–18, 22, 25, 27, 28] to investigate the exact solutions of nonlinear wave equations, among which the dynamical system method has been well applied to study the bifurcations and exact traveling wave solutions of some nonlinear wave equations, especially integrable equations and singular wave equations [7, 12, 16, 17, 19, 26, 27].

In the current paper, we pay attentions to the problem how to define or extend the solutions of nonlinear wave equations with singularities which approach or reach the straight lines and examine the exact singular traveling wave solutions. We will show that the general nonlinear wave equation (1.7) possess singular traveling wave solutions as well as one of its specific case when $b = (p+1)c$, $a = (p+2)c$ and $c = 1$, namely, the integrable equation (1.8) by using the singular line and their solvable sub-equation.

This paper is arranged as follows. In Section 2, we introduce the definition of weak traveling wave solutions for the general four-parameter equation (1.7) and the definition of singular traveling wave solutions of a specific case of its, namely equation (1.8), which include solitary peakon, periodic peakon, solitary cuspon and periodic cuspon. The explicit peakon solutions of equation (1.8) are investigated via (2.5) in Section 3. For some specific case of (1.7) with $a = (p+1)c$ and $b = (p+2)c$, motivated by the derived exact peakon solutions for (1.8), we find a sub-equation of the associated traveling wave equation (1.9) from which we obtain some explicit peakon solutions of equation (1.7) in Section 4.

2. Preliminary

The weak singular traveling wave solutions for a wave equation are interpreted as functions satisfying the model equation in the sense of distributions [7, 12, 13]. That is, after multiplying the higher-order differential equation in ϕ by a test function $\psi \in \mathbb{C}_0^\infty(\mathbb{R})$, and integrating over $\mathbb{R}$, reducing the orders of the derivatives in ϕ by using integral by parts and leaving at most first-order derivative of ϕ in the integral, the higher-order differential equation of ϕ is transformed into a first-order differential equation involving an arbitrary test function. For the general 4-parameter family equation (1.7) we present the definition of weak traveling wave solutions as follows.

Definition 2.1. We say that $\phi(\xi) = \phi(x - vt)$ is a weak traveling wave solution of (1.7), if $\phi(\xi) \in W_{loc}^{1,3}(\mathbb{R})$ and satisfies

$$\int_{\mathbb{R}} (v(\psi'' - \psi)\phi' + (a\psi - c\psi'')\phi^p\phi') \, d\xi + \frac{1}{2}(b - 3cp) \int_{\mathbb{R}} \phi^{p-1}\phi'^2\psi' d\xi \\ + \frac{1}{2}(p-1)(b-cp) \int_{\mathbb{R}} \phi^{p-2}\phi'^3\psi d\xi = 0, \tag{2.1}$$

for any test function $\psi \in \mathbb{C}_0^\infty(\mathbb{R})$.

It is easy to see that this definition generalizes the classical traveling wave solutions significantly. However, it turns out that this concept is too rude to make the weak solutions meaningful in the physical point of view. For instance, it is shown in [13] that the Camassa-Holm equation can have traveling wave solutions (by this definition) in the form $u = \phi(\xi)$ such that some of its level sets $\{\phi(\xi) = k\}$ are cantor sets, which might be only mathematical meaningful. Consequently, besides the condition (2.1), some extra restriction should be imposed on the definition of traveling wave solutions.

For equation (1.10), that is, the specific case of equation (1.9) with $a = (p+1)c$, $b = (p+2)c$ and $c = 1$, after multiplying by ϕ on its both sides and then integrating once with respect to the new variable ξ, one has the following second-order ODE:

$$2\phi(v - \phi^p)\phi'' - v(\phi')^2 + \phi^2(2\phi^p - v) = g, \tag{2.2}$$

where g is the constant of integration. For any $\phi \in C^2(\mathbb{R})$ having $\phi(v - \phi^p) \neq 0$, it admits that

$$\begin{aligned}
&\frac{d}{d\xi}\left(\frac{1}{\phi}[v - \phi^p]^{\frac{1}{p}}(\phi'^2 + \frac{g}{v} - \phi^2)\right) \\
&= \frac{\phi'}{\phi^2[v - \phi^p]^{\frac{p-1}{p}}}\left(2\phi(v - \phi^p)\phi'' - v(\phi')^2 + \phi^2(2\phi^p - v) - g\right).
\end{aligned} \tag{2.3}$$

Here and hereafter we denote $[v - \phi^p] = |v - \phi^p|$ if p is an even number and $[v - \phi^p] = v - \phi^p$ if p is an odd integer. Consequently, any classical solution of (1.10) with $\phi(v - \phi^p) \neq 0$ naturally satisfies

$$\frac{1}{\phi}[v - \phi^p]^{\frac{1}{p}}(\phi'^2 + \frac{g}{v} - \phi^2) = h \tag{2.4}$$

for some constant $h \in \mathbb{R}$. We rewrite (2.4) as

$$[v - \phi^p]^{\frac{1}{p}}(\phi'^2 + \frac{g}{v} - \phi^2) - h\phi = 0 \tag{2.5}$$

to include the case that $\phi = 0$. Clearly for a solution $\phi(\xi)$ of (2.2), we can see that if $\phi(\xi_0) = 0$ then $\phi'(\xi_0) = -\frac{g}{v}$, so it satisfies (2.5) too. Therefore, any classical solution of (1.10) or (2.2) with $\phi^p(\xi) \neq v$ must satisfy (2.5) for some constant h automatically.

Inversely, we consider the solutions of (2.5). If $\phi^p(\xi) \neq v$, (2.5) can be rewritten as

$$\phi'^2 = \frac{h\phi}{[v - \phi^p]^{\frac{1}{p}}} + \phi^2 - \frac{g}{v}. \tag{2.6}$$

Furthermore, for arbitrary real number h and an open interval $I \subset \mathbb{R}$, if $\phi(\xi) \in C^1(I)$ is not identically a constant on any subinterval of I, then we have

$$\phi'' = \phi + sgn(v - \phi^p)\frac{hv}{2[v - \phi^p]^{\frac{p+1}{p}}} \tag{2.7}$$

by differentiating (2.6) once with respect to ξ (refer to [12] for more detail). Substituting (2.6) and (2.7) into (2.2) makes (2.2) an identity, which implies that $\phi(\xi)$ defined by (2.4) solves (2.2). Consequently, we have the following conclusion.

Lemma 2.1. *Suppose that I is an open interval and $\phi \in C^2(I)$. If $\phi^p \neq v$, then ϕ solves (2.2) if and only if there are some values of g and h such that ϕ satisfies (2.5).*

From Lemma 2.1, one knows that the classical solutions of (2.5) with $\phi' \not\equiv 0$ on any interval are also the classical solutions of (2.2). Therefore, (2.5) can be applied to study the classical solutions of (2.2) and then derive the smooth traveling wave solutions of (1.8). Since (2.5) is well defined when $\phi = 0$ for the case when $vg < 0$, the singularity appears only at $\phi = \phi_e$, where $\phi_e^p = v$.

Based on the previous analysis and Lemma 2.1, we impose further conditions on the definition of traveling wave solutions of (1.8).

Definition 2.2. We say that a non-constant function $\phi(\xi) = \phi(x - vt)$ is a traveling wave solution of (1.8), if $\phi(\xi) \in W_{loc}^{1,3}(\mathbb{R})$ and satisfies the following statements:

(1) It satisfies (2.1) with $a = p + 2$, $b = p + 1$ and $c = 1$;

(2) There exist some g and h such that (2.5) holds for $\phi(\xi)$ in the limit sense, that is, $\phi(\xi)$ satisfies

$$\lim_{\xi \to \xi_0} \left(|v - \phi^p|^{\frac{1}{p}} (\phi'^2 + \frac{g}{v} - \phi^2) - h\phi \right) = 0 \tag{2.8}$$

for arbitrary $\xi_0 \in \mathbb{R}$.

Furthermore, we say that $\phi(\xi) = \phi(x - vt)$ is a singular traveling wave solution if $\phi(\xi) \in W_{loc}^{1,3}(\mathbb{R}) \backslash C^2(\mathbb{R})$, namely, there exists ξ_0 where $\phi(\xi)$ loses its smoothness.

Obviously, the classical solutions of (1.8) with $v - \phi^p \neq 0$ satisfy the above definition naturally. It follows from Definition 2.2 that for a singular traveling wave solution of (1.8) $\phi(\xi) = \phi(x - vt)$, if there is $\xi_0 \in \mathbb{R}$ such that $\phi^p(\xi_0) = v$, then $\lim_{\xi \to \xi_0} \phi'^2 = \infty$ for $h \neq 0$, and $\lim_{\xi \to \xi_0} (\phi'^2 - \phi^2 + \frac{g}{v}) = 0$ for $h = 0$.

There are two possible cases that this definition extend the classical solution set of (1.8):

Case (1) $\phi'(\xi) \to \pm A$ $(A \neq 0)$ as $\xi \to \xi_0^\pm$, where $|A| = \sqrt{v^{\frac{2}{p}} - \frac{g}{v}}$;

Case (2) $\phi'(\xi) \to \pm\infty$ as $\xi \to \xi_0^\pm$.

Here $\xi \to \xi_0^+$ (resp. $\xi \to \xi_0^-$) means $\xi > \xi_0$ (resp. $\xi > \xi_0$) and $\xi \to \xi_0$. For Case (1) and Case (2), we know that $\phi'(\xi_0)$ fails to exist but (2.5) holds for some h in the sense of limit and thus ϕ might be a singular traveling wave solution of (1.8). Note that some well-known singular traveling wave solutions, such as solitary peakon, solitary cuspon, periodic peaked and periodic cusped wave solutions, are right the traveling wave solutions having some points where the derivatives satisfy Case (1) or Case (2). Here we recall the definitions.

Definition 2.3. For nonlinear wave equation, we say that a traveling wave solution $\phi(\xi) = \phi(x - vt)$ is a

(1) solitary peakon if there exist $\xi_0 \in \mathbb{R}$ and some constants A and A' such that

$$\lim_{\xi \to \xi_0^+} \phi'(\xi) = -\lim_{\xi \to \xi_0^-} \phi'(\xi) = A \neq 0; \tag{2.9}$$

and

$$\lim_{\xi \to \infty} \phi'(\xi) = 0, \quad \lim_{\xi \to \infty} \phi(\xi) = A' \tag{2.10}$$

(2) periodic peakon if $\phi(\xi)$ is periodic solution which attains its peak (or valley) at $\xi = \xi_0 + kT, (k \in \mathbb{Z})$, and satisfies (2.9);

(3) solitary cuspon if $\phi(\xi)$ satisfies (2.10) and

$$\lim_{\xi \to \xi_0^+} \phi'(\xi) = -\lim_{\xi \to \xi_0^-} \phi'(\xi) = \infty; \tag{2.11}$$

(4) periodic cuspon if $\phi(\xi)$ is periodic solution which attains its peak (or valley) at $\xi = \xi_0 + kT, (k \in \mathbb{Z})$, and satisfies (2.11).

In this paper, we consider the peakon solutions of the generalized Camassa-Holm-Novikov equation (1.8) as well as the more general equation (1.7) based on the definitions and analysis presented in this section.

3. Explicit peakon solutions of the generalized Camassa-Holm-Novikov equation (1.8)

In this section, we focus on the singular traveling wave solutions of the generalized Camassa-Holm-Novikov equation (1.8). For two specific cases when $p = 2$ and $p = 3$, bifurcations and all possible phase portraits of equation (2.2) as well as peakon and cuspon solutions of equation (1.8) have been studied by using planar dynamical system method in [26]. We have noticed from literature, for instance [12, 16, 17, 26], that the singular traveling wave solutions of nonlinear wave equations always appear with the presence of singular lines in their associated traveling wave systems.

Let ϕ_e be a zero of $v - \phi^p = 0$, then the line $\phi = \phi_e$ will be a singular line of ODE (2.2) as well as (2.5) for arbitrary positive integer p. Consequently, we predict that the singular traveling wave solutions of equation (1.8), namely the singular solutions of third-order ODE (1.10) or second-order ODE (2.2) with certain value of g, have singularities when ϕ approaches ϕ_e. We know from the definition of peakon solution that $\phi \to \phi_e$ and $\phi'(\xi) \to \pm A$ $(A \neq 0)$ as $\xi \to \xi_0^\pm$. Substituting these limits into (2.4) yields that $h = 0$. It implies that peakon solutions appear only when $h = 0$. When $h = 0$, equation (2.6) reduces to

$$\phi'^2 = \phi^2 - \frac{g}{v}. \tag{3.1}$$

Clearly, solving (3.1) gives

$$\phi(\xi) = \begin{cases} \pm\sqrt{-\frac{g}{v}}\,\sinh(\xi) & vg < 0, \\ \pm\sqrt{\frac{g}{v}}\,\cosh(\xi) & vg > 0, \\ \alpha\,e^{\pm\xi} & g = 0, \end{cases} \tag{3.2}$$

for $\xi \in \mathbb{R}$ and arbitrary α. The solutions defined by (3.2) are unbounded traveling wave solutions of (1.8) passing through the singular straight line $\phi = \phi_e$. The phase orbits of (3.1) which intersect with singular lines are presented in Figure 1 and Figure 2. Integrating along these bounded orbits (orbits not passing through the singular lines) yields the peakon solutions of equation (1.8).
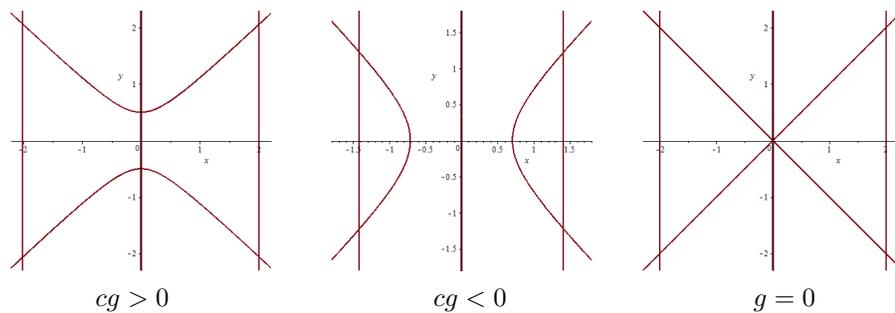
Figure 1. The orbits determined by (3.1) and the singular lines for $p = 2n$.
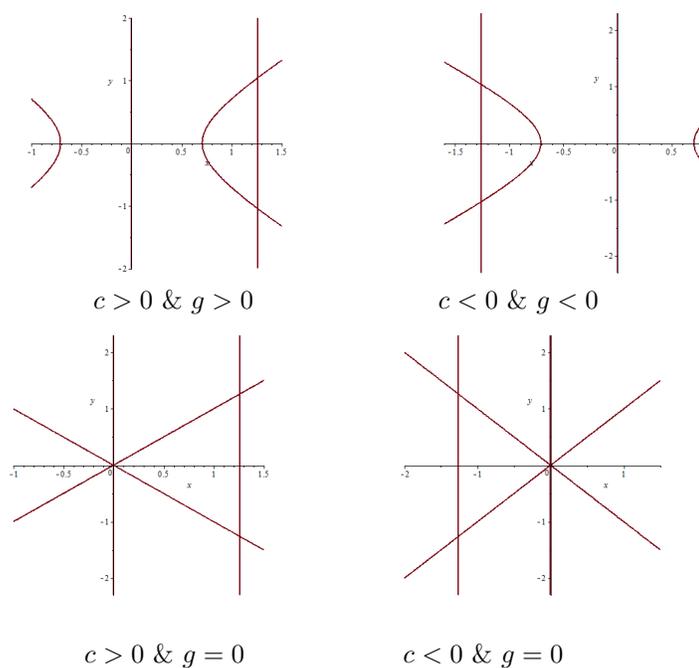

Figure 2. The orbits determined by (3.1) and the singular line for $p = 2n + 1$.

Theorem 3.1. *For the peakon solutions of the generalized Camassa-Holm-Novikov equation (1.8), we have the following conclusions.*

(1) for arbitrary $v > 0$ and $p = 2n$, $(n \in \mathbb{Z}^+)$, equation (1.8) has two solitary peakons

$$\phi(\xi) = \pm v^{\frac{1}{p}} e^{-|x-vt|}; \tag{3.3}$$

a family of periodic peakons

$$\phi(\xi) = \begin{cases} \sqrt{-\frac{g}{v}} \, \sinh(\xi - 4k\xi_0) & (4k-1)\xi_0 < \xi \le (4k+1)\xi_0, \\ -\sqrt{-\frac{g}{v}} \, \sinh(\xi - (4k+2)\xi_0) & (4k+1)\xi_0 < \xi \le (4k+3)\xi_0, \end{cases} \tag{3.4}$$

where $k \in \mathbb{Z}$, $\xi = x - vt$ and $\xi_0 = \operatorname{arcsinh}\left(v^{\frac{1}{p}}(-\frac{v}{g})^{\frac{1}{2}}\right)$ for arbitrary $g < 0$; and two families of periodic peakons

$$\phi(\xi) = \pm\sqrt{\frac{g}{v}}\,\cosh(\xi - 2k\xi_0), \quad (2k-1)\xi_0 < \xi \le (2k+1)\xi_0, \tag{3.5}$$

where $\xi_0 = \operatorname{arccosh}\left(v^{\frac{1}{p}}(\frac{v}{g})^{\frac{1}{2}}\right)$ for arbitrary $0 < g < v^{1+\frac{2}{p}}$;

(2) *for arbitrary $v \ne 0$ and $p = 2n - 1$, $(n \in \mathbb{Z}^+)$, equation (1.1) has a solitary peakon*

$$\phi(\xi) = v^{\frac{1}{p}} e^{-|x-vt|}, \tag{3.6}$$

and two families of periodic peakons

$$\phi(\xi) = \operatorname{sign}(v)\sqrt{\frac{g}{v}}\,\cosh(\xi - 2k\xi_0), \quad (2k-1)\xi_0 < \xi \le (2k+1)\xi_0, \tag{3.7}$$

where $\xi_0 = \operatorname{arccosh}\left((v/g)^{\frac{1}{2}}v^{\left(\frac{1}{p}\right)}\right)$ for arbitrary g satisfying $0 < vg < v^{2+\frac{2}{p}}$.

Clearly, to prove that the functions defined above are peakon solutions of (1.8), we only need to prove that they satisfy (2.1), which will be accomplished in the following section.

4. Explicit peakon solutions of the general 4-parameter family equation (1.7)

Inspired by the peakon solutions of the generalized Camassa-Holm-Novikov equation (1.8) derived by solving the first-order ODE (3.1), we show now that equation (3.1) also defines peakon solutions for the general 4-parameter family equation (1.7) with $a = b + c$.

Lemma 4.1. *Let $\phi = \phi(\xi)$ be a non-trivial solution of $\phi'' = \phi$, then it solves equation (1.9) if and only if $a = b + c$.*

Proof. If $\phi = \phi(\xi)$ satisfies $\phi'' = \phi$, then $\phi''' = \phi'$. Substituting $\phi'' = \phi$ and $\phi''' = \phi'$ into (1.9) gives $(a - b - c)\phi^p\phi' = 0$. Consequently, $\phi = \phi(\xi)$ defined by $\phi'' = \phi$ is a non-trival solution of equation (1.9) if and only if $a = b + c$. $\quad\square$

Theorem 4.1. *Equation (1.7) with $a = b+c$ admits the peakon solutions as follows:*
(1) *for arbitrary $cv > 0$ and $p = 2n$, $(n \in \mathbb{Z}^+)$, (1.7) has two solitary peakons:*

$$\phi(\xi) = \pm\left(\frac{v}{c}\right)^{\frac{1}{p}} e^{-|x-vt|}; \tag{4.1}$$

a family of periodic peakons:

$$\phi(\xi) = \begin{cases} A\,\sinh(\xi - 4k\xi_0) & (4k-1)\xi_0 < \xi \le (4k+1)\xi_0, \\ -A\,\sinh(\xi - (4k+2)\xi_0) & (4k+1)\xi_0 < \xi \le (4k+3)\xi_0, \end{cases} \tag{4.2}$$

where $k \in \mathbb{Z}$, $\xi = x - vt$ and $\xi_0 = \operatorname{arcsinh}\left(\frac{1}{A}\left(\frac{v}{c}\right)^{\frac{1}{p}}\right)$ for arbitrary $A > 0$; and two families of periodic peakons:

$$\phi(\xi) = \pm A\,\cosh(\xi - 2k\xi_0), \quad (2k-1)\xi_0 < \xi \le (2k+1)\xi_0, \tag{4.3}$$

where $\xi_0 = \operatorname{arccosh}\left(\frac{1}{A}\left(\frac{v}{c}\right)^{\frac{1}{p}}\right)$ for arbitrary $0 < A < \left(\frac{v}{c}\right)^{\frac{1}{p}}$;

(2) for arbitrary $v \neq 0$ and $p = 2n - 1$, $(n \in \mathbb{Z}^+)$, (1.7) has a solitary peakon:

$$\phi(\xi) = \left(\frac{v}{c}\right)^{\frac{1}{p}} e^{-|x - vt|};\tag{4.4}$$

and a family of periodic peakons:

$$\phi(\xi) = \operatorname{sign}(cv) A \cosh(\xi - 2k\xi_0), \quad (2k - 1)\xi_0 < \xi \leq (2k + 1)\xi_0,\tag{4.5}$$

where $\xi_0 = \operatorname{arccosh}\left(\frac{1}{A}\left(\frac{v}{c}\right)^{\frac{1}{p}}\right)$ for arbitrary $0 < A < |\frac{v}{c}|^{\frac{1}{p}}$.

Proof. It has been proven in [1] that functions (3.3) and (3.6) are solitary peakons for equation (1.7) with $c = 1$. In a similar way, one can easily see that both (4.1) and (4.4) are solitary peakons for equation (1.7). To shown that (4.2), (4.3) and (4.5) are singular traveling wave solutions of (1.7), we examine that (4.2), (4.3) and (4.5) satisfy (2.1). Now, we testify that it holds for solution (4.2). The other conclusions can be proved in a similar way. Notice that (4.2) satisfies $\phi'^2 = \phi^2 + A^2$, $\phi'' = \phi$ and $\phi''' = \phi'$. One can see that it holds for ϕ defined by (4.2) that $\phi^p((2k + 1)\xi_0) = \frac{v}{c}$ and $\phi^p((2k)\xi_0) = 0$ for arbitrary $k \in \mathbb{Z}$. By using all these information, we can prove that (4.2) satisfies (2.1).

The first term in equation (2.1) yields, after integration by parts,

$$
\begin{aligned}
&\sum_{k=-\infty}^{\infty} \left(\int_{(4k-1)\xi_0}^{(4k+1)\xi_0} v(\psi'' - \psi)\phi' d\xi + \int_{(4k+1)\xi_0}^{(4k+3)\xi_0} v(\psi'' - \psi)\phi' d\xi \right) \\
={}& v \sum_{k=-\infty}^{\infty} (\phi'\psi' - \phi''\psi)\big|_{(4k-1)\xi_0}^{(4k+1)\xi_0} + \sum_{k=-\infty}^{\infty} (\phi'\psi' - \phi''\psi)\big|_{(4k+1)\xi_0}^{(4k+3)\xi_0} \\
={}& v \sum_{k=-\infty}^{\infty} (\phi'\psi' - \phi\psi)\big|_{(4k-1)\xi_0}^{(4k+1)\xi_0} + \sum_{k=-\infty}^{\infty} (\phi'\psi' - \phi\psi)\big|_{(4k+1)\xi_0}^{(4k+3)\xi_0} \\
={}& 2v \sum_{k=-\infty}^{\infty} \phi'\psi'\big|_{(4k-1)\xi_0}^{(4k+1)\xi_0}.
\end{aligned}
\tag{4.6}
$$

The second term in (2.1) gives

$$
\begin{aligned}
&\sum_{k=-\infty}^{\infty} \int_{(2k-1)\xi_0}^{(2k+1)\xi_0} a\phi^p \phi' \psi d\xi - 2c \sum_{k=-\infty}^{\infty} \phi^p \phi' \psi'\big|_{(4k-1)\xi_0}^{(4k+1)\xi_0} \\
&+ \sum_{k=-\infty}^{\infty} c \int_{(2k-1)\xi_0}^{(2k+1)\xi_0} \psi'(\phi^p \phi')' d\xi = -2c \sum_{k=-\infty}^{\infty} \phi^p \phi' \psi'\big|_{(4k-1)\xi_0}^{(4k+1)\xi_0} \\
&\sum_{k=-\infty}^{\infty} \int_{(2k-1)\xi_0}^{(2k+1)\xi_0} (a\phi^p - c(p+1)^2 \phi^p - cp(p-1)A^2 \phi^{p-2})\phi'\psi d\xi.
\end{aligned}
\tag{4.7}
$$

Substituting $\phi'^2 = \phi^2 + A^2$ in the sum of the third and fourth term of (2.1) leads to

$$
\begin{aligned}
&\frac{1}{2}(b - 3cp) \sum_{k=-\infty}^{\infty} \int_{(2k-1)\xi_0}^{(2k+1)\xi_0} \phi^{p-1}(\phi^2 + A^2)\psi' d\xi \\
&+ \frac{1}{2}(p-1)(b-cp) \int_{(2k-1)\xi_0}^{(2k+1)\xi_0} \phi^{p-2}(\phi^2 + A^2)\phi'\psi d\xi \\
&= -\frac{1}{2}(b - 3cp) \sum_{k=-\infty}^{\infty} \int_{(2k-1)\xi_0}^{(2k+1)\xi_0} ((p+1)\phi^p + A^2(p-1)\phi^{p-2})\phi'\psi d\xi \\
&+ \frac{1}{2}(p-1)(b-cp) \int_{(2k-1)\xi_0}^{(2k+1)\xi_0} (\phi^p + A^2\phi^{p-2})\phi'\psi d\xi.
\end{aligned} \tag{4.8}
$$

Combining (4.6)-(4.8) and substituting them in the left side of (2.1), we have

$$
\sum_{k=-\infty}^{\infty} 2(v - c\phi^p)\phi'\psi'\Big|_{(4k-1)\xi_0}^{(4k+1)\xi_0} + \sum_{k=-\infty}^{\infty} \int_{(2k-1)\xi_0}^{(2k+1)\xi_0} (a - b - c)\phi^p\phi'\psi d\xi,
$$

which is equivalent to 0 since $a - b - c = 0$ and $\phi^p((2k+1)\xi_0) = \frac{v}{c}$ for arbitrary $k \in \mathbb{Z}$. Similarly, we can prove that (4.2) and (4.3) also satisfy (2.1), which implies that functions (4.1)-(4.3) satisfy (2.9) in the sense of distribution, that is, they are weak traveling wave solutions of (1.7). One can easily see from the formula of function (4.2) that it satisfies Definition 2.3 for periodic peakon. It completes the proof. $\qquad\square$

Acknowledgement

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