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DYNAMIC PROPERTIES FOR NONLINEAR VISCOELASTIC KIRCHHOFF-TYPE EQUATION WITH ACOUSTIC CONTROL BOUNDARY CONDITIONS II*

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Abstract In this paper, we consider the nonlinear viscoelastic Kirchhoff-type equation with initial conditions and acoustic boundary conditions. Under suitable conditions on the initial data, the relaxation function $h(\cdot)$ and $M(\cdot)$, we prove that the solution blows up in finite time and give the upper bound of the blow-up time T^*.

Keywords Kirchhoff-type equation, acoustic boundary condition, blow-up.

MSC(2010) 35L05, 35L15, 35L70.

1. Introduction

In this paper, we consider the nonlinear viscoelastic Kirchhoff-type equation with acoustic control boundary conditions

$$u_{tt}-M(\|\nabla u\|_2^2)\Delta u+\int_0^t h(t-s)\Delta u(s)ds+a|u_t|^{m-2}u_t=|u|^{p-2}u \quad \text{in } \Omega\times(0,\infty), \tag{1.1}$$

$$u=0 \quad \text{on } \Gamma_1\times(0,\infty), \tag{1.2}$$

$$M(\|\nabla u\|_2^2)\frac{\partial u}{\partial \nu}-\int_0^t h(t-s)\frac{\partial u(s)}{\partial \nu}ds=y_t \quad \text{on } \Gamma_0\times(0,\infty), \tag{1.3}$$

$$u_t+\alpha(x)y_t+\beta(x)y=0 \quad \text{on } \Gamma_0\times(0,\infty), \tag{1.4}$$

$$u(x,0)=u_0, u_t(x,0)=u_1 \quad \text{in } \Omega, \tag{1.5}$$

$$y(x,0)=y_0 \quad \text{on } \Gamma_0, \tag{1.6}$$

where Ω is a bounded domain in $\mathbb{R}^n (n\geq 1)$ with C^2 boundary $\Gamma=\Gamma_0\cup\Gamma_1$, Γ_0 and Γ_1 are closed and disjoint, meas$(\Gamma_0)>0$ and meas$(\Gamma_1)>0$. $a\geq 0$, $m\geq 2$ and $p>2$ are constants. ν is the unit outward normal to Γ, $u_t=\frac{\partial u}{\partial t}$, $y_t=\frac{\partial y}{\partial t}$, $\Delta u=\sum_{i=1}^{n}\frac{\partial^2 u}{\partial x_i^2}$. $M(\cdot)$ is a positive C^1-function and h represents the kernel of the memory term. y

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is the normal displacement to the boundary at time t with the boundary point x. α, β will be specified later.

When $h = 0$ and $M \equiv 1$, Eq.(1.1) becomes a nonlinear wave equation which has been extensively studied and several results concerning existence and nonexistence have been established. When $M(\cdot)$ is not a constant function, Eq.(1.1) is the Kirchhoff-type wave equation. This type model was introduced by Kirchhoff in order to study the nonlinear vibrations of an elastic string. Kirchhoff is the first one to study the oscillations of stretched strings and plates. In this case the existence and nonexistence of solutions with homogeneous Dirichlet boundary condition have been discussed by many authors (see Wu and Tsai [36]).

For Eq.(1.1) with $h \neq 0$ and $M \equiv 1$, Cavalcanti etc [8] studied the case of $m = 2$, and a localized damping $a(x)u_t$. They obtained an exponential rate of decay with the assumption that the kernel h is exponential decay. And they [9] also studied the case of $m \geq 2$. This work was later improved by Cavalcanti and Oquendo [10], Berrimi and Messaoudi [6].

The *homogeneous Dirichlet boundary value* problems for Kirchhoff-type equations have been considered by many mathematicians. Nishihara and Yamada [31] considered the global solvability of the homogeneous Dirichlet boundary value problem for

$$\frac{\partial^2 u}{\partial t^2} - a\left(\int_\Omega |\nabla u|^2 dx\right)\Delta u + 2\gamma\frac{\partial u}{\partial t} = 0$$

and showed the global existence, uniqueness and asymptotic decay of solutions provided that the initial data u_0 ($u_0 \neq 0$) and u_1 are small and u_1 is much smaller than u_0 in some sense. Aassila and Benaissa [1] extended the global existence part of Christensen [11] to the case where $\varphi(s) > 0$ with $\varphi(\|\nabla u_0\|^2) \neq 0$ and the nonlinear dissipative term $|u_t|^{\alpha-2}u_t$. Ono [32] and Ye [40] obtained the global existence of the solution to homogeneous Dirichlet boundary value problem

$$u_{tt} - \varphi(\|\nabla u\|_2^2)\Delta u - au_t = b|u|^{\beta-2}u,$$

where $a, b > 0$ and $\beta > 2$ are constants, $\varphi(s)$ is a C^1-class function on $[0, +\infty)$ satisfying

$$\varphi(s) \geq m_0,\ s\varphi(s) \geq \int_0^s \varphi(\tau)d\tau,\ \forall s \in [0, \infty)$$

with $m_0 \geq 1$. Wu and Tsai [36] verified the general Kirchhoff-type equation

$$u_{tt} - M(\|\nabla u(t)\|_2^2)\Delta u + |u_t|^{m-2}u_t = |u|^{p-2}u$$

with homogeneous Dirichlet boundary condition and positive upper bounded initial energy blows up. Applying the Banach contraction mapping principle, Gao etc [14] obtained the local existence and blow-up property of the solution to homogeneous Dirichlet boundary value problem for the higher-order nonlinear Kirchhoff-type equation

$$u_{tt} + M(\|D^m u(t)\|_2^2)(-\Delta)^m u + |u_t|^{q-2}u_t = |u|^{p-2}u,$$

where $p > q \geq 2, m \geq 1$. Using Galerkin method, Ono and Nishihara [33] proved the global existence and decay structure of solutions of the homogeneous Dirichlet boundary value problem for

$$u_{tt} - \varphi(\|\nabla u\|_2^2)\triangle u - a\triangle u_t = b|u|^{\beta-2}u$$

without small condition of data. In Wu [37], Wu considered the strong damping integro-differential equation

$$u_{tt} - M(\|\nabla u\|_2^2)\Delta u + \int_0^t h(t-s)\Delta u(s)ds - \Delta u_t = |u|^{p-2}u$$

with homogeneous Dirichlet boundary and showed that under some conditions on h the solution is global in time and energy decays exponentially.

The *mixed Dirichlet and Neumann homogenous boundary value* problems for Kirchhoff-type equations have been considered in Gorain [15]. Gorain [15] studied the uniform stability of two mixed Dirichlet and Neumann homogenous boundary value problems for

$$u_{tt} + 2\delta u_t = \left(a^2 + b\int_\Omega |\nabla u|^2 dx\right)\Delta u,$$

and

$$u_{tt} = \left(a^2 + b\int_\Omega |\nabla u|^2 dx\right)\Delta u + 2\lambda\Delta u_t.$$

Beale and Rosencrans [3] introduced *acoustic boundary conditions* of general form

$$\frac{\partial u}{\partial \nu} = y_t \qquad \text{on } \Gamma_0 \times (0,\infty), \tag{1.7}$$

$$\gamma u_t + m(x)y_{tt} + \alpha(x)y_t + \beta(x)y = 0 \qquad \text{on } \Gamma_0 \times (0,\infty) \tag{1.8}$$

and then Beale [4, 5] investigated global existence and regularity of solutions for wave equation

$$u_{tt} - \Delta u = 0$$

with (1.7)-(1.8) by means of semigroup methods. Recently, wave equations with acoustic boundary conditions have been treated by many authors. Frota and Goldstein [13] studied the nonlinear wave equation

$$u_{tt} - M\left(\int_\Omega u^2 dx\right)\Delta u + |u_t|^\alpha u_t = 0$$

with (1.2), (1.7) and (1.8), proved the existence and uniqueness of global solution, but the dynamic properties are not given. Park [34] considered a wave equation of memory type with acoustic boundary conditions

$$\begin{cases} u_{tt} - \Delta u + \displaystyle\int_0^t h(t-s)\Delta u(s)ds = 0 & \text{in } \Omega \times (0,\infty), \\ u = 0 & \text{on } \Gamma_1 \times (0,\infty), \\ \dfrac{\partial u}{\partial \nu} - \displaystyle\int_0^t h(t-s)\frac{\partial u(s)}{\partial \nu}ds = y_t & \text{on } \Gamma_0 \times (0,\infty), \\ u_t + \alpha(x)y_t + \beta(x)y = 0 & \text{on } \Gamma_0 \times (0,\infty), \\ u(x,0) = u_0, u_t(x,0) = u_1 & \text{in } \Omega \end{cases}$$

and investigated the influence of kernel function h and proved general decay rates of solutions when h does not necessarily decay exponentially. For the recent results on the wave equations with acoustic boundary conditions, one can see [16, 17, 39].

In Li etc [18–20], the authors proved the existence uniqueness, uniform energy decay rates and limit behavior of the solution to nonlinear viscoelastic Marguerre-von Kármán shallow shells system, respectively. Li etc [21, 22, 24, 25, 28, 29] showed the global existence uniqueness and decay estimates for nonlinear viscoelastic equation with boundary dissipation. The authors studied the blow-up phenomenon for some evolution equations in Li etc [23, 26, 27, 38]. In Li and Xi [30], the authors gave the asymptotic stability result of the solution to (1.1)-(1.6) under suitable assumptions.

Motivated by the above work, we intend to study the blow-up property of the solution to problem (1.1)-(1.6). To our knowledge, blow-up phenomenon of the solution to viscoelastic Kirchhoff-type equation with acoustic boundary conditions hasn't been studied. By using contraction mapping principle, we will prove that under some conditions on $M, h, \alpha, \beta, \gamma$ and the initial data, the problem has a unique local solution and the solution blows up in finite time. The main contributions of this paper are: (a) the problem considered in this paper is nonlinear viscoelastic Kirchhoff-type equation with acoustic boundary conditions. To our knowledge, the blow up phenomena to (1.1)-(1.6) has not been considered by predecessors and is studied firstly as a new problem in this paper; (b) acoustic boundary conditions, propagation velocity function $M(\cdot)$ and relaxation function h bring great difficulties to the estimations; (c) the construction of auxiliary functions is ingenious, the estimates are precise.

The present work is organized as follows. In section 2, we present some notations and material needed for our work. Section 3 is devoted to state and prove our main result.

2. Preliminaries

Throughout this paper, we define

$$V = \{u \in H^1(\Omega) | u = 0 \text{ on } \Gamma_1\},$$

and the following scalar products

$$(u, \upsilon) = \int_\Omega u(x)\upsilon(x)dx, \quad (u, \upsilon)_{\Gamma_0} = \int_{\Gamma_0} u(x)\upsilon(x)dS,$$

and the following norms

$$\|u\|_{L^p(\Omega)} = \left(\int_\Omega |u|^p dx\right)^{\frac{1}{p}}, \ \|u\|_{L^p(\Gamma_0)} = \left(\int_{\Gamma_0} |u|^p dS\right)^{\frac{1}{p}}.$$

To simplify the notations, we denote $\|u\|_{L^p(\Omega)}$, $\|u\|_{L^p(\Gamma_0)}$ by $\|u\|_p$, $\|u\|_{p,\Gamma_0}$ respectively.

For $h \in C^1(\mathbb{R})$ and $u \in H^1(0,T)$, the symbol $h * u$ stands for convolution, that is

$$h * u = \int_0^t h(t-s)u(s)ds,$$

and by $\circ$, we denote

$$h \circ \nabla u = \int_0^t h(t-s) \int_\Omega |\nabla u(s) - \nabla u(t)|^2 dx ds = \int_0^t h(t-s)\|\nabla u(s) - \nabla u(t)\|_2^2 ds.$$

We state the general hypotheses on functions M, α, β, h and the constants p, m, γ.
(A_1) *$M(s)$ is a positive C^1-function for $s \geq 0$ satisfying $M(s) = m_0 + bs^\gamma$, $m_0 > 0$, $b \geq 0$ and* $\gamma : \begin{cases} 0 \leq \gamma < \frac{2}{n-2}, & n \geq 3, \\ \gamma \geq 0, & n = 1, 2. \end{cases}$
(A_2) *$h(t)$: $[0, \infty) \to [0, \infty)$ are nonincreasing C^1-function such that*

$$m_0 - \int_0^\infty h(s)ds = l > 0. \tag{2.1}$$

(A_3) *For the functions α and β, we assume that α, $\beta \in C(\Gamma_0)$ and $\alpha(x) > 0$ and $\beta(x) > 0$ for all $x \in \Gamma_0$. This assumptions imply that there exist positive constants α_i, $\beta_i (i = 0, 1)$, such that*

$$\alpha_0 \leq \alpha(x) \leq \alpha_1, \quad \beta_0 \leq \beta(x) \leq \beta_1, \textit{ for all } x \in \Gamma_0.$$

(A_4) $p > \max\left\{1 + \sqrt{1 + \frac{m_0 - l}{l}}, 2(\gamma + 1)\right\}$, $2 \leq m < p \leq \frac{2n}{n-2}$, $n \geq 3$, *or* $p > m \geq 2$, $n = 1$, 2.

Remark 2.1. If $h \equiv 0$, $M \equiv const$, then $\gamma = 0, l = m_0$, the assumptions $(A_1)-(A_4)$ is the assumptions for wave equation in some known literatures.

By using contraction mapping principle and the similar procedure in [3, 14], we can have the following existence theorem for (1.1)-(1.6) under the conditions $(A_1) - (A_4)$ as above.

Theorem 2.1. *Let the assumptions $(A_1)-(A_4)$ hold and $(u_0, u_1, y_0) \in \left(V \cap H^2(\Omega)\right) \times V \times L^2(\Gamma_0)$. Then there exists a unique local solution u of (1.1)-(1.6) satisfying*

$$u \in L^\infty\left(0, T; V \cap H^2(\Omega)\right), \, u_t \in L^\infty(0, T; V),$$
$$u_{tt} \in L^\infty\left(0, T; L^2(\Omega)\right), \, y, \, y_t \in L^2\left(0, T; L^2(\Gamma_0)\right).$$

Moreover, we have

$$u \in C\left([0, T); V\right), u_t \in C\left([0, T); L^2(\Omega)\right).$$

In the following, we give some lemmas which will be used in this paper.

Lemma 2.1 (General poincaré inequality)**.** *If $2 \leq q \leq \frac{2n}{n-2}$, $n \geq 3$, or $q \geq 2$, $n = 1, 2$, then there exists an optimal constant B such that*

$$\|u\|_q \leq B\|\nabla u\|_2, \ \forall \ u \in V. \tag{2.2}$$

Moreover, using the trace theorem, we have

$$\|u\|_{2,\Gamma_0} \leq c_*\|\nabla u\|_2, \ \forall \ u \in V. \tag{2.3}$$

Proof. The proof can be found in [2, 12]. □

Lemma 2.2. *If $z \geq 0$, $0 < \theta \leq 1$, $a \geq 0$, we have the following inequality*

$$z^\theta \leq \left(1 + \frac{1}{a}\right)(z + a), \tag{2.4}$$

Proof. In fact,

$$z^{\theta} \leq (z+1) \leq \left(1+\frac{1}{a}\right)(z+a). \qquad \square$$

3. The main result

In order to define the energy function $E(t)$ of the problem (1.1)-(1.6), we give the following computation. Multiplying u_t on both sides of (1.1), integrating the resulting equation over Ω, using Green formula, (A_1) and (1.2)-(1.4), we have

$$\begin{aligned}
&(u_{tt}, u_t) + \left(M(\|\nabla u\|_2^2)\nabla u, \nabla u_t\right) - \int_0^t h(t-s)\left(\nabla u(s), \nabla u_t(t)\right) ds + a\|u_t\|_m^m \\
&\quad - \int_{\Gamma_0} u_t \left[M(\|\nabla u\|_2^2)\frac{\partial u}{\partial \nu} - \int_0^t h(t-s)\frac{\partial u(s)}{\partial \nu} ds\right] dS \\
=&\frac{1}{2}\frac{d}{dt}\|u_t\|_2^2 + m_0\left(\nabla u, \nabla u_t\right) + b\|\nabla u\|_2^{2\gamma}\left(\nabla u, \nabla u_t\right) \\
&\quad - \int_0^t h(t-s)\left(\nabla u(s), \nabla u_t(t)\right) ds + a\|u_t\|_m^m \\
&\quad - \int_{\Gamma_0} u_t \left[M(\|\nabla u\|_2^2)\frac{\partial u}{\partial \nu} - \int_0^t h(t-s)\frac{\partial u(s)}{\partial \nu} ds\right] dS \\
=&\frac{1}{2}\frac{d}{dt}\|u_t\|_2^2 + \frac{m_0}{2}\frac{d}{dt}\|\nabla u\|_2^2 + \frac{b}{2}\|\nabla u\|_2^{2\gamma}\frac{d}{dt}\|\nabla u\|_2^2 \\
&\quad - \int_0^t h(t-s)\left(\nabla u(s), \nabla u_t(t)\right) ds + a\|u_t\|_m^m \\
&\quad - \int_{\Gamma_0} u_t \left[M(\|\nabla u\|_2^2)\frac{\partial u}{\partial \nu} - \int_0^t h(t-s)\frac{\partial u(s)}{\partial \nu} ds\right] dS \\
=&\left(|u|^{p-2}u, u_t\right) = \frac{1}{p}\frac{d}{dt}\|u\|_p^p,
\end{aligned}$$

that is

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|u_t\|_2^2 + m_0\|\nabla u\|_2^2 + \frac{b}{\gamma+1}\|\nabla u\|_2^{2\gamma+2} - \frac{2}{p}\|u\|_p^p\right) \\
&- \int_{\Gamma_0} u_t \left(M(\|\nabla u\|_2^2)\frac{\partial u}{\partial \nu} - \int_0^t h(t-s)\frac{\partial u(s)}{\partial \nu} ds\right) dS \\
&- \int_0^t h(t-s)\left(\nabla u(s), \nabla u_t(t)\right) ds = -a\|u_t\|_m^m,
\end{aligned} \tag{3.1}$$

for any regular solution. This result remains valid for weak solutions by a simple density argument.

Using (1.2)-(1.4), we get

$$\begin{aligned}
&- \int_{\Gamma_0} u_t \left[M(\|\nabla u\|_2^2)\frac{\partial u}{\partial \nu} - \int_0^t h(t-s)\frac{\partial u(s)}{\partial \nu} ds\right] dS \\
=&-\left(u_t, y_t\right)_{\Gamma_0} = \left(\alpha(x), y_t^2\right)_{\Gamma_0} + \frac{1}{2}\frac{d}{dt}\left(\beta(x), y^2\right)_{\Gamma_0}.
\end{aligned} \tag{3.2}$$

Direct calculation shows

$$\begin{aligned}
&-\int_0^t h(t-s)(\nabla u(s),\nabla u_t(t))ds\\
=&-\int_0^t h(t-s)(\nabla u_t(t),\nabla u(s)-\nabla u(t))ds-\int_0^t h(t-s)(\nabla u_t(t),\nabla u(t))ds\\
=&\frac{1}{2}\int_0^t h(t-s)\frac{d}{dt}\|\nabla u(t)-\nabla u(s)\|_2^2ds-\frac{1}{2}\int_0^t h(s)ds\frac{d}{dt}\|\nabla u\|_2^2\\
=&\frac{1}{2}\frac{d}{dt}\left[\int_0^t h(t-s)\|\nabla u(t)-\nabla u(s)\|_2^2ds\right]-\frac{1}{2}\frac{d}{dt}\left[\int_0^t h(s)ds\|\nabla u\|_2^2\right]\\
&-\frac{1}{2}\int_0^t h'(t-s)\|\nabla u(t)-\nabla u(s)\|_2^2ds+\frac{1}{2}h(t)\|\nabla u\|_2^2.
\end{aligned}\tag{3.3}$$

Inserting (3.2) and (3.3) into (3.1), we obtain

$$\begin{aligned}
\frac{d}{dt}E(t)=&\frac{1}{2}\frac{d}{dt}\left(\|u_t\|_2^2+m_0\|\nabla u\|_2^2+\frac{b}{\gamma+1}\|\nabla u\|_2^{2\gamma+2}-\frac{2}{p}\|u\|_p^p\right)\\
&-\frac{1}{2}\frac{d}{dt}\left[\int_0^t h(s)ds\|\nabla u\|_2^2-h\circ\nabla u\right]+\frac{1}{2}\frac{d}{dt}\left(\beta(x),y^2\right)_{\Gamma_0}\\
=&-a\|u_t\|_m^m-\frac{1}{2}h(t)\|\nabla u\|_2^2+\frac{1}{2}h'\circ\nabla u-\left(\alpha(x),y_t^2\right)_{\Gamma_0},
\end{aligned}$$

where

$$\begin{aligned}
E(t)=&\frac{1}{2}\|u_t\|_2^2+\frac{1}{2}\left[m_0-\int_0^t h(s)ds\right]\|\nabla u\|_2^2+\frac{b}{2(\gamma+1)}\|\nabla u\|_2^{2\gamma+2}\\
&+\frac{1}{2}h\circ\nabla u+\frac{1}{2}\left(\beta(x),y^2\right)_{\Gamma_0}-\frac{1}{p}\|u\|_p^p.
\end{aligned}\tag{3.4}$$

Lemma 3.1. *$E(t)$ is a non-increasing functional.*
Proof. Note that

$$E'(t)=-a\|u_t\|_m^m-\frac{1}{2}h(t)\|\nabla u\|_2^2+\frac{1}{2}h'\circ\nabla u-(\alpha(x),y_t^2)_{\Gamma_0}\le 0.\tag{3.5}$$

□

Theorem 3.2. *Assume the assumptions in Theorem 2.1 hold. Then the local solution of (1.1)-(1.6) with initial condition $E(0)<0$ blows up in finite time. In other words, there exists a positive constant T^* such that $\lim_{t\to T^*}\|u\|_p=\infty$.*
Proof. At first, we set

$$H(t)=-E(t).$$

By Lemma 3.1 we have

$$H'(t)=-E'(t)=a\|u_t\|_m^m+\frac{1}{2}h(t)\|\nabla u\|_2^2-\frac{1}{2}h'\circ\nabla u+(\alpha(x),y_t^2)_{\Gamma_0}\ge a\|u_t\|_m^m\ge 0.\tag{3.6}$$

Then we obtain

$$0<-E(0)=H(0)\le H(t).\tag{3.7}$$

Using (3.4),(3.5) and (3.7), we get

$$
\begin{aligned}
0 < H(0) \leq H(t) = &-\frac{1}{2}\|u_t\|_2^2 - \frac{1}{2}\left[m_0 - \int_0^t h(s)ds\right]\|\nabla u\|_2^2 - \frac{b}{2(\gamma+1)}\|\nabla u\|_2^{2\gamma+2} \\
&- \frac{1}{2}h \circ \nabla u - \frac{1}{2}\left(\beta(x), y^2\right)_{\Gamma_0} + \frac{1}{p}\|u\|_p^p \\
&\leq \frac{1}{p}\|u\|_p^p,
\end{aligned}
$$

that is

$$
0 < H(0) \leq H(t) \leq \frac{1}{p}\|u\|_p^p. \tag{3.8}
$$

Define

$$
L(t) = H^{1-\sigma}(t) + \varepsilon(u, u_t) - \frac{\varepsilon}{2}\left(\alpha(x), y^2\right)_{\Gamma_0} - \varepsilon(u, y)_{\Gamma_0}, \tag{3.9}
$$

for small $\varepsilon > 0$ to be chosen later and for

$$
0 < \sigma \leq \min\left\{\frac{1}{2} - \frac{1}{p}, \frac{p-m}{p(m-1)}\right\}. \tag{3.10}
$$

By taking the derivative of (3.9) and using (1.1)-(1.4), we obtain

$$
\begin{aligned}
L'(t) =& (1-\sigma)H^{-\sigma}(t)H'(t) + \varepsilon\|u_t\|_2^2 + \varepsilon(u, u_{tt}) - \varepsilon\left(\alpha(x), yy_t\right)_{\Gamma_0} - \varepsilon(u_t, y)_{\Gamma_0} - \varepsilon(u, y_t)_{\Gamma_0} \\
=& (1-\sigma)H^{-\sigma}(t)H'(t) + \varepsilon\left[\|u_t\|_2^2 - a\left(u, |u_t|^{m-2}u_t\right) + \|u\|_p^p + \left(\beta(x), y^2\right)_{\Gamma_0}\right] \\
&+ \varepsilon\left[M(\|\nabla u\|_2^2)\|(u, \Delta u) - \int_0^t h(t-s)(u(t), \Delta u(s))ds - (u, y_t)_{\Gamma_0}\right] \\
=& (1-\sigma)H^{-\sigma}(t)H'(t) + \varepsilon\left[\|u_t\|_2^2 - a(u, |u_t|^{m-2}u_t) + \|u\|_p^p + \left(\beta(x), y^2\right)_{\Gamma_0}\right] \\
&+ \varepsilon\left[-m_0\|\nabla u\|_2^2 - b\|\nabla u\|_2^{2\gamma+2} + \int_0^t h(t-s)(\nabla u(t), \nabla u(s))ds\right] \\
=& (1-\sigma)H^{-\sigma}(t)H'(t) + \varepsilon\left[\|u_t\|_2^2 - a(u, |u_t|^{m-2}u_t) + \|u\|_p^p + \left(\beta(x), y^2\right)_{\Gamma_0}\right] \\
&+ \varepsilon\left[-m_0\|\nabla u\|_2^2 - b\|\nabla u\|_2^{2\gamma+2} + \int_0^t h(t-s)(\nabla u(t), \nabla u(s))ds\right] \\
&+ 2\varepsilon\left[H(t) + E(t))\right] \\
=& (1-\sigma)H^{-\sigma}(t)H'(t) + \varepsilon\left[\|u_t\|_2^2 - a(u, |u_t|^{m-2}u_t) + \|u\|_p^p + \left(\beta(x), y^2\right)_{\Gamma_0}\right] \\
&+ \varepsilon\left[-m_0\|\nabla u\|_2^2 - b\|\nabla u\|_2^{2\gamma+2} + \int_0^t h(t-s)(\nabla u(t), \nabla u(s))ds\right] \\
&+ 2\varepsilon H(t) \\
&+ 2\varepsilon\left[\frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}\left[m_0 - \int_0^t h(s)ds\right]\|\nabla u\|_2^2 + \frac{b}{2(\gamma+1)}\|\nabla u\|_2^{2\gamma+2} + \frac{1}{2}h \circ \nabla u\right] \\
&+ \varepsilon\left(\beta(x), y^2\right)_{\Gamma_0} - \frac{2\varepsilon}{p}\|u\|_p^p \\
=& (1-\sigma)H^{-\sigma}(t)H'(t) + 2\varepsilon H(t)
\end{aligned}
$$

$$+\varepsilon\left[2\|u_t\|_2^2-a(u,|u_t|^{m-2}u_t)+\left(1-\frac{2}{p}\right)\|u\|_p^p-\frac{b\gamma}{(\gamma+1)}\|\nabla u\|_2^{2\gamma+2}+2\left(\beta(x),y^2\right)_{\Gamma_0}\right]$$
$$+\varepsilon\left[\int_0^t h(t-s)(\nabla u(t),\nabla u(s))ds-\int_0^t h(s)ds\|\nabla u\|_2^2+h\circ\nabla u\right]. \tag{3.11}$$

Next, we estimate the terms in (3.11). Applying Cauchy-Schwarz and Hölder inequalities, we know that

$$-a\left(u,|u_t|^{m-2}u_t\right)\geq -a\frac{\delta^m}{m}\|u\|_m^m-a\frac{m-1}{m}\delta^{\frac{-m}{m-1}}\|u_t\|_m^m. \tag{3.12}$$

Choosing $\delta^{-\frac{m}{m-1}}=kH^{-\sigma}(t)$ (by (3.6)), where $k>0$ is a positive constant to be specified later, and inserting (3.9) and (3.12) into (3.11), we see

$$\begin{aligned}L'(t)\geq&\left(1-\sigma-\varepsilon\frac{m-1}{m}k\right)H^{-\sigma}(t)H'(t)+2\varepsilon H(t)\\&+\varepsilon\left[2\|u_t\|_2^2+\left(1-\frac{2}{p}\right)\|u\|_p^p-\frac{b\gamma}{(\gamma+1)}\|\nabla u\|_2^{2\gamma+2}+2\left(\beta(x),y^2\right)_{\Gamma_0}+h\circ\nabla u\right]\\&+\varepsilon\left[\int_0^t h(t-s)(\nabla u(t),\nabla u(s))ds-\int_0^t h(s)ds\|\nabla u\|_2^2\right]\\&-a\varepsilon\frac{k^{-(m-1)}H^{\sigma(m-1)}(t)}{m}\|u\|_m^m.\end{aligned} \tag{3.13}$$

Since $p>m\geq 2$ and using inequality $\|u\|_m^m\leq C_1\|u\|_p^m$ (where $C_1=|\Omega|^{\frac{p-m}{p}}$, and $|\Omega|$ denotes the Lebesgue measure of Ω, we obtain from (3.8) that

$$H^{\sigma(m-1)}(t)\|u\|_m^m\leq\left(\frac{1}{p}\|u\|_p^p\right)^{\sigma(m-1)}C_1\|u\|_p^m=\frac{C_1}{p^{\sigma(m-1)}}\|u\|_p^{p\left(\frac{m}{p}+\sigma(m-1)\right)}.$$

From $0<\sigma\leq\min\left\{\frac{p-m}{p(m-1)},\frac{1}{2}-\frac{1}{p}\right\}$, we easily get $\sigma(m-1)+\frac{m}{p}\leq 1$. Then, using Lemma 2.2 and (3.7) we obtain

$$\|u\|_p^{p\left(\sigma(m-1)+\frac{m}{p}\right)}\leq d\left(\|u\|_p^p+H(0)\right)\leq d\left(\|u\|_p^p+H(t)\right), \tag{3.14}$$

where $d=1+\frac{1}{H(0)}$. Substituting (3.14) into (3.13), we get

$$\begin{aligned}L'(t)\geq&\left(1-\sigma-\varepsilon\frac{m-1}{m}k\right)H^{-\sigma}(t)H'(t)+\varepsilon\left(2-a\frac{1}{k^{m-1}m}\frac{C_1}{p^{\sigma(m-1)}}d\right)H(t)\\&+\varepsilon\left[2\|u_t\|_2^2+\left(1-\frac{2}{p}-a\frac{1}{k^{m-1}m}\frac{C_1}{p^{\sigma(m-1)}}d\right)\|u\|_p^p+2\left(\beta(x),y^2\right)_{\Gamma_0}+h\circ\nabla u\right]\\&+\varepsilon\left[\int_0^t h(t-s)(\nabla u(t),\nabla u(s))ds-\int_0^t h(s)ds\|\nabla u\|_2^2-\frac{b\gamma}{(\gamma+1)}\|\nabla u\|_2^{2\gamma+2}\right].\end{aligned} \tag{3.15}$$

Note that

$$\int_0^t h(t-s)(\nabla u(t),\nabla u(s))ds$$

$$
\begin{aligned}
&= \int_0^t h(t-s)(\nabla u(t), \nabla u(s) - \nabla u(t))ds + \int_0^t h(s)ds\|\nabla u\|_2^2 \\
&\geq -\int_0^t h(t-s)\left(\frac{\delta_1}{2}\|\nabla u(s) - \nabla u(t)\|_2^2 + \frac{1}{2\delta_1}\|\nabla u\|_2^2\right) ds + \int_0^t h(s)ds\|\nabla u\|_2^2 \\
&= -\frac{\delta_1}{2} h \circ \nabla u + \left(1 - \frac{1}{2\delta_1}\right)\int_0^t h(s)ds\|\nabla u\|_2^2, \qquad (3.16)
\end{aligned}
$$

and

$$
\begin{aligned}
\|u\|_p^p = {} & pH(t) + \frac{p}{2}\|u_t\|_2^2 + \frac{p}{2}\left[m_0 - \int_0^t h(s)ds\right]\|\nabla u\|_2^2 + \frac{pb}{2(\gamma+1)}\|\nabla u\|_2^{2\gamma+2} \\
& + \frac{p}{2} h \circ \nabla u + \frac{p}{2}\left(\beta(x)y^2\right)_{\Gamma_0}. \qquad (3.17)
\end{aligned}
$$

Inserting the estimates (3.16)-(3.17) into (3.15), we obtain

$$
\begin{aligned}
L'(t) \geq {} & \left(1 - \sigma - \varepsilon\frac{m-1}{m}k\right) H^{-\sigma}(t)H'(t) \\
& + \varepsilon\left[2 - a\frac{1}{k^{m-1}m}\frac{C_1}{p^{\sigma(m-1)}}d + p\left(1 - \frac{2}{p} - a\frac{1}{k^{m-1}m}\frac{C_1}{p^{\sigma(m-1)}}d\right)\right] H(t) \\
& + \varepsilon\left[\frac{p}{2}\left(1 - \frac{2}{p} - a\frac{1}{k^{m-1}m}\frac{C_1}{p^{\sigma(m-1)}}d\right) + 2\right]\left(\|u_t\|_2^2 + \left(\beta(x), y^2\right)_{\Gamma_0}\right) \\
& + \varepsilon\left[\frac{p}{2}\left(1 - \frac{2}{p} - a\frac{1}{k^{m-1}m}\frac{C_1}{p^{\sigma(m-1)}}d\right) + 1 - \frac{\delta_1}{2}\right] h \circ \nabla u \\
& + \varepsilon\left[\frac{pb}{2(\gamma+1)}\left(1 - \frac{2}{p} - a\frac{1}{k^{m-1}m}\frac{C_1}{p^{\sigma(m-1)}}d\right) - \frac{b\gamma}{(\gamma+1)}\right]\|\nabla u\|_2^{2\gamma+2} \\
& + \left[\frac{p}{2}\left(m_0 - \int_0^t h(s)ds\right)\left(1 - \frac{2}{p} - a\frac{1}{k^{m-1}m}\frac{C_1}{p^{\sigma(m-1)}}d\right)\right. \\
& \left. - \frac{1}{2\delta_1}\int_0^t h(s)ds\right]\|\nabla u\|_2^2.
\end{aligned}
$$

Noting $(A_1) - (A_4)$, and choosing $\delta_1 \in \left(\frac{m_0 - l}{(p-2)l}, p\right)$, for fixed $k > 0$ large enough and fixed $\varepsilon > 0$ small enough, we get

$$
\begin{aligned}
& 1 - \sigma - \varepsilon\frac{m-1}{m}k > 0, \\
& 2 - a\frac{1}{k^{m-1}m}\frac{C_1}{p^{\sigma(m-1)}}d + p\left(1 - \frac{2}{p} - a\frac{1}{k^{m-1}m}\frac{C_1}{p^{\sigma(m-1)}}d\right) > 0, \\
& \frac{p}{2}\left(1 - \frac{2}{p} - a\frac{1}{k^{m-1}m}\frac{C_1}{p^{\sigma(m-1)}}d\right) + 2 > 0, \\
& \frac{p}{2}\left(1 - \frac{2}{p} - a\frac{1}{k^{m-1}m}\frac{C_1}{p^{\sigma(m-1)}}d\right) + 1 - \frac{\delta_1}{2} > 0, \\
& \frac{pb}{2(\gamma+1)}\left(1 - \frac{2}{p} - a\frac{1}{k^{m-1}m}\frac{C_1}{p^{\sigma(m-1)}}d\right) - \frac{b\gamma}{(\gamma+1)} > 0, \\
& \frac{p}{2}\left(m_0 - \int_0^t h(s)ds\right)\left(1 - \frac{2}{p} - a\frac{1}{k^{m-1}m}\frac{C_1}{p^{\sigma(m-1)}}d\right) - \frac{1}{2\delta_1}\int_0^t h(s)ds
\end{aligned}
$$

$$\geq \frac{p}{2}l\left(1-\frac{2}{p}-a\frac{1}{k^{m-1}m}\frac{C_1}{p^{\sigma(m-1)}}d\right)-\frac{1}{2\delta_1}(m_0-l)>0.$$

Therefore, there exists a positive constant $\eta>0$ such

$$L'(t)\geq \varepsilon\eta\left[H(t)+\|u_t\|_2^2+\|y\|_{2,\Gamma_0}^2+h\circ\nabla u+\|\nabla u\|_2^{2\gamma+2}+\|\nabla u\|_2^2\right]. \tag{3.18}$$

Noting $\left(\alpha(x),y^2\right)_{\Gamma_0}\geq 0$, we obtain from (3.9) that

$$L(t)\leq H^{1-\sigma}(t)+\varepsilon(u,u_t)-\varepsilon(u,y)_{\Gamma_0}. \tag{3.19}$$

Using Hölder inequalities, Lemma 2.1 and noting (A_1) we know that

$$|(u,u_t)|\leq \|u\|_2\|u_t\|_2\leq C_2\|u\|_{2\gamma+2}\|u_t\|_2\leq C_3\|\nabla u\|_2\|u_t\|_2, \tag{3.20}$$

where $C_2=|\Omega|^{\frac{\gamma}{2(\gamma+1)}}$, $C_3=BC_2$. Furthermore,

$$|(u,u_t)|^{\frac{1}{1-\sigma}}\leq C_3^{\frac{1}{1-\sigma}}\|\nabla u\|_2^{\frac{1}{1-\sigma}}\|u_t\|_2^{\frac{1}{1-\sigma}}.$$

Applying Young inequality, we see

$$|(u,u_t)|^{\frac{1}{1-\sigma}}\leq C_3^{\frac{1}{1-\sigma}}\left(\frac{1}{\mu_1}\|\nabla u\|_2^{\frac{\mu_1}{1-\sigma}}+\frac{1}{\mu_2}\|u_t\|_2^{\frac{\mu_2}{1-\sigma}}\right)\leq C_3^{\frac{1}{1-\sigma}}\left(\|\nabla u\|_2^{\frac{\mu_1}{1-\sigma}}+\|u_t\|_2^{\frac{\mu_2}{1-\sigma}}\right),$$

where $\frac{1}{\mu_1}+\frac{1}{\mu_2}=1$. Then choosing $\mu_2=2(1-\sigma)$, we get $\frac{\mu_1}{1-\sigma}=\frac{2}{1-2\sigma}$, and substituting it into the above inequality, we have

$$|(u,u_t)|^{\frac{1}{1-\sigma}}\leq C_3^{\frac{1}{1-\sigma}}\left(\|\nabla u\|_2^{\frac{2}{1-2\sigma}}+\|u_t\|_2^2\right)=C_3^{\frac{1}{1-\sigma}}\left(\|\nabla u\|_2^{(2\gamma+2)\frac{1}{(1-2\sigma)(\gamma+1)}}+\|u_t\|_2^2\right). \tag{3.21}$$

Owing to (A_4) and (3.10), we see $0<\sigma\leq\frac{1}{2}-\frac{1}{p}<\frac{1}{2}-\frac{1}{2(\gamma+1)}=\frac{\gamma}{2(\gamma+1)}$ which implies $\frac{1}{(1-2\sigma)(\gamma+1)}<1$. Using Lemma 2.2, we have

$$\|\nabla u\|_2^{(2\gamma+2)\frac{1}{(1-2\sigma)(\gamma+1)}}\leq d\left(\|\nabla u\|_2^{2\gamma+2}+H(0)\right)\leq d\left(\|\nabla u\|_2^{2\gamma+2}+H(t)\right), \tag{3.22}$$

where $d=1+\frac{1}{H(0)}$. Inserting (3.22) into (3.21), we get

$$|(u,u_t)|^{\frac{1}{1-\sigma}}\leq C_4\left(H(t)+\|\nabla u\|_2^{2\gamma+2}+\|u_t\|_2^2\right). \tag{3.23}$$

Furthermore, applying Cauchy inequality and Lemma 2.1, we have

$$\begin{aligned}\left|(u,y)_{\Gamma_0}\right|=\left|\left(\frac{1}{\beta(x)}u,\beta(x)y\right)_{\Gamma_0}\right|&\leq\frac{\sqrt{\|\beta(x)\|_\infty}}{\beta_0}\sqrt{(\beta(x),y^2)_{\Gamma_0}}\cdot\sqrt{(u,u)_{\Gamma_0}}\\&\leq c_*\frac{\sqrt{\|\beta(x)\|_\infty}}{\beta_0}\sqrt{(\beta(x),y^2)_{\Gamma_0}}\|\nabla u\|_2\\&\leq c_*\frac{\beta_1\sqrt{\|\beta(x)\|_\infty}}{\beta_0}\|y\|_{2,\Gamma_0}\|\nabla u\|_2.\end{aligned}$$

Imitating the above process, we have

$$\left|(u,y)_{\Gamma_0}\right|^{\frac{1}{1-\sigma}} \le C_5\left(H(t)+\|\nabla u\|_2^{2\gamma+2}+\|y\|_{2,\Gamma_0}^2\right). \tag{3.24}$$

Using the inequality $(a_1+a_2+a_3)^l \le 3^{(j-1)}(a_1^j+a_2^j+a_3^j)$, where $a_1\ge 0$, $a_2\ge 0$, $a_3\ge 0$, $j\ge 1$, we can obtain

$$\begin{aligned} L^{\frac{1}{1-\sigma}}(t) &\le \left(H^{1-\sigma}(t)+\varepsilon\,(u,u_t)-\varepsilon\,(u,y)_{\Gamma_0}\right)^{\frac{1}{1-\sigma}} \\ &\le \left(H^{1-\sigma}(t)+\varepsilon\,|(u,u_t)|+\varepsilon\left|(u,y)_{\Gamma_0}\right|\right)^{\frac{1}{1-\sigma}} \\ &\le C_6\left[H(t)+\varepsilon^{\frac{1}{1-\sigma}}\left(|(u,u_t)|^{\frac{1}{1-\sigma}}+\left|(u,y)_{\Gamma_0}\right|^{\frac{1}{1-\sigma}}\right)\right] \\ &\le C_7\left[H(t)+\|\nabla u\|_2^{2\gamma+2}+\|u_t\|_2^2+\|y\|_{2,\Gamma_0}^2)\right]. \end{aligned} \tag{3.25}$$

Combining (3.18) and (3.25), we conclude that for some $\tau>0$

$$L'(t)\ge \tau L^{\frac{1}{1-\sigma}}(t),\ \forall\, t\in[0,T^*).$$

Integrating the above inequality from 0 to t yields

$$L(t)\ge\left[\frac{1}{L^{-\frac{\sigma}{1-\sigma}}(0)-\frac{\sigma\tau}{1-\sigma}t}\right]^{\frac{1-\sigma}{\sigma}}. \tag{3.26}$$

Since $L(0)>0$, (3.26) shows that $\lim\limits_{t\to T^*}L(t)=\infty$, where $T^*\le \frac{1-\sigma}{\sigma\tau L^{\frac{\sigma}{1-\sigma}}(0)}$. From the definition of $L(t)$, we have $\lim\limits_{t\to T^*}H(t)=\infty$. According to (3.8), we obtain that $\lim\limits_{t\to T^*}\|u\|_p^p=\infty$, that is, the solution blows up at finite time in L^p norm.

4. Conclusions

In this paper, we consider the nonlinear viscoelastic Kirchhoff-type equation with acoustic control boundary conditions. Under suitable conditions on the initial data, the relaxation function $h(\cdot)$ and $M(\cdot)$, we prove that the solution blows up in finite time and give the upper bound of the blow-up time T^*. One can further consider the blow-up phenomenon and the lower bound estimation of T^* for the solution under the condition $E(0)<E_1$ (E_1 is some positive constant).

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References

[1] M. Aassila and A. Benaissa, *Existence globale et comportement asymptotique des solutions des equations de Kirchhoff moyennement degenerees avce un terme nonlinear dissipatif*, Funkc. Ekvacioj, 2000, 43, 309–333.

[2] R. A. Admas, *Sobolev Space*, New York: Academac press, 1975.

[3] J. T. Beale and S. I. Rosencrans, *Acoustic boundary conditions*, Bull. Amer. Math. Soc., 1974, 80, 1276–1278.

[4] J. T. Beale, *Spectral properties of an acoustic boundary condition*, Indiana Univ. Math. J., 1976, 25, 895–917.

[5] J. T. Beale, *Acoustic scattering from locally reacting surfaces*, Indiana Univ. Math. J., 1977, 26, 199–222.

[6] S. Berrimi and S. A. Messaoudi, *Exponential decay of solutions to a viscoelastic equation with nonlinear localized damping*, Electron. J. Differ. Eq., 2004, 88, 1–10.

[7] F. A. Boussouira, P. Cannarsa and D. Sforza, *Decay estimates for second order evolution equations with memory*, J. Funct. Anal., 2008, 254, 1342–1372.

[8] M. M. Cavalcanti, V. N. Domingos Cavalcanti and J. A. Soriano, *Exponential decay for the solution of semilinear viscoelastic wave equations with localized damping*, Electron. J. Differ. Eq., 2002, 44, 1–14.

[9] M. M. Cavalcanti, V. N. Domingos Cavalcanti and J. A. Soriano, *Global existence and asymptotic stability for viscoelastic probiems*, Differ. Integral Equ., 2002, 15, 731–748.

[10] M. M. Cavalcanti and H. P. Oquendo, *Frictional versus viscoelastic damping in a semilinear wave equation*, SIAM J. Control Optim., 2003, 42, 1310–1324.

[11] R. M. Christensen, *Theory of viscoelasticity*, Academic Press, New York, 1971.

[12] L. C. Evans, *Partial Differential Equations(Second Edition)*, Rhode Island: American Mathematical Society Providence, 2010.

[13] C. L. Frota and J. A. Goldstein, *Some nonlinear wave equations with acoustic boundary conditions*, J. Differ. Equations, 2000, 164, 92–109.

[14] Q. Gao, F. Li and Y. Wang, *Blow-up of the solution for higher-order Kirchhoff-type equations with nonlinear dissipation*, Cent. Eur. J. Math., 2011, 9(3), 686–698.

[15] G. C. Gorain, *Exponential energy decay estimates for the solutions of n-dimensional Kirchhoff type wave equation*, Appl. Math. Comp., 2006, 177, 235–242.

[16] P. J. Graber and B. Said-Houari, *On the wave equation with semilinear porous acoustic boundary conditions*, J. Differ. Equations, 2012, 252, 4898–4941.

[17] J. Jeong, J. Park, and Y. H.Kang, *Global nonexistence of solutions for a nonlinear wave equation with time delay and acoustic boundary conditions*, Comput. Math. Appl., 2018, 76, 661–671.

[18] F. Li, *Global existence and uniqueness of weak solution to nonlinear viscoelastic full Marguerre-von Kármán shallow shell equations*, Acta. Math. Sini., 2009, 25, 2133–2156.

[19] F. Li, *Limit behavior of the solution to nonlinear viscoelastic Marguerre-von Kármán shallow shell system*, J. Differ. Equations, 2010, 249, 1241–1257.

[20] F. Li and Y. Bai, *Uniform decay rates for nonlinear viscoelastic Marguerre-von Kármán equations*, J. Math. Anal. Appl. 2009, 351, 522–535.

[21] F. Li and Y. Bao, *Uniform Stability of the Solution for a Memory-Type Elasticity System with Nonhomogeneous Boundary Control Condition*, J. Dyn. Control. Syst., 2017, 23, 301–315.

[22] F. Li, and G. Du, *General energy decay for a degenerate viscoelastic Petrovsky-type plate equation with boundary feedback*, J. Appl. Anal. Comput. 2018, 8(1), 390–401.

[23] F. Li and Q. Gao, *Blow-up of solution for a nonlinear Petrovsky type equation with memory*, Appl. Math. Comput., 2016, 274, 383–392.

[24] F. Li and F. Hu, *Weighted integral inequality and applications in general energy decay estimate for a variable density wave equation with memory*, Bound. Value. Probl. 2018, 2018:164.

[25] F. Li and Z. Jia, *Global existence and stability of a class of nonlinear evolution equations with hereditary memory and variable density*, Bound. Value. Probl. 2019, 2019:37.

[26] F. Li and J. Li, *Global existence and blow-up phenomena for nonlinear divergence form parabolic equations with inhomogeneous Neumann boundary conditions*, J. Math. Anal. Appl., 2012, 385, 1005–1014.

[27] F. Li and J Li, *Global existence and blow-up phenomena for p-Laplacian heat equation with inhomogeneous Neumann boundary conditions*, Bound. Value Probl. 2014, 2014:219.

[28] F. Li, Z. Zhao and Y. Chen, *Global existence uniqueness and decay estimates for nonlinear viscoelastic wave equation with boundary dissipation*, Nonlinear Anal.: Real World Applications, 2011, 12, 1770–1784.

[29] F. Li and C. Zhao, *Uniform energy decay rates for nonlinear viscoelastic wave equation with nonlocal boundary damping*, Nonlinear Anal., 2011, 74, 3468–3477.

[30] F. Li and S. Xi, *Dynamic properties for nonlinear viscoelastic Kirchhoff-type equation with acoustic control boundary conditions I*, Mathematical Notes, Accepted.

[31] K. Nishihara and Y. Yamada, *On global solutions of some degenerate quasilinear hyperbolic equations with dissipative terms*, Funkc. Ekvacioj, 1990, 33, 151–159.

[32] K. Ono, *Global existence, decay and blowup of solutions for some mildly degenerate nonlinear Kirchhoff strings*, J. Differ. Equations, 1997, 137, 273–301.

[33] K. Ono and K. Nishihara, *On a nonlinear degenerate integro-differential equation of hyperbolic type with a strong dissipation*, Adv. Math. Seciencesn and Applications, 1995, 5, 457–476.

[34] J. Y. Park and S. H. Park, *Decay rate estimates for wave equations of memory type with acoustic boundary conditions*, Nonlinear Anal. 74 (2011) 993–998.

[35] J. Y. Park and J. J. Bae and Pusan, *On the existence of solutions for some nondegenerate nonlinear wave equations of kirchhoff type*, Czechoslovak Mathematical Journal, 2002, 52(127), 781–795.

[36] S. T. Wu and L. Y. Tsai, *Blow-up of solutions for some non-linear wave equations of Kirchhoff-type with some dissipation*, Nonlinear Anal., 2006, 65, 243–264.

[37] S. T. Wu, *Exponential energy decay of solutions for an integro-differential equation with strong damping*, J. Math. Anal. Appl., 2010, 364, 609–617.

[38] S. Xi and S. Zhu, *Blow-Up Criterion for the 3D Non-resistive Compressible Magnetohydrodynamic Equations*, J. Dyn. Diff. Equat., 2019, 1–22.

[39] B. Yamna and B. Benyattou, *Polynomial decay and blow up of solutions for variable coefficients viscoelastic wave equation with acoustic boundary conditions*, Acta. Math. Sini., 2016, 32(2), 153–174.

[40] Y. J. Ye, *On the exponential decay of solutions for some Kirchhoff-type modelling equations with strong dissipation*, Appl. Math., 2010, 1, 529–533.