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INTEGRABILITY CONDITIONS FOR COMPLEX SYSTEMS WITH HOMOGENEOUS QUINTIC NONLINEARITIES

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Abstract In this paper we obtain conditions for the existence of a local analytic first integral for four eight-parameter families of quintic complex system. We also discuss computational difficulties arising in the study of the problem of integrability for these systems.

Keywords Integrability, center problem, time-reversibility, polynomial vector field, Darboux integral.

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1. Introduction

The problem of integrability of systems of differential equations is one of central problems in the theory of ODE's. Although integrability is a rare phenomena and a generic system is not integrable, integrable systems are important in studying various mathematical models, since often perturbations of integrable systems exhibit rich picture of bifurcations.

Starting from the beginning of the last century many studies have been devoted to investigating the integrability of two-dimensional autonomous polynomial systems of differential equations. Most of the works (see, for instance, [1, 3, 4, 12, 17] and the references therein) deal with local integrability of the real systems of the form

$$\dot{u} = -v + \sum_{i+j=2}^{n} \alpha_{ij} u^i v^j = -v + U(u,v), \quad \dot{v} = u + \sum_{i+j=2}^{n} \beta_{ij} u^i v^j = u + V(u,v). \tag{1.1}$$

If in a neighborhood of the origin system (1.1) admits a local analytic first integral of the form

$$\phi(u,v) = u^2 + v^2 + \sum_{j+k=3}^{\infty} \phi_{jk}\, u^j v^k, \tag{1.2}$$

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then all solutions close to $u = 0, v = 0$ are periodic, that means the corresponding trajectories are closed, and the singular point at the origin is called a *center*.

Setting $x = u + iv$ we can write system (1.1) as a single complex equation

$$\dot{x} = ix + X(x, \bar{x}) = ix + \sum_{k=2}^{n} X_k(x, \bar{x}), \tag{1.3}$$

where each X_k is a homogeneous polynomial of degree k. It turns out it is easier and more convenient from the computational point of view to study the integrability problem not only for system (1.3), but also for more general complex system

$$\begin{aligned} \dot{x} &= ix + X(x, y) = ix + \sum_{j+k=2}^{n} X_{jk}\ x^j y^k, \\ \dot{y} &= -iy + Y(x, y) = -iy + \sum_{j+k=2}^{n} Y_{jk}\ x^j y^k, \end{aligned} \tag{1.4}$$

where X_{jk} and Y_{jk} are complex parameters. System (1.4) is equivalent to system (1.3) in the case $x = \bar{y}$ and $X_{jk} = \bar{Y}_{kj}$. If we apply the change of time $t \mapsto it$, (1.4) becomes a system of the form

$$\dot{x} = x - \sum_{j+k=2}^{n} a_{jk}\ x^j y^k = x + P(x, y), \quad \dot{y} = -y + \sum_{j+k=2}^{n} b_{jk}\ x^j y^k = -y + Q(x, y). \tag{1.5}$$

For system (1.5) one can always find a function of the form

$$\Psi(x, y) = xy + \sum_{j+k \geq 3} \psi_{j-1,k-1}\ x^j y^k, \tag{1.6}$$

such that

$$\dot{\Psi} := \frac{\partial \Psi}{\partial x}(x + P) + \frac{\partial \Psi}{\partial y}(-y + Q) = \sum_{s \geq 1} g_{s,s}(xy)^{s+1},$$

where $g_{s,s}$ $(s = 1, 2, \dots)$ are polynomials of a_{jk} and b_{kj} with rational coefficients. The polynomial $g_{s,s}$ is called the *s-th focus quantity*. In the case that $\dot{\Psi} \equiv 0$ we say that the origin of system (1.5) is a *(complex) center*. If (1.5) is a complexification of a real system (1.1), then going back to the coordinates u, v we obtain from Ψ a first integral of (1.1) of the form (1.2).

Conditions for existence of a complex center (that is, for existence of a first integral (1.6)) have been found for the case of quadratic system ($n = 2$ in (1.5)) [5, 8], for the case when P and Q in (1.5) are homogeneous cubic polynomials [5, 18], for some particular cubic systems [6, 13].

It was found that for the case of the linear center with homogeneous quadratic or cubic perturbations (P and Q in (1.5) are homogeneous polynomials of degree 2 and 3, respectively) each system with a center at the origin is of one of the following three types: Hamiltonian, time-reversible or Darboux integrable. Computations for these cases are very difficult but still can be performed without making use of a computer.

With the center problem for these cases resolved the next problem arising naturally is the classification of centers for systems (1.5) with P and Q being homogeneous polynomials of degree 4 and 5. As it is known, the case of homogeneous perturbations of odd degree is easer to tackle than the case of homogeneous perturbations of even degree, so it appears worthwhile to study the system (1.5) with P and Q being homogeneous quintic polynomials first, that is to study the center problem for the system

$$\begin{aligned}\dot{x} &= x - a_{40}x^5 - a_{31}x^4y - a_{22}x^3y^2 - a_{13}x^2y^3 - a_{04}xy^4 - a_{-15}y^5,\\ \dot{y} &= -y + b_{5,-1}x^5 + b_{40}x^4y + b_{31}x^3y^2 + b_{22}x^2y^3 + b_{13}xy^4 + b_{04}y^5,\end{aligned} \tag{1.7}$$

where $x, y, a_{ij}, b_{ji} \in \mathbb{C}$. It turns out the computations involved to the determination of the necessary conditions of integrability for the full family (1.7) are so heavy that they cannot be completed even using powerful computers and the modern computer algebra systems. Thus, it is reasonable to study some subfamilies of system (1.7). Recently, the center conditions for the subfamily of (1.7), where $a_{-15} = b_{5,-1} = 0$, have been obtained in [9]. By a linear transformation, system (1.7) with $a_{-15}b_{5,-1} \neq 0$ can be written as

$$\begin{aligned}\dot{x} &= x - a_{40}x^5 - a_{31}x^4y - a_{22}x^3y^2 - a_{13}x^2y^3 - a_{04}xy^4 - y^5,\\ \dot{y} &= -y + x^5 + b_{40}x^4y + b_{31}x^3y^2 + b_{22}x^2y^3 + b_{13}xy^4 + b_{04}y^5.\end{aligned} \tag{1.8}$$

In this paper we find necessary conditions for existence of a center at the origin for the following eight-parameter subfamilies of system (1.8):

(C_1) $a_{40} = b_{04} = 0$, (C_2) $a_{31} = b_{13} = 0$, (C_3) $a_{13} = b_{31} = 0$, (C_4) $a_{04} = b_{40} = 0$.

For the most of cases we show that the obtained conditions are sufficient conditions for the existence of center.

2. Preliminaries

We remind briefly some results related to two main mechanisms which provide local integrability of polynomial systems.

By the definition the Darboux factor of system (1.5) is a polynomial $f(x, y)$ such that

$$\frac{\partial f}{\partial x}(x + P) + \frac{\partial f}{\partial y}(-y + Q) = Kf,$$

where $K(x, y)$ is a polynomial of degree $n - 1$ ($K(x, y)$ is called the *cofactor*). A simple computation shows that if there are Darboux factors $f_1, f_2, \ldots, f_k$ with the cofactors $K_1, K_2, \ldots, K_k$ satisfying

$$\sum_{i=1}^{k} \alpha_i K_i = 0,$$

then $H = f_1^{\alpha_1} \cdots f_k^{\alpha_k}$ is a first integral of (1.5), and if

$$\sum_{i=1}^{k} \alpha_i K_i + P'_x + Q'_y = 0,$$

then the equation admits the integrating factor $\mu = f_1^{\alpha_1} \cdots f_k^{\alpha_k}$.

If a system (1.5) has an integrating factor of the form $\mu = 1 + h.o.t$ then it has also a first integral of the form (1.6) (see, e.g. [1, 3]) and, therefore, a center at the origin.

It is said that a system of the form (1.5) is time-reversible if for some $\alpha \in \mathbb{C}$

$$\alpha Q(\alpha y, x/\alpha) = -P(x, y), \quad \alpha Q(x, y) = -P(\alpha y, x/\alpha)\,.$$

It is well-known (see e.g. [14, 15, 16, 17]) that time-reversible systems (1.5) have a center at the origin.

Computing with the algorithm of [16] we find that the Zariski closure of all time-reversible systems in the family (1.7), denoted by $\overline{\mathcal{R}}$, is the variety of the ideal

$$\begin{aligned} S \;=\; & \langle a_{22} - b_{22}, a_{40}a_{04} - b_{40}b_{04}, a_{31}a_{13} - b_{31}b_{13}, a_{40}b_{13}^2 - a_{31}^2 b_{04}, \\ & a_{13}^2 b_{40} - a_{04}b_{31}^2, a_{31}^2 a_{04} - b_{40}b_{13}^2, a_{40}a_{13}^2 - b_{31}^2 b_{04}, a_{13}^3 b_{5,-1} - a_{-1,5}b_{31}^3, \\ & a_{31}^3 a_{-1,5} - b_{5,-1}b_{13}^3, a_{04}^3 b_{5,-1}^2 - a_{-1,5}^2 b_{20}^3, a_{40}^3 a_{-1,5}^2 - b_{5,-1}^2 b_{04}^3\rangle. \end{aligned}$$

3. The center conditions

In this section we give conditions for the existence of the first integral (1.6) for subfamilies $(C_1) - (C_4)$ of system (1.8), that is, for the existence of a center at the origin for the systems. To obtain the conditions, for system (1.8) using the algorithm of [17] we computed 22 first focus quantities, $g_{1,1}, ..., g_{22,22}$, and found that $g_{s,s} = 0$ for s odd and

$$\begin{aligned} g_{2,2} &= a_{22} - b_{22}, \\ g_{4,4} &= a_{13}a_{31} + a_{04}a_{40} - b_{13}b_{31} - b_{04}b_{40}, \\ g_{6,6} &= 192a_{13}a_{22}a_{31} + 36a_{13}^2 a_{40} + 57a_{04}a_{22}a_{40} + 8a_{31}a_{40} - 27a_{22}a_{40}b_{04} \\ & -144a_{22}a_{31}b_{13} - 12a_{13}a_{40}b_{13} - 168a_{13}a_{31}b_{22} - 33a_{04}a_{40}b_{22} \\ & +27a_{40}b_{04}b_{22} + 144a_{31}b_{13}b_{22} - 216a_{13}a_{22}b_{31} + 36a_{04}a_{31}b_{31} \\ & +36a_{40}b_{31} + 12a_{31}b_{04}b_{31} + 168a_{22}b_{13}b_{31} + 216a_{13}b_{22}b_{31} + 32b_{22} \\ & -192b_{13}b_{22}b_{31} - 12a_{04}b_{31}^2 - 36b_{04}b_{31}^2 + 12a_{13}^2 b_{40} - 63a_{04}a_{22}b_{40} \\ & +24a_{31}b_{40} + 33a_{22}b_{04}b_{40} - 36a_{13}b_{13}b_{40} + 63a_{04}b_{22}b_{40} - 57b_{04}b_{22}b_{40} \\ & -20b_{31}b_{40} + 20a_{04}a_{13} - 32a_{22} - 36a_{13}b_{04} - 24a_{04}b_{13} - 8b_{04}b_{13}. \end{aligned}$$

The size of the polynomials $g_{k,k}$ sharply increases so we do not present the other polynomials here, but the interested reader can easily compute them using any available computer algebra system.

3.1. Case (C_1)

Conditions for existence of a center for systems of this case are given in the following theorem.

Theorem 1. *For case (C_1), system (1.8) is integrable if $a_{22} = b_{22}$ and one of the following conditions holds:*
(α) $a_{04} - b_{40} = a_{13} - b_{31} = a_{31} - b_{13} = 0$;
(β) $b_{40} = a_{04} = 2b_{13} - a_{13} = 2a_{31} - b_{31} = 0$;
(γ) $a_{04}^2 + a_{04}b_{40} + b_{40}^2 = b_{31}a_{04} + a_{13}b_{40} + b_{31}b_{40} = a_{13}a_{04} - b_{31}b_{40} = a_{13}^2 + a_{13}b_{31} + b_{31}^2 = a_{31}b_{40} - b_{13}a_{04} = a_{31}a_{04} + b_{13}a_{04} + b_{13}b_{40} = b_{13}a_{13} + a_{31}b_{31} + b_{13}b_{31} = a_{31}a_{13} - b_{13}b_{31} = a_{31}^2 + a_{31}b_{13} + b_{13}^2 = 0$.

Proof. Since $g_{2,2} = a_{22} - b_{22} = 0$ is the necessary condition for integrability of system (1.8) *from now on we assume that* $a_{22} = b_{22}$. Then, using the routine *minAssGTZ*[*] [7] of Singular [11] and performing computations over the field of characteristic 32003 (we were not able to complete computations over the field of characteristic 0 on our computational facilities), we found that the minimal associate primes of the ideal $I_{18} = \langle g_{2,2}, ..., g_{18,18}\rangle$ are
$J_1 = \langle a_{04} - b_{40}, a_{13} - b_{31}, a_{31} - b_{13}\rangle$,
$J_2 = \langle b_{13} + 16001a_{13}, a_{31} + 16001b_{31}, b_{40}, a_{04}\rangle$,
$J_3 = \langle a_{04}^2 + a_{04}b_{40} + b_{40}^2, b_{31}a_{04} + a_{13}b_{40} + b_{31}b_{40}, a_{13}a_{04} - b_{31}b_{40}, a_{13}^2 + a_{13}b_{31} + b_{31}^2, -b_{13}a_{04} + a_{31}b_{40}, a_{31}a_{04} + b_{13}a_{04} + b_{13}b_{40}, b_{13}a_{13} + a_{31}b_{31} + b_{13}b_{31}, a_{31}a_{13} - b_{13}b_{31}, a_{31}^2 + a_{31}b_{13} + b_{13}^2\rangle$.
Since $16001 \equiv -1/2 \mod 32003$ we obtain from J_1, J_2, J_3 the conditions (α)–(γ) of Theorem 1. However as it is known modular computations provide a result which is not necessary correct, but it is correct with high probability. To perform a check we computed the intersection J of the ideals J_1, J_3 and $\langle 2b_{13} - a_{13}, 2a_{31} - b_{31}, b_{40}, a_{04}\rangle$ in the ring $\mathbb{Q}[a_{31}, b_{13}, a_{13}, b_{31}, a_{04}, b_{40}, b_{22}]$ and using the radical membership test (see e.g. [17]) we checked that $\mathbf{V}(J) \subset \mathbf{V}(I_{18})$. This means that under conditions (α)–(γ) of Theorem 1 all polynomials $g_{2,2}, ..., g_{18,18}$ vanish.

To check the opposite inclusion, $\mathbf{V}(I_{18}) \subset \mathbf{V}(J)$, we need to show that for each $f \in J$ the reduced Groebner basis of the ideal $\langle 1-wf, I\rangle \subset \mathbb{Q}[w, a_{31}, b_{13}, a_{13}, b_{31}, a_{04}, b_{40}, b_{22}]$ is $\{1\}$. We were not able to complete computations of the Groebner bases over $\mathbb{Q}$. However the computations over fields of characteristics 32003 and 4236233 shows that all bases are $\{1\}$ in the corresponding rings. That means that $\mathbf{V}(I_{18}) \subset \mathbf{V}(J)$ with the probability close to one.

We now prove that under each of conditions (α)–(γ) there is a center at the origin.

If (α) or (γ) holds, then all polynomials defining the ideal S vanish. It means the corresponding systems are time-reversible.

When (β) holds, system (1.8) is written as

$$\dot{x} = x - a_{31}x^4y - b_{22}x^3y^2 - 2b_{13}x^2y^3 - y^5,$$
$$\dot{y} = -y + x^5 + 2a_{31}x^3y^2 + b_{22}x^2y^3 + b_{13}xy^4.$$

It is Hamiltonian with the Hamiltonian

$$\Phi(x, y) = (6xy - x^6 - 3a_{31}x^4y^2 - 2b_{22}x^3y^3 - 3b_{13}x^2y^4 - y^6)/6.$$

□

3.2. Case (C_2)

In case (C_2), that is when $a_{31} = b_{13} = 0$, substituting in the polynomials of the ideal I_{18} $a_{22} = b_{22}$ and computing with *minAssGTZ* of Singular in the field of characteristic 32003 using the degree reverse lexicographic order with $a_{13} \succ b_{31} \succ a_{04} \succ b_{40} \succ a_{40} \succ b_{04} \succ b_{22}$ we found the following components of the variety $\mathbf{V}(I_{18})$:
1) $a_{40}^2 + a_{40}b_{04} + b_{04}^2 = b_{40}a_{40} + a_{04}b_{04} + b_{40}b_{04} = a_{04}a_{40} - b_{40}b_{04} = a_{04}^2 + a_{04}b_{40} + b_{40}^2 = -b_{31}a_{40} + a_{13}b_{04} = a_{13}a_{40} + b_{31}a_{40} + b_{31}b_{04} = b_{31}a_{04} + a_{13}b_{40} + b_{31}b_{40} =$

[*] which finds the minimal associate primes of a polynomial ideal using the algorithm by Gianni, Trager and Zacharias [10]

$a_{13}a_{04} - b_{31}b_{40} = a_{13}^2 + a_{13}b_{31} + b_{31}^2 = 0$;
2) $a_{40} - b_{04} = a_{04} - b_{40} = a_{13} - b_{31} = 0$;
3) $b_{22} = a_{40} + b_{04} = b_{40} = a_{04} = b_{31} + 1 = a_{13} + 1 = 0$;
4) $a_{40}^2 - a_{40}b_{04} + b_{04}^2 = b_{31}b_{04} - a_{40} = b_{31}a_{40} - a_{40} + b_{04} = b_{31}^2 - b_{31} + 1 = b_{22} = b_{40} = a_{04} = b_{31}^2 + a_{13} = 0$;
5) $b_{40} - 5a_{40} = a_{04} - 5b_{04} = b_{31} = a_{13} = 0$;
6) $a_{40}b_{04} + 5999 = b_{22} = b_{40} - 6400a_{40} = a_{04} - 6400b_{04} = b_{31} = a_{13} = 0$;
7) $8686b_{04}^3b_{22}^5 + b_{04}^6 - 9149b_{04}^3b_{22}^3 - 11558b_{22}^6 + 13631b_{04}^3b_{22} + 4254b_{22}^4 - 5589b_{22}^2 + 439 = -5120b_{04}^2b_{22}^5 - 14532b_{04}^5 + 12806b_{04}^2b_{22}^3 + a_{40}b_{22}^4 + 12878b_{04}^2b_{22} + 6407a_{40}b_{22}^2 - 2550a_{40} = a_{40}b_{04} - 9112b_{22}^2 - 3556 = 6934b_{04}b_{22}^5 + 12830b_{04}^4 + a_{40}^2b_{22}^2 + 5334b_{04}b_{22}^3 - 12798a_{40}^2 - 10665b_{04}b_{22} = 8686b_{22}^5 + a_{40}^3 + b_{04}^3 - 9149b_{22}^3 + 13631b_{22} = b_{31}b_{22}^2 + 48a_{40}^2 - 8b_{04}b_{22} + 16b_{31} = 5482b_{04}b_{22}^4 + b_{31}b_{04}^3 + 8890a_{40}^2b_{22} + 3630b_{04}b_{22}^2 - 8297b_{31}b_{22} + 11853b_{04} = -4000b_{31}b_{04}^2b_{22} - 3b_{04}^3 + 4806b_{22}^3 + b_{31}a_{40} + 5778b_{22} = 12001b_{31}b_{04}b_{22} + b_{31}^2 + b_{04}^2 - 12000a_{40}b_{22} = b_{40} - 3a_{40} = a_{04} - 3b_{04} = 4000b_{31}b_{04}b_{22} + 3b_{04}^2 - 4001a_{40}b_{22} + a_{13} = 0$.

Performing the rational reconstruction with the algorithm of [19] we obtain the components defined by the ideals
(α) $J_1 = \langle a_{40}^2 + a_{40}b_{04} + b_{04}^2, b_{40}a_{40} + a_{04}b_{04} + b_{40}b_{04}, a_{04}a_{40} - b_{40}b_{04}, a_{04}^2 + a_{04}b_{40} + b_{40}^2, -b_{31}a_{40} + a_{13}b_{04}, a_{13}a_{40} + b_{31}a_{40} + b_{31}b_{04}, b_{31}a_{04} + a_{13}b_{40} + b_{31}b_{40}, a_{13}a_{04} - b_{31}b_{40}, a_{13}^2 + a_{13}b_{31} + b_{31}^2\rangle$;
(β) $J_2 = \langle a_{40} - b_{04}, a_{04} - b_{40}, a_{13} - b_{31}\rangle$;
(γ) $J_3 = \langle b_{22}, a_{40} + b_{04}, b_{40}, a_{04}, b_{31} + 1, a_{13} + 1\rangle$;
(δ) $J_4 = \langle a_{40}^2 - a_{40}b_{04} + b_{04}^2, b_{31}b_{04} - a_{40}, b_{31}a_{40} - a_{40} + b_{04}, b_{31}^2 - b_{31} + 1, b_{22}, b_{40}, a_{04}, a_{13} + b_{31} - 1\rangle$;
(ϵ) $J_5 = \langle b_{40} - 5a_{40}, a_{04} - 5b_{04}, b_{31}, a_{13}\rangle$;
(ζ) $J_6 = \langle 16a_{40}b_{04} - 25, b_{22}, 5b_{40} + 3a_{40}, 5a_{04} + 3b_{04}, b_{31}, a_{13}\rangle$
and the component defined by
$U = \langle -\frac{37}{70}b_{04}^3b_{22}^5 + b_{04}^6 - \frac{37}{7}b_{04}^3b_{22}^3 - \frac{49}{36}b_{22}^6 + \frac{5}{54}b_{04}^3b_{22} + \frac{69}{158}b_{22}^4 - \frac{74}{63}b_{22}^2 + \frac{44}{73}, \frac{12}{25}b_{04}^2b_{22}^5 + \frac{163}{11}b_{04}^5 + \frac{24}{5}b_{04}^2b_{22}^3 + a_{40}b_{22}^4 - \frac{103}{82}b_{04}^2b_{22} + \frac{32}{5}a_{40}b_{22}^2 - \frac{123}{113}a_{40}, a_{40}b_{04} - \frac{5}{144}b_{22}^2 - \frac{1}{9}, \frac{1}{60}b_{04}b_{22}^5 + \frac{144}{5}b_{04}^4 + a_{40}^2b_{22}^2 + \frac{1}{6}b_{04}b_{22}^3 + \frac{16}{5}a_{40}^2 + \frac{8}{3}b_{04}b_{22}, -\frac{37}{70}b_{22}^5 + a_{40}^3 + b_{04}^3 - \frac{37}{7}b_{22}^3 + \frac{5}{54}b_{22}, b_{31}b_{22}^2 + 48a_{40}^2 - 8b_{04}b_{22} + 16b_{31}, -\frac{148}{35}b_{04}b_{22}^4 + b_{31}b_{04}^3 + \frac{5}{18}a_{40}^2b_{22} + \frac{77}{97}b_{04}b_{22}^2 + \frac{2}{27}b_{31}b_{22} + \frac{1}{27}b_{04}, \frac{3}{8}b_{31}b_{04}^2b_{22} - 3b_{04}^3 + \frac{111}{20}b_{22}^3 + b_{31}a_{40} - \frac{23}{72}b_{22}, -\frac{1}{8}b_{31}b_{04}b_{22} + b_{31}^2 + b_{04}^2 + \frac{9}{8}a_{40}b_{22}, b_{40} - 3a_{40}, a_{04} - 3b_{04}, -\frac{3}{8}b_{31}b_{04}b_{22} + 3b_{04}^2 - \frac{5}{8}a_{40}b_{22} + a_{13}\rangle$.

Using the radical membership test and computing over the field of characteristic zero we have checked that $\mathbf{V}(J_i) \subset \mathbf{V}(I_{18})$ for $i = 1, \ldots, 6$, however $\mathbf{V}(U) \not\subset \mathbf{V}(I_{18})$, that is, $\mathbf{V}(U)$ is not a component of the center variety.

A usual recipe in such situations is to look for the decomposition of the variety over a field of other final characteristic. We repeated the calculations in the fields of characteristics 4256233 and 7368787. In both cases the decomposition of the variety consists of 9 components, however after the reconstruction we obtained the components, among which the true ones are only (β), (γ), (ϵ) and (ζ). So, the result is worse than the one obtained computing over 32003. From our empiric observation [2] in such cases when a component of a modular decomposition is not a true one, the most simple polynomials defining the associate primes are usually the correct polynomials. So we recomputed with the condition $b_{40} - 3a_{40} = a_{04} - 3b_{04} = 0$ over the field of characteristic zero and found the components:
$U_1 = \langle 3b_{31}^2 + 4a_{40}b_{22} - a_{13}, 9a_{13}b_{31} - 2b_{22}^2 - 1, 6b_{04}b_{31} + a_{13}b_{22} + 2a_{40}, 3a_{13}^2 + 4b_{04}b_{22} - b_{31}, 6a_{40}a_{13} + b_{31}b_{22} + 2b_{04}, 144a_{40}b_{04} - 5b_{22}^2 - 16, b_{22}^3 - 36b_{04}a_{13} - 36a_{40}b_{31} +$

$14b_{22}, b_{31}b_{22}^2+48a_{40}^2-8b_{04}b_{22}+16b_{31}, a_{13}b_{22}^2+48b_{04}^2-8a_{40}b_{22}+16a_{13}, b_{40}-3a_{40}, a_{04}-3b_{04}\rangle$;
$U_2 = \langle a_{13} - b_{31} = a_{40} - b_{04} = b_{40} - 3a_{40} = a_{04} - 3b_{04} = 0\rangle$;
$U_3 = \langle a_{13}^2 + a_{13}b_{31} + b_{31}^2, b_{04}a_{13} - a_{40}b_{31}, a_{40}a_{13} + a_{40}b_{31} + b_{04}b_{31}, a_{40}^2 + a_{40}b_{04} + b_{04}^2, b_{40} - 3a_{40}, a_{04} - 3b_{04}\rangle$.

Then, with *intersect* of Singular we computed $J = \cap_{i=1}^6 J_i \cap U_1 \cap U_2 \cap U_3$ and with *minAssChar* found that the minimal associate primes of J are the ideals $J_1, \dots, J_6$ and

(η) $J_7 = \langle 4a_{13}^3 + 3a_{13}^2b_{31}^2 + 6a_{13}b_{31} + 4b_{31}^3 - 1, 2b_{22}^2 - 9a_{13}b_{31} + 1, 108b_{40}b_{31}^3 - 4b_{40} - 12b_{22}^3a_{13} - 9b_{22}^3b_{31}^2 - 12b_{22}a_{13} - 108b_{22}b_{31}^2, 4b_{40}a_{13} - 12b_{40}b_{31}^2 + 5b_{22}^3b_{31} + 6b_{22}a_{13}^2 - 18b_{22}a_{13}b_{31}^2 + 16b_{22}b_{31}, 4b_{40}b_{22} - 3a_{13} + 9b_{31}^2, 16b_{40}^2 + 24b_{40} * b_{22}a_{13} + 15b_{22}^2b_{31} + 48b_{31}, 640a_{04} - 384b_{40}^3b_{31} + 32b_{40}^2b_{22}^3 + 2576b_{40}^2b_{22} + 4608b_{40}b_{31}^2 + 75b_{22}^3b_{31} + 1200b_{22}b_{31}, 3b_{04} - a_{04}, 3a_{40} - b_{40}\rangle$.

Using the radical membership test we checked that $\sqrt{I_{18}} \subset \sqrt{J}$ (where $J = \cap_{i=1}^7 J_7 \subset Q_2 = \mathbb{Q}[a_{40}, b_{04}, a_{13}, b_{31}, a_{04}, b_{40}, b_{22}]$) yielding $\mathbf{V}(J) \subset \mathbf{V}(I_{18})$. However we failed to complete computation of Groebner bases in Q_2 to check that

$$\sqrt{J} \subset \sqrt{I_{18}}. \tag{3.1}$$

Nevertheless computations show that (3.1) holds in the rings $k[a_{40}, b_{04}, a_{13}, b_{31}, a_{04}, b_{40}, b_{22}]$ with k being the fields of characteristic 32003 and 4256233. This means that with a very high probability (3.1) holds also in Q_2.

Theorem 2. *For case (C_2), system (1.8) is integrable if $a_{22} = b_{22}$ and one of the following conditions holds:*
(α) $a_{40}^2 + a_{40}b_{04} + b_{04}^2 = b_{40}a_{40} + a_{04}b_{04} + b_{40}b_{04} = a_{04}a_{40} - b_{40}b_{04} = a_{04}^2 + a_{04}b_{40} + b_{40}^2 = -b_{31}a_{40} + a_{13}b_{04} = a_{13}a_{40} + b_{31}a_{40} + b_{31}b_{04} = b_{31}a_{04} + a_{13}b_{40} + b_{31}b_{40} = a_{13}a_{04} - b_{31}b_{40} = a_{13}^2 + a_{13}b_{31} + b_{31}^2 = 0$;
(β) $a_{40} - b_{04} = a_{04} - b_{40} = a_{13} - b_{31} = 0$;
(ϵ) $b_{40} - 5a_{40} = a_{04} - 5b_{04} = b_{31} = a_{13} = 0$.

Proof. When condition (α) or (β) holds the system is time-reversible.

When condition (ϵ) holds, the corresponding system is Hamiltonian and has a first integral

$$\Phi(x, y) = (6xy - x^6 - 6a_{40}x^5y - 2b_{22}x^3y^3 - 6b_{04}xy^5 - y^6)/6. \tag{3.2}$$

□

We note that systems corresponding to the ideal J_3 are written in the form

$$\begin{aligned} \dot{x} &= x + b_{04}x^5 + x^2y^3 - y^5, \\ \dot{y} &= -y + x^5 - x^3y^2 + b_{04}y^5. \end{aligned} \tag{3.3}$$

and those corresponding to J_4 are

$$\begin{aligned} \dot{x} &= x - b_{31}b_{04}x^5 - (1 - b_{31})x^2y^3 - y^5, \\ \dot{y} &= -y + x^5 + b_{31}x^3y^2 + b_{04}y^5, \end{aligned} \tag{3.4}$$

where $b_{31} = (1 \pm i\sqrt{3})/2$. It is easy to see that any system of the form (3.4) is transformed to a system (3.3) by the substitution $x \to \alpha x$, $y \to \beta y$, where $\alpha = \beta^5$ and $\beta = (b_{31}-1)^{-1/8}$. Therefore to complete the study of case (C_2) it is necessary to prove integrability of system (3.3) and of the systems corresponding to the ideals J_6 and J_7. We have tried to prove integrability of these systems using various methods but we failed. In particular we looked for algebraic invariant curves in the form $\ell = 1+\sum_{i=1}^{3} f_{4i}(x,y)$, $\ell = x+\sum_{i=1}^{4} f_{4i+1}(x,y)$ and $\ell = y+\sum_{i=1}^{4} f_{4i+1}(x,y)$ where f_{4i} is a homogeneous polynomial of degree $4i$ and f_{4i+1} is a homogeneous polynomial of degree $4i+1$. For system (3.3) there is an invariant curve $\ell = 1+b_{04}x^4 - b_{04}y^4$ however it is impossible to find a Darboux integral or a Darboux integrating factor using this curve. For the other systems we were not able to find any curve.

3.3. Case (C_3)

Similarly as above computing with *minAssGTZ* the minimal associate primes of the ideal I_{18} we obtain the following ten components:
(α) $a_{40}^2+a_{40}b_{04}+b_{04}^2 = b_{40}a_{40}+a_{04}b_{04}+b_{40}b_{04} = a_{04}a_{40}-b_{40}b_{04} = a_{04}^2+a_{04}b_{40}+b_{40}^2 = b_{13}a_{40}+a_{31}b_{04}+b_{13}b_{04} = a_{31}a_{40}-b_{13}b_{04} = -b_{13}a_{04}+a_{31}b_{40} = a_{31}a_{04}+b_{13}a_{04}+b_{13}b_{40} = a_{31}^2+a_{31}b_{13}+b_{13}^2 = 0$;
(β) $a_{40}-b_{04} = a_{04}-b_{40} = a_{31}-b_{13} = 0$;
(γ) $b_{22}+1 = b_{13}+2a_{40} = -2a_{40}b_{22}^2+b_{40}+a_{40} = -2b_{04}b_{22}^2+a_{04}+b_{04} = 8a_{40}b_{04}^2b_{22}-4b_{13}b_{04}^2-2b_{04}b_{22}+a_{31} = 0$;
(δ) $b_{22}-1 = b_{13}-2a_{40} = -2a_{40}b_{22}^2+b_{40}+a_{40} = -2b_{04}b_{22}^2+a_{04}+b_{04} = 8a_{40}b_{04}^2b_{22}-4b_{13}b_{04}^2-2b_{04}b_{22}+a_{31} = 0$;
(ϵ) $4a_{40}b_{04}-1 = b_{13}-2a_{40} = b_{22} = b_{40}+a_{40} = a_{04}+b_{04} = -4b_{13}b_{04}^2+a_{31} = 0$;
(ζ) $4a_{40}b_{04}-1 = b_{13}+2a_{40} = b_{22} = b_{40}+a_{40} = a_{04}+b_{04} = -4b_{13}b_{04}^2+a_{31} = 0$;
(η) $b_{40}-5a_{40} = a_{04}-5b_{04} = b_{13} = a_{31} = 0$;
(θ) $16a_{40}b_{04}-25 = b_{22} = 5b_{40}+3a_{40} = 5a_{04}+3b_{04} = b_{13} = a_{31} = 0$;
(ι) $b_{22}+1 = a_{40}b_{04}-1 = b_{40} = a_{04} = -a_{40}b_{22}+b_{13} = -b_{04}b_{22}+a_{31} = 0$;
(κ) $b_{22}-1 = a_{40}b_{04}-1 = b_{40} = a_{04} = -a_{40}b_{22}+b_{13} = -b_{04}b_{22}+a_{31} = 0$.
Note that no fake component (like the one defined by the ideal U in case (C_2)) appears for this case (and for case (C_4) below), so the components 1)–10) are obtained after applying the rational reconstruction algorithm of [19] to the ideals returned by *minAssGTZ* when computed modulo 32003.

Theorem 3. *For case (C_3), system (1.8) is integrable if $a_{22} = b_{22}$ and the coefficients of the system satisfy one of conditions (α) – (η), (ι), (κ).*

Proof. When condition (α) or (β) holds the corresponding system (1.8) is time-reversible.

In case (γ) the system (1.8) is written as

$$\begin{aligned} \dot{x} &= x - a_{40}x^5 + 2b_{04}yx^4 + y^2x^3 - b_{04}y^4x - y^5, \\ \dot{y} &= -y + x^5 + a_{40}yx^4 - y^3x^2 - 2a_{40}y^4x + b_{04}y^5. \end{aligned} \tag{3.5}$$

It admits an algebraic invariant curve of degree twelve†

$$\begin{aligned} l_1 \;=\; & x^{12} + 8a_{40}yx^{11} + 24a_{40}^2y^2x^{10} - 8b_{04}y^2x^{10} + 32a_{40}^3y^3x^9 - 48a_{40}b_{04}y^3x^9 \\ & -4y^3x^9 + 16a_{40}^4y^4x^8 + 24b_{04}^2y^4x^8 - 24a_{40}y^4x^8 - 96a_{40}^2b_{04}y^4x^8 - 4b_{04}x^8 \end{aligned}$$

† note that finding an invariant algebraic curve of degree 12 is a difficult computational problem, so to obtain this curve we used computations in the field of characteristic 32003

$$
\begin{aligned}
&-48a_{40}^2y^5x^7+96a_{40}b_{04}^2y^5x^7-64a_{40}^3b_{04}y^5x^7+24b_{04}y^5x^7-16a_{40}b_{04}yx^7\\
&-12yx^7-32a_{40}^3y^6x^6-32b_{04}^3y^6x^6+96a_{40}^2b_{04}^2y^6x^6+96a_{40}b_{04}y^6x^6\\
&+6y^6x^6+16b_{04}^2y^2x^6-52a_{40}y^2x^6-16a_{40}^2b_{04}y^2x^6-64a_{40}b_{04}^3y^7x^5\\
&-48b_{04}^2y^7x^5+24a_{40}y^7x^5+96a_{40}^2b_{04}y^7x^5-64a_{40}^2y^3x^5+32a_{40}b_{04}^2y^3x^5\\
&+56b_{04}y^3x^5+16b_{04}^4y^8x^4+24a_{40}^2y^8x^4-24b_{04}y^8x^4-96a_{40}b_{04}^2y^8x^4\\
&-16a_{40}^3y^4x^4-16b_{04}^3y^4x^4+128a_{40}b_{04}y^4x^4+24y^4x^4-8a_{40}x^4\\
&+32b_{04}^3y^9x^3-48a_{40}b_{04}y^9x^3-4y^9x^3-64b_{04}^2y^5x^3+56a_{40}y^5x^3\\
&+32a_{40}^2b_{04}y^5x^3+32b_{04}yx^3+24b_{04}^2y^{10}x^2-8a_{40}y^{10}x^2+16a_{40}^2y^6x^2\\
&-16a_{40}b_{04}^2y^6x^2-52b_{04}y^6x^2+16a_{40}b_{04}y^2x^2+36y^2x^2+8b_{04}y^{11}x\\
&-16a_{40}b_{04}y^7x-12y^7x+32a_{40}y^3x+y^{12}-4a_{40}y^8-8b_{04}y^4+8
\end{aligned}
$$

yielding the integrating factor $\mu = l_1^{-1}$.

When condition (δ) holds, the system (1.8) is of the form

$$
\begin{aligned}
\dot{x} &= x - a_{40}x^5 - 2b_{04}yx^4 - y^2x^3 - b_{04}y^4x - y^5,\\
\dot{y} &= -y + x^5 + a_{40}yx^4 + y^3x^2 + 2a_{40}y^4x + b_{04}y^5.
\end{aligned} \tag{3.6}
$$

It can be transformed to (3.5) by the substitution $x \to \alpha x$, $y \to \beta y$, where $\alpha = \beta^5$ and $\beta = (-1)^{1/12}$.

(ϵ) In this case the system has two invariant curves

$$
\begin{aligned}
l_1 &= 4b_{04} - x^4 - 4b_{04}y^2x^2 - 4b_{04}^2y^4,\\
l_2 &= 16b_{04}^4x^8 + 2b_{04}x^8 + 64b_{04}^5y^2x^6 + 8b_{04}^2y^2x^6 + 64b_{04}^6y^4x^4 + 16b_{04}^3y^4x^4 + y^4x^4\\
&\quad -16b_{04}^2x^4 + 4b_{04}^2y^8 - 128b_{04}^4yx^3 + 32b_{04}^4y^6x^2 + 4b_{04}y^6x^2 + 64b_{04}^3y^2x^2\\
&\quad -32b_{04}^2y^3x + 32b_{04}^5y^8 - 64b_{04}^4y^4 + 32b_{04}^3,
\end{aligned}
$$

which allow to construct the integrating factor $\mu = l_1^{\frac{1}{2}}l_2^{-1}$.

(ζ) This system can be transformed to system (ϵ) by the substitution $x \to \alpha x$, $y \to \beta y$, where $\alpha = \beta^5$ and $\beta = (-1)^{1/12}$. Thus, it is integrable.

(η) In this case the system is Hamiltonian with a first integral of the form (3.2).

(ι) When this condition holds, system (1.8) is written as

$$
\begin{aligned}
\dot{x} &= x + \frac{x^5}{a_{31}} - a_{31}yx^4 + y^2x^3 - y^5,\\
\dot{y} &= -y + x^5 - y^3x^2 + \frac{y^4x}{a_{31}} - a_{31}y^5.
\end{aligned} \tag{3.7}
$$

It admits two invariant curves

$$
\begin{aligned}
l_1 &= x^4 + 2a_{31}y^2x^2 + a_{31}^2y^4 + a_{31},\\
l_2 &= a_{31}^4x^8 - 4a_{31}^3yx^7 + 2a_{31}^5y^2x^6 + 6a_{31}^2y^2x^6 - 8a_{31}^4y^3x^5 - 4a_{31}y^3x^5\\
&\quad +a_{31}^6y^4x^4 + 12a_{31}^3y^4x^4 + y^4x^4 + 4a_{31}^2x^4 - 4a_{31}^5y^5x^3 - 8a_{31}^2y^5x^3\\
&\quad -8a_{31}^4yx^3 + 6a_{31}^4y^6x^2 + 2a_{31}y^6x^2 + 8a_{31}^3y^2x^2 - 4a_{31}^3y^7x - 8a_{31}^2y^3x\\
&\quad +a_{31}^2y^8 + 4a_{31}^4y^4 + 4a_{31}^3,
\end{aligned}
$$

which yield the integrating factor $\mu = l_1^{-\frac{1}{4}}l_2^{-1}$.

(κ) The system of this case is written as

$$\begin{aligned}\dot{x} &= x - \frac{x^5}{a_{31}} - a_{31}yx^4 - y^2x^3 - y^5,\\ \dot{y} &= -y + x^5 + y^3x^2 + \frac{y^4x}{a_{31}} + a_{31}y^5.\end{aligned} \tag{3.8}$$

It can be transformed into (3.7) by the change $x \to x(-1-i)/\sqrt{2}$, $y \to y(1+i)/\sqrt{2}$. Thus, (3.8) is integrable. □

Note, that the remaining open case (θ) is the same as case (ζ) of Subsection 3.2.

3.4. Case (C_4)

Similarly as above, using modular computations and then the rational reconstruction we found the following components of the variety of the ideal I_{18} generated by the first 18 focus quantities of system (1.8) with $a_{04} = b_{40} = 0$ and $a_{22} = b_{22}$:
(α) $a_{40}^2 + a_{40}b_{04} + b_{04}^2 = -b_{31}a_{40} + a_{13}b_{04} = a_{13}a_{40} + b_{31}a_{40} + b_{31}b_{04} = a_{13}^2 + a_{13}b_{31} + b_{31}^2 = b_{13}a_{40} + a_{31}b_{04} + b_{13}b_{04} = a_{31}a_{40} - b_{13}b_{04} = b_{13}a_{13} + a_{31}b_{31} + b_{13}b_{31} = a_{31}a_{13} - b_{13}b_{31} = a_{31}^2 + a_{31}b_{13} + b_{13}^2 = 0$;
(β) $a_{40} - b_{04} = a_{13} - b_{31} = a_{31} - b_{13} = 0$;
(γ) $2b_{22} + 1 = a_{40}^2 - a_{40}b_{04} + b_{04}^2 + a_{40} + b_{04} + 1 = 8b_{04}b_{22}^3 - 5b_{04}b_{22} + 3b_{31} = 8a_{40}b_{22}^3 - 5a_{40}b_{22} + 3a_{13} = 8a_{40}b_{22}^3 - 5a_{40}b_{22} + b_{13} = 8b_{04}b_{22}^3 - 5b_{04}b_{22} + a_{31} = 0$;
(δ) $2b_{22} - 1 = a_{40}^2 - a_{40}b_{04} + b_{04}^2 - a_{40} - b_{04} + 1 = 8b_{04}b_{22}^3 - 5b_{04}b_{22} + 3b_{31} = 8a_{40}b_{22}^3 - 5a_{40}b_{22} + 3a_{13} = 8a_{40}b_{22}^3 - 5a_{40}b_{22} + b_{13} = 8b_{04}b_{22}^3 - 5b_{04}b_{22} + a_{31} = 0$;
(ϵ) $2b_{22} + 1 = a_{40} + b_{04} - 1 = 8b_{04}b_{22}^3 - 5b_{04}b_{22} + 3b_{31} = 8a_{40}b_{22}^3 - 5a_{40}b_{22} + 3a_{13} = 8a_{40}b_{22}^3 - 5a_{40}b_{22} + b_{13} = 8b_{04}b_{22}^3 - 5b_{04}b_{22} + a_{31} = 0$;
(ζ) $2b_{22} - 1 = a_{40} + b_{04} + 1 = 8b_{04}b_{22}^3 - 5b_{04}b_{22} + 3b_{31} = 8a_{40}b_{22}^3 - 5a_{40}b_{22} + 3a_{13} = 8a_{40}b_{22}^3 - 5a_{40}b_{22} + b_{13} = 8b_{04}b_{22}^3 - 5b_{04}b_{22} + a_{31} = 0$;
(η) $a_{40}b_{04} - 1 = b_{13} + a_{40} = b_{22} + 1 = b_{31} = a_{13} = -b_{13}b_{04}^2 + a_{31} = 0$;
(θ) $a_{40}b_{04} - 1 = b_{13} - a_{40} = b_{22} - 1 = b_{31} = a_{13} = -b_{13}b_{04}^2 + a_{31} = 0$;
(ι) $2b_{13} - a_{13} = 2a_{31} - b_{31} = b_{04} = a_{40} = 0$;
(κ) $b_{22} = a_{40} + b_{04} = b_{31} + 1 = a_{13} + 1 = b_{13} = a_{31} = 0$;
(λ) $a_{40}^2 - a_{40}b_{04} + b_{04}^2 = b_{31}b_{04} - a_{40} = b_{31}a_{40} - a_{40} + b_{04} = b_{31}^2 - b_{31} + 1 = b_{22} = a_{13} + b_{31} - 1 = b_{13} = a_{31} = 0$.

Theorem 4. *For case (C_4), system (1.8) is integrable if $a_{22} = b_{22}$ and one of conditions (α), (β), (η), (θ), (ι) holds.*

Proof. When condition (α) or (β) holds, the corresponding system (1.8) is time-reversible. Condition (η) is condition (ι) of Theorem 3, condition (θ) is condition (κ) of Theorem 3 and condition (ι) is condition (β) of Theorem 1. Thus, these systems are integrable. □

Conditions (κ) and (λ) are conditions (γ) and (δ), respectively, of Subsection 3.2. However these cases are open. Note also that system of case (δ) can be transformed to system of case (γ) by the substitution $x \to \alpha x$, $y \to -\alpha y$, where $\alpha = (-1)^{1/4}$ and system of case (ζ) can be transformed to system of case (ϵ) by the same substitution.

To summarize, we have found conditions for local integrability of systems (C_1)–(C_4), that is, the conditions of Theorem 1, conditions (α)–(η) of Subsection 3.2, (α)–(κ) of Subsection 3.3 and (α)–(λ) of Subsection 3.4. We believe that these conditions are the necessary and sufficient conditions for integrability of systems

(C_1)–(C_4). However to prove this there remains to show that no component was lost under the computations with modular arithmetic (that is, all Groebner bases arising in the radical membership test are $\{1\}$) and to prove integrability of system (3.3), the systems corresponding to (ζ) and (η) of Subsection 3.2 and the systems corresponding to (γ) and (ϵ) of Subsection 3.4.

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