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# CYCLICITY OF SEVERAL QUADRATIC REVERSIBLE SYSTEMS WITH CENTER OF GENUS ONE

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**Abstract** By using the Chebyshev criterion to study the number of zeros of Abelian integrals, developed by M. Grau, F. Mañosas and J. Villadelprat in [2], we prove that the cyclicity of period annulus of the quadratic reversible systems with center of genus one, classified as (r8), (r13) and (r16) by S. Gautier, L. Gavrilov and I. D. Iliev in [1], under quadratic perturbations is two. These results partially give a positive answer to the conjecture 1 in [1].

**Keywords** Cyclicity of period annulus, Quadratic reversible center of genus one, Abelian integrals, Chebyshev property.

**MSC(2000)** 34C07, 34C08, 37G15.

## 1. Introduction and Main Result

It is well known that all quadratic systems with at least one center can be classified as four types: Hamiltonian ($Q_3^H$), reversible ($Q_3^R$), generalized Lotka-Volterra ($Q_3^{LV}$) and of codimension four ($Q_4$). One of interesting questions is that how many limit cycles could be produced from the annulus (or annuli) surrounding the center (or centers) under small quadratic perturbations ? Such a maximal number is called the *cyclicity of the annulus (or annuli)*. If we restrict our attention to the limit cycles bifurcated from the open annulus (not from the graphics), then the problem is solved completely for $Q_3^H$, and partially for other 3 classes. S. Gautier, L. Gavrilov and I. D. Iliev presented a program in [1] to solve this problem for $Q_3^R$ and $Q_3^{LV}$ with centers of genus one, i. e. centers whose (generic complexified) period orbits are elliptic curves. Note that an algebraic phase curve is called generic, if it does not contain a singular point of the vector field in its closure, and the generic level sets of the first integral of $Q_4$ are always elliptic. At the end of section 5 of [1] there is also a program to solve this problem for $Q_4$.

A quadratic reversible system with a center at the origin has the form

$$\dot{z} = -iz + az^2 + 2|z|^2 + b\bar{z}^2, \quad a, b \in \mathbb{R}, \quad z = x + iy. \tag{1.1}$$

The authors of [1] classify quadratic reversible centers of genus one into 18 cases (r1)-(r18), in terms of the values $a$ and $b$ in (1.1), and gave the following conjecture.

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**Conjecture** ([1]) The period annulus around the center at the origin in (r1)-(r18) has the following cyclicity under small quadratic perturbations: three for cases (r1) with $a^* < a < 4$, (r3) with $\frac{7}{3} < a < 4$, (r4) with $4 < a < 5$, (r5) with $a = 4$, (r6) with $a > 4$ and (r10), and two otherwise.

Here $a^* = 2.0655\cdots$ is determined from a transcendental equation in [3].

This conjecture was proved completely for cases (r1)-(r3), (r7), (r9)-(r12), (r14)-(r15), (r17)-(r18), and partially for cases (r4)-(r6), see the recent survey paper [4] and references therein for detailed introduction. The main result in this paper is the following theorem.

**Theorem 1.1.** *The cyclicity of the period annulus around the center at the origin in cases (r8), (r13) and (r16) under small quadratic perturbations is two.*

This theorem gives a positive answer to the above conjecture for the open cases (r8), (r13) and (r16).

## 2. Preliminaries

To study the cyclicity of period annulus, we need to estimate the number of zeros of corresponding (generalized) Abelian integrals, and normally the procedure involves some quite complicated computations, such as Picard-Fuchs equations and Riccati equations. In [2] the authors generalized an idea of [5], and use Shebyshev criterion of certain functions to study the number of zeros of Abelian integrals by some purely algebraic computations. Note that this method is valid for some restricted forms of the first integrals.

We first introduce the definition of Chebyshev property, and show how to use it for studying Abelian integrals.

**Definition 2.1.** (see [6] and [2] for instance) Let $f_0, f_1, \ldots, f_{n-1}$ be analytic functions on an open interval L of $\mathbb{R}$.

(a) $(f_0, f_1, \ldots, f_{n-1})$ is a Chebyshe system (for short, a T-system) on L if any nontrivial linear combination

$$\alpha_0 f_0(x) + \alpha_1 f_1(x) + \cdots + \alpha_{n-1} f_{n-1}(x)$$

has at most n-1 isolated zeros for $x \in L$.

(b) $(f_0, f_1, \ldots, f_{n-1})$ is a complete Chebyshev system (for short, a CT-system) on L if $(f_0, f_1, \ldots, f_{k-1})$ is a T-system for all $k = 1, 2, \ldots, n$.

(c) $(f_0, f_1, \ldots, f_{n-1})$ is an extend complete Chebyshev system (for short, an ECT-system) on L, if for all $k = 1, 2...n$, any nontrivial linear combination

$$\alpha_0 f_0(x) + \alpha_1 f_1(x) + \cdots + \alpha_{k-1} f_{k-1}(x)$$

has at most $k-1$ isolated zeros for $x \in L$, counted with multiplicities.

**Definition 2.2.** (see [6] and [2] for instance) Let $f_0, f_1, \ldots, f_{k-1}$ be analytic functions on an open interval L of $\mathbb{R}$. The continuous Wronskian of $(f_0, f_1, \ldots, f_{k-1})$ at $x \in L$ is

$$W[f]_k(x) := det(f_j^i(x))_{0 \leqslant i,j \leqslant k-1} = \begin{vmatrix} f_0(x) & \ldots & \ldots & f_{k-1}(x) \\ f_0^{'}(x) & \ldots & \ldots & f_{k-1}^{'}(x) \\ \ldots & \ldots & \ldots & \ldots \\ f_0^{(k-1)}(x) & \ldots & \ldots & f_{k-1}^{(k-1)}(x) \end{vmatrix}$$

**Lemma 2.1.** *(see [6] for instance) $(f_0, f_1, \ldots, f_{n-1})$ is an ECT-system on L if and only if for each $k = 1, 2, \ldots, n$,*

$$W[f]_k(x) \neq 0 \ \ \textit{for all} \ \ x \in L.$$

Now we consider a Hamiltonian function with the form

$$H(x, y) = A(x) + B(x)y^{2m}, \tag{2.1}$$

where $H$ is an analytic function in some open subset of the plane that has a local minimum at the origin. Without loss of generality, we fix $H(0, 0) = 0$, and then we get a period annulus by the set of ovals $\gamma_h \subset \{(x, y) \,|\, H(x, y) = h\}$, which is parameterized by the energy levels $h \in (0, h_0)$ for $h_0 \in (0, \infty]$. We denote the projection of $\gamma_h$ on the $x$-axis by $(x_\ell, x_r)$, it is easy to verify that $xA'(x) > 0$ for any $x \in (x_\ell, x_r)\backslash\{0\}$. According to this, we define $\sigma$ as an analytic involution ($\sigma \circ \sigma = Id$ and $\sigma \neq Id$), which satisfies

$$A(x) = A(\sigma(x)) \quad \text{for all } x \in (x_\ell, x_r). \tag{2.2}$$

**Lemma 2.2.** *(Theorem B of [2]) Let us consider the Ablian integral,*

$$I_i(h) = \int_{\gamma_h} f_i(x) y^{2s-1} dx, i = 0, 1, \ldots, n-1$$

*where, for each $h \in (0, h_0)$, $\gamma_h$ is the oval surrounding the origin inside the level curve $\{A(x)+B(x)y^{2m} = h\}$. Let $\sigma$ be the involution associated to $A$, and we define*

$$l_i(x) = \left(\frac{f_i}{A' B^{\frac{2s-1}{2m}}}\right)(x) - \left(\frac{f_i}{A' B^{\frac{2s-1}{2m}}}\right)(\sigma(x)).$$

*Then $(I_0, I_1, \ldots, I_{n-1})$ is an ECT-system on $(0, h_0)$ if $(l_0, l_1, \ldots, l_{n-1})$ is a CT-system on $(0, x_r)$ and $s > m(n-2)$.*

**Lemma 2.3.** *(Lemma 4.1 of [2]) Let $\gamma_h$ be an oval inside the level curve $\{A(x) + B(x)y^2 = h\}$, and we consider a function $F$ such that $F/A^{'}$ is analytic at $x = 0$. Then, for any $k \in \mathbf{N}$,*

$$\int_{\gamma_h} F(x) y^{k-2} dx = \int_{\gamma_h} G(x) y^k dx,$$

*where $G(x) = \frac{2}{k}(\frac{BF}{A'})^{'}(x) - (\frac{B^{'}F}{A'})(x)$.*

We will use Lemma 2.5 to change the form of Abelian integrals such that the condition $s > m(n-2)$ in Lemma 2.4 is satisfied.

Besides, the concepts of the resultant between two polynomials and the Sturm's Theorem are used in our proof. More details about them are given in the appendix of [2]. For polynomials with integer coefficients the computation of resultant and making use of the Sturm's Theorem can be done by Maple precisely.

# 3. Proof of Theorem 1.1

According to the formulas derived in page 520 of [1], the number of limit cycles of such perturbations is bounded by the number of zeros of the related *generating integral* (Abelian integrals). For cases (r8), (r13) and (r16) the corresponding generating integrals are listed as follows:

$$\text{(r8)} \quad I(t) = \int_{H=0} x^{-2}(\mu_1 + \mu_2 x^{-3} + \mu_3 x^3) y dx,$$

$$H = \tfrac{1}{2}y^2 + \tfrac{1}{12} - \tfrac{1}{3}x^3 - tx^4,\ t \in (-\tfrac{1}{4}, 0).$$

$$\text{(r13)} \quad I(t) = \int_{H=0} (\mu_1 + \mu_2 x^{-4} + \mu_3 x^4) y dx,$$

$$H = \tfrac{1}{2}y^2 + \tfrac{1}{12} + \tfrac{1}{4}x^4 - tx^3,\ t \in (\tfrac{1}{3}, \infty).$$

$$\text{(r16)} \quad I(t) = \int_{H=0} x^{-2}(\mu_1 + \mu_2 x^{4} + \mu_3 x^{-4}) y dx,$$

$$H = \tfrac{1}{2}y^2 + \tfrac{1}{4} + \tfrac{1}{12}x^4 - tx,\ t \in (\tfrac{1}{3}, \infty).$$

In above list $\mu_1, \mu_2$ and $\mu_3$ are arbitrary constants. It is easy to know that in all above three cases the center of the corresponding vector field is located at the point $(x, y) = (1, 0)$.

## 3.1. Proof for case (r8)

To make use of the Chebyshev criterion (Lemma 2.4), we replace $x$ by $x+1$, which moves the center to the origin. Thus, equivalently to (r8), we consider

$$I(h) = \int_{\tilde{H}=h} [\mu_3(x+1) + \mu_1(x+1)^{-2} + \mu_2(x+1)^{-5}]\, y dx,$$

$$\tilde{H}(x, y) = A(x) + B(x)y^2 = h,\ h \in (0, \frac{1}{4}).$$

where

$$A = \frac{1}{12(x+1)^4} - \frac{1}{3(x+1)} + \frac{1}{4} = \frac{x^2(3x^2+8x+6)}{12(x+1)^4},\ B = \frac{1}{2(x+1)^4}.$$

We rewrite the Abelian integral as $I(h) = \alpha_0 I_0 + \alpha_1 I_1 + \alpha_2 I_2$, where $\alpha_0, \alpha_1$ and $\alpha_2$ are arbitrary constants, and

$$I_j(h) = \int_{\gamma_h} (x+1)^{3j-5} y dx, \quad j = 0, 1, 2.$$

The parameter $h \in (0, \frac{1}{4})$ and the energy level of the polycycle in its outer boundary is $h_0 = \frac{1}{4}$. It is clear that $\tilde{H}(x, y)$ has a local minimum at $(0, 0)$. Then there exists a punctured neighborhood $\rho$ of the origin foliated by $\gamma_h \subset H^{-1}(h)$. Denote the projection of $\rho$ on the $x$-axis by $(x_\ell, x_r) = (x_\ell, +\infty)$, where $x_\ell = -0.37\cdots$. Then we can find a involution $\sigma(x)$ such that $A(x) = A(\sigma(x))$ for all $x \in (x_\ell, +\infty)$.

We are going to apply Lemma 2.4 to prove that $(I_0, I_1, I_2)$ is an ECT-system on $(0, \frac{1}{4})$, which implies that $I(h)$ has at most most 2 zeros for $h \in (0, \frac{1}{4})$. However, in this case $m = 1, n = 3$ and $s = 1$, so the hypothesis $s > m(n-2)$ is not satisfied. To solve this problem, we note that $\tilde{H}(x, y) = A(x) + B(x)y^2 = h$ along $\gamma_h$, so we can rewrite $I_j(h)$ as

$$\begin{aligned} I_j(h) &= \tfrac{1}{h} \int_{\gamma_h} (A(x) + B(x)y^2)(x+1)^{3j-5} y dx \\ &= \tfrac{1}{h} \int_{\gamma_h} A(x)(x+1)^{3j-5} y dx + \tfrac{1}{h} \int_{\gamma_h} B(x)(x+1)^{3j-5} y^3 dx. \end{aligned}$$

Then, we apply Lemma 2.5 to the first integral above with $k = 3$ and $F(x) = A(x)(x+1)^{3j-5}$, and finally we obtain

$$I_j(h) = \frac{1}{h}\tilde{I}_j(h) = \frac{1}{h}\int_{\gamma_h} f_j(x)y^3 dx,$$

where

$$\begin{aligned} f_0(x) &= \frac{(3x^2+7x+6)(x^3-9x-12)}{12(x+1)^9(x^2+3x+3)^2}, \\ f_1(x) &= \frac{6x^5+44x^4+138x^3+225x^2+192x+72}{12(x+1)^6(x^2+3x+3)^2}, \\ f_2(x) &= \frac{15x^5+95x^4+255x^3+351x^2+246x+72}{12(x+1)^3(x^2+3x+3)^2}. \end{aligned}$$

It is clear that $(I_0, I_1, I_2)$ is an ECT-system on $(0, \frac{1}{4})$ if and only if $(\tilde{I}_0, \tilde{I}_1, \tilde{I}_2)$ is as well. Now we apply Lemma 2.4 to $(\tilde{I}_0, \tilde{I}_1, \tilde{I}_2)$ with $m = 1, n = 3$ and $s = 2$. Setting

$$L_j(x) = \left(\frac{f_j}{A'B^{\frac{3}{2}}}\right)(x),$$

then by direct computations we find

$$\begin{aligned} L_0(x) &= -\frac{\sqrt{2}(x+1)^2(3x^2+7x+6)(x^3-9x-12)}{2x(x^2+3x+3)^3}, \\ L_1(x) &= \frac{\sqrt{2}(x+1)^5(6x^5+44x^4+138x^3+225x^2+192x+72)}{2x(x^2+3x+3)^3}, \\ L_2(x) &= \frac{\sqrt{2}(x+1)^8(15x^5+95x^4+255x^3+351x^2+246x+72)}{x(x^2+3x+3)^3}. \end{aligned}$$

Now, according to Lemma 2.4, we define

$$l_j(x) = L_j(x) - L_j(\sigma(x)), \quad j = 0, 1, 2,$$

we need only to check that $(l_0, l_1, l_2)$ is an ECT-system on $(0, +\infty)$. Here $\sigma$ is the involution associated with $A$.

In order to compute Wronskians, we let

$$l_j(x) = L_j(x) - L_j(z),$$

with $z = \sigma(x)$ as an implicit function.

On the other hand, we have $A(x) - A(z) = (x-z)q(x,z)$, where

$$\begin{aligned} q(x,z) = \quad & x^3(4z^3+12z^2+12z+3) + x^2(12z^3+36z^2+35z+8) \\ & x(4z+1)(3z^2+8z+6) + z(3z^2+8z+6). \end{aligned}$$

Hence, from $A(x) - A(z) = 0$ with $z < 0 < x$ we find that $x$ and $z = \sigma(x)$ satisfy $q(x,z) = 0$, and

$$\sigma'(x) = \frac{dz}{dx} = -\frac{q'_x(x,z)}{q'_z(x,z)}.$$

By using above expressions, we get that $W[l]_k(x) = \omega_k(x, \sigma(x))$, with $\omega_k(x,z)$ being a rational function for $k = 1, 2, 3$. Let us prove that for case (r8)

$$W[l]_k(x) \neq 0 \quad \text{for } x \in (0, +\infty) \text{ and } k = 1, 2, 3. \tag{3.1}$$

A sufficient condition to obtain (3.1) is to show that $\omega_k(x,z) = 0$ and $q(x,z) = 0$ have no common solutions for $x_\ell < z < 0 < x < +\infty$ and $k = 1, 2, 3$.

The resultant with respect to $z$ between $q(x,z)$ and the numerator of $\omega_1(x,z)$ is $r_1(x) = 8\sqrt{2}x(x^2+3x+3)^3(x+1)^9 p_1(x)$ with

$$\begin{aligned} p_1(x) = \;& 30233088 + 655050240x + 6643406160x^2 + 42041838240x^3 \\ & +186803282970x^4 + 622012446972x^5 + 1618576036119x^6 \\ & +3391133671314x^7 + 5853400628247x^8 + 8482308483096x^9 \\ & +10488391859088x^{10} + 11223353230344x^{11} + 10515406708386x^{12} \\ & +8697571272720x^{13} + 6374037383604x^{14} + 4132005936624x^{15} \\ & +2353669697952x^{16} + 1165527404352x^{17} + 494911700199x^{18} \\ & +177277496754x^{19} + 52530785745x^{20} + 12566323920x^{21} \\ & +2349011270x^{22} + 327342936x^{23} + 31550796x^{24} \\ & +1832760x^{25} + 46008x^{26}. \end{aligned}$$

Applying Sturm's Theorem, we can assert that $p_1(x) \neq 0$ for all $x \in (0, +\infty)$. Thus, $\omega_1(x,z) = 0$ and $q(x,z) = 0$ have no common roots.

In the same way, we can calculate the resultant with respect to $z$ between $q(x,z)$ and the numerator of $\omega_2(x,z)$ and get

$$r_2(x) = 27648x^3(x^2+3x+3)^5(x+1)^{33}p_2(x),$$

where $p_2(x)$ is a polynomial of degree 56. By applying Sturm's Theorem, we can assert that $p_2(x) \neq 0$ for all $x \in (0, +\infty)$. Thus, $\omega_2(x,z) = 0$ and $q(x,z) = 0$ have no common roots.

Similar computation shows that the resultant with respect to z between $q(x,z)$ and the numerator of $\omega_3(x,z)$ is

$$r_3(x) = 8916100448256000\sqrt{2}x^6(x^2+3x+3)^8(x+1)^{78}p_3(x),$$

where $p_3(x)$ is a polynomial of degree 92. By applying Sturm's Theorem again, we can check that $p_3(x)$ does not vanish for $x \in (0, +\infty)$. Thus, $\omega_3(x,z) = 0$ and $q(x,z) = 0$ have no common roots.

Therefore, we have proved that (3.1) holds. As we explained above, by using Lemmas 2.3 and 2.4 we give a proof of Theorem 1.1 in case (r8).

## 3.2. Proof for case (r13)

The proof of Theorem 1.1 for case (r13) is basically the same as for case (r8) above, so we only list the expressions of $f_j(x)$ and $L_j(x)$ for $j = 0, 1, 2$. Note that the denominator of $L_j(x)$ contains some fractional power, so we need to use a technique to solve this problem, and this technique is borrowed from [2].

Changing $x$ by $(x+1)$, we find out that the case (r13) is equivalent to

$$I(h) = \alpha_0 I_0(h) + \alpha_1 I_1(h) + \alpha_2 I_2(h),$$

where $I_j(h) = \int_{\gamma_h} (x+1)^{4j-4} y dx,\ j = 0, 1, 2$; and

$$H(x, y) = A(x) + B(x)y^2 = h,\ h \in (0, \infty),$$

where

$$A(x) = \frac{1}{12(x+1)^3} + \frac{1}{4}(x+1) - \frac{1}{3} = \frac{x^2(3x^2+8x+6)}{12(x+1)^3},\ B(x) = \frac{1}{2(x+1)^3}.$$

By the similar procedure we find

$$f_0(x) = \frac{2(5x^5+35x^4+105x^3+164x^2+134x+48)}{9(x+1)^7(x+2)^2(x^2+2x+2)^2},$$

$$f_1(x) = \frac{2(6x^6+45x^5+147^4+273x^3+300x^2+182x+48)}{9(x+2)^2(x^2+2x+2)^2(x+1)^3},$$

$$f_2(x) = \frac{2(x+1)(12x^6+85x^5+259x^4+441x^3+436x^2+230x+48)}{9(x+2)^2(x^2+2x+2)^2}.$$

From $A(x) = A(z)$ we find $x$ and $z = \sigma(x)$ satisfy

$$\begin{aligned} q(x, z) = \quad & 3x^3(z^3+3z^2+3z+1) + x^2(9z^3+27z^2+27z+8) \\ & + x(9z^3+27z^2+26z+6) + 3z^3+8z^2+6z = 0, \end{aligned}$$

where $-1 < z < 0 < x < +\infty$.

Since the denominator of $L_j(x)$ contains some fractional power, we let

$$u = \sqrt{x+1}, \quad v = \sqrt{z+1},$$

where $0 < v < 1 < u < +\infty$, and $u$ and $v$ satisfy

$$\tilde{q}(u, v) = 3(uv)^6 - u^4 - (uv)^2 - v^4 = 0, \tag{3.2}$$

and

$$\frac{du}{dv} = -\frac{u(-2u^2 - v^2 + 9u^4v^6)}{v(-u^2 - 2v^2 + 9u^6v^4)}.$$

Thus, instead of $L_j(x)$, we use $L_j(u)$, where

$$\begin{aligned} L_0(u) \quad &= \frac{16\sqrt{2}u^3(5u^{10}+10u^8+15u^6+9u^4+6u^2+3)}{9(u^2-1)(u^2+1)^3(u^4+1)^3}, \\ L_1(u) \quad &= \frac{16\sqrt{2}u^{11}(6u^{12}+9u^{10}+12u^8+15u^6+3u^4+2u^2+1)}{9(u^2-1)(u^2+1)^3(u^4+1)^3}, \\ L_2(u) \quad &= \frac{16\sqrt{2}u^{19}(12u^{12}+13u^{10}+14u^8+15u^6-3u^4-2u^2-1)}{9(u^2-1)(u^2+1)^3(u^4+1)^3}. \end{aligned}$$

We similarly define $l_j(u) = L_j(u) - L_j(v)$, where $v = \tilde{\sigma}(u)$ is determined implicitly by (3.2), and check that $(l_0, l_1, l_2)$ is an ECT-system on $u \in (1, +\infty)$ and $v \in (0, 1)$.

Accordingly, we have that $W[l]_k(u) = \omega_k(u, \tilde{\sigma}(u))$, with $\omega_k(u, v)$ being a rational function of $u$ and $v$.

The resultant with respect to $u$ between $\tilde{q}(u,v)$ and the numerator of $\omega_k(u,v)$ is $r_k(v)$, where

$$\begin{aligned} r_1(v) &= c_1 v^{34}(v-1)^3(v+1)^3(v^2+1)^9(v^4+1)^9 p_1(v),\\ r_2(v) &= c_2 v^{106}(v-1)^8(v+1)^8(v^2+1)^{16}(v^4+1)^{16} p_2(v),\\ r_3(v) &= c_3 v^{244}(v-1)^{15}(v+1)^{15}(v^2+1)^{25}(v^4+1)^{25} p_3(v), \end{aligned}$$

where $c_1, c_2$ and $c_3$ are nonzero constants, $p_1(v), p_2(v)$ and $p_3(v)$ are polynomials in $v$ of degree 140, 264 and 452 respectively. Applying Sturm's Theorem by Maple, we can assert that $p_k(v) \neq 0$ for $v \in (0,1)$ and $k = 1,2,3$. This finishes the proof of Theorem 1.1 for case (r13).

## 3.3. Proof for case (r16)

The procedure is the same as case (r13), so we only list the different expressions.

Replace $x$ by $(x+1)$, the case (r16) is equivalent to

$$I(h) = \alpha_0 I_0(h) + \alpha_1 I_1(h) + \alpha_2 I_2(h),$$

where $I_j(h) = \int_{\gamma_h} (x+1)^{4j-6} y dx$, $j = 0,1,2$; and

$$H(x,y) = A(x) + B(x)y^2 = h, \ h \in (0,\infty),$$

where

$$A(x) = \frac{(x+1)^3}{12} + \frac{1}{4(x+1)} - \frac{1}{3} = \frac{x^2(x^2+4x+6)}{12(x+1)}, \ B(x) = \frac{1}{2(x+1)}.$$

By similar computations we find

$$\begin{aligned} f_0(x) &= \frac{2(15x^3+58x^2+82x+48)}{9(x+1)^7(x+2)^2(x^2+2x+2)^2},\\ f_1(x) &= \frac{2(2x^6+16x^5+56x^4+119x^3+162x^2+130x+48)}{9(x+1)^3(x+2)^2(x^2+2x+2)^2},\\ f_2(x) &= \frac{2(4x^6+32x^5+112x^4+223x^3+266x^2+178x+48)(x+1)}{9(x+2)^2(x^2+2x+2)^2}. \end{aligned}$$

From $A(x) = A(z)$ we find $x$ and $z = \sigma(x)$ satisfy

$$q(x,z) = x^3(z+1) + x^2(z^2+5z+4) + x(z^3+5z^2+10z+6) + z^3+4z^2+6z = 0,$$

where $-1 < z < 0 < x < +\infty$.

Let

$$u = \sqrt{x+1}, \quad v = \sqrt{z+1},$$

where $0 < v < 1 < u < +\infty$, and $u$ and $v$ satisfy

$$\tilde{q}(u,v) = (uv)^2(u^4+u^2v^2+v^4) - 3 = 0, \tag{3.3}$$

and

$$\frac{du}{dv} = -\frac{v(3u^4+2u^2v^2+v^4)}{u(3v^4+2u^2v^2+u^4)}.$$

Instead of $L_j(x)$, we use $L_j(u)$, where

$$\begin{aligned} L_0(u) &= \frac{16\sqrt{2}(15u^6+13u^4+11u^2+9)}{9u^7(u^2-1)(u^2+1)^3(u^4+1)^3}, \\ L_1(u) &= \frac{16\sqrt{2}u(2u^{12}+4u^{10}+6u^8+15u^6+11u^4+7u^2+3)}{9(u^2-1)(u^2+1)^3(u^4+1)^3}, \\ L_2(u) &= \frac{16\sqrt{2}u^9(4u^{12}+8u^{10}+12u^8+15u^6+9u^4+3u^2-3)}{9(u^2-1)(u^2+1)^3(u^4+1)^3}. \end{aligned}$$

We similarly define $l_j(u) = L_j(u) - L_j(v)$, where $v = \tilde{\sigma}(u)$ is determined implicitly by (3.3), and to check that $(l_0, l_1, l_2)$ is an ECT-system on $u \in (1, +\infty)$ and $v \in (0, 1)$.

We also have that $W[l]_k(u) = \omega_k(u, \tilde{\sigma}(u))$, with $\omega_k(u, v)$ being a rational function of $u$ and $v$.

The resultant with respect to $u$ between $\tilde{q}(u, v)$ and the numerator of $\omega_k(u, v)$ is $r_k(v)$, where

$$\begin{aligned} r_1(v) &= c_1(v-1)^3(v+1)^3(v^2+1)^9(v^4+1)^9 p_1(v), \\ r_2(v) &= c_2\, v^6(v-1)^8(v+1)^8(v^2+1)^{16}(v^4+1)^{16} p_2(v), \\ r_3(v) &= c_3\, v^6(v-1)^{15}(v+1)^{15}(v^2+1)^{25}(v^4+1)^{25} p_3(v), \end{aligned}$$

where $c_1, c_2$ and $c_3$ are some nonzero constants, $p_1(v), p_2(v)$ and $p_3(v)$ are polynomials in $v$ of degree 176, 284 and 456 respectively. Applying Sturm's Theorem by Maple, we can assert that $p_k(v) \neq 0$ for $v \in (0, 1)$ and $k = 1, 2, 3$. This finishes the proof of Theorem 1.1 for case (r16).

Thus, we have proved Theorem 1.1 completely.

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