

THE C^1 SOLUTION OF THE HIGH DIMENSIONAL FEIGENBAUM-LIKE FUNCTIONAL EQUATION

Ying Liang^a, XiaoPei Li^{a,†} and YuZhen Mi^a

Abstract By constructing a structure operator quite different from before, and using the Schauder's fixed point theory, the existence and uniqueness of the C^1 solutions of the high dimensional Feigenbaum-like functional equations are discussed.

Keywords Feigenbaum-like functional equations; Schauder's fixed point; existence; uniqueness

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1. Introduction

The second kind of Feigenbaum functional equation (1.1) with initial value condition in interval $[0, 1]$ is an important form of iterative equations .

$$\begin{cases} f(x) = \frac{1}{\lambda} f(f(\lambda x)), 0 < \lambda < 1; \\ f(0) = 1; \\ 0 \leq f(x) \leq 1, x \in [0, 1] \end{cases} \quad (1.1)$$

The case of the continuous single-peak solutions was considered in [1-3] and [5-7], also the precise solution was considered in some functional space. Certainly, these discussions are very important to the dynamical systems, but being iterative equations, we also focus on the no-monotone continuous case. In this paper, we consider the C^1 solution of the Feigenbaum-like functional equation

$$f(x) = \frac{1}{\lambda + 1} f^2(\lambda x) + \frac{\lambda}{\lambda + 1} g(x) \quad (1.2)$$

where $g(x)$ is a given function, $f(x)$ is an unknown function, and $f(x)$, $g(x)$ are the differentiable function of the high dimensional space $\mathbf{R}^n$, $0 < \lambda < 1$. We use the different methods from [1-3] and [5-7].

2. Preliminaries

Suppose $C^0(B, \mathbf{R}^n) = \{f : B \rightarrow \mathbf{R}^n, f \text{ is continuous}\}$, where B is a compact convex subset of $\mathbf{R}^n$. let $|\cdot|_n$ be a norm of $\mathbf{R}^n$, $|\cdot|_{n \times n}$ be a norm of $\mathbf{R}^{n \times n}$. Clearly

[†]the corresponding author: Lixp27333@sina.com(X.Li)

^aMathematics and Computational School, Zhanjiang Normal University, Zhanjiang, Guangdong, 524048, China

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$C^0(B, \mathbf{R}^n)$ is a Banach space with the norm $\|\cdot\|_{c^0}$, where $\|f\|_{c^0} = \sup_{x \in B} |f(x)|_n$, for $f \in C^0(B, \mathbf{R}^n)$. Let $C^1(B, \mathbf{R}^n) = \{f : B \rightarrow \mathbf{R}^n, f \text{ is continuous and continuously differentiable}\}$, then $C^1(B, \mathbf{R}^n)$ is a Banach space with the norm $\|\cdot\|_{c^1}$, where $\|f\|_{c^1} = \sup_{x \in B} |f(x)|_n + \sup_{x \in B} |f'(x)|_{n \times n}$, for $f \in C^1(B, \mathbf{R}^n)$.

Being a closed subset of $C^1(B, \mathbf{R}^n)$, $C^1(B, B)$ defined by $C^1(B, B) = \{f \in C^1(B, \mathbf{R}^n), f(B) \subseteq B, \forall x \in B\}$ is a complete space.

Lemma 2.1. Suppose that $\varphi \in C^1(B, B)$ and

$$|\varphi'(x)|_{n \times n} \leq M, \forall x \in B, \quad (2.1)$$

$$|\varphi'(x_1) - \varphi'(x_2)|_{n \times n} \leq M'|x_1 - x_2|_n, \forall x_1, x_2 \in B, \quad (2.2)$$

where M and M' are positive constants. Then

$$|(\varphi^n(x_1))' - (\varphi^n(x_2))'|_{n \times n} \leq M' \left(\sum_{i=n-1}^{2n-2} M^i \right) |x_1 - x_2|_n, \quad (2.3)$$

$$|(\varphi^2(\lambda x_1))' - (\varphi^2(\lambda x_2))'|_{n \times n} \leq \lambda^2 M' (M + M^2) |x_1 - x_2|_n, \quad (2.4)$$

where any x_1, x_2 in B , $(\varphi^n)'$ is the Jacobian matrix denoting $d\varphi^n/dx$.

Proof. The methods of proof in (2.3) are similar to paper [8], but the conditions are weaker than that of [8]. Now we proof (2.4).

For any x_1, x_2 in B ,

$$\begin{aligned} & |(\varphi^2(\lambda x_1))' - (\varphi^2(\lambda x_2))'|_{n \times n} \\ &= |\varphi'(\varphi(\lambda x_1))(\varphi(\lambda x_1))' - \varphi'(\varphi(\lambda x_2))(\varphi(\lambda x_2))'|_{n \times n} \\ &\leq |\varphi'(\varphi(\lambda x_1))(\varphi(\lambda x_1))' - \varphi'(\varphi(\lambda x_2))(\varphi(\lambda x_1))'|_{n \times n} \\ &\quad + |\varphi'(\varphi(\lambda x_2))(\varphi(\lambda x_1))' - \varphi'(\varphi(\lambda x_2))(\varphi(\lambda x_2))'|_{n \times n} \\ &\leq \lambda M |(\varphi'(\varphi(\lambda x_1)) - (\varphi'(\varphi(\lambda x_2)))|_{n \times n} + M |(\varphi(\lambda x_1))' - (\varphi(\lambda x_2))'|_{n \times n} \\ &\leq \lambda M M' |\varphi(\lambda x_1) - \varphi(\lambda x_2)|_n + \lambda^2 M M' |x_1 - x_2|_n \\ &\leq \lambda^2 M^2 M' |x_1 - x_2|_n + \lambda^2 M M' |x_1 - x_2|_n \\ &\leq \lambda^2 M' (M + M^2) |x_1 - x_2|_n \end{aligned}$$

Thus (2.4) holds. $\square$

Lemma 2.2. Suppose that $\varphi_1, \varphi_2 \in C^0(B, \mathbf{R}^n)$, $\varphi_1(x), \varphi_2(x)$ satisfy (2.1) and $\varphi(\lambda x) := \varphi^{(\lambda)}(x)$. Then

$$|\varphi_1^{2(\lambda)} - \varphi_2^{2(\lambda)}|_n \leq (M + 1) |\varphi_1^{(\lambda)} - \varphi_2^{(\lambda)}|_n \quad (2.5)$$

Proof. Since

$$\begin{aligned} |\varphi_1^{2(\lambda)} - \varphi_2^{2(\lambda)}|_n &= \sup_{x \in B} |\varphi_1^2(\lambda x) - \varphi_2^2(\lambda x)|_n \\ &\leq \sup_{x \in B} |(\varphi_1(\varphi_1(\lambda x)) - \varphi_1(\varphi_2(\lambda x)))|_n \\ &\quad + \sup_{x \in B} |\varphi_1(\varphi_2(\lambda x)) - \varphi_2(\varphi_2(\lambda x))|_n \\ &\leq M |\varphi_1^{(\lambda)} - \varphi_2^{(\lambda)}|_n + |\varphi_1^{(\lambda)} - \varphi_2^{(\lambda)}|_n \\ &= (M + 1) |\varphi_1^{(\lambda)} - \varphi_2^{(\lambda)}|_n \end{aligned}$$

Thus (2.5) holds. $\square$

Lemma 2.3. Suppose that $\varphi_1, \varphi_2 \in C^1(B, B)$. Then

$$\begin{aligned} & |(\varphi_1^{(2(\lambda))})' - (\varphi_2^{(2(\lambda))})'|_{n \times n} \\ & \leq 2\lambda M |(\varphi_1^{(\lambda)})' - (\varphi_2^{(\lambda)})'|_{n \times n} + \lambda M M' |\varphi_1^{(\lambda)} - \varphi_2^{(\lambda)}|_n \end{aligned} \quad (2.6)$$

Proof. Because

$$\begin{aligned} & |(\varphi_1^{(2(\lambda))})' - (\varphi_2^{(2(\lambda))})'|_{n \times n} \\ & \leq \sup_{x \in B} \{ |\varphi_1'(\varphi_1(\lambda x))(\varphi_1(\lambda x))' - \varphi_2'(\varphi_2(\lambda x))(\varphi_1(\lambda x))'|_{n \times n} \} \\ & \quad + \sup_{x \in B} \{ |\varphi_2'(\varphi_2(\lambda x))(\varphi_1(\lambda x))' - \varphi_2'(\varphi_2(\lambda x))(\varphi_2(\lambda x))'|_{n \times n} \} \\ & \leq \lambda M \sup_{x \in B} \{ |\varphi_1'(\varphi_1(\lambda x)) - \varphi_2'(\varphi_2(\lambda x))|_{n \times n} + \lambda^n M |\varphi_1'(\lambda x) - \varphi_2'(\lambda x)|_{n \times n} \} \\ & \leq \lambda M \sup_{x \in B} |\varphi_1'(\varphi_1(\lambda x)) - \varphi_1'(\varphi_2(\lambda x))|_{n \times n} \\ & \quad + \lambda M \sup_{x \in B} |\varphi_1'(\varphi_2(\lambda x)) - \varphi_2'(\varphi_2(\lambda x))|_{n \times n} + \lambda M |\varphi_1'(\lambda x) - \varphi_2'(\lambda x)|_{n \times n} \\ & \leq 2\lambda M |(\varphi_1^{(\lambda)})' - (\varphi_2^{(\lambda)})'|_{n \times n} + \lambda M M' |\varphi_1^{(\lambda)} - \varphi_2^{(\lambda)}|_n. \end{aligned}$$

Thus (2.6) holds. $\square$

3. Main results

For given constants $M_1 > 0$ and $M_2 > 0$, let

$$\mathcal{A}(M_1, M_2) = \{ \varphi \in C^1(B, B) : |\varphi'(x)|_{n \times n} \leq M_1, \forall x \in B,$$

$$|\varphi'(x_1) - \varphi'(x_2)|_{n \times n} \leq M_2 |x_1 - x_2|_n \forall x_1, x_2 \in B \},$$

Theorem 3.1. (Existence) Given positive constants M_1, M_2 and $g \in \mathcal{A}(M_1, M_2)$, if there exists constants K_1, K_2 , $0 \leq K_1 \leq 1$ and $K_2 \geq \frac{1}{2}$, such that

$$(P_1) \quad K_1^2 + M_1 \leq K_1,$$

$$(P_2) \quad K_2(K_1 + K_1^2) + M_2 \leq K_2,$$

then equation (1.2) has a solution f in $\mathcal{A}(K_1, K_2)$.

Proof. Define $T : \mathcal{A}(K_1, K_2) \rightarrow C^1(B, B)$ as follow:

$$Tf(x) = \frac{1}{\lambda + 1} f^2(\lambda x) + \frac{\lambda}{\lambda + 1} g(x), \quad \forall x \in B$$

where $f \in \mathcal{A}(K_1, K_2)$, then $f(\lambda x) \in \mathcal{A}(\lambda K_1, \lambda K_2)$. Because f , $f(\lambda x)$ and g are continuously differentiable for all $x \in B$, then Tf is also continuously differentiable for all $x \in B$. By condition (P₁) and (P₂), for any x_1, x_2 in B , thus

$$\begin{aligned} |(Tf(x))'|_{n \times n} & \leq \frac{1}{\lambda + 1} |(f^2(\lambda x))'|_{n \times n} + \frac{\lambda}{\lambda + 1} |g'(x)|_{n \times n} \\ & \leq \frac{1}{\lambda + 1} |(f'(f(\lambda x)))(f(\lambda x))'|_{n \times n} + \frac{\lambda}{\lambda + 1} |g'(x)|_{n \times n} \\ & \leq \frac{\lambda}{\lambda + 1} (K_1^2 + M_1) \leq \lambda K_1 \leq K_1 \end{aligned}$$

Furthermore

$$\begin{aligned}
& |(Tf(x_1))' - (Tf(x_2))'|_{n \times n} \\
= & \left| \frac{1}{\lambda+1}(f^2(\lambda x_1))' + \frac{\lambda}{\lambda+1}g'(x_1) - \frac{1}{\lambda+1}(f^2(\lambda x_2))' - \frac{\lambda}{\lambda+1}g'(x_2) \right|_{n \times n} \\
\leq & \frac{\lambda}{\lambda+1} |(f^2(\lambda x_1))' - (f^2(\lambda x_2))'|_{n \times n} + \frac{\lambda}{\lambda+1} |g'(x_1) - g'(\lambda x_2)|_{n \times n} \\
\leq & \frac{\lambda^2}{\lambda+1} K_2(K_1 + K_1^2) |x_1 - x_2|_n + \frac{\lambda}{\lambda+1} M_2 |x_1 - x_2|_n \\
\leq & \frac{\lambda}{\lambda+1} (K_2(K_1 + K_1^2) + K_2) |x_1 - x_2|_n \\
\leq & \lambda K_2 |x_1 - x_2|_n \\
\leq & K_2 |x_1 - x_2|_n
\end{aligned}$$

Thus $T : \mathcal{A}(K_1, K_2) \rightarrow \mathcal{A}(K_1, K_2)$ is a self-diffeomorphism.

Now we prove the continuity of T under the norm $\|\cdot\|_{c^1}$. For arbitrary $f_1, f_2 \in \mathcal{A}(K_1, K_2)$, $f_i(\lambda x) \in \mathcal{A}(\lambda K_1, \lambda K_2)$ $i = 1, 2$ and $f(\lambda x) := f^{(\lambda)}(x)$,

$$\begin{aligned}
& |Tf_1 - Tf_2|_n \\
\leq & \sup_{x \in B} \left\{ \left| \frac{1}{\lambda+1} f_1^2(\lambda x) + \frac{\lambda}{\lambda+1} g(x) - \frac{1}{\lambda+1} f_2^2(\lambda x) - \frac{\lambda}{\lambda+1} g(x) \right|_n \right\} \\
= & \frac{1}{\lambda+1} \sup_{x \in B} \{ |f_1^2(\lambda x) - f_2^2(\lambda x)|_n \} \\
\leq & \frac{1}{\lambda+1} (K_1 + 1) |f_1^{(\lambda)} - f_2^{(\lambda)}|_n, \\
& |(Tf_1)' - (Tf_2)'|_{n \times n} \\
\leq & \sup_{x \in B} \left\{ \left| \frac{1}{\lambda+1} (f_1^2(\lambda x))' + \frac{\lambda}{\lambda+1} g'(x) - \frac{1}{\lambda+1} (f_2^2(\lambda x))' - \frac{\lambda}{\lambda+1} g'(x) \right|_{n \times n} \right\} \\
= & \sup_{x \in B} \left\{ \frac{1}{\lambda+1} |(f_1^2(\lambda x))' - (f_2^2(\lambda x))'|_{n \times n} \right\} \\
\leq & \frac{1}{\lambda+1} \{ \lambda K_1 K_2 |f_1^{(\lambda)} - f_2^{(\lambda)}|_n + 2\lambda K_1 |(f_1^{(\lambda)})' - (f_2^{(\lambda)})'|_{n \times n} \}.
\end{aligned}$$

Let

$$E_1 = \frac{1}{\lambda+1} (K_1 + 1) + \frac{\lambda}{\lambda+1} K_1 K_2, \quad E_2 = \frac{2\lambda}{\lambda+1} K_1, \quad E = \max\{E_1, E_2\}.$$

Then we have

$$\begin{aligned}
\|Tf_1 - Tf_2\|_{c^1} &= \sup_{x \in B} |Tf_1(x) - Tf_2(x)|_n + \sup_{x \in B} |(Tf_1(x))' - (Tf_2(x))'|_{n \times n} \\
&\leq E_1 |f_1^{(\lambda)} - f_2^{(\lambda)}|_n + E_2 |(f_1^{(\lambda)})' - (f_2^{(\lambda)})'|_{n \times n} \\
&\leq E |f_1 - f_2|_n + E |(f_1)' - (f_2)'|_{n \times n} \\
&= E \|f_1 - f_2\|_{c^1},
\end{aligned} \tag{3.1}$$

which gives continuity of T .

It is easy to show that $\mathcal{A}(K_1, K_2)$ is a compact convex subset of $C^1(B, B)$. By the

Schauder's fixed point theorem, we assert that there is a mapping $f \in \mathcal{A}(K_1, K_2)$, such that

$$f(x) = Tf(x) = \frac{1}{\lambda + 1}f^2(\lambda x) + \frac{\lambda}{\lambda + 1}g(x), \quad \forall x \in B$$

This completes the proof. $\square$

Theorem 3.2. (Uniqueness) *Suppose that (P_1) and (P_2) are satisfied, also we suppose that $(P_3) : E < 1$, then for arbitrary function g in $\mathcal{A}(M_1, M_2)$, equation (1.2) has a unique solution $f \in \mathcal{A}(K_1, K_2)$.*

Proof. The existence of equation(1.2) in $\mathcal{A}(K_1, K_2)$ is given by theorem 3.1, from the proof of theorem 3.1, we see that $\mathcal{A}(K_1, K_2)$ is a closed subset of $C^1(B, B)$, by (3.1) and (P_3) , we see that

$$T : \mathcal{A}(K_1, K_2) \rightarrow \mathcal{A}(K_1, K_2)$$

is a contraction. Therefore T has a unique fixed point $f(x)$ in $\mathcal{A}(K_1, K_2)$, that is, equation(1.2) has a unique solution in $\mathcal{A}(K_1, K_2)$, this proves theorem. $\square$

4. Example

Suppose $B := \{(x_1, x_2) : x_1^2 + x_2^2 \leq 1, x_i \in \mathbf{R}, i = 1, 2\} \subset \mathbf{R}^2$. Clearly B is a compact convex subset of $\mathbf{R}^2$. We can choose

$$g(x) := g(x_1, x_2) = (g_1(x_1, x_2), g_2(x_1, x_2)) = \left(\frac{1}{24}x_1^2, \frac{1}{16}x_1x_2\right),$$

then $g(x) \in C^1(B, B)$.

Consider the equation

$$f(x) = \frac{2}{3}f^2\left(\frac{1}{2}x\right) + \frac{1}{3}g(x), x \in B. \quad (4.1)$$

It is in the form of (1.2) where $\lambda = \frac{1}{2}$.

Since

$$g'(x) = \begin{pmatrix} \frac{1}{12}x_1 & 0 \\ \frac{1}{16}x_2 & \frac{1}{16}x_1 \end{pmatrix},$$

then

$$|g'(x)|_{2 \times 2} = \max_{x \in B} \left(\frac{1}{12}|x_1|, \frac{1}{16}(|x_1| + |x_2|) \right) \leq \frac{1}{8}.$$

For any $x, y \in B$, let $x = (x_1, x_2), y = (y_1, y_2)$, we can obtain

$$g'(x) = \begin{pmatrix} \frac{1}{12}x_1 & 0 \\ \frac{1}{16}x_2 & \frac{1}{16}x_1 \end{pmatrix}, g'(y) = \begin{pmatrix} \frac{1}{12}y_1 & 0 \\ \frac{1}{16}y_2 & \frac{1}{16}y_1 \end{pmatrix},$$

then

$$|g'(x) - g'(y)|_{2 \times 2} = \max_{x, y \in B} \left(\frac{1}{12}|x_1 - y_1|, \frac{1}{16}(|x_1 - y_1| + |x_2 - y_2|) \right) \leq \frac{\sqrt{2}}{12}|x - y|_2,$$

thus $g \in \mathcal{A}(\frac{1}{8}, \frac{\sqrt{2}}{12})$.

By condition (P_1) , there is

$$K_1^2 - K_1 + \frac{1}{8} \leq 0,$$

then $K_1 \in (\frac{1-\sqrt{2}}{2}, \frac{1+\sqrt{2}}{2})$, we can choose $K_1 = 0, 5$.
 Simily,by condition (P_2) , there is

$$K_2(\frac{1}{2} + \frac{1}{4}) + \frac{\sqrt{2}}{12} \leq K_2,$$

then $K_2 \in (\frac{\sqrt{2}}{3}, +\infty)$, we can choose $K_2 = 0.6$. Then by theorem 3.1, there is a continuously differentiable solution of equation(4.1) in $\mathcal{A}(0.5, 0.6)$.

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