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ROBUST SYNCHRONIZATION OF PARAMETRIZED NONAUTONOMOUS DISCRETE SYSTEMS WITH APPLICATIONS TO COMMUNICATION SYSTEMS

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Abstract We study synchronization of a coupled discrete system consisting of a Master System and a Slave System. The Master System usually exhibits chaotic or complicated behavior and transmits a signal with a chaotic component to the Slave System. The Slave System then recovers the original signal and removes the chaotic component. To ensure secured communication, the Master and the Slave systems must synchronize independent of the variation of the systems parameters and initial conditions. Here we develop a general approach and obtain some general results for synchronization of such coupled systems naturally arising from discretization of well-know continuous systems, and we illustrate general results with two specific examples: the discretized Lorenz system and a discretized nonlinear oscillator. We also present some simulations using MatLab to illustrate our discussions.

Keywords Discrete system, attractor, synchronization, communication system, Liapunov function.

MSC(2000) 39A30, 39A60, 93D05.

1. Introduction

Synchronization of two coupled continuous nonlinear systems has been studied by many authors, including Rodrigues [12], Affraimovich & Rodrigues [2], Carvalho, Dlotko & Rodrigues [3], Rodrigues, Alberto & Bretas [13, 14], to name a few. Synchronization has also been used by Labouriau & Rodrigues to study the coupled system of Hodgkin-Huxley equations [7].

For continuous systems, Gameiro & Rodrigues [5] studied the uniform dissipativeness and synchronization for a coupled system arising from the application of secured communication.

In a series of papers, initiating with Rodrigues, Wu & Gabriel [15], we address issues related to the synchronization of two coupled chaotic discrete systems arising from secured communication. In the first paper of this series, Rodrigues, Wu &

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Gabriel [15] studied uniform dissipativeness with respect to parameter variation via the Liapunov direct method. We obtained uniform estimates of the global attractor for a general discrete non-autonomous system and established a uniform invariance principle in the autonomous case. The Liapunov function used there was allowed to have positive derivative along solutions of the system inside a bounded set, and this reduces substantially the difficulty of constructing a Liapunov function for a given system. In particular, we developed an approach that incorporates the classical Lagrange multiplier into the Liapunov function method to naturally extend those Liapunov functions from continuous dynamical systems to their discretizations, so that the corresponding uniform dissipativeness results are valid when the step size of the discretization is small. Applications to the discretized Lorenz system and the discretization of a time-periodic chaotic system were given to illustrate the general results.

In the present paper, we study some specific discrete systems obtained by discretizing corresponding continuous systems via the Euler method.

The coupled discrete system we consider is composed of a usually chaotic or complicated system (master system) to be used to codify a signal (a sequence of real numbers) by the addition of a component of a solution of the master system, and a slave system that will be used to de-codify and to recover the original signal. The central issue for such a procedure to be effective is the synchronization of the master and the slave systems. The main and general result (Theorem 2.1) is obtained by using a Liapunov function associated to both systems with identical fixed value of the parameters, and then by some perturbation argument. We should mention here that the Liapunov functions used are similar to those used previously for continuous systems in Gameiro & Rodrigues [5]. This result is then applied first to coupled discretized Lorenz systems and to coupled forced nonlinear oscillators. Some simulations using Matlab are also presented to give more evidences to support our theoretical results. In Section 2 we present our main results. In Section 3.1 and in Section 3.2, respectively, we discuss the discretized coupled Lorenz systems and the discretized coupled oscillators.

2. Main Results

Let $f\colon (X,\ell,\lambda,n)\in\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}^p\times\mathbb{Z}\mapsto f(X,\ell,\lambda,n)\in\mathbb{R}^n$ be a $\mathcal{C}^1$-function. Consider the following discrete system

$$\begin{cases} X(n+1)=X(n)+h\big[f(X(n),x_1(n)+\ell_n,\lambda_0,n)\big] \\ U(n+1)=U(n)+h\big[f(U(n),x_1(n)+\ell_n,\lambda,n)\big], \end{cases} \tag{1}$$

where h is a small step-size, $\{\ell_n\}_{n\in\mathbb{N}}$ plays the role of a secret message to be transmitted by the *master system* (first equation) to the *slave system* (second equation) and λ is close to λ_0.

We make the following hypotheses:

(H1) System (1) is globally dissipative, that is, there exists a bounded convex set $B\subset\mathbb{R}^n$, such that for any initial condition (X_0,U_0) there exists a $n_0\in\mathbb{N}$ such that the solution $(X(n),U(n))$ belongs to $B\times B$ for $n\geq n_0$.

(H2) There exists $k_0=k_0(B)$ and $\ell_0>0$ such that

$$|f(U,\ell,\lambda,n)-f(U,\ell,\lambda_0,n)|\leq k_0|\lambda-\lambda_0| \tag{2}$$

for every $n \in \mathbb{N}$, every $U \in B$ and every $\ell \in [0, \ell_0]$.

(H3) There exists $F\colon \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ linear in the first variable, such that, for fixed λ_0,

$$f(X, \ell, \lambda_0, n) - f(U, \ell, \lambda_0, n) = F(X - U, \ell),$$

for every $(\ell, n) \in \mathbb{R} \times \mathbb{N}$ and every $X, U \in \mathbb{R}^n$.

Consider now the discrete system

$$Z(n+1) = Z(n) + h[F(Z(n), \ell)]. \tag{3}$$

We associate to this system a $\mathcal{C}^1$ Lyapunov Function $V\colon \mathbb{R}^n \to \mathbb{R}$. We define the derivative of this function along the solutions of (3) by

$$\dot{V}(Z) := V(Z + hF(Z, \ell)) - V(Z).$$

We assume the following additional hypothesis:

(H4) There exists a constant $c_1 > 0$ such that

$$c_1 \|Z\|^2 \leq V(Z),$$

for every $Z \in \mathbb{R}^n$.

(H5) Let $B_1 := B - B := \{z \in \mathbb{R}^n \mid z = x - y,\ x, y\ \in B\}$ and $\ell_0 > 0$. There exists $h_0 > 0$, $\rho > 0$ and $k_1 > 0$ such that

$$-\dot{V}(Z, \ell) - \rho h V(Z, \ell) \geq -k_1 h^2,$$

for every $(Z, \ell) \in B_1 \times [0, \ell_0]$ and $0 \leq h \leq h_0$.

We now present our main result:

Theorem 2.1. *Under the above assumptions, system* (1) *synchronizes. That is, given $\varepsilon > 0$ there exist $h_1 > 0$ and $\delta > 0$ such that for any initial condition $(X_0, U_0) \in \mathbb{R}^n \times \mathbb{R}^n$ we have*

$$\limsup_{n\to\infty} \|X(n) - U(n)\| \leq \varepsilon, \quad \textit{if } 0 < h < h_1 \textit{ and } |\lambda - \lambda_0| < \delta.$$

Proof. We start with

$$\begin{aligned} f(X, \ell, \lambda_0, n) - f(U, \ell, \lambda, n) &= f(X, \ell, \lambda_0, n) - f(U, \ell, \lambda_0, n) \\ &\quad + f(U, \ell, \lambda_0, n) - f(U, \ell, \lambda, n) \\ &= F(X - U, \ell) + G, \end{aligned}$$

where $G := f(U, \ell, \lambda_0, n) - f(U, \ell, \lambda, n)$. Back to system (1), we have

$$\begin{aligned} & f(X(n), x_1(n) + \ell_n), \lambda_0, n) - f(U(n), x_1(n) + \ell_n), \lambda, n) \\ = \;& F(X(n) - U(n), x_1(n) + \ell_n) + G_n, \end{aligned}$$

where

$$G_n := f(U(n), x_1(n) + \ell_n, \lambda_0, n) - f(U(n), x_1(n) + \ell_n, \lambda, n).$$

From system (1) we obtain

$$X(n+1) - U(n+1) = X(n) - U(n) + h[F(X(n) - U(n), x_1(n) + \ell_n) + G_n].$$

Now if we introduce a new variable $Z := X - U$, we see that $Z(n) = X(n) - U(n)$ is a solution of the system

$$Z(n+1) = Z(n) + h[F(Z(n), x_1(n) + \ell_n) + G_n]. \tag{4}$$

Now we consider the Lyapunov Function associated to **(H5)**

$$\begin{aligned}
& -\Big[V\big(Z(n+1)\big) - V\big(Z(n)\big)\Big] \\
= \; & -\Big[V\big(Z(n) + h[F(Z(n), x_1(n) + \ell_n) + G_n]\big) - V\big(Z(n)\big)\Big] \\
= \; & -\Big[V\big(Z(n) + h[F(Z(n), x_1(n) + \ell_n)]\big) - V\big(Z(n)\big)\Big] \\
& -V\big(Z(n) + h[F(Z(n), x_1(n) + \ell_n) + G_n]\big) \\
& +V\big(Z(n) + h[F(Z(n), x_1(n) + \ell_n)]\big).
\end{aligned}$$

Since V is of class $\mathcal{C}^1$, if $B_3 \subset \mathbb{R}^n$ is bounded, closed and convex then there exists a constant k_2 such that

$$|V(X_1) - V(X_2)| \leq k_2|X_1 - X_2|$$

for every $X_1, X_2 \in B_3$. Therefore, if we let

$$D_n := V\big(Z(n) + h[F(Z(n), x_1(n) + \ell_n) + G_n]\big) - V\big(Z(n) + h[F(Z(n), x_1(n) + \ell_n)]\big),$$

then we have $|D_n| \leq k_2 h|G_n| = hO(|\lambda - \lambda_0|)$ for sufficiently large n. Then

$$-\Big[V\big(Z(n+1)\big) - V\big(Z(n)\big)\Big] = -\dot{V}(Z(n), \ell) - D_n,$$

and so,

$$-\Big[V\big(Z(n+1)\big) - V\big(Z(n)\big)\Big] - \rho h V(Z(n)) + D_n = -\dot{V}(Z(n), \ell) - \rho h V(Z(n)) \geq -k_1 h^2.$$

This gives

$$V\big(Z(n+1)\big) \leq (1 - \rho h) V\big(Z(n)\big) + D_n + k_1 h^2.$$

Then for sufficiently large n, since $|D_n| \leq hO(|\lambda - \lambda_0|)$, it follows that

$$\begin{aligned}
V\big(Z(n)\big) \;&\leq\; (1 - \rho h)^n V\big(Z(0)\big) + \big[(1 - \rho h)^{n-1} + \cdots + (1 - \rho h) + 1\big] k\delta h \\
&\leq\; (1 - \rho h)^n V\big(Z(0)\big) + \tfrac{k\delta}{\rho},
\end{aligned}$$

for some constant $k > 0$. Finally, choosing h sufficiently small so that $0 < 1 - \rho h < 1$ and using (**H4**), we have

$$\limsup_{n \to \infty} \|Z(n)\|^2 \leq \limsup_{n \to \infty} \frac{1}{c_1} V\big(Z(n)\big) \leq \frac{k\delta}{\rho c_1} \leq \varepsilon,$$

for δ sufficiently small. $\square$

3. Applications

Example 3.1. The discretized Lorenz System This application to communication systems is motivated by the synchronization of the coupled continuos Lorenz system

$$\begin{cases} \dot{x}(t) = -a_0 x(t) + a_0 y(t) \\ \dot{y}(t) = -y(t) - r_0(x(t) + \alpha(t)) - (x(t) + \alpha(t))z \\ \dot{z}(t) = -b_0 z(t) + (x(t) + \alpha(t))y(t) \end{cases}$$

and a *Slave-System*

$$\begin{cases} \dot{u}(t) = -au(t) + av(t) \\ \dot{v}(t) = -v(t) - r(x(t) + \alpha(t)) - (x(t) + \alpha(t))w(t) \\ \dot{w}(t) = -bw(t) + (x(t) + \alpha(t))v(t), \end{cases}$$

where $\alpha(t)$ plays the role of the signal to be transmitted. An equivalent system was studied in [5].

We define

$$f(X, \ell, \lambda) := \begin{pmatrix} -\sigma x_1 + \sigma x_2 \\ -x_2 + r\ell - \ell x_3 \\ -bx_3 + \ell x_2 \end{pmatrix}, \quad X := \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

and $\lambda := (\sigma, r, b)$, and consider the following discrete system:

$$\begin{cases} X(n+1) = X(n) + h[f(X(n), x_1(n) + \ell_n, \lambda_0)] \\ U(n+1) = U(n) + h[f(U(n), x_1(n) + \ell_n, \lambda)], \end{cases} \tag{5}$$

where $\lambda_0 := (\sigma_0, r_0, b_0) = (10, 28, 8/3)$ and $(\ell_n), n \in \mathbb{N}$ is a sequence of real numbers.

Let us first verify hypothesis **(H2)**:

$$\begin{aligned} f(U, \ell, \lambda) - f(U, \ell, \lambda_0) &= \begin{pmatrix} -\sigma u_1 + \sigma u_2 \\ -u_2 + r\ell - \ell u_3 \\ -bu_3 + \ell u_2 \end{pmatrix} - \begin{pmatrix} -\sigma_0 u_1 + \sigma_0 u_2 \\ -u_2 + r_0\ell - \ell u_3 \\ -b_0 u_3 + \ell u_2 \end{pmatrix} \\ &= \begin{pmatrix} -(\sigma - \sigma_0)u_1 + (\sigma - \sigma_0)u_2 \\ (r - r_0)\ell \\ -(b - b_0)u_3 \end{pmatrix} \end{aligned}$$

From the last expression it follows that **(H2)** is satisfied.

Let us consider now hypothesis **(H3)**:

$$\begin{aligned} f(X, \ell, \lambda_0) - f(U, \ell, \lambda_0) &= \begin{pmatrix} -\sigma_0 x_1 + \sigma_0 x_2 \\ -x_2 + r_0\ell - \ell x_3 \\ -b_0 x_3 + \ell x_2 \end{pmatrix} - \begin{pmatrix} -\sigma_0 u_1 + \sigma_0 u_2 \\ -u_2 + r_0\ell - \ell u_3 \\ -b_0 u_3 + \ell u_2 \end{pmatrix} \\ &= \begin{pmatrix} -\sigma_0(x_1 - u_1) + \sigma_0(x_2 - u_2) \\ -(x_2 - u_2) - \ell(x_3 - u_3) \\ -b_0(x_3 - u_3) + \ell(x_2 - u_2) \end{pmatrix} = F(X - U, \ell), \end{aligned}$$

where

$$F(Z, \ell) := \begin{pmatrix} -\sigma_0 z_1 + \sigma_0 z_2 \\ -z_2 - \ell z_3 \\ -b_0 z_3 + \ell z_2 \end{pmatrix}.$$

Defining $V(Z) := z_1^2 + \sigma_0 z_2^2 + \sigma_0 z_3^2$, we see that (**H4**) is easy to verify. As for (**H5**), we have

$$\begin{aligned}
& -\dot{V}(Z,\ell) := V(Z) - V(Z + hF(Z,\ell)) \\
= \quad & z_1^2 + \sigma_0 z_2^2 + \sigma_0 z_3^2 - (z_1 + h(-\sigma_0 z_1 + \sigma_0 z_2))^2 \\
& -[\sigma_0(z_2 + h(-z_2 - \ell z_3))^2 + \sigma_0(z_3 + h(-b_0 z_3 + \ell z_2))^2] \\
= \quad & z_1^2 + \sigma_0 z_2^2 + \sigma_0 z_3^2 - (z_1^2 + 2z_1 h(-\sigma_0 z_1 + \sigma_0 z_2)) - h^2(-\sigma_0 z_1 + \sigma_0 z_2)^2 \\
& -[\sigma_0(z_2^2 + 2hz_2(-z_2 - \ell z_3) + h^2(-z_2 - \ell z_3)^2) + \\
& \sigma_0(z_3^2 + 2z_3 h(-b_0 z_3 + \ell z_2) + h^2(-b_0 z_3 + \ell z_2)^2)] \\
= \quad & -[(2z_1 h(-\sigma_0 z_1 + \sigma_0 z_2) + h^2(-\sigma_0 z_1 + \sigma_0 z_2)^2 + \sigma_0(2hz_2(-z_2 - \ell z_3) + \\
& h^2(-z_2 - \ell z_3)^2) + \sigma_0(2z_3 h(-b_0 z_3 + \ell z_2) + h^2(-b_0 z_3 + \ell z_2)^2))] \\
= \quad & -2h[z_1(-\sigma_0 z_1 + \sigma_0 z_2) + \sigma_0 z_2(-z_2 - \ell z_3) + \sigma_0 z_3(-b_0 z_3 + \ell z_2)] \\
& -h^2[(-\sigma_0 z_1 + \sigma_0 z_2)^2 + \sigma_0(-z_2 - \ell z_3)^2 + \sigma_0(-b_0 z_3 + \ell z_2)^2] \\
= \quad & -2h[-\sigma_0 z_1^2 + \sigma_0 z_1 z_2 - \sigma_0 z_2^2 - \sigma_0 \ell z_2 z_3 - \sigma_0 b_0 z_3^2 + \sigma_0 z_3 \ell z_2)] \\
& -h^2[(-\sigma_0 z_1 + \sigma_0 z_2)^2 + \sigma_0(-z_2 - \ell z_3)^2 + \sigma_0(-b_0 z_3 + \ell z_2)^2] \\
= \quad & 2h\sigma_0[z_1^2 - z_1 z_2 + z_2^2 + b_0 z_3^2)] - h^2[(-\sigma_0 z_1 + \sigma_0 z_2)^2 + \\
& \sigma_0(-z_2 - \ell z_3)^2 + \sigma_0(-b_0 z_3 + \ell z_2)^2].
\end{aligned}$$

Let $g(Z,\ell) := (-\sigma_0 z_1 + \sigma_0 z_2)^2 + \sigma_0(-z_2 - \ell z_3)^2 + \sigma_0(-b_0 z_3 + \ell z_2)^2$. There exists $\ell_0 > 0$ such that $|x_1(n) + \ell_n| \leq \ell_0$. Let $k_1 := \sup_{Z \in B_1,\ \ell \in [0,\ell_0]} g(Z,\ell)$. Therefore,

$$-\dot{V}(Z,\ell) \geq 2h\sigma_0[z_1^2 - z_1 z_2 + z_2^2 + b_0 z_3^2)] - k_1 h^2.$$

$$\begin{aligned}
-\dot{V}(Z,\ell) - \rho h V(Z) &\geq 2h\sigma_0[z_1^2 - z_1 z_2 + z_2^2 + b_0 z_3^2] - \rho h[z_1^2 + \sigma_0 z_2^2 + \sigma_0 z_3^2] - k_1 h^2 \\
&= h[(2\sigma_0 - \rho) z_1^2 - z_1 z_2 + \sigma_0(2 - \rho) z_2^2 + \sigma_0(2b_0 - \rho) z_3^2] - k_1 h^2.
\end{aligned}$$

If we take $\rho := \min\{\sigma_0, 1, b_0\}$, then

$$-\dot{V}(Z,\ell) - \rho h V(Z) \geq h[\sigma_0 z_1^2 - z_1 z_2 + \sigma_0 z_2^2 + \sigma_0 b_0 z_3^2] - k_1 h^2 \geq -k_1 h^2.$$

The last inequality follows from the Sylvester Criterion, since the quadratic form $\sigma_0 z_1^2 - z_1 z_2 + \sigma_0 z_2^2 + \sigma_0 b_0 z_3^2$ is positive definite.

In Figure 1, we show some simulations for this system with $h = 0.01$.

Example 3.2. A Discretized Oscillator This application to communication systems is motivated by the synchronization of the coupled continuous oscillators system

$$\begin{cases} \dot{x} = y \\ \dot{y} = -\omega_0 x - c_0 y - q_0 \left(g(x + m(t))\right)^3 - r_0 g(x + m(t)) \cos(t) \end{cases} \tag{6}$$

$$\begin{cases} \dot{u} = v \\ \dot{v} = -\omega u - cv - q \left(g(x + m(t))\right)^3 - r g(x + m(t)) \cos(t). \end{cases} \tag{7}$$

In this case, $m(t)$ plays the role of the signal transmitted and was studied in [5].

We define the function

$$X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad f(X,\ell,\lambda) := \begin{pmatrix} x_2 \\ -cx_2 - \omega x_1 - q(g(\ell))^3 - rg(\ell)\cos(nh) \end{pmatrix},$$

where $\lambda = (c, \omega, q, r)$, with c, ω, q, and r being positive and $g\colon \mathbb{R} \to \mathbb{R}$ being a $\mathcal{C}^1$, bounded and globally Lipschitz function.

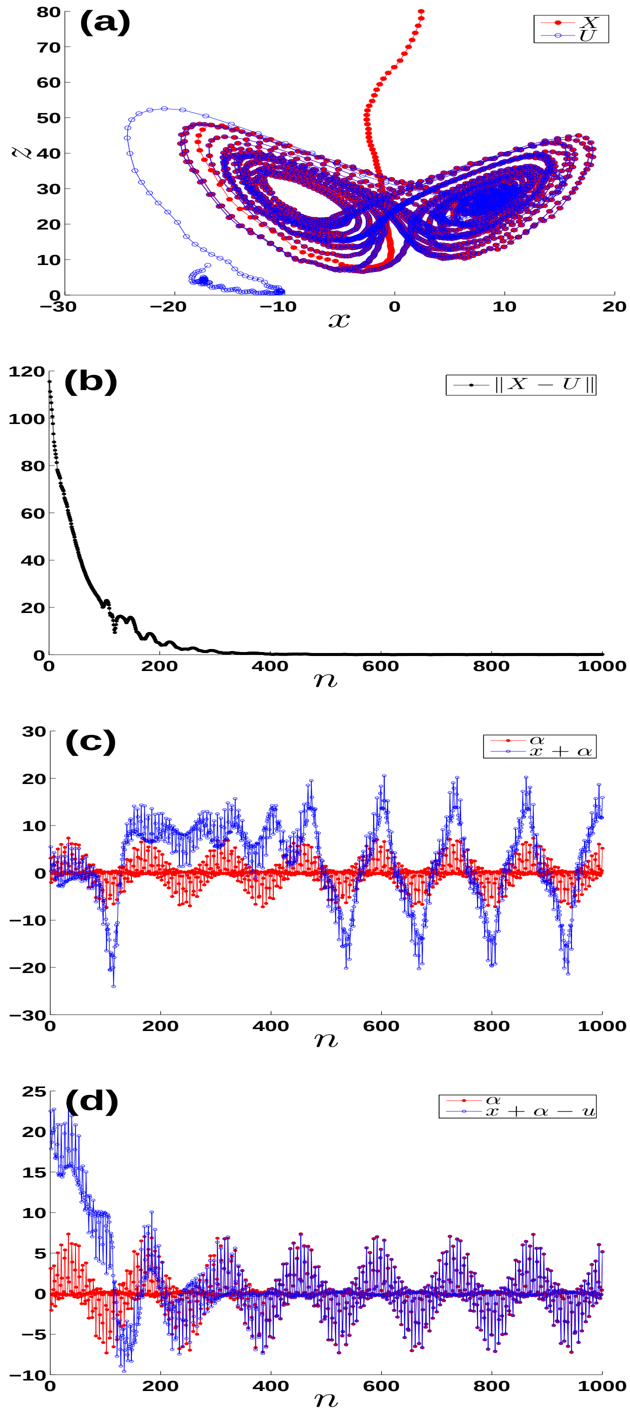

Figure 1. Simulations for the discretized Lorenz system. In **(a)** we plot the solutions $(x_1(n), x_3(n))$ and $(u_1(n), u_3(n))$; in **(b)** we show $|x_1(n)-u_1(n)|+|x_2(n)-u_2(n)|+|x_3(n)-u_3(n)|$; in **(c)** we plot the original message $\alpha(n)$ and the coded message $x(n)+\alpha(n)$; and in **(d)** we plot the original message $\alpha(n)$ and the decoded message $x(n)+\alpha(n)-u(n)$.

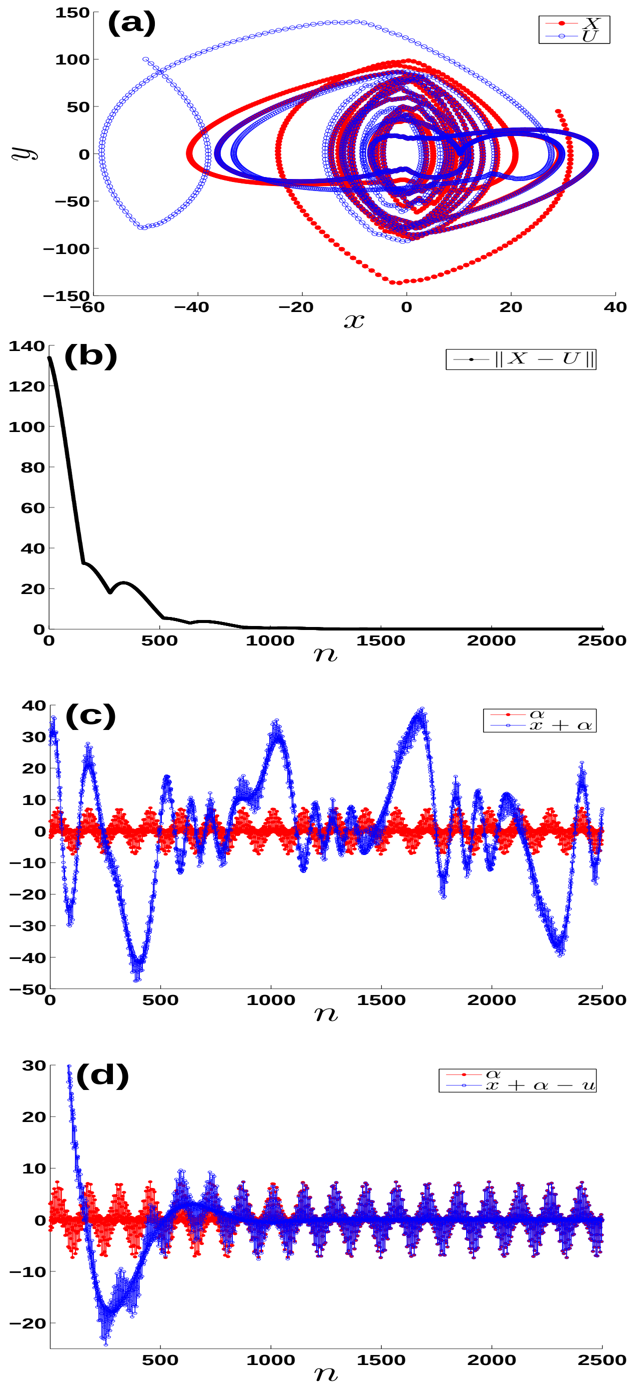

Figure 2. Simulations for the discretized oscillator. In **(a)** we plot the solutions $(x_1(n), x_2(n))$ and $(u_1(n), u_2(n))$; in **(b)** we show $|x_1(n) - u_1(n)| + |x_2(n) - u_2(n)|$; in **(c)** we plot the original message $\alpha(n)$ and the coded message $x(n) + \alpha(n)$; and in **(d)** we plot the original message $\alpha(n)$ and the decoded message $x(n) + \alpha(n) - u(n)$.

We consider the following discrete system

$$\begin{cases} X(n+1) = X(n) + h[f(X(n), x_1(n) + \ell_n, \lambda_0)] \\ U(n+1) = U(n) + h[f(U(n), x_1(n) + \ell_n, \lambda)], \end{cases} \tag{8}$$

where $\lambda_0 := (c_0, \omega_0, q_0, r_0)$, with $\omega_0 = c_0$, and $\{\ell_n\}_{n\in\mathbb{N}}$ is a sequence of real numbers.

Using results of Rodrigues, Wu & Gabriel [15] and the ideas of Gameiro & Rodrigues [5], with similar Lyapunov functions, one can prove that the above system is globally dissipative.

Let us verify that hypothesis (**H2**):

$$\begin{aligned} f(U,\ell,\lambda) - f(U,\ell,\lambda_0) &= \begin{pmatrix} u_2 \\ -cu_2 - \omega u_1 - q(g(\ell))^3 - rg(\ell)\cos(nh) \end{pmatrix} \\ &\quad - \begin{pmatrix} u_2 \\ -c_0u_2 - c_0u_1 - q_0(g(\ell))^3 - r_0g(\ell)\cos(nh) \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ -(c-c_0)u_2 - (\omega - c_0)u_1 - (q-q_0)(g(\ell))^3 - (r-r_0)g(\ell)\cos(nh) \end{pmatrix} \end{aligned}$$

From the last expression it follows that hypothesis (**H2**) is satisfied.

Let us consider hypothesis (**H3**):

$$\begin{aligned} f(X,\ell,\lambda_0) - f(U,\ell,\lambda_0) &= \begin{pmatrix} x_2 \\ -c_0x_2 - c_0x_1 - q_0(g(\ell))^3 - r_0g(\ell)\cos n \end{pmatrix} \\ &\quad - \begin{pmatrix} u_2 \\ -c_0u_2 - c_0u_1 - q_0(g(\ell))^3 - r_0g(\ell)\cos n \end{pmatrix} \\ &= \begin{pmatrix} x_2 - u_2 \\ -c_0(x_2-u_2) - c_0(x_1-u_1) \end{pmatrix} = F(X-U), \end{aligned}$$

where

$$F(X) := \begin{pmatrix} x_2 \\ -c_0x_2 - c_0x_1 \end{pmatrix}.$$

For hypothesis (**H4**) and (**H5**), we consider the Lyapunov function

$$V(X) := \frac{1}{2}[c_0x_1^2 + x_2^2 + \frac{c_0}{2}x_1x_2].$$

Like in the previous example, (**H4**) is easy to verify. As for (**H5**) we have

$$\begin{aligned} & -\dot{V}(X) = V(X) - V(X + hF(X)) \\ = \ & \tfrac{1}{2}\left[c_0x_1^2 + x_2^2 + \tfrac{c_0}{2}x_1x_2\right] - \tfrac{1}{2}\left[c_0(x_1+hx_2)^2 + (x_2 + h(-c_0x_2 - c_0x_1))^2\right] \\ & -\tfrac{1}{2}\left[\tfrac{c_0}{2}(x_1+hx_2)(x_2 + h(-c_0x_2 - c_0x_1))\right] \\ = \ & \tfrac{1}{2}\left(\left[c_0x_1^2 + x_2^2 + \tfrac{c_0}{2}x_1x_2\right] - \left[c_0(x_1^2 + 2hx_1x_2 + h^2x_2^2) + x_2^2 - 2hc_0x_2(x_2+x_1)\right]\right) \\ & -\tfrac{1}{2}\left[h^2c_0^2(x_2+x_1)^2 + \tfrac{c_0}{2}(x_1x_2 - hc_0x_1(x_2+x_1) + hx_2^2 - h^2((c_0x_2 + c_0x_1))\right] \\ = \ & -\tfrac{1}{2}\left[c_0(2hx_1x_2 + h^2x_2^2) - 2hc_0x_2(x_2+x_1) + h^2c_0^2(x_2+x_1)^2\right] \\ & -\tfrac{1}{2}\left[\tfrac{c_0}{2}(-hc_0x_1(x_2+x_1) + hx_2^2 - h^2(c_0x_2 + c_0x_1))\right] \\ = \ & -\tfrac{1}{2}\left[-2hc_0x_2^2 + \tfrac{c_0}{2}(-hc_0(x_1x_2 + x_1^2) + hx_2^2)\right] - \tfrac{1}{2}h^2\left[c_0x_2^2 + c_0^2(x_2+x_1)^2\right. \\ & \left. -\tfrac{c_0}{2}(c_0x_2 + c_0x_1)\right] \\ = \ & -\tfrac{h}{2}\left[-2c_0x_2^2 - \tfrac{c_0^2}{2}x_1x_2 - \tfrac{c_0^2}{2}x_1^2 + \tfrac{c_0}{2}x_2^2\right] - \tfrac{h^2}{2}\left[c_0x_2^2 + c_0^2(x_2+x_1)^2\right. \\ & \left. +\tfrac{c_0}{2}(c_0x_2 + c_0x_1)\right] \\ = \ & -\tfrac{h}{2}\left[-\tfrac{3}{2}c_0x_2^2 - \tfrac{c_0^2}{2}x_1x_2 - \tfrac{c_0^2}{2}x_1^2\right] - \tfrac{h^2}{2}\left[c_0x_2^2 + c_0^2(x_2+x_1)^2 - \tfrac{c_0}{2}(c_0x_2 + c_0x_1)\right] \\ = \ & \tfrac{h}{4}\left[3c_0x_2^2 + c_0^2x_1x_2 + c_0^2x_1^2\right] - \tfrac{h^2}{2}\left[c_0x_2^2 + c_0^2(x_2+x_1)^2 - \tfrac{c_0}{2}(c_0x_2 + c_0x_1)\right]. \end{aligned}$$

Now, setting $\rho = \frac{c_0}{4}$, we have

$$\begin{aligned}
& -\dot{V}(X) - \rho h V(X) = V(X) - V(X + hF(X)) - \rho h V(X) \\
= \quad & \tfrac{h}{4}\left[3c_0 x_2^2 + c_0^2 x_1 x_2 + c_0^2 x_1^2\right] - \tfrac{c_0}{4}\tfrac{h}{2}\left[c_0 x_1^2 + x_2^2 + \tfrac{c_0}{2} x_1 x_2\right] \\
& -\tfrac{h^2}{2}\left[c_0 x_2^2 + c_0^2 (x_2 + x_1)^2 - \tfrac{c_0}{2}(c_0 x_2 + c_0 x_1)\right] \\
= \quad & \tfrac{h}{4}\left(\left[3c_0 x_2^2 + c_0^2 x_1 x_2 + c_0^2 x_1^2\right] - \left[\tfrac{1}{2} c_0^2 x_1^2 + \tfrac{1}{2} c_0 x_2^2 + \tfrac{c_0^2}{4} x_1 x_2\right]\right) \\
& -\tfrac{h^2}{2}\left[c_0 x_2^2 + c_0^2 (x_2 + x_1)^2 - \tfrac{c_0^2}{2}(x_2 + x_1)\right] \\
= \quad & \tfrac{h}{4}\left[\tfrac{5}{2} c_0 x_2^2 + \tfrac{3}{4} c_0^2 x_1 x_2 + \tfrac{1}{2} c_0^2 x_1^2\right] - \tfrac{h^2}{2}\left[c_0 x_2^2 + c_0^2 (x_2 + x_1)^2 - \tfrac{c_0^2}{2}(x_2 + x_1)\right] \\
= \quad & \tfrac{h}{8}\left[5c_0 x_2^2 + \tfrac{3}{2} c_0^2 x_1 x_2 + c_0^2 x_1^2\right] - \tfrac{h^2}{2}\left[c_0 x_2^2 + c_0^2 (x_2 + x_1)^2 - \tfrac{c_0}{2}(c_0 x_2 + c_0 x_1)\right] \\
\geq \quad & -\tfrac{h^2}{2}\left|c_0 x_2^2 + c_0^2 (x_2 + x_1)^2 - \tfrac{c_0^2}{2}(x_2 + x_1)\right|.
\end{aligned}$$

The last inequality follows from Sylvester Criterion since the quadratic form $5c_0 x_2^2 + \frac{3}{2} c_0^2 x_1 x_2 + c_0^2 x_1^2$ is positive definite.

Since this system is globally dissipative, there exists a bounded set $B_2 \subset \mathbb{R}^2$ such that the function $|c_0 x_2^2 + c_0^2 (x_2 + x_1)^2 - \frac{c_0^2}{2}(x_2 + x_1)|$ is bounded in B_2. Therefore,

$$-\dot{V}(X) - \rho h V(X) \geq -k_1 h^2,$$

where $k_1 := \sup_{X \in B_2} |c_0 x_2^2 + c_0^2 (x_2 + x_1)^2 - \frac{c_0}{2}(c_0 x_2 + c_0 x_1)|$. This completes the verification of hypothesis (**H5**).

In Figure 2 we present simulations for this system using $h = 0.01$ and $g(t) = \arctan(t)$.

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