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PEAKON SOLITON SOLUTIONS OF $K(2, -2, 4)$ EQUATION*

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Abstract In this paper, the qualitative analysis methods of a dynamical system are used to investigate the peakon soliton solutions of $K(2,-2,4)$ equation: $u_t + a(u^2)_x + b[u^{-2}(u^4)_{xx}]_x = 0$. The phase portrait bifurcation of the traveling wave system corresponding to the equation is given. The explicit expressions of the peakon soliton solutions are obtained by using the portraits. The graph of the solutions are given with the numerical simulation. This supplements the results obtained in [4].

Keywords Peakon, $K(2,-2,4)$ equation, bifurcation method.

MSC(2000) 35C08, 37K40, 74J35, 35Q51.

1. Introduction

Recently, by using the sine-cosine method and the tanh method, A. M. Wazwaz [5] investigated the compact and noncompact solutions for the $K(n,-n,2n)$ equations,

$$u_t + a(u^n)_x + b[u^{-n}(u^{2n})_{xx}]_x = 0, \tag{1.1}$$

where a, b are two non-zero real number. More recently, by using the bifurcation theory of dynamical systems S. Tang and W. Huang [4] considered some solitary wave solutions, kink and anti-kink wave solutions and uncountably infinite many smooth and non-smooth periodic wave solutions for $K(n,-n,2n)$ equations. Unfortunately, their results are not complete. In the present paper, we shall continue their work and obtain the peakon soliton solutions of Eq.(1.1) for $n = 2$ (simply called $K(2,-2,4)$).

Making the transformations $u(x,t) = \phi(x-ct) = \phi(\xi)$, where c is the wave speed. (1.1) becomes

$$-c\phi' + a(\phi^2)' + b[\phi^{-2}(\phi^4)'']' = 0, \tag{1.2}$$

where "$\prime$" is the derivative with respect to ξ. Taking the integration once on both sides leads to

$$-c\phi + a\phi^2 + b[\phi^{-2}(\phi^4)''] = g, \tag{1.3}$$

where g is integration constant.

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Clearly, (1.3) is equivalent to the following two-dimensional systems

$$\frac{d\phi}{d\xi} = y, \quad \frac{dy}{d\xi} = \frac{1}{4b\phi}(g + c\phi - a\phi^2 - 12by^2). \tag{1.4}$$

Systems (1.4) have the first integrals

$$y^2 = \frac{1}{2b\phi^6}\left(\frac{g}{6}\phi^6 + \frac{c}{7}\phi^7 - \frac{a}{8}\phi^8 + h\right) \tag{1.5}$$

and

$$H(\phi, y) = \phi^6\left(2by^2 - \left(\frac{g}{6} + \frac{c}{7}\phi - \frac{a}{8}\phi^2\right)\right) = h. \tag{1.6}$$

The rest of this paper is organized as follows. In Section 2, we discuss the bifurcations of phase portraits of system (1.4), where explicit parametric conditions will be derived. In Section 3, we give exact explicit parametric representations for peakon soliton solutions of Eq. (1.1) for the case $n = 2$. Section 4 contains the concluding remarks.

2. Bifurcations and phase portraits of systems (1.4)

Throughout we assume that $c > 0$. Otherwise, we can make a transformation $a \to -a,\ b \to -b,\ c \to -c,\ g \to -g,\ y \to y,\ \phi \to \phi,\ \xi \to \xi$ to reduce (1.4) to this case. Systems (1.4) have the same phase orbits as the following systems,

$$\frac{d\phi}{d\tau} = 4b\phi y, \quad \frac{dy}{d\tau} = -12by^2 + g + c\phi - a\phi^2, \tag{2.1}$$

except for the straight line $\phi = 0$, where $d\xi = 4b\phi d\tau$.

When $bg > 0, c^2 + 4ag > 0$, system (1.4) has four equilibrium points: $A_\pm\,(\phi_\pm, 0)$ and $S_\pm\,(0, \pm Y)$, where $\phi_\pm = \frac{-c\pm\sqrt{c^2+4ag}}{-2a}$ and $Y = \sqrt{\frac{g}{12b}}$.

For the function defined by (1.6), we denote that

$$h_\pm = H(\phi_\pm, 0) = -(\phi_\pm)^6\left(\frac{g}{24} + \frac{c}{56}\phi_\pm\right), h_s = H\left(0, \pm Y\right) = 0.$$

When $\frac{g}{24} + \frac{c}{56}\phi_- = 0$, i.e., $ag = -\frac{12}{49}c^2$, we have $H(\phi_-, 0) = H\left(0, \pm Y\right) = 0$.

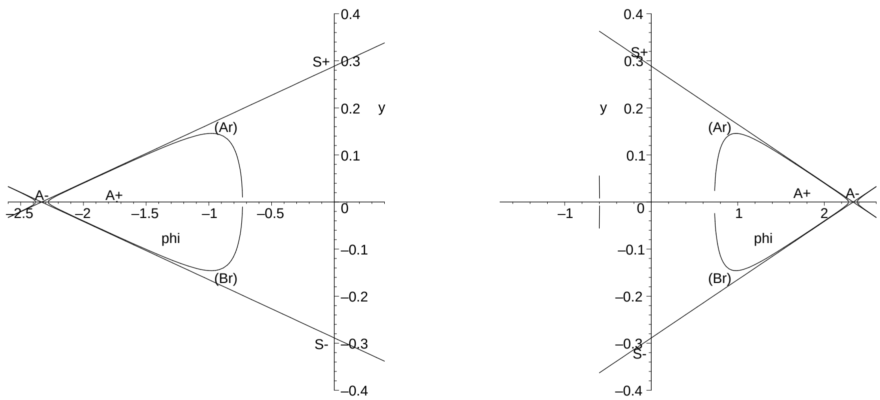
Let $M(\phi_i, y_i)$ be the coefficient matrix of the linearized system of (2.1) at an equilibrium point (ϕ_i, y_i). Then, for $c > 0,\ ab < 0,\ ag < 0, ag = -\frac{12}{49}c^2$, we have

$$\begin{aligned} J(\phi_-, 0) &= \frac{2b\sqrt{c^2 + 4ag}}{a}(c + \sqrt{c^2 + 4ag}) < 0,\ J(0, \pm Y) = -48b^2Y^2 < 0, \\ J(\phi_+, 0) &= \frac{2b\sqrt{c^2 + 4ag}}{a}(-c + \sqrt{c^2 + 4ag}) > 0,\ Trace(M(\phi_+, 0)) = 0. \end{aligned}$$

By the theory of planar dynamical systems, we know that for an equilibrium point (ϕ_i, y_i) of a planar integrable system, if $J < 0$ then the equilibrium point is a saddle point; if $J > 0$ and $Trace(M(\phi_i, y_i)) = 0$ then it is a center point; if $J > 0$ and $(Trace(M(\phi_i, y_i)))^2 - 4J(\phi_i, y_i) > 0$ then it is a node; if $J = 0$ and the index of the equilibrium point is zero then it is a cusp; if $J = 0$ and the index of the equilibrium point is not zero then it is a high order equilibrium point.

Thus, the equilibrium points $S_\pm$ and A_- are saddle points, the equilibrium point A_+ is a center.

For our purpose, in the parameter region: $bg>0, c^2+4ag>0, ag=-\frac{12}{49}c^2$, we shown the phase portraits of system (1.4) in Fig. 1.



(1-1) $a<0,\ b>0,\ g>0$. (1-2) $a>0,\ b<0,\ g<0$.

Figure 1. The phase portraits of system (1.4) for $bg>0, c^2+4ag>0, ag=-\frac{12}{49}c^2$.

3. Exact explicit peakon soliton solutions of Eq. (1.1)

To discuss the existence of peakon soliton solution, we need to use the following two lemmas relating to the singular straight line (see [2,3]).

Lemma 3.1. *When $h\to h_-$, the periodic orbits of the periodic annulus surrounding $(\phi_+,0)$ approach to the boundary curves. Let $(\phi, y=\phi')$ be a point in a periodic orbit γ of system (1.4). Then, along the segment $B_{\gamma\epsilon}A_{\gamma\epsilon}$ near the straight line $\phi=0$, in a very short time interval of $\xi, y=\phi'_\xi$ jumps up rapidly.*

Lemma 3.2. *(Existence of finite time interval(s) of solutions with respect to ξ in the positive or (and) negative direction(s)). Let $(y=\phi'(\xi))$ be the parametric representation of an orbit γ of system (1.4) and $S_\pm$ be two points on the singular straight line $\phi=0$. Suppose that the following conditions holds: $\frac{g}{12b}>0$ and, along the orbit γ, as ξ increases or (and) decreases, the phase point $(\phi(\xi), y(\xi))$ tends to the points $(0,\pm Y)$, respectively.*

Then, there is a finite value $\xi=\widetilde{\xi}$ such that $\lim_{\xi\to\widetilde{\xi}}\phi(\xi)=0$.

We next consider the curve triangle $S_+A_-S_-$ in Fig. 1(1-1). There exists an equilibrium point A_- of system (1.4) at the vertex of the triangle $S_+A_-S_-$, which is far from the singular straight line $\phi=0$. If and only if $\xi\to\pm\infty$, along two curves S_+A_- and A_-S_- as ξ varies the phase point $(\phi(\xi), y(\xi))$ of system (1.4) tends to the equilibrium point A_-. Because the curve triangle $S_+A_-S_-$ in Fig. 1(1-1) defined by $H(\phi,y)=0$ is the limit curve of the $\{\Gamma^h\}$ of periodic orbits of system (1.4) given by $H(\phi,y)=h,\ h\in(h_+,h_s)$, as h is varied from h_s to h_+ the period of the periodic orbit $\{\Gamma^h\}$ tends to ∞. Thus, by using Lemma 3.1 and Lemma 3.2, we know that, as the limit curve of the family of periodic cusp wave, this curve triangle $S_+A_-S_-$ gives rise to a solitary cusp wave of $\phi(\xi)$ (so called "peakon" [1]).

Thus, we have the following conclusion.

Theorem 3.1. *As limit boundary of a family of periodic orbits of system (1.4), the curved triangle gives rise to a peakon soliton solution for Eq. (1.1).*

In the following, we give exact explicit parametric representations of peakon soliton solutions.

Case (I). $a < 0, b > 0, g > 0, ag = -\frac{12}{49}c^2$.

In this case, we have $\phi_- = \frac{4c}{7a}$ and the phase portrait of system (1.4) shown in Fig. 1(1-1). Notice that $H(A_-) = H(S_\pm) = 0$, the upper and lower straight lines of the boundary triangle of the periodic annulus with center $A_+\left(\frac{3c}{7a}, 0\right)$ are

$$y = \pm\sqrt{\frac{-a}{16b}}\left(\phi - \frac{4c}{7a}\right). \tag{3.2}$$

Substituting (3.2) into the first equation of (1.4) and integrating it, we get

$$\mid \xi \mid = \int_{\phi}^{\frac{3c}{7a}} \sqrt{\frac{b}{-a}}\frac{4d\phi}{\left(\phi - \frac{4c}{7a}\right)}, \tag{3.3}$$

from which one have the following parametric representations:

$$\phi_1(\xi) = \frac{c}{7a}\left(4 - \exp\left(-\frac{1}{4}\sqrt{\frac{-a}{b}}|\xi|\right)\right). \tag{3.4}$$

$\phi_1(\xi)$ gives a peakon soliton solution satisfying

$$\begin{aligned} &\phi_1(0) = \frac{3c}{7a}, &&\lim_{\xi\to\pm\infty}\phi_1(\xi) = \frac{4c}{7a}, \\ &\phi_1'(0+) = -\frac{c}{28a}\sqrt{\frac{-a}{b}}, &&\phi_1'(0-) = \frac{c}{28a}\sqrt{\frac{-a}{b}}. \end{aligned} \tag{3.5}$$

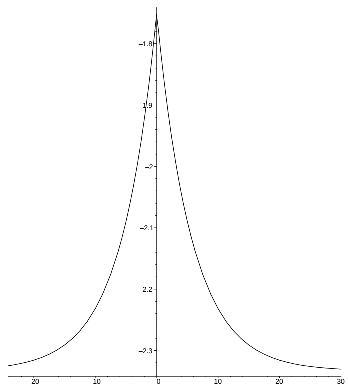
The profile of peakon soliton solution is shown in Fig. 2(2-1).

Remark 3.1. To the best of our knowledge, the solution (3.5) of Eq. (1.1) has not been reported in literature.

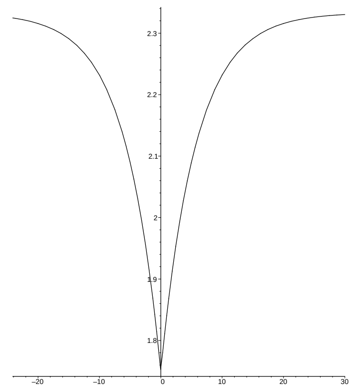
Case (II). $a > 0, b < 0, g < 0, ag = -\frac{12}{49}c^2$.

In this case, we have the phase portrait of system (1.4) shown in Fig. 1(1-2). Similar to the above cases, system (1.4) has a the parametric representation of this arch as (3.1).

The profile of peakon soliton solution is shown in Fig. 2(2-2).



(2-1) $a = \frac{12}{49}, b = g = -1, c = 1.$



(2-2) $a = -\frac{12}{49}, b = g = c = 1.$

Figure 2. Peakon soliton solutions of Eq.(1.1) for $bg > 0, c^2 + 4ag > 0, ag = -\frac{12}{49}c^2$.

4. Discussion

In this paper, we used the qualitative analysis methods of a dynamical system to investigate the peakon soliton solutions of K(2, -2, 4) equation. As a result, we obtained two of new exact peakon soliton solutions. The phase portrait bifurcation of the traveling wave system corresponding to the equation is given. The graph of the solutions are given with the numerical simulation.

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