STABILITY AND NEIMARK-SACKER BIFURCATION OF A SEMI-DISCRETE POPULATION MODEL*

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Abstract   In this paper, a semi-discrete model is derived for a nonlinear simple population model, and its stability and bifurcation are investigated by invoking a key lemma we present. Our results display that a Neimark-Sacker bifurcation occurs in the positive fixed point of this system under certain parametric conditions. By using the Center Manifold Theorem and bifurcation theory, the stability of invariant closed orbits bifurcated is also obtained. The numerical simulation results not only show the correctness of our theoretical analysis, but also exhibit new and interesting dynamics of this system, which do not exist in its corresponding continuous version.

Keywords   Semi-discrete population model, stability, Neimark-Sacker bifurcation, Lyapunov exponent, Chaos.

MSC(2000)   39A11, 37F45, 37G35.

1. Introduction

In order to describe the control of a single population of cells, Nazarenko [23] proposed the nonlinear delay differential equation

$$\frac{\mathrm{d}\xi}{\mathrm{d}t} = -p\xi(t) + \frac{q\xi(t)}{r + \xi^m(t-\omega)}, \quad t \geq 0, \tag{1.1}$$

where p, q, r, $\omega \in (0, +\infty)$, $m \in \{1, 2, ...\}$ and $q > pr$, $\xi(t)$ is the size of the population at time t, p is the death rate, the feedback is given by the function $f(z, z(t - \omega)) = \frac{qz(t)}{r+z^m(t-\omega)}$, and ω is the generation time. Since then, Eq.(1.1) has been well studied by several authors (see [5, 16, 27, 33]).

In recent years, the modern theories of difference equations have been widely applied in the discrete systems of computer science, economy, neutral net, ecology and control theory etc., especially, in the applications of population dynamics. Many authors (see [1, 22]) have argued that the discrete systems governed by difference equations are more appropriate than the continuous counterparts, particularly, when the populations have nonoverlapping generations. Li [17] studied the dynamics of a discrete food-limited population model with time delay. Saker [26] investigated the nonlinear periodic solutions, oscillation and attractivity of discrete nonlinear delay

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population model. Liz [18] considered the global stability for a discrete population model. Song and Peng [28] discussed the periodic solutions of a nonautonomous periodic model of population with continuous and discrete time. Li [19] studied the global stability and oscillation in nonlinear difference equations of population dynamics. Zhang etc. [34] studied the periodic solutions of a single species discrete population model with periodic harvest/stock. It is well known that the dynamics including stability, bifurcations and chaos etc. of a system have been a popular subject (see $[2, 4, 6-8, 10-14, 20, 21, 24, 30-32, 35]$).

In this paper, motivated by the above work we discuss the analogue of the Eq. (1.1). Without loss of generality, we may assume $\omega = 1$ in (1.1). In fact, by letting $s = \frac{t}{\omega}$, namely, $\xi(t) = \xi(s\omega) \overset{\Delta}{=} \eta(s)$, (1.1) is reduced to

$$\frac{\mathrm{d}\eta}{\mathrm{d}s} = -p\omega\eta(s) + \frac{q\omega\eta(s)}{r + \eta^m(s-1)}, \quad s \geq 1. \tag{1.2}$$

By resetting p by $\frac{p}{\omega}$ and q by $\frac{q}{\omega}$, Eq.(1.2) becomes

$$\frac{\mathrm{d}\eta}{\mathrm{d}s} = -p\eta(s) + \frac{q\eta(s)}{r + \eta^m(s-1)}, \quad s \geq 1. \tag{1.3}$$

This is just (1.1) with $\omega = 1$.

Suppose that the average growth rate in (1.3) changes at regular intervals of time, then we may incorporate this aspect into (1.3) and obtain the following version of (1.3)

$$\frac{1}{\eta(s)}\frac{\mathrm{d}\eta(s)}{\mathrm{d}s} = -p + \frac{q}{r + \eta^m([s-1])}, \quad s \neq 1, 2, 3, ..., \tag{1.4}$$

where $[s-1]$ denotes the integer part of $s-1$, $s \in [1, +\infty)$. Equation of type (1.4) is known as differential equation with piecewise constant arguments and these equations occupy a position midway between differential and difference equations. By a solution of Eq. (1.4), we mean a function $\eta(s)$, which is defined for $s \in [1, +\infty)$, and possesses the following properties:

(i) $\eta(s)$ is continuous on $[1, +\infty)$;

(ii) the derivative $\mathrm{d}\eta(s)/\mathrm{d}s$ exists at each point $s \in [1, +\infty)$ with the possible exception of the point $s \in \{1, 2, 3, ...\}$, where the left-side derivative exists;

(iii) the Eq. (1.4) is satisfied on each internal $[k, k+1)$ with $k = 1, 2, 3, ...$

By integrating (1.4) on any interval $[n, n+1)$, $n = 1, 2, 3, ...$, we can get

$$\eta(s) = \eta(n)\exp\left(-p + \frac{q}{r + \eta^m(n-1)}\right)(s-n). \tag{1.5}$$

Letting $s \to n+1$, we have

$$\eta(n+1) = \eta(n)\exp\left(-p + \frac{q}{r + \eta^m(n-1)}\right), \tag{1.6}$$

which is the discrete analogy of Eq. (1.3) without delay.

Let

$$\begin{cases} x(n) = \eta(n-1), \\ y(n) = \eta(n), \end{cases} \tag{1.7}$$

then we arrive in a discrete system as follows:

$$\begin{cases} x(n+1) = y(n), \\ y(n+1) = y(n) \exp\left(-p + \dfrac{q}{r + x^m(n)}\right), \end{cases} \tag{1.8}$$

where p, q, r and m are defined as in Eq. (1.1).

The main aim of this paper is to investigate the dynamics of the system (1.8) by using the Center Manifold Theorem, bifurcation theory ([3,9,15], [25,29]) and numerical simulations. It is shown that the system (1.8) possesses a Neimark-Sacker bifurcation and other complex dynamics under certain parametric conditions, which have not been considered in any known literature.

The rest of this paper is organized as follows. The existence and stability of the fixed points for the system (1.8) are analyzed in the next section. In Section 3, the sufficient conditions of the existence for Neimark-Sacker bifurcation are obtained. In Section 4, numerical simulations are presented, which not only illustrate our theoretical results, but also exhibit other complex dynamics of the system (1.8), and the Lyapunov exponents are computed numerically to confirm some of its dynamics. A brief conclusion is given in Section 5.

2. Existence and stability of fixed point

In this section, we first determine the existence of the fixed points of the system (1.8), then investigate their stability.

The fixed points of the system (1.8) satisfy the following equations:

$$\begin{cases} y = x, \\ y \exp\left(-p + \dfrac{q}{r + x^m}\right) = y. \end{cases} \tag{2.1}$$

By some computations to the system (2.1), it is easy to obtain:

(i) the trivial fixed point $E_0(0,0)$, which always exists for all parameter values;

(ii) the unique positive fixed point $E_+(x_*, y_*)$(feasible because of $q > pr$), where

$$x_* = \left(\frac{q - pr}{p}\right)^{1/m}, \qquad y_* = \left(\frac{q - pr}{p}\right)^{1/m}. \tag{2.2}$$

Now investigate the local stability of every fixed point of the system (1.8). The Jacobian matrix of the system (1.8) at a fixed point $\overline{E}(\overline{x}, \overline{y})$ is

$$J = \begin{pmatrix} 0 & 1 \\ -\overline{y} \exp\left(-p + \dfrac{q}{r + \overline{x}^m}\right) \dfrac{mq\overline{x}^{m-1}}{(r + \overline{x}^m)^2} & \exp\left(-p + \dfrac{q}{r + \overline{x}^m}\right) \end{pmatrix}. \tag{2.3}$$

The characteristic equation associated with (2.3) is

$$\lambda^2 - \mathrm{Tr}(J)\lambda + \mathrm{Det}(J) = 0, \tag{2.4}$$

where λ is the eigenvalue, $\mathrm{Tr}(J)$ and $\mathrm{Det}(J)$ are the trace and determinant of (2.3) respectively, namely,

$$\mathrm{Tr}(J) = \exp\left(-p + \frac{q}{r + \overline{x}^m}\right) \tag{2.5}$$

and

$$\mathrm{Det}(J) = -\overline{y}\exp\left(-p + \frac{q}{r + \overline{x}^m}\right)\frac{mq\overline{x}^{m-1}}{(r + \overline{x}^m)^2}. \tag{2.6}$$

Hence the system (1.8) is (see [14])

(i) a dissipative dynamical system if and only if

$$\left|-\overline{y}\exp\left(-p + \frac{q}{r + \overline{x}^m}\right)\frac{mq\overline{x}^{m-1}}{(r + \overline{x}^m)^2}\right| < 1;$$

(ii) a conservative dynamical system if and only if

$$\left|-\overline{y}\exp\left(-p + \frac{q}{r + \overline{x}^m}\right)\frac{mq\overline{x}^{m-1}}{(r + \overline{x}^m)^2}\right| = 1;$$

(iii) an undissipated dynamical system otherwise.

In order to study the local stability and bifurcation for a fixed point of a general 2D system, the following lemma will be very useful and even essential.

Lemma 2.1. *Let $F(\lambda) = \lambda^2 + B\lambda + C$, where B and C are two real constants. Suppose λ_1 and λ_2 are two roots of $F(\lambda) = 0$. Then the following statements hold.*

(i) *If $F(1) > 0$, then*
  - (i.1) *$|\lambda_1| < 1$ and $|\lambda_2| < 1$ if and only if $F(-1) > 0$ and $C < 1$;*
  - (i.2) *$\lambda_1 = -1$ and $\lambda_2 \neq -1$ if and only if $F(-1) = 0$ and $B \neq 2$;*
  - (i.3) *$|\lambda_1| < 1$ and $|\lambda_2| > 1$ if and only if $F(-1) < 0$;*
  - (i.4) *$|\lambda_1| > 1$ and $|\lambda_2| > 1$ if and only if $F(-1) > 0$ and $C > 1$;*
  - (i.5) *λ_1 and λ_2 are a pair of conjugate complex roots and $|\lambda_1| = |\lambda_2| = 1$ if and only if $-2 < B < 2$ and $C = 1$;*
  - (i.6) *$\lambda_1 = \lambda_2 = -1$ if and only if $F(-1) = 0$ and $B = 2$.*

(ii) *If $F(1) = 0$, namely, 1 is one root of $F(\lambda) = 0$, then the other root λ satisfies $|\lambda| = (<,>)1$ if and only if $|C| = (<,>)1$.*

(iii) *If $F(1) < 0$, then $F(\lambda) = 0$ has one root lying in $(1, \infty)$. Moreover,*
  - (iii.1) *the other root λ satisfies $\lambda < (=) -1$ if and only if $F(-1) < (=)0$;*
  - (iii.2) *the other root λ satisfies $-1 < \lambda < 1$ if and only if $F(-1) > 0$.*

Proof.   The proof for Lemma 2.1 is simple and omitted here.   □

Remark 2.1. (i) When $F(1) > 0$, our results are the same as the ones in [21] except the cases (ii) and (vi).

Corresponding to the above (i.2), the conclusion in [21] is stated as:

$\lambda_1 = -1$ and $|\lambda_2| \neq 1$ if and only if $F(-1) = 0$ and $B \neq 0, 2$.

We think, $B \neq 0$ is redundant. Otherwise, $\lambda_1 + \lambda_2 = 0$, together with $\lambda_1 = -1$, implies $\lambda_2 = 1$, which is contrary to $F(1) > 0$. Therefore, $B \neq 0$ should be kicked out.

So, our results correct the case (iv) of Lemma 2.2. in [21] and give a new conclusion (vi) which is not considered in any known literature.

(ii) The results for the cases $F(1) = 0$ and $F(1) < 0$ are completely new.

Next, we recall the definition of topological types for a fixed point $(\overline{x}, \overline{y})$.

Definition 2.1. Let $\overline{E}(\overline{x}, \overline{y})$ be a fixed point of the system (1.8) with multipliers λ_1 and λ_2.

(i) A fixed point $\overline{E}(\overline{x}, \overline{y})$ is called sink if $|\lambda_1| < 1$ and $|\lambda_2| < 1$, so sink is locally asymptotically stable.

(ii) A fixed point $\overline{E}(\overline{x}, \overline{y})$ is called source if $|\lambda_1| > 1$ and $|\lambda_2| > 1$, so source is locally asymptotically unstable.

(iii) A fixed point $\overline{E}(\overline{x}, \overline{y})$ is called saddle if $|\lambda_1| < 1$ and $|\lambda_2| > 1$(or $|\lambda_1| > 1$ and $|\lambda_2| < 1$).

(iv) A fixed point $\overline{E}(\overline{x}, \overline{y})$ is called to be non-hyperbolic if either $|\lambda_1| = 1$ or $|\lambda_2| = 1$.

Now, we discuss the local dynamics for the fixed points of the system (1.8). The result for the stability of the fixed point $E_0(x_0, y_0)$ is as follows.

Theorem 2.1. *The fixed point $E_0(0,0)$ of the system* (1.8) *is a saddle.*

Proof. The Jacobian matrix J of the system (1.8) at E_0 is given by

$$J(E_0) = \begin{pmatrix} 0 & 1 \\ 0 & \exp\left(\dfrac{q - pr}{r}\right) \end{pmatrix}. \tag{2.7}$$

Obviously, the eigenvalues of (2.7) are $\lambda_1 = 0$ and $\lambda_2 = \exp(\frac{q-pr}{r})$ with $|\lambda_1| < 1$ and $|\lambda_2| > 1$ (because of $q > pr$). Thus E_0 is a saddle. $\qquad\square$

In the following we deduce the local dynamics of the fixed point $E_+(x_*, y_*)$.

Theorem 2.2. *The system* (1.8) *has a unique positive fixed point $E_+(x_*, y_*)$, where $x_* = y_* = (\frac{q-pr}{p})^{1/m}$.*

(i) *When $0 < mp \leq 1$, E_+ is a sink.*

(ii) *When $mp > 1$, there exist three different topological types of E_+ for all permissible values of parameters:*

(ii.1) *E_+ is a sink if $q < \frac{mp^2 r}{mp-1} \overset{\Delta}{=} q_0$;*

(ii.2) *E_+ is a source if $q > q_0$;*

(ii.3) *E_+ is non-hyperbolic if $q = q_0$.*

Proof. The Jacobian matrix J of the system (1.8) at E_+ is given by

$$J(E_+) = \begin{pmatrix} 0 & 1 \\ -\dfrac{mp(q - pr)}{q} & 1 \end{pmatrix}. \tag{2.8}$$

The corresponding characteristic equation of (2.8) can be written as

$$F(\lambda) = \lambda^2 - \lambda + \frac{mp(q - pr)}{q} = 0. \tag{2.9}$$

It is easy to verify that

$$F(1) = \frac{mp(q-pr)}{q} > 0 \tag{2.10}$$

and

$$F(-1) = 2 + \frac{mp(q-pr)}{q} > 0. \tag{2.11}$$

When $0 < mp \leq 1$, $\frac{mp(q-pr)}{q} < 1$. By using Lemma 2.1 (i.1), Eq. (2.9) has two eigenvalues λ_1 and λ_2 with $|\lambda_1| < 1$ and $|\lambda_2| < 1$, so E_+ is a sink.

When $mp > 1$, $q < (>, =) q_0$ is equivalent to $\frac{mp(q-pr)}{q} < (>, =) 1$. By Lemma 2.1 and Ddfinition 2.1, it is easy to see E_+ is a sink for $q < q_0$, a source for $q > q_0$ and non-hyperbolic for $q = q_0$.   □

Remark 2.2. Theorem 2.2 shows that there exists a 2D locally stable manifold W_{loc}^s in E_+ for $q < q_0$ whereas a 2D locally unstable manifold W_{loc}^u for $q > q_0$. Hence, one can see that there will be an occurrence of bifurcation at E_+ for $q = q_0$.

3. Neimark-Sacker bifurcation

From Theorem 2.2 (ii.3), it is easy to see that two eigenvalues of the fixed point $E_+((\frac{q-pr}{p})^{1/m}, (\frac{q-pr}{p})^{1/m})$ are $\frac{1\pm\sqrt{3}i}{2}$. Notice at this time that all the parameters locate in the following set:

$$S_{E_+} = \{(p,q,r,m) \in (0,+\infty) : m \in \{1,2,...\}, q > pr, mp > 1, q = q_0 \triangleq \frac{mp^2r}{mp-1}\}.$$

The fixed point $E_+(x_*, y_*)$ can pass through a Neimark-Sacker bifurcation when the parameters $(p,q,r,m) \in S_{E_+}$ and q varies in the small neighborhood of q_0.

Based on the previous analysis, we choose the parameter q as a bifurcation parameter to study the Neimark-Sacker bifurcation for the unique positive fixed point $E_+(x_*, y_*)$ of the system (1.8) by using the Center Manifold Theorem and bifurcation theory in [3, 9, 15, 25, 29] in this section.

We consider the system (1.8) with parameters $(p,q,r,m) \in S_{E_+}$, which is described by

$$\begin{cases} x \to y, \\ y \to y\exp\left(-p + \dfrac{q_0}{r+x^m}\right). \end{cases} \tag{3.1}$$

The first step.   Giving a perturbation q^* of parameter q_0, we consider a perturbation of the system (3.1) as follows:

$$\begin{cases} x \to y, \\ y \to y\exp\left(-p + \dfrac{q_0+q^*}{r+x^m}\right), \end{cases} \tag{3.2}$$

where $|q^*| \ll 1$.

The second step.   Let $u = x - x_*$ and $v = y - y_*$, which transforms the fixed point $E_+(x_*, y_*)$ to the origin $O(0,0)$ and system (3.2) into

$$\begin{cases} u \to v, \\ v \to (v+y_*)\exp\left(-p + \dfrac{q_0+q^*}{r+(u+x_*)^m}\right) - y_*. \end{cases} \tag{3.3}$$

The characteristic equation associated with the linerization of the system (3.3) at $(u, v) = (0, 0)$ is given by

$$\lambda^2 - a(q^*)\lambda + b(q^*) = 0, \tag{3.4}$$

where

$$a(q^*) = \exp\left(-p + \frac{q_0 + q^*}{r + x_*^m}\right),$$

and

$$b(q^*) = \exp\left(-p + \frac{q_0 + q^*}{r + x_*^m}\right) \frac{(q_0 + q^*)mx_*^m}{(r + x_*^m)^2}.$$

Correspondingly, when q^* varies in a small neighborhood of $q^* = 0$, the roots of the characteristic equation are

$$\lambda_{1,2} = \frac{1}{2}\left[a(q^*) \pm i\sqrt{4b(q^*) - a^2(q^*)}\right]. \tag{3.5}$$

Hence

$$|\lambda_{1,2}| = (b(q^*))^{1/2} \tag{3.6}$$

and

$$\frac{d|\lambda_{1,2}|}{dq^*}\bigg|_{q^*=0} = \frac{mx_*^m(x_*^m + q_0 + r)}{2(r + x_*^m)^2(mq_0x_*^m)^{1/2}} > 0. \tag{3.7}$$

In addition, it is required that $\lambda_{1,2}^i \neq 1$, $i = 1, 2, 3, 4$ when $q^* = 0$. Since $a(q^*)|_{q^*=0} = 1$ and $b(q^*)|_{q^*=0} = 1$, we have $\lambda_{1,2} = \frac{1}{2}(1 \pm i\sqrt{3}) = e^{\pm i\frac{\pi}{3}}$, which obviously satisfy

$$(\lambda_{1,2})^m \neq 1, \, m = 1, 2, 3, 4. \tag{3.8}$$

The third step. Study the normal form of the system (3.3) when $q^* = 0$. Expanding the system (3.3) as Taylor series at $(u, v) = (0, 0)$ to the third order, we obtain

$$\begin{cases} u \to a_{10}u + a_{01}v + a_{20}u^2 + a_{11}uv + a_{02}v^2 + a_{30}u^3 \\ \quad + a_{21}u^2v + a_{12}uv^2 + a_{03}v^3 + O((\sqrt{|u|^2 + |v|^2})^4), \\ v \to b_{10}u + b_{01}v + b_{20}u^2 + b_{11}uv + b_{02}v^2 + b_{30}u^3 \\ \quad + b_{21}u^2v + b_{12}uv^2 + b_{03}v^3 + O((\sqrt{|u|^2 + |v|^2})^4), \end{cases} \tag{3.9}$$

where

$$a_{10} = 0, \quad a_{01} = 1, \quad a_{20} = 0, \quad a_{11} = 0, \quad a_{02} = 0,, \quad a_{30} = 0,$$
$$a_{21} = 0, \quad a_{12} = 0 \quad a_{03} = 0, \quad b_{10} = -1, \quad b_{01} = 1, \quad b_{12} = 0, \quad b_{03} = 0,$$
$$b_{20} = \frac{2m^2q_0x_*^{2m-1}}{(r + x_*^m)^3} + \frac{m^2q_0^2x_*^{2m-1}}{(r + x_*^m)^4} - \frac{mq_0(m-1)x_*^{m-1}}{(r + x_*^m)^2}, \quad b_{11} = -\frac{mq_0x_*^{m-1}}{(r + x_*^m)^2},$$
$$b_{21} = \frac{2m^2q_0x_*^{2m-2}}{(r + x_*^m)^3} + \frac{m^2q_0^2x_*^{2m-2}}{(r + x_*^m)^4} - \frac{mq_0(m-1)x_*^{m-2}}{(r + x_*^m)^2}, \quad b_{02} = 0,$$
$$b_{30} = \frac{3m^2(m-1)q_0^2x_*^{2m-2} - 6m^3q_0x_*^{3m-2}}{(r + x_*^m)^4} - \frac{6m^3q_0^2x_*^{3m-2}}{(r + x_*^m)^5}$$
$$\quad - \frac{m^3q_0^3x_*^{3m-2}}{(r + x_*^m)^6} - \frac{mq_0(m-1)(m-2)x_*^{m-2}}{(r + x_*^m)^2} + \frac{6m^2q_0(m-1)x_*^{2m-2}}{(r + x_*^m)^3}.$$

Let

$$J(E_+) = \begin{pmatrix} a_{10} & a_{01} \\ b_{10} & b_{01} \end{pmatrix}, \quad \text{namely,} \quad J(E_+) = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}.$$

By some computations we obtain the eigenvalues of the matrix $J(E_+)$ are

$$\lambda_1 = \frac{1 + i\sqrt{3}}{2} \quad \text{and} \quad \lambda_2 = \frac{1 - i\sqrt{3}}{2}.$$

The fourth step. Find the normal form of (3.3). Let matrix

$$T = \begin{pmatrix} 0 & 1 \\ \dfrac{\sqrt{3}}{2} & \dfrac{1}{2} \end{pmatrix}, \quad \text{then} \quad T^{-1} = \begin{pmatrix} -\dfrac{1}{\sqrt{3}} & \dfrac{2}{\sqrt{3}} \\ 1 & 0 \end{pmatrix}.$$

Using transformation

$$(u, v)^T = T(X, Y)^T,$$

the system (3.9) is transformed into the following form

$$\begin{cases} X \to \dfrac{1}{2}X - \dfrac{\sqrt{3}}{2}Y + F(X,Y) + O((\sqrt{|X|^2 + |Y|^2})^4), \\ Y \to \dfrac{\sqrt{3}}{2}X + \dfrac{1}{2}Y + G(X,Y) + O((\sqrt{|X|^2 + |Y|^2})^4), \end{cases} \tag{3.10}$$

where

$$F(X,Y) = \left(\frac{2}{\sqrt{3}}b_{20} + \frac{1}{\sqrt{3}}b_{11} \right) Y^2 + b_{11}XY + b_{21}XY^2 + \left(\frac{2}{\sqrt{3}}b_{30} + \frac{1}{\sqrt{3}}b_{21} \right) Y^3$$

and

$$G(X,Y) = 0.$$

The fifth step. Compute some coefficients. On the center manifold the system (3.9) has the above norm form (3.10). For convenience, for a function $F(x_1, x_2, ..., x_n)$, denote F_{x_i}, $F_{x_i x_j}$, and $F_{x_i x_j x_k}$ as the first order, the second order and the third order partial derivative of $F(x_1, x_2, ..., x_n)$, respectively. Then,

$$F_{XX}|_{(0,0)} = 0, \quad F_{XY}|_{(0,0)} = b_{11}, \quad F_{YY}|_{(0,0)} = \frac{4}{\sqrt{3}}b_{20} + \frac{2}{\sqrt{3}}b_{11},$$

$$F_{XXX}|_{(0,0)} = 0, \quad F_{XXY}|_{(0,0)} = 0, \quad F_{XYY}|_{(0,0)} = 2b_{21},$$

$$F_{YYY}|_{(0,0)} = \frac{12}{\sqrt{3}}b_{30} + \frac{6}{\sqrt{3}}b_{21}, \quad G_{XX}|_{(0,0)} = 0, \quad G_{XY}|_{(0,0)} = 0,$$

$$G_{YY}|_{(0,0)} = 0, \quad G_{XXX}|_{(0,0)} = 0, \quad G_{XXY}|_{(0,0)} = 0,$$

$$G_{XYY}|_{(0,0)} = 0, \quad G_{YYY}|_{(0,0)} = 0.$$

The sixth step. Compute the discriminating quantity a^*, which determines the stability of the invariant circle bifurcated from Nemark-Sacker bifurcation of the system (3.10) and can be computed via the formulae (see [25])

$$a^* = -Re\left[\frac{(1 - 2\lambda)\bar{\lambda}^2}{1 - \lambda}L_{11}L_{20} \right] - \frac{1}{2}|L_{11}|^2 - |L_{02}|^2 + Re(\bar{\lambda}L_{21}), \tag{3.11}$$

where

$$L_{20} = \frac{1}{8}[(F_{XX} - F_{YY} + 2G_{XY}) + i(G_{XX} - G_{YY} - 2F_{XY})],$$

$$L_{11} = \frac{1}{4}[(F_{XX} + F_{YY}) + i(G_{XX} + G_{YY})],$$

$$L_{02} = \frac{1}{8}[(F_{XX} - F_{YY} - 2G_{XY}) + i(G_{XX} - G_{YY} + 2F_{XY})], \tag{3.12}$$

$$L_{21} = \frac{1}{16}[(F_{XXX} + F_{XYY} + G_{XXY} + G_{YYY})$$
$$+ i(G_{XXX} + G_{XYY} - F_{XXY} - F_{YYY})].$$

Some computations produce

$$L_{20} = -\frac{1}{8}\left[\left(\frac{4}{\sqrt{3}}b_{20} + \frac{2}{\sqrt{3}}b_{11}\right) + i2b_{11}\right], \ L_{11} = \frac{1}{4}\left(\frac{4}{\sqrt{3}}b_{20} + \frac{2}{\sqrt{3}}b_{11}\right),$$

$$L_{02} = \frac{1}{8}\left[-\left(\frac{4}{\sqrt{3}}b_{20} + \frac{2}{\sqrt{3}}b_{11}\right) + i2b_{11}\right], \ L_{21} = \frac{1}{16}\left[2b_{21} - i\left(\frac{12}{\sqrt{3}}b_{30} + \frac{6}{\sqrt{3}}b_{21}\right)\right]. \tag{3.13}$$

Hence,

$$a^* = \frac{1}{8}(b_{11}b_{20} - b_{21} - 3b_{30})$$

$$= \frac{16m^3 q_0^2 x_*^{3m-2}}{(r + x_*^m)^5} + \frac{18m^3 q_0 x_*^{3m-2} - (8m - 7)m^2 q_0^2 x_*^{2m-2}}{(r + x_*^m)^4}$$

$$- \frac{2m^2(9m - 8)q_0 x_*^{2m-2}}{(r + x_*^m)^3} + \frac{3mq_0(m - 1)(m - 2)x_*^{m-2}}{(r + x_*^m)^2}.$$

Clearly, (3.7) and (3.8) demonstrate that the transversal condition and the non-degenerate condition of the system (1.8) are satisfied. So, summarizing the above discussions, we obtain the following conclusion.

Theorem 3.1. *If $a^* \neq 0$, then the system (1.8) undergoes a Neimark-Sacker bifurcation at the fixed point $E_+(x_*, y_*)$ when the parameter q^* varies in the small neighborhood of origin. Moreover, if $a^* < 0$ (resp., $a^* > 0$), then an attracting (resp., repelling) invariant closed curve bifurcates from the fixed point for $q^* > 0$ (resp., $q^* < 0$).*

Two examples, which illustrate the above Theorem 3.1, are given below.

Example 3.1. Consider the system (1.8) with $r = 0.19$, $m = 2$, $p = 10$, $q = q_0 = 2$. Then, there is a unique positive fixed point $E_+(0.1, 0.1)$ with the multipliers $\lambda = \frac{1}{2} + \frac{\sqrt{3}}{2}i$ and $\overline{\lambda} = \frac{1}{2} - \frac{\sqrt{3}}{2}i$, $|\lambda| = 1$, $\left.\frac{d|\lambda_{1,2}|}{dq^*}\right|_{q^*=0} = 5.5 > 0$, and $a^* = -77.75 < 0$. Hence, according to Theorem 3.1, an attracting invariant closed curve bifurcates from the fixed point for $q^* > 0$.

Example 3.2. Consider the system (1.8) with $r = 9$, $m = 1$, $p = 10$, $q = q_0 = 100$. The unique positive fixed point is $E_+(1, 1)$ for the system (1.8), whose multipliers are $\lambda = \frac{1}{2} + \frac{\sqrt{3}}{2}i$ and $\overline{\lambda} = \frac{1}{2} - \frac{\sqrt{3}}{2}i$ with $|\lambda| = 1$, $\left.\frac{d|\lambda_{1,2}|}{dq^*}\right|_{q^*=0} = 0.11 > 0$ and

$a^* = 0.3225 > 0$. Hence, Theorem 3.1 tells us that an repelling invariant closed curve bifurcates from the fixed point for $q^* < 0$.

4. Numerical simulation

In this section, by using numeral simulation, we give the bifurcation diagrams, phase portraits and Lyapunov exponents of the system (1.8) to confirm the previous theoretical analysis and show some new interesting complex dynamical behaviors existing in the system (1.8). Without lose generality, the bifurcation parameters are considered in the following two cases:

Case 1. Fix the parameters $p = 0.1, r = 10, m = 2$, the initial value $(x_0, y_0) = (0.001, \ 0.001)$ and assume that q varies in the interval $[1.8, 2.8]$.

Evidently, $0 < mp < 1$. We see that the system (1.8) has the unique positive fixed point $E_+(\sqrt{\frac{q-1}{0.1}}, \sqrt{\frac{q-1}{0.1}})$. Figures 1, 2 and 3 show the correctness of the Theorem 2.1 (i) in Section 2.
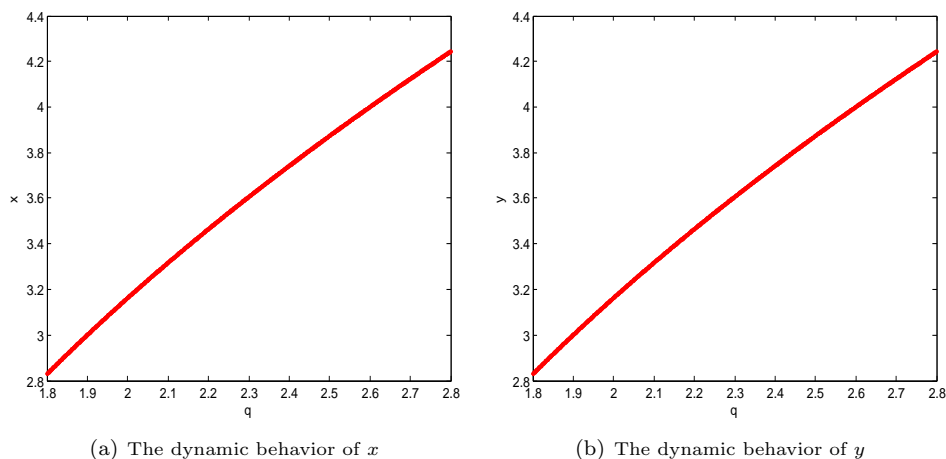

(a) The dynamic behavior of x (b) The dynamic behavior of y

Figure 1. The dynamic behavior for the system (1.8) which exist for $p = 0.1, r = 10, m = 2$ and $q \in [1.8, 2.8]$.

From Figure 1 we see that the fixed point $E_+(\sqrt{\frac{q-1}{0.1}}, \sqrt{\frac{q-1}{0.1}})$ is asymptotically stable.

Taking $q = 1.8,\ 2.0,\ 2.2$ and 2.8 and submitting it into $E_+(\sqrt{\frac{q-1}{0.1}}, \sqrt{\frac{q-1}{0.1}})$, the positive fixed point is $(2.8284,\ 2.8284)$, $(3.1623,\ 3.1623)$, $(3.4641,\ 3.4641)$ and $(4.2426,\ 4.2426)$, respectively. The phase portraits corresponding to Figure 1 are plotted in Figure 2 which show that the fixed point is asymptotically stable.

The maximum Lyapunov exponents corresponding to Figure 1 and 2 are computed and plotted in Figure 3 in which we can easily see that the maximal Lyapunov exponents are negative for the parameter $q \in [1.8,\ 2.8]$, that is to say, the fixed point $E_+(\sqrt{\frac{q-1}{0.1}}, \sqrt{\frac{q-1}{0.1}})$ is stable.
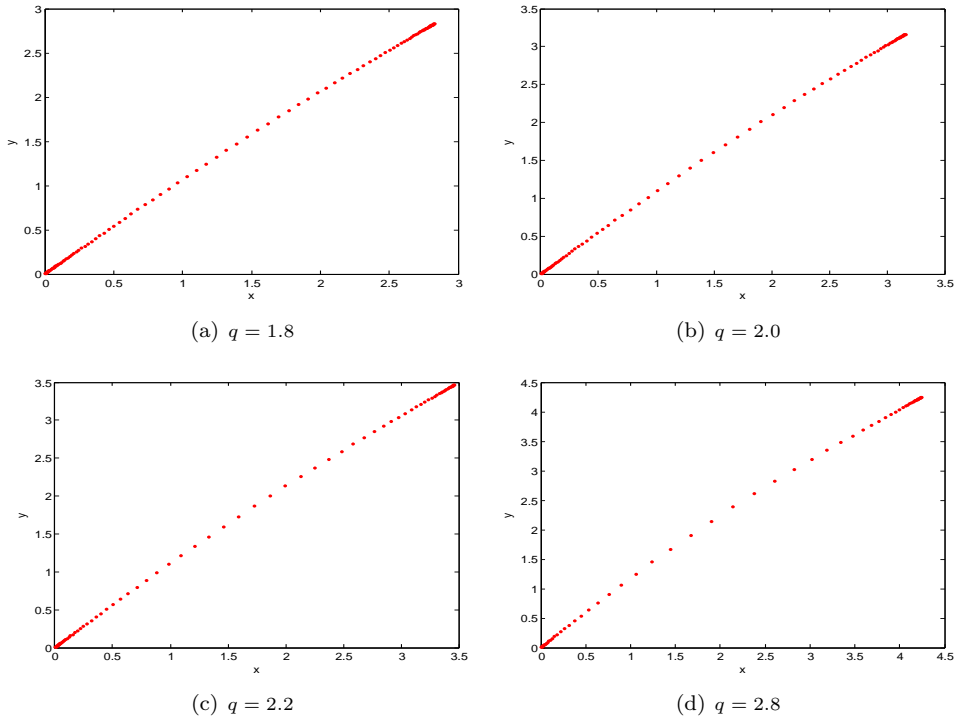
(a) $q = 1.8$

(b) $q = 2.0$

(c) $q = 2.2$

(d) $q = 2.8$

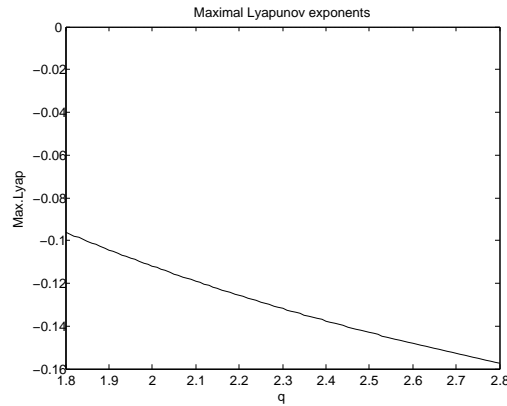**Figure 2.**   Phase portrait of the system (1.8) versus q.

Figure 3. Maximal Lyapunov exponent versus q corresponding to Figure 1 and 2.

Case 2. Choose the parameters $p = 10, r = 0.19, m = 2$, the initial values $(x_0, y_0) = (0.001, 0.001)$ and assume that q varies in the interval $[1.9, 4.8]$.

We see that $mp > 1$ and the unique positive fixed point is $E_+(\sqrt{\frac{q-1}{0.1}}, \sqrt{\frac{q-1}{0.1}})$.

After calculation for the positive fixed point of the system (1.8), we find that the Neimark-Sacker bifurcation emerges from the fixed point (0.1, 0.1) at $q = 2$, whose multipliers are $\lambda_{1,2} = \frac{1 \pm i\sqrt{3}}{2}$ with $|\lambda_{1,2}| = 1$.
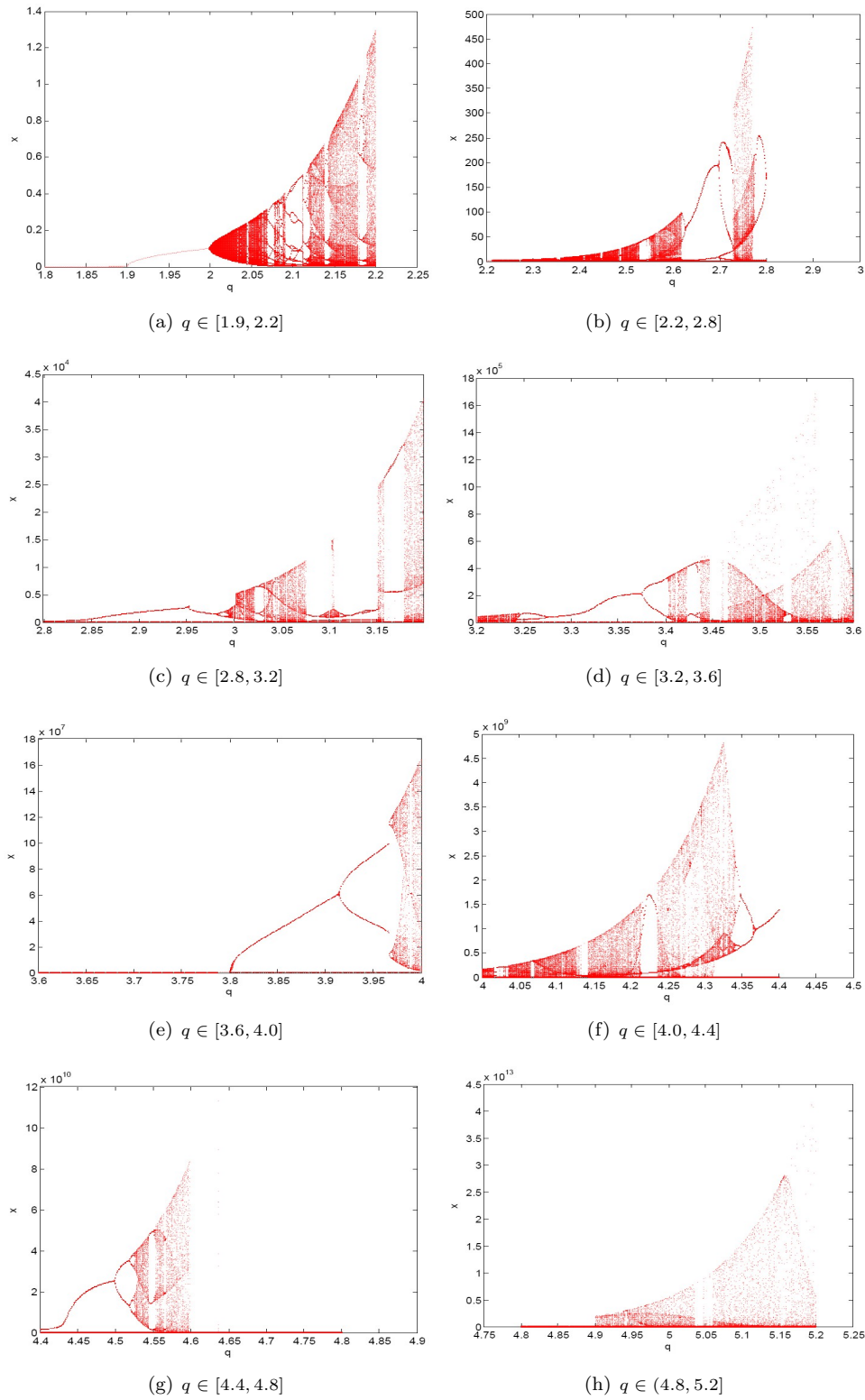
(a) $q \in [1.9, 2.2]$

(b) $q \in [2.2, 2.8]$

(c) $q \in [2.8, 3.2]$

(d) $q \in [3.2, 3.6]$

(e) $q \in [3.6, 4.0]$

(f) $q \in [4.0, 4.4]$

(g) $q \in [4.4, 4.8]$

(h) $q \in (4.8, 5.2]$

Figure 4. Bifurcation diagrams of component x for the system (1.8) versus q.

(a) $q \in [1.8, 2.2]$

(b) $q \in [2.2, 2.8]$

(c) $q \in [2.8, 3.2]$

(d) $q \in [3.2, 3.6]$

(e) $q \in [3.6, 4.0]$

(f) $q \in [4.0, 4.4]$
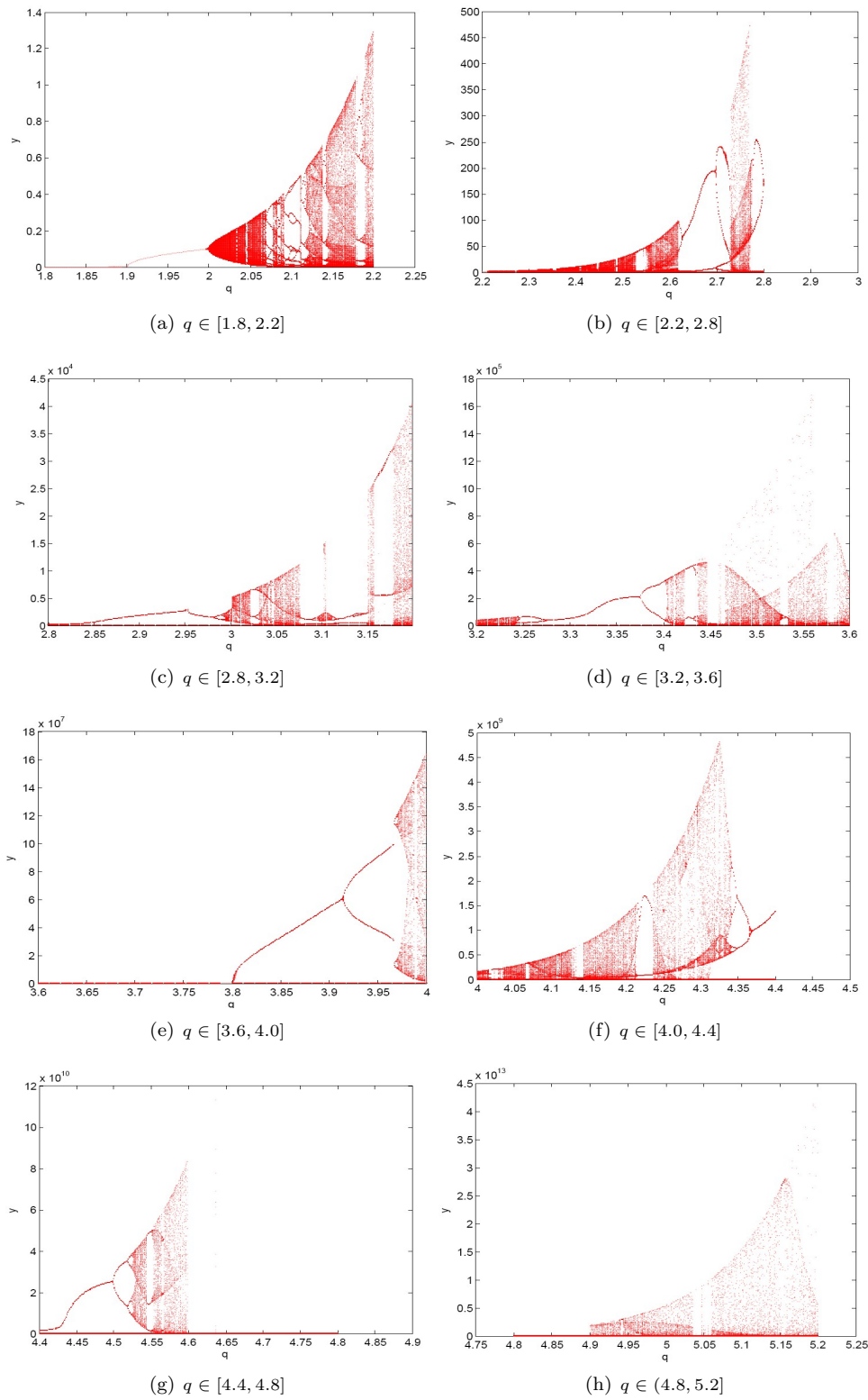
(g) $q \in [4.4, 4.8]$

(h) $q \in (4.8, 5.2]$

Figure 5. Bifurcation diagrams of component y for the system (1.8) versus q.

In Figures 4 and 5, the bifurcation diagrams for the system (1.8) are plotted as a function of the control parameter q for $1.8 \leq q \leq 5.2$. From Figures 4(a) and 5(a), it is clear that the fixed point is stable for $q < 2$, and loses its stability at the Neimark-Sacker bifurcation parameter value $q = 2$. An attracting invariant circle appears when the parameter q exceeds 2. This shows the correctness of the Theorem 3.1. Figures 4 and 5 also display the new and interesting dynamics as q increases.

The maximum Lyapunov exponents corresponding to Figures 4 and 5 are computed and plotted in Figure 6, in which we can easily see that the maximal Lyapunov exponents are negative for the parameter $q \in (1.9, \ 2.0)$, that is to say that fixed point is stable for $q < 2$. For $q \in (2.0, \ 5.2)$, some Lyapunov exponents are positive and some are negative, so there exist stable fixed point or stable period windows in the chaotic region. In general, when the maximal Lyapunov exponent is positive, this can be considered to be one of the characteristics for the existence of chaos.
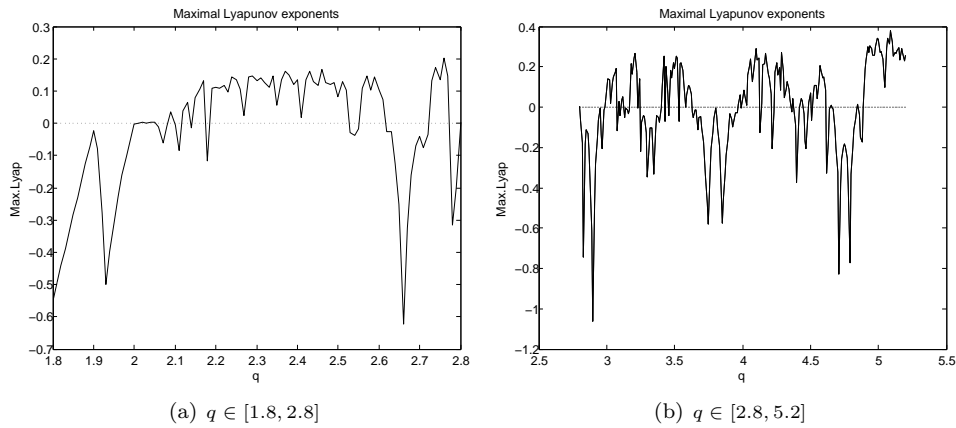

(a) $q \in [1.8, 2.8]$ (b) $q \in [2.8, 5.2]$

Figure 6. Maximal Lyapunov exponent versus q corresponding to Figure 4 and 5.

The phase portraits are considered in the following:

An attractive fixed point takes place for $q = 1.98$, which means that the system orbit is a fixed point, as shown in Figure 7(a).

Figure 7(b) shows that fixed point E_+ is a stable attractor at $q = 1.998$. For this parameter value, the fixed point E_+ occurs with $x_* = 0.0995, y_* = 0.0995$ and the associated complex conjugate eigenvalues are $\lambda_{1,2} = 0.5 \pm 0.8605i$ with $|\lambda_{1,2}| = 0.9952$, which means that the fixed point E_+ is asymptotically stable.

Figure 7(c) demonstrates the behavior of the system (1.8) before the Neimark-Sacker bifurcation when $q = 1.9996$ while Figure 7(d) demonstrates the behavior of the system (1.8) after the Neimark-Sacker bifurcation when $q = 2.001$. From Figure 7(c) and Figure 7(d), we deduce that the fixed point E_+ loses its stability through a Neimark-Sacker bifurcation when the parameter q varies from 1.9996 to 2.001.

(a) $q = 1.98$ (b) $q = 1.998$ (c) $q = 1.9996$

(d) $q = 2.001$ (e) $q = 2.02$ (f) $q = 2.034$

(g) $q = 2.04615$ (h) $q = 2.15$ (i) $q = 2.3$

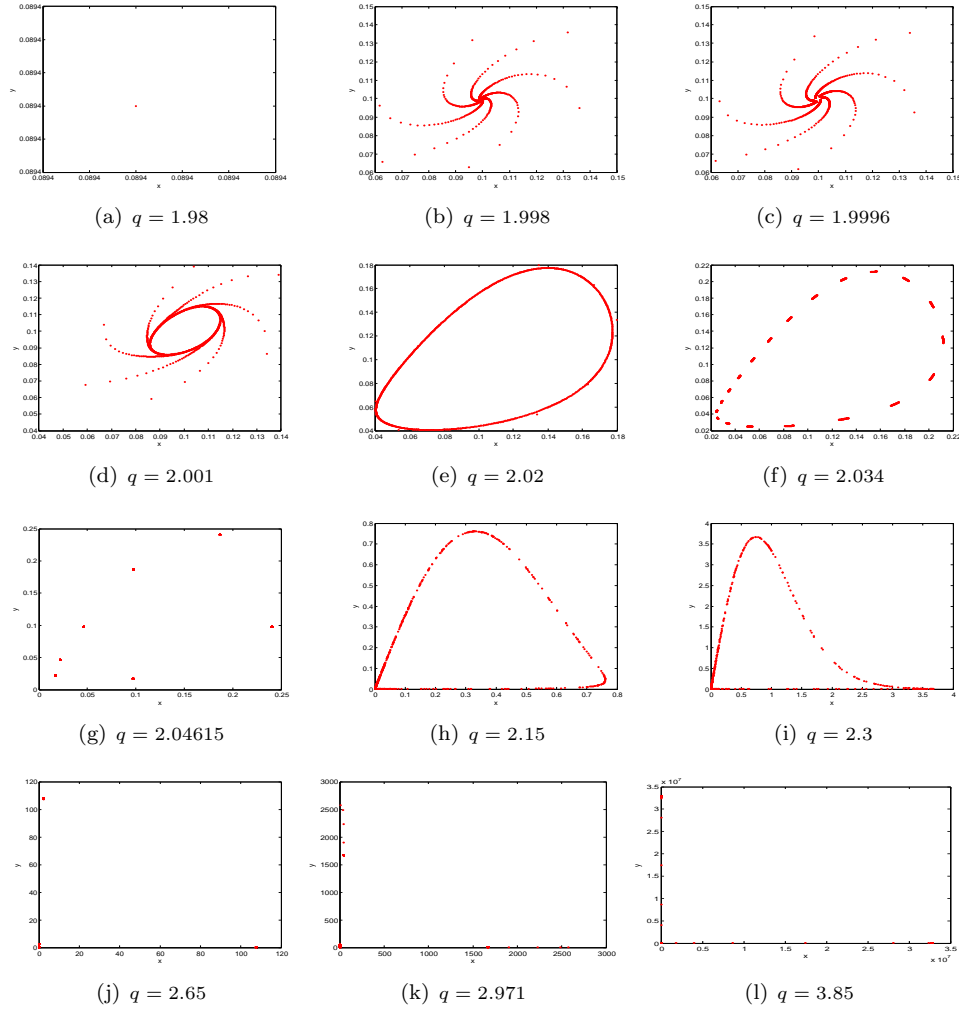(j) $q = 2.65$ (k) $q = 2.971$ (l) $q = 3.85$

Figure 7. Phase portraits for the system (1.8) versus q.

Increasing the control parameter $q(q = 2.02)$, the system (1.8) has the fixed point $E_+(0.1095, 0.1095)$, whose associated eigenvalues are $\lambda_{1,2} = 0.5 \pm i0.9686$ with $|\lambda_{1,2}| = 1.0900$. So, one can conclude that the fixed point becomes unstable and invariant closed curve is created around the fixed point. Figure 7(e) and 7(f) confirms the above argument. Continuing to increase the value of q, we observe that the dynamics of the fixed point E_+ becomes complex from Figure 7(g-l). There exist chaotic sets.

5. Conclusion

In this paper, a semi-discrete model is derived for a nonlinear simple population model and its stability and bifurcation have been investigated. Our results display that a Neimark-Sacker bifurcation phenomenon occurs in the positive fixed point

of this system under certain parametric conditions. Some other basic dynamical properties of the system (1.8) have been analyzed by means of bifurcation diagrams, phase portraits, Lyapunov exponents. Numerical simulations show that the system has more complex dynamical behaviors than its corresponding continuous case.

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