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TRAVELING WAVEFRONTS OF A DELAYED LATTICE REACTION-DIFFUSION MODEL*

Li Shu[1], Peixuan Weng[1,†], and Yanling Tian[1]

Abstract We investigate a system of delayed lattice differential system which is a model of pioneer-climax species distributed on one dimensional discrete space. We show that there exists a constant $c^* > 0$, such that the model has traveling wave solutions connecting a boundary equilibrium to a co-existence equilibrium for $c \geq c^*$. We also argue that c^* is the minimal wave speed and the delay is harmless. The Schauder's fixed point theorem combining with upper-lower solution technique is used for showing the existence of wave solution.

Keywords Pioneer-climax model, lattice differential system, harmless delay, traveling wave solution, minimal wave speed.

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1. Introduction

In an ecosystem, the development of the population models depends on the species' per capita growth rate which is called as the fitness. Some species thrive best at low density. For example, certain varieties of pine and poplar have a fitness which is decreasing monotonically with the total tree density of the forest. Species whose fitness decreases with population density and have a sole equilibrium are often refereed to "pioneer species". On the contrast, some other species may have survival and reproduction rates which benefit from increased population densities. Such a species is called a climax species if its fitness increases up to a maximum value and then decreases on its total density. Hence, a climax population is assumed to have a non-monotone, "one-humped" smooth fitness function. Oak and maple are the examples of climax species.

From the above introduction, we assume that the pioneer fitness function f satisfies,

$$f'(z) < 0, \quad f(z_0) = 0 \tag{1.1}$$

for some $z_0 > 0$, and the climax fitness function g satisfies,

$$\begin{cases} g(w_1) = g(w_2) = 0, \ 0 < w_1 < w_2, \\ (w^* - w)g'(w) > 0 \ \text{ for } w \neq w^* \in (w_1, w_2). \end{cases} \tag{1.2}$$

[†]the corresponding author. Email address: wengpx@scnu.edu.cn(P.X. Weng)
[1]School of Mathematics, South China Normal University, 510631 Guangzhou, P. R. China
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Please see Hassell & Comins [9] and Cushing [8] for some typical examples of fitness functions such as

$$f(u) = \frac{r}{(1+bu)^p} - a, \quad g(u) = ue^{r(1-u)} - a.$$

Selgrade *et al.* [13, 14] examined the dynamics and the Hopf bifurcation of a pioneer-climax ecosystem with the ordinary differential form,

$$\frac{du}{dt} = uf(c_{11}u + v), \quad \frac{dv}{dt} = vg(u + c_{22}v), \tag{1.3}$$

Because of the rich equilibria possibility and the various range of parameters, the dynamics of system (1.3) is complex. Some works and a survey review could be found in Sumner [15, 16] and Buchanan [2]).

Assuming a random dispersal mechanism and by incorporating a spatial variable x into (1.3), we are given the following reaction-diffusion system

$$\begin{cases} \dfrac{\partial u}{\partial t} = d_1 \dfrac{\partial^2 u}{\partial x^2} + uf(c_{11}u + v), \\ \dfrac{\partial v}{\partial t} = d_2 \dfrac{\partial^2 v}{\partial x^2} + vg(u + c_{22}v). \end{cases} \tag{1.4}$$

There are two key elements to the developmental process and study of reaction-diffusion systems. One is the stability, another is the appearance of a traveling wave and the spread speed. We mention here some important works on the stability and instability dynamics of (1.4) from Buonomo and Rionero [4] and Buchanan [3]. Brown et al [1] studied the existence of traveling wave solution of (1.4) connecting two boundary equilibria, and another kind of traveling wave connecting the pioneer-existence equilibrium to the co-existence equilibrium for (1.4) is published recently by Yuan and Zou [20].

Assume that the pioneer and climax species locate on the integer notes of a one-dimensional lattice, and $u_n(t)$ and $v_n(t)$ denote the densities of the pioneer and climax species at the n-th patch and time t, respectively. If the spatial diffusion occurs only at the nearest neighbourhood and is proportional to the difference of the densities of the population at adjacent patches, the diffusion equation can be derived via scaling of deterministic spatially discrete model as follows,

$$\begin{cases} \dfrac{du_n(t)}{dt} =& d_1[u_{n+1}(t) + u_{n-1}(t) - 2u_n(t)] \\ & + u_n(t)f(c_{11}u_n(t) + v_n(t-\tau)), \\ \dfrac{dv_n(t)}{dt} =& d_2[v_{n+1}(t) + v_{n-1}(t) - 2v_n(t)] \\ & + v_n(t)g(u_n(t-\tau) + c_{22}v_n(t)), \quad n \in \mathbb{Z}, \end{cases} \tag{1.5}$$

where $d_1 > 0$, $d_2 > 0$ are diffusion coefficients and $c_{11} > 0$, $c_{22} > 0$. The delay τ in the system may be interpreted as the interaction retard between the two species.

(1.5) is in fact a lattice differential system (LDEs) which is composed with infinite number of differential equations. LDEs arise from mathematical models in many scientific disciplines, such as materials science [5,6], pattern formation [7] and especially population biology [11, 12, 17]. As Keener [11] shows, LDEs may exhibit some different behaviors such as "propagation failure" (see the precise definition in [11]), which generally does not exist in the associated PDEs. Please also see other literatures on the study of various of lattice differential systems, Huang, Lu & Ruan [10], Ma, Wu & Zou [12], Weng, Huang & Wu [17], Wu & Zou [18], Zinner [21].

To the best of our knowledge, the existence of traveling wave solutions for (1.5) has not yet been conducted. Furthermore, as we shall see in the next section that the wave profile system (2.6) is in fact a functional differential system of mixed type (with retarded and advanced variables). Let $c^* = 2\sqrt{d_2 g(\frac{z_0}{c_{11}})}$. Assume that $d_2 \geq \frac{d_1}{2}$. Yuan & Zou [20] showed that c^* is the minimal wave speed of (1.4) in the sense that for $c \geq c^*$, there exists a traveling wave solution connecting the pioneer-existence equilibrium to the co-existence equilibrium, and there is no such a traveling wave solution for $c \in (0, c^*)$. It is natural to ask whether this conclusion is true or false for (1.5)? In this article, we confirm that a similar conclusion holds for (1.5) with $d_2 \geq d_1$, and thus "propagation failure" will not appear for this lattice differential system.

The rest of this paper is organized as follows. In section 2, there are preliminaries and the statement of the main theorem. In section 3, we prove the main theorem by using Schauder's fixed point theorem combining with a pair of upper and lower solutions. We also identify the minimal wave speed in this section. We end this article by a concluding discussion.

2. Preliminaries and main result

The existence of nonnegative steady states depends on the following two systems composed with null-clines,

$$c_{11}u + v = z_0, \quad u + c_{22}v = w_1, \tag{2.1}$$

or

$$c_{11}u + v = z_0, \quad u + c_{22}v = w_2. \tag{2.2}$$

The long term behavior of solutions to (1.5) can be qualitatively determined by the number, distribution and type of the equilibria. In this article, we will only consider the following case,

$$z_0 > \frac{w_1}{c_{22}}, \quad w_1 < \frac{z_0}{c_{11}} < w_2. \tag{2.3}$$

The conclusion $c_{11}c_{22} > 1$ then follows. Under the above assumption (2.3), (1.5) has four nontrivial equilibria, $(\frac{z_0}{c_{11}}, 0)$, $(0, \frac{w_1}{c_{22}})$, $(0, \frac{w_2}{c_{22}})$ and (u^*, v^*) except for $(0, 0)$, where

$$u^* = \frac{c_{22}z_0 - w_2}{c_{11}c_{22} - 1}, \quad v^* = \frac{c_{11}w_2 - z_0}{c_{11}c_{22} - 1}.$$

It is obvious that $u^* < \frac{z_0}{c_{11}}$. In the present article, we only consider a simple case,

$$w^* \leq u^*. \tag{2.4}$$

By making changes of variables $\widetilde{u} = \frac{z_0}{c_{11}} - u$, $\widetilde{v} = v$ and dropping the tildes, system (1.5) becomes

$$\begin{cases} \dfrac{du_n(t)}{dt} =& d_1[u_{n+1}(t)+u_{n-1}(t)-2u_n(t)] \\ & +(u_n(t)-\dfrac{z_0}{c_{11}})f(z_0-c_{11}u_n(t)+v_n(t-\tau)), \\ \dfrac{dv_n(t)}{dt} =& d_2[v_{n+1}(t)+v_{n-1}(t)-2v_n(t)] \\ & +v_n(t)g(\dfrac{z_0}{c_{11}}-u_n(t-\tau)+c_{22}v_n(t)), \quad n\in\mathbb{Z}. \end{cases} \tag{2.5}$$

We are interested in seeking the traveling wavefronts (monotone traveling wave solutions) connecting equilibria $(\frac{z_0}{c_{11}},0)$ and (u^*,v^*), which can be changed into finding a traveling wave of (2.5) connection $(0,0)$ and (u^+,v^+), where $u^+=\frac{z_0}{c_{11}}-u^*$, $v^+=v^*$. Therefore, we now consider a solution of (2.5) with the form $u_n(t)=\phi(n+ct)$ and $v_n(t)=\varphi(n+ct)$, where $c>0$ is a wave speed. Denote the traveling wave coordinate $n+ct$ still by t. We derive the wave profile system from (2.5)

$$\begin{cases} c\phi^{'}(t) =& d_1[\phi(t+1)+\phi(t-1)-2\phi(t)] \\ & +[\phi(t)-\dfrac{z_0}{c_{11}}]f(z_0-c_{11}\phi(t)+\varphi(t-c\tau)), \\ c\varphi^{'}(t) =& d_2[\varphi(t+1)+\varphi(t-1)-2\varphi(t)] \\ & +\varphi(t)g(\dfrac{z_0}{c_{11}}-\phi(t-c\tau)+c_{22}\varphi(t)). \end{cases} \tag{2.6}$$

Associated with (2.6), we consider its solutions subjecting to the following asymptotic boundary value conditions,

$$\lim_{t\to-\infty}(\phi(t),\varphi(t))=(0,0), \quad \lim_{t\to+\infty}(\phi(t),\varphi(t))=(u^+,v^+). \tag{2.7}$$

Remark 2.1. We see from (2.6) that $\phi(t+1)$ and $\phi(t-1)$ appear in the first equation, and $\varphi(t+1)$ and $\varphi(t-1)$ appear in the second equation. This implies that the wave profile system (2.6) is a functional differential system of fixed type. This is an important characteristic of lattice differential system.

The main result is given in the following theorem.

Theorem 2.1. *Let $d_2\geq d_1$. Then there is a $c^*>0$ such that for $c\geq c^*$, there exists a co-invasion traveling wave solution of (1.5) connecting $(z_0/c_{11},0)$ and (u^*,v^*) with speed c.*

3. Proof of the main result

For some positive constants β_1, β_2, let

$$\begin{cases} H_1(\phi,\varphi)(t) := d_1[\phi(t+1)+\phi(t-1)-2\phi(t)] \\ \qquad +[\phi(t)-\dfrac{z_0}{c_{11}}]f(z_0-c_{11}\phi(t)+\varphi(t-c\tau))+\beta_1\phi(t), \\ H_2(\phi,\varphi)(t) := d_2[\varphi(t+1)+\varphi(t-1)-2\varphi(t)] \\ \qquad +\varphi(t)g(\dfrac{z_0}{c_{11}}-\phi(t-c\tau)+c_{22}\varphi(t))+\beta_2\varphi(t). \end{cases}$$

Then, (2.6) can be written as

$$\begin{cases} c\phi^{'}(t) = -\beta_1\phi(t) + H_1(\phi,\varphi)(t), \\ c\varphi^{'}(t) = -\beta_2\varphi(t) + H_2(\phi,\varphi)(t). \end{cases} \tag{3.1}$$

Define a set

$$D = \{(\phi,\varphi) \in C(\mathbb{R},\mathbb{R}^2) \mid 0 \le \phi(t) \le u^+, 0 \le \varphi(t) \le v^+ \text{ for } t \in \mathbb{R}\},$$

and an operator $Q = (Q_1, Q_2), D \to C(\mathbb{R},\mathbb{R}^2)$ by

$$\begin{cases} Q_1(\phi,\varphi)(t) = \dfrac{1}{c}e^{-\frac{\beta_1}{c}t}\displaystyle\int_{-\infty}^{t} e^{\frac{\beta_1}{c}s}H_1(\phi,\varphi)(s)ds, \\ Q_2(\phi,\varphi)(t) = \dfrac{1}{c}e^{-\frac{\beta_2}{c}t}\displaystyle\int_{-\infty}^{t} e^{\frac{\beta_2}{c}s}H_2(\phi,\varphi)(s)ds. \end{cases} \tag{3.2}$$

Then for any $(\phi,\varphi) \in D$, we have

$$\begin{cases} Q_1^{'}(\phi,\varphi)(t) = -\dfrac{\beta_1}{c}Q_1(\phi,\varphi)(t) + \dfrac{1}{c}H_1(\phi,\varphi)(t), \\ Q_2^{'}(\phi,\varphi)(t) = -\dfrac{\beta_2}{c}Q_2(\phi,\varphi)(t) + \dfrac{1}{c}H_2(\phi,\varphi)(t). \end{cases} \tag{3.3}$$

Therefore a fixed point of Q is a solution of (2.6), and vice verse.

In the following, we introduce an exponential decay norm. Let $\mu \in (0, \min\{\frac{\beta_1}{c}, \frac{\beta_2}{c}\})$ and equip $C(\mathbb{R},\mathbb{R}^2)$ with the norm $|\cdot|_\mu$ defined by $|\Phi|_\mu = \sup_{t\in\mathbb{R}} |\Phi(t)|e^{-\mu|t|} < \infty$. Let

$$B_\mu(\mathbb{R},\mathbb{R}^2), = \{\Phi \in C(\mathbb{R},\mathbb{R}^2) \mid \sup_{t\in\mathbb{R}} |\Phi(t)|e^{-\mu|t|} < \infty\}.$$

Then it is easy to check that $(B_\mu(\mathbb{R},\mathbb{R}^2), |\cdot|_\mu)$ is a Banach space.

For the above H, it is easy to show that for sufficiently large β_1, $\beta_2 > 0$ and $(\phi_i,\varphi_i) \in D, i = 1,2$ with $\phi_1(t) \le \phi_2(t)$ and $\varphi_1(t) \le \varphi_2(t)$, $t \in \mathbb{R}$, we have the following monotonic properties,

(i) $H_1(\phi_2,\varphi_2)(t) \ge H_1(\phi_1,\varphi_1)(t); H_2(\phi_2,\varphi_2)(t) \ge H_2(\phi_1,\varphi_1)(t)$ for $t \in \mathbb{R}$;

(ii) $H_1(\phi,\varphi)(t), H_2(\phi,\varphi)(t)$ are nondecreasing on $t \in \mathbb{R}$, if $\phi(t), \varphi(t)$ are nondecreasing on $t \in \mathbb{R}$.

Similarly, $Q = (Q_1, Q_2)$ also enjoys the same properties as those for $H = (H_1, H_2)$ settled above, and we shall use these monotonic properties of H and Q directly in the following arguments.

Definition 3.1. A function $\Phi(t) = (\phi(t), \varphi(t))$ is called an upper solution of (2.6), if there exists a set with finite of numbers $S = \{t_i \mid i = 1, 2, \cdots, p\}$, such that $\Phi(t)$ is differentiable in $\mathbb{R}\backslash S$ and satisfies,

$$\begin{cases} c\dfrac{d\phi(t)}{dt} \ge d_1[\phi(t+1) + \phi(t-1) - 2\phi(t)] \\ \qquad\qquad + [\phi(t) - \dfrac{z_0}{c_{11}}]f(z_0 - c_{11}\phi(t) + \varphi(t - c\tau)), \\ c\dfrac{d\varphi(t)}{dt} \ge d_2[\varphi_(t+1) + \varphi_(t-1) - 2\varphi_(t)] \\ \qquad\qquad + \varphi(t)g(\dfrac{z_0}{c_{11}} - \phi(t - c\tau) + c_{22}\varphi(t)), \end{cases}$$

for $t \in \mathbb{R} \backslash S$. A lower solution of (2.6) can be defined by reversing the above inequalities.

Remark 3.1. We have from (3.3) that an upper solution of (2.6) satisfies an e-quivalent system with two inequalities for $t \in \mathbb{R} \backslash S$,

$$\begin{cases} c\phi^{'}(t) \geq -\beta_1 \phi(t) + H_1(\phi, \varphi)(t), \\ c\varphi^{'}(t) \geq -\beta_2 \varphi(t) + H_2(\phi, \varphi)(t), \end{cases}$$

and two reversed inequalities hold for a lower solution of (2.6).

In what follows, we assume that exist an upper solution $\overline{\Phi}(t) = (\overline{\phi}(t), \overline{\varphi}(t))$ and a lower solution $\underline{\Phi}(t) = (\underline{\phi}(t), \underline{\varphi}(t))$ of (2.6) satisfying

$(P1)$ $(0,0) < (\underline{\phi}(t), \underline{\varphi}(t)) \leq (\overline{\phi}(t), \overline{\varphi}(t)) \leq (u^+, v^+)$ for $t \in \mathbb{R}$;

$(P2)$ $\lim_{t \to -\infty} (\overline{\phi}(t), \overline{\varphi}(t)) = (0,0), \quad \lim_{t \to +\infty} (\overline{\phi}(t), \overline{\varphi}(t)) = (u^+, v^+)$;

$(P3)$ $\sup_{s \leq t} \underline{\phi}(s) \leq \overline{\phi}(t), \quad \sup_{s \leq t} \underline{\varphi}(s) \leq \overline{\varphi}(t)$.

Remark 3.2. For $(0,0) < (\underline{\phi}(t), \underline{\varphi}(t))$ for $t \in \mathbb{R}$, we mean that either $\underline{\phi}(t) \not\equiv 0$ or $\underline{\varphi}(t) \not\equiv 0$ holds.

Define a set $\Omega = \Omega(\underline{\Phi}, \overline{\Phi})$ by

$$\Omega(\underline{\Phi}, \overline{\Phi}) = \left\{ (\phi, \varphi) \in C(\mathbb{R}, \mathbb{R}^2) \,\middle|\, \begin{array}{l} (1)\underline{\phi}(t) \leq \phi(t) \leq \overline{\phi}(t),\ \underline{\varphi}(t) \leq \varphi(t) \leq \overline{\varphi}(t) \text{ for } t \in \mathbb{R}; \\ (2)\phi(t), \varphi(t) \text{ are nondecreasing for } t \in \mathbb{R}. \end{array} \right\}$$

Let $\phi(t) = \sup_{s \leq t} \underline{\phi}(s)$, $\varphi(t) = \sup_{s \leq t} \underline{\varphi}(s)$, it's easy to know that $(\phi(t), \varphi(t)) \in \Omega$, therefore, $\Omega(\underline{\Phi}, \overline{\Phi})$ is nonempty.

Lemma 3.1. *For sufficiently large $\beta_1, \beta_2 > 0$, we have $Q(\Omega) \subset \Omega$. Furthermore, if $(P1) - (P3)$ hold, then $Q(\Omega)$ is a compact set in $B_\mu(\mathbb{R}, \mathbb{R}^2)$.*

Proof. We only show the compactness of $Q(\Omega)$. For any $(\phi, \varphi) \in \Omega(\underline{\Phi}, \overline{\Phi})$, we have

$$\begin{aligned} Q_1^{'}(\phi, \varphi)(t) &= -\frac{\beta_1}{c^2} e^{-\frac{\beta_1}{c}t} \int_{-\infty}^{t} e^{\frac{\beta_1}{c}s} H_1(\phi, \varphi)(s) ds + \frac{1}{c} H_1(\phi, \varphi)(t) \\ &\geq -\frac{\beta_1}{c^2} e^{-\frac{\beta_1}{c}t} \int_{-\infty}^{t} e^{\frac{\beta_1}{c}s} H_1(\overline{\phi}, \overline{\varphi})(s) ds \geq -\frac{\beta_1}{c} \overline{\phi}(t) \geq -\frac{\beta_1}{c} u^+, \\ Q_1^{'}(\phi, \varphi)(t) &= -\frac{\beta_1}{c^2} e^{-\frac{\beta_1}{c}t} \int_{-\infty}^{t} e^{\frac{\beta_1}{c}s} H_1(\phi, \varphi)(s) ds + \frac{1}{c} H_1(\phi, \varphi)(t) \\ &\leq \frac{1}{c} H_1(\overline{\phi}, \overline{\varphi})(t) \leq \frac{1}{c} H_1(u^+, v^+). \end{aligned}$$

It implies that $Q_1^{'}(\phi, \varphi)(t)$ is uniformly bounded. Similarly, $Q_2^{'}(\phi, \varphi)(t)$ is uniformly bounded. Hence $Q(\Omega)$ is equi-continuous with respect to the supremum norm. Furthermore, it is easy to see that $Q(\Omega)$ is also uniformly bounded.

Now we can verify that $Q(\Omega)$ is a compact set in $B_\mu(\mathbb{R}, \mathbb{R})$. In fact, assume that $\{\Phi^{(n)}(t) = (\phi^{(n)}(t), \varphi^{(n)}(t))\} \subset Q(\Omega)$ is a sequence. For any given $\epsilon > 0$, choose

$M_1 > 0$ large enough such that

$$\begin{aligned}&\sup_{|t|\geq M_1}[|\phi^{(n)}(t)-\phi^{(m)}(t)|+|\varphi^{(n)}(t)-\varphi^{(m)}(t)|]e^{-\mu|t|}\\ \leq&2(u^{+}+v^{+})e^{-\mu M_1}<\frac{\epsilon}{2}.\end{aligned}\tag{3.4}$$

Since $\{\Phi^{n(t)}\}$ is uniformly bounded and equi-continuous on $[-M_1, M_1]$, by Arzéra-Ascoli theorem, $\{\Phi^{(n)}(t)\}$ has a subsequence which is convergent on $[-M_1, M_1]$ with respect to the supremum norm. Without loss of generality, we still denote this subsequence by $\{\Phi^{(n)}(t)\}$. This leads to the conclusion that $\{\Phi^{(n)}(t)\}$ is a Cauchy sequence on $[-M_1, M_1]$ with respect to the supremum norm. Therefore, there exists $K > 0$ such that

$$\begin{aligned}&\sup_{|t|\leq M_1}[|\phi^{(n)}(t)-\phi^{(m)}(t)|+|\varphi^{(n)}(t)-\varphi^{(m)}(t)|]e^{-\mu|t|}\\ \leq&\sup_{|t|\leq M_1}[|\phi^{(n)}(t)-\phi^{(m)}(t)|+|\varphi^{(n)}(t)-\varphi^{(m)}(t)|]<\frac{\epsilon}{2}\quad\text{for } n,m>K.\end{aligned}$$

This, together with (3.4), leads to the conclusion that $\{\Phi^{(n)}(t)\}$ is a Cauchy sequence in $B_\mu(\mathbb{R},\mathbb{R}^2)$. As $B_\mu(\mathbb{R},\mathbb{R}^2)$ is a Banach space, thus $\{\Phi^{(n)}(t)\}$ is convergent in $B_\mu(\mathbb{R},\mathbb{R}^2)$. The proof is complete. □

Lemma 3.2. *Assume that* $(\overline{\phi},\overline{\varphi})$, $(\underline{\phi},\underline{\varphi})$ *satisfy* $(P1)-(P3)$, *then* (2.6) *has a monotone solution* (ϕ,φ) *in* $\Omega(\underline{\Phi},\overline{\Phi})$ *satisfying* (2.7).

Proof. Note that the set $\Omega(\underline{\Phi},\overline{\Phi})$ is closed, bounded and convex in the space $B_\mu(R,R^2)$, and the map $Q=(Q_1,Q_2), D\to C(\mathbb{R},\mathbb{R}^2)$ is continuous with respect to the norm $|\cdot|_\mu$ in $B_\mu(\mathbb{R},\mathbb{R}^2)$. Schauder's fixed point theorem is applicable to Q for obtaining a fixed point $\Phi^*=(\phi^*,\varphi^*)$ in $\Omega(\underline{\Phi},\overline{\Phi})$. That is, (2.6) has a solution (ϕ^*,φ^*) in $\Omega(\underline{\Phi},\overline{\Phi})$ satisfying

$$\begin{aligned}&0\leq\phi^*_{-}:=\lim_{t\to-\infty}\phi^*(t)\leq\lim_{t\to-\infty}\overline{\phi}(t),\\ &0\leq\varphi^*_{-}:=\lim_{t\to-\infty}\varphi^*(t)\leq\lim_{t\to-\infty}\overline{\varphi}(t),\\ &\sup_{t\in\mathbb{R}}\underline{\phi}(t)\leq\phi^*_{+}:=\lim_{t\to\infty}\phi^*(t)\leq u^{+},\\ &\sup_{t\in\mathbb{R}}\underline{\varphi}(t)\leq\varphi^*_{+}:=\lim_{t\to\infty}\varphi^*(t)\leq v^{+}.\end{aligned}\tag{3.5}$$

Therefore we can obtain from (P2) that

$$\phi^*_{-}=0,\quad\varphi^*_{-}=0.$$

On the other hand, we can show that $\phi^*_{+}=u^{+},\varphi^*_{+}=v^{+}$. In fact, since (ϕ^*,φ^*) is a fixed point of Q, we have

$$\phi^*(t)=Q_1(\phi^*,\varphi^*)(t),\quad\varphi^*(t)=Q_2(\phi^*,\varphi^*)(t).$$

By using L'Hôspital's rule, we obtain

$$\begin{aligned}\phi^*_{+}=\lim_{t\to\infty}\phi^*(t)&=\lim_{t\to\infty}Q_1(\phi^*(t),\varphi^*(t))\\ &=\lim_{t\to\infty}\frac{\frac{1}{c}\int_{-\infty}^{t}e^{\frac{\beta_1}{c}s}H_1(\phi^*,\varphi^*)(s)ds}{e^{\frac{\beta_1}{c}t}}\\ &=\frac{1}{\beta_1}H_1(\phi^*_{+},\varphi^*_{+}).\end{aligned}$$

Similarly, one can obtain $\varphi_+^* = \frac{1}{\beta_2} H_2(\phi_+^*, \varphi_+^*)$. That is, (ϕ_+^*, φ_+^*) is a nonnegative equilibrium of (2.6) in D. Furthermore, we have from Remark 3.2 and (3.5) that (ϕ_+^*, φ_+^*) is a positive equilibrium. Note that the assumption (2.4) implies that there is only one positive equilibrium (u^+, v^+) of (2.6) in D. Thus (ϕ^*, φ^*) is a monotone solution of (2.6) satisfying (2.7) in $\Omega(\underline{\Phi}, \overline{\Phi})$. The proof is complete. □

Now, summarizing the above discussion, we obtain a theorem.

Theorem 3.1. *If* (2.6) *has a pair of upper and lower solutions satisfy* $(P1)-(P3)$, *then system* (2.5) *has a traveling wave solution satisfying* (2.7).

In order to construct appropriate upper-lower solutions for (2.6), we linearize (2.6) at $(0,0)$ and obtain

$$\begin{cases} d_1[\phi(t+1) + \phi(t-1) - 2\phi(t)] - c\phi^{'}(t) + z_0 f^{'}(z_0)\phi(t) - \dfrac{z_0}{c_{11}} f^{'}(z_0)\varphi(t - c\tau) = 0, \\ d_2[\varphi(t+1) + \varphi(t-1) - 2\varphi(t)] - c\varphi^{'}(t) + \varphi(t) g(\dfrac{z_0}{c_{11}}) = 0. \end{cases} \tag{3.6}$$

Consider the following characteristic equation,

$$F(\lambda, c), = d_2(e^{\lambda} + e^{-\lambda} - 2) - c\lambda + g(\frac{z_0}{c_{11}}) = 0.$$

Note the following facts,

$$\begin{aligned} &F(\lambda, 0) > 0 \quad \text{for any} \quad \lambda \in \mathbb{R}, \\ &F(0, c) = g(\frac{z_0}{c_{11}}) > 0, \ F(\infty, c) = \infty \quad \text{for any } c > 0, \\ &\frac{\partial F(\lambda, c)}{\partial \lambda} = d_2(e^{\lambda} - e^{-\lambda}) - c, \\ &\frac{\partial^2 F(\lambda, c)}{\partial \lambda^2} = d_2(e^{\lambda} + e^{-\lambda}) > 0, \end{aligned}$$

and we can obtain the following Lemma.

Lemma 3.3. *The following conclusions are true.*

(i) There exists a (λ^*, c^*) *such that* $\lambda^* > 0, c^* > 0$, $F(\lambda^*, c^*) = 0$, $\frac{\partial F(\lambda,c)}{\partial \lambda}|_{(\lambda^*, c^*)} = 0$;

(ii) for $0 < c < c^*, F(\lambda, c) > 0$, *for* $\lambda \in \mathbb{R}$;

(iii) for $c > c^*$, *the equation* $F(\lambda, c) = 0$ *has two zeros* $0 < \lambda_1, = \lambda_1(c) < \lambda_2, = \lambda_2(c)$ *such that*

$$F(\lambda, c) < 0 \textit{ for } \lambda_1 < \lambda < \lambda_2. \tag{3.7}$$

In what follows, we shall use the above conclusions to construct a pair of upper and lower solutions of (2.6). Although the form of this pair of upper-lower solutions have similar forms as that in [20], the verification has its own feature and difficulties because of the advanced time $t+1$ and the decay time $t-1$ and $t - c\tau$.

Lemma 3.4. *Let* $c > c^*, d_1 \leq d_2$. *Define*

$$\overline{\phi}(t) = \min\{e^{\lambda_1 t}, u^+\}, \quad \overline{\varphi}(t) = \min\{c_{11} e^{\lambda_1 t}, v^+\}.$$

Then $(\overline{\phi}(t), \overline{\varphi}(t))$ *is an upper solution of* (2.6).

Proof. Let t_1, t_2 be such that $e^{\lambda_1 t_1} = u^+, c_{11}e^{\lambda_1 t_2} = v^+$. Notice that $v^+ = c_{11}u^+$. We have $t_0 := t_1 = t_2 = \frac{1}{\lambda_1}\ln\frac{v^+}{c_{11}}$. Firstly, note the facts,

$$\begin{aligned} &\overline{\phi}(t) \le e^{\lambda_1 t}, \quad \overline{\phi}(t) \le u^+ \quad &&\text{for } t \in \mathbb{R}, \\ &\overline{\varphi}(t) \le c_{11}e^{\lambda_1 t}, \quad \overline{\varphi}(t) \le v^+ \quad &&\text{for } t \in \mathbb{R}. \end{aligned} \tag{3.8}$$

For $t < t_0$, $\overline{\phi}(t) = e^{\lambda_1 t}, \overline{\varphi}(t - c\tau) = e^{\lambda_1(t-c\tau)}$. Since that $f(w)$ is nonincreasing for $w \ge 0$, and $e^{\lambda_1 t} - \frac{z_0}{c_{11}} \le 0$, and thus

$$\begin{aligned} (\overline{\phi}(t) - \frac{z_0}{c_{11}})f(z_0 - c_{11}\overline{\phi}(t) + \overline{\varphi}(t - c\tau)) &= (e^{\lambda_1 t} - \frac{z_0}{c_{11}})f(z_0 - c_{11}e^{\lambda_1 t} + c_{11}e^{\lambda_1(t-c\tau)}) \\ &\le (e^{\lambda_1 t} - \frac{z_0}{c_{11}})f(z_0 - c_{11}e^{\lambda_1 t} + c_{11}e^{\lambda_1 t}) \\ &= (e^{\lambda_1 t} - \frac{z_0}{c_{11}})f(z_0) = 0, \end{aligned}$$

we obtain

$$\begin{aligned} &d_1[\overline{\phi}(t+1) + \overline{\phi}(t-1) - 2\overline{\phi}(t)] - c\overline{\phi}^{'}(t) + (\overline{\phi}(t) - \frac{z_0}{c_{11}})f(z_0 - c_{11}\overline{\phi}(t) + \overline{\varphi}(t - c\tau)) \\ \le& d_1[e^{\lambda_1(t+1)} + e^{\lambda_1(t-1)} - 2e^{\lambda_1 t}] - c\lambda_1 e^{\lambda_1 t} = \{d_1[e^{\lambda_1} + e^{-\lambda_1} - 2] - c\lambda_1\}e^{\lambda_1 t} \\ \le& \{d_2[e^{\lambda_1} + e^{-\lambda_1} - 2] - c\lambda_1\}e^{\lambda_1 t} = -g(\frac{z_0}{c_{11}})e^{\lambda_1 t} \le 0. \end{aligned}$$

For $t \ge t_0$, $\overline{\phi}(t) = u^+, \overline{\varphi}(t - c\tau) \le v^+$. Since that $f(w)$ is nonincreasing for $w \ge 0$, $u^+ - \frac{z_0}{c_{11}} \le 0$, and thus

$$\begin{aligned} (\overline{\phi}(t) - \frac{z_0}{c_{11}})f(z_0 - c_{11}\overline{\phi}(t) + \overline{\varphi}(t - c\tau)) &\le (u^+ - \frac{z_0}{c_{11}})f(z_0 - c_{11}u^+ + v^+) \\ &\le (u^+ - \frac{z_0}{c_{11}})f(z_0) = 0, \end{aligned}$$

we obtain

$$\begin{aligned} &d_1[\overline{\phi}(t+1) + \overline{\phi}(t-1) - 2\overline{\phi}(t)] - c\overline{\phi}^{'} + (\overline{\phi}(t) - \frac{z_0}{c_{11}})f(z_0 - c_{11}\overline{\phi}(t) + \overline{\varphi}(t - c\tau)) \\ \le& d_1[u^+ + u^+ - 2u^+] - c \cdot 0 = 0. \end{aligned}$$

For $t \le t_0$, $\overline{\varphi}(t) = c_{11}e^{\lambda_1 t}, \overline{\phi}(t - c\tau) = e^{\lambda_1(t-c\tau)}$. Since that $g(w)$ is nonincreasing for $w \ge w^*$, $\frac{z_0}{c_{11}} - e^{\lambda_1 t}e^{-\lambda_1 c\tau} + c_{22}c_{11}e^{\lambda_1 t} \ge \frac{z_0}{c_{11}} + (c_{22}c_{11} - 1)e^{\lambda_1 t} \ge \frac{z_0}{c_{11}} \ge w^*$, and thus

$$\begin{aligned} \overline{\varphi}(t)g(\frac{z_0}{c_{11}} - \overline{\phi}(t - c\tau) + c_{22}\overline{\varphi}(t)) &= c_{11}e^{\lambda_1 t}g(\frac{z_0}{c_{11}} - e^{\lambda_1(t-c\tau)} + c_{22}c_{11}e^{\lambda_1 t}) \\ &= c_{11}e^{\lambda_1 t}g(\frac{z_0}{c_{11}} - e^{\lambda_1 t}e^{-\lambda_1 c\tau} + c_{22}c_{11}e^{\lambda_1 t}) \\ &\le c_{11}e^{\lambda_1 t}g(\frac{z_0}{c_{11}}), \end{aligned}$$

we obtain

$$\begin{aligned} &d_2[\overline{\varphi}(t+1) + \overline{\varphi}(t-1) - 2\overline{\varphi}(t)] - c\overline{\varphi}^{'} + \overline{\varphi}(t)g(\frac{z_0}{c_{11}} - \overline{\phi}(t - c\tau) + c_{22}\overline{\varphi}(t)) \\ \le& d_2[c_{11}e^{\lambda_1(t+1)} + c_{11}e^{\lambda_1(t-1)} - 2c_{11}e^{\lambda_1 t}] - cc_{11}\lambda_1 e^{\lambda_1 t} + c_{11}e^{\lambda_1 t}g(\frac{z_0}{c_{11}}) \\ =& \{d_2(e^{\lambda_1} + e^{-\lambda_1} - 2) - c\lambda_1 + g(\frac{z_0}{c_{11}})\}c_{11}e^{\lambda_1 t} = 0. \end{aligned}$$

For $t > t_0$, $\overline{\phi}(t-c\tau) \le u^+, \overline{\varphi}(t) = v^+$. Since that $g(w)$ is nonincreasing for $w \ge w^*$, $\frac{z_0}{c_{11}} - \overline{\phi}(t-c\tau) + c_{22}\overline{\varphi}(t) \ge \frac{z_0}{c_{11}} - u^+ + 0 = u^* \ge w^*$, and thus

$$\begin{aligned}\overline{\varphi}(t)g(\frac{z_0}{c_{11}} - \overline{\phi}(t-c\tau) + c_{22}\overline{\varphi}(t)) &\le v^+ g(\frac{z_0}{c_{11}} - u^+ + c_{22}v^+)\\ &= v^+ g(w_2) = 0,\end{aligned}$$

we obtain

$$\begin{aligned}&d_2[\overline{\varphi}(t+1) + \overline{\varphi}(t-1) - 2\overline{\varphi}(t)] - c\overline{\varphi}^{'} + \overline{\varphi}(t)g(\frac{z_0}{c_{11}} - \overline{\phi}(t-c\tau) + c_{22}\overline{\varphi}(t))\\ \le& d_2[v^+ + v^+ - 2v^+] - c\cdot 0 + v^+ g(w_2) = 0.\end{aligned}$$

The proof is complete. □

Lemma 3.5. *Let $c > c^*$ and $q > 1$ is large, and $\eta = 1+\epsilon > 1$ ($\epsilon > 0$ is small). Define*

$$\underline{\phi}(t) = 0, \quad \underline{\varphi}(t) = max\{c_{11}(e^{\lambda_1 t} - qe^{\eta\lambda_1 t}), 0\}.$$

Then $(\underline{\phi}(t), \underline{\varphi}(t))$ is a lower solution of (2.6) .

Proof. Since $\underline{\phi}(t) = 0$ and $\underline{\varphi}(t-c\tau_4) \ge 0$, we always have

$$\begin{aligned}&d_1[\underline{\phi}(t+1) + \underline{\phi}(t-1) - 2\underline{\phi}(t)] - c\underline{\phi}^{'} + (\underline{\phi}(t) - \frac{z_0}{c_{11}})f(z_0 - c_{11}\underline{\phi}(t) + \underline{\varphi}(t-c\tau))\\ \ge& -\frac{z_0}{c_{11}}f(z_0) = 0.\end{aligned}$$

Let t_3 be such that $c_{11}(e^{\lambda_1 t} - qe^{\eta\lambda_1 t}) = 0$. It follows $t_3 = \frac{1}{(\eta-1)\lambda_1}\ln\frac{1}{q} < 0$. Denote $m = \min_{t\in[\frac{z_0}{c_{11}}, \frac{z_0}{c_{11}} + c_{22}v^+]} g^{'}(t) < 0$. For $t < t_3$, we have $e^{\lambda_1 t} - qe^{\eta\lambda_1 t} \ge 0$, and thus

$$\begin{aligned}&g(\frac{z_0}{c_{11}} + c_{11}c_{22}(e^{\lambda_1 t} - qe^{\eta\lambda_1 t}))\\ =&[g(\frac{z_0}{c_{11}} + c_{11}c_{22}(e^{\lambda_1 t} - qe^{\eta\lambda_1 t})) - g(\frac{z_0}{c_{11}})] + g(\frac{z_0}{c_{11}})\\ \ge& mc_{11}c_{22}(e^{\lambda_1 t} - qe^{\eta\lambda_1 t}) + g(\frac{z_0}{c_{11}}),\\ &c_{11}(e^{\lambda_1 t} - qe^{\eta\lambda_1 t})g(\frac{z_0}{c_{11}} + c_{11}c_{22}(e^{\lambda_1 t} - qe^{\eta\lambda_1 t}))\\ \ge& mc_{11}^2c_{22}(e^{\lambda_1 t} - qe^{\eta\lambda_1 t})^2 + c_{11}(e^{\lambda_1 t} - qe^{\eta\lambda_1 t})g(\frac{z_0}{c_{11}}).\end{aligned}$$

Note that $\underline{\varphi}(t) \ge c_{11}(e^{\lambda_1(t)} - qe^{\eta\lambda_1(t)})$ for $t \in \mathbb{R}$. For $t < t_3$, we obtain

$$\begin{aligned}&d_2[\underline{\varphi}(t+1) + \underline{\varphi}(t-1) - 2\underline{\varphi}(t)] - c\underline{\varphi}^{'}(t) + \underline{\varphi}(t)g(\frac{z_0}{c_{11}} - \underline{\phi}(t-c\tau) + c_{22}\underline{\varphi}(t))\\ \ge& d_2[c_{11}(e^{\lambda_1(t+1)} - qe^{\eta\lambda_1(t+1)}) + c_{11}(e^{\lambda_1(t-1)} - qe^{\eta\lambda_1(t-1)}) - 2c_{11}(e^{\lambda_1 t} - qe^{\eta\lambda_1 t})]\\ &- cc_{11}\lambda_1(e^{\lambda_1 t} - q\eta e^{\eta\lambda_1 t}) + c_{11}(e^{\lambda_1 t} - qe^{\eta\lambda_1 t})g(\frac{z_0}{c_{11}} + c_{11}c_{22}(e^{\lambda_1 t} - qe^{\eta\lambda_1 t}))\\ \ge& d_2c_{11}[(e^{\lambda_1(t+1)} - qe^{\eta\lambda_1(t+1)}) + (e^{\lambda_1(t-1)} - qe^{\eta\lambda_1(t-1)}) - 2(e^{\lambda_1 t} - qe^{\eta\lambda_1 t})]\\ &- cc_{11}(\lambda_1 e^{\lambda_1 t} - q\eta\lambda_1 e^{\eta\lambda_1 t}) + mc_{11}^2c_{22}(e^{\lambda_1 t} - qe^{\eta\lambda_1 t})^2 + c_{11}(e^{\lambda_1 t} - qe^{\eta\lambda_1 t})g(\frac{z_0}{c_{11}})\end{aligned}$$

$$
\begin{aligned}
&=c_{11}e^{\lambda_1 t}F(\lambda_1,c)-c_{11}qe^{\eta\lambda_1 t}F(\eta\lambda_1,c)+mc_{11}^2c_{22}(e^{\lambda_1 t}-qe^{\eta\lambda_1 t})^2\\
&=c_{11}[-qe^{\eta\lambda_1 t}F(\eta\lambda_1,c)+mc_{11}c_{22}(e^{\lambda_1 t}-qe^{\eta\lambda_1 t})^2]\\
&=c_{11}e^{\eta\lambda_1 t}[-qF(\eta\lambda_1,c)+mc_{11}c_{22}(e^{\lambda_1(1-\frac{\eta}{2})t}-qe^{\frac{\eta}{2}\lambda_1 t})^2],
\end{aligned}
$$

where $F(\lambda,c)=d_2[e^{\lambda}+e^{-\lambda}-1]-c\lambda+g(\frac{z_0}{c_{11}})$. Let $\omega(t):=e^{\lambda_1(1-\frac{\eta}{2})t}-qe^{\frac{\eta}{2}\lambda_1 t}$ and solve the equation

$$
\omega'(t)=\lambda_1\{(1-\frac{\eta}{2})e^{\lambda_1 t}-q\frac{\eta}{2}e^{\frac{\eta}{2}\lambda_1 t}\}=0.
$$

We obtain that $\bar{t}=\frac{1}{(\frac{\eta}{2}-1)\lambda_1}\ln(\frac{2-\eta}{\eta q})$. Thus

$$
\max_{t\in\mathbb{R}}\omega(t)=\omega(\bar{t})=e^{\frac{1-\frac{\eta}{2}}{(\frac{\eta}{2}-1)}\ln(\frac{2-\eta}{\eta q})}-qe^{\frac{\eta}{\eta-2}\ln(\frac{2-\eta}{\eta q})}=\frac{\eta q}{2-\eta}-q(\frac{2-\eta}{\eta q})^{\frac{\eta}{\eta-2}}.
$$

Note $\eta=1+\epsilon$, and thus one can choose $\epsilon>0$ small enough such that $F(\eta\lambda_1,c)<0$. On the other hand, we have $m<0$, so that we can choose $q>1$ large enough such that

$$
\begin{aligned}
&-qF(\eta\lambda_1,c)+mc_{11}c_{22}(e^{\lambda_1(1-\frac{\eta}{2})t}-qe^{\frac{\eta}{2}\lambda_1 t})^2\\
\geq&-qF(\eta\lambda_1,c)+mc_{11}c_{22}\omega^2(\bar{t})\geq 0,
\end{aligned}
$$

which leads to

$$
d_2[\underline{\varphi}(t+1)+\underline{\varphi}(t-1)-2\underline{\varphi}(t)]-c\underline{\varphi}'(t)+\underline{\varphi}(t)g(\frac{z_0}{c_{11}}-\underline{\phi}(t-c\tau)+c_{22}\underline{\varphi}(t))\geq 0.
$$

Note $\underline{\varphi}(t)\geq 0$ for $t\in\mathbb{R}$ and $\underline{\varphi}(t)=0$ for $t>t_3$. Thus we obtain,

$$
\begin{aligned}
&d_2[\underline{\varphi}(t+1)+\underline{\varphi}(t-1)-2\underline{\varphi}(t)]-c\underline{\varphi}'(t)+\underline{\varphi}(t)g(\frac{z_0}{c_{11}}-\underline{\phi}(t-c\tau)+c_{22}\underline{\varphi}(t))\\
\geq&d_2[0+0-2\cdot 0]-c\cdot 0+0\cdot g(\frac{z_0}{c_{11}}+c_{22}\cdot 0)=0.
\end{aligned}
$$

The proof is complete. □

Remark 3.3. It is obvious that the upper-lower solutions defined in Lemmas 3.4 and 3.5 satisfy (P1)-(P3).

Theorem 3.2. *Assume that $d_1\leq d_2$. Then for any $c\geq c^*$, system* (2.5) *has a traveling wave solution with speed c, which connects* $(0,0)$ *with* (u^+,v^+)

Proof. The conclusion for $c>c^*$ can be obtained from the above discussion. Thus we only need to established the existence of traveling wave solution when $c=c^*$. For this case, let $c_k\subset(c^*,c^*+1)$ be a decreasing series with $\lim_{k\to\infty}c_k=c^*$. For $c_k>c^*$, equation (2.6) with $c=c_k$ admits a nondecreasing solution $(\phi_k(t),\varphi_k(t))$ such that

$$
\lim_{t\to-\infty}(\phi_k(t),\varphi_k(t))=(0,0),\quad \lim_{t\to+\infty}(\phi_k(t),\varphi_k(t))=(u^+,v^+).
$$

By using a limit argument similar to Zhao and Wang [22], we are able to obtain the existence result. We omit the details. The proof is complete. □

Summarizing the results in Theorem 3.2 and noting the equivalence between (1.5) and (2.5), we in fact obtain the conclusion in Theorem 2.1.

We say that c^* is the minimal wave speed in the sense that (2.5) has no traveling wave solution with $c \in (0, c^*)$. In the following, we argue that c^* is the minimal wave speed. In fact, the linearized system of (2.6) at zero solution is (3.6) and the function $F(\lambda, c)$ is obtained by substituting $e^{\lambda t}$ into the second equation of (3.6). For $0 < c < c^*$, we have from (ii) of Lemma 3.3 that $F(\lambda, c) = 0$ has no real root. Furthermore, assume that $F(\lambda, c) = 0$ has a root $\lambda = \alpha + \mathrm{i}\beta$ with $\beta > 0$, then we obtain

$$F(\alpha + \mathrm{i}\beta) = 2d_2(e^{\alpha}\cos\beta - 1) - c(\alpha + \mathrm{i}\beta) + g(\frac{z_0}{c_{11}}) = 0,$$

which is impossible. Therefore, $F(\lambda, c) = 0$ has no root in $\mathbb{C}$. That is, the second equation of (3.6) has no solution with the form $\varphi(t) = e^{\lambda t}$ for $\lambda \in \mathbb{C}$.

Assume that (2.6) has a nondecreasing solution $(\phi(t), \varphi(t))$ satisfying $\lim_{t\to-\infty}(\phi(t), \varphi(t)) = (0, 0)$. We have from (2.6) that $\lim_{t\to-\infty}(\phi'(t), \varphi'(t)) = (0, 0)$. If $(\phi(t), \varphi(t))$ is smooth enough, one can obtain $\lim_{t\to-\infty}(\phi^{(k)}(t), \varphi^{(k)}(t)) = (0, 0)$ for any integer $k > 0$. For second equation of the linearized system (3.6), its solution with the property, $\lim_{t\to-\infty}\varphi^{(k)}(t) = 0$ could only be a function with the form $e^{\lambda t}$. But this is unavailable.

4. Concluding discussions

Recently, Yu, Weng and Huang [19] considered a corresponding model of (1.4) with nonlocal diffusion terms $\int_{-\infty}^{+\infty} J(y-x)[u(t,y)-u(t,x)]dy$ and $\int_{-\infty}^{+\infty} J(y-x)[v(t,y)-v(t,x)]dy$. They showed that the sufficient condition for the existence of minimal wave speed c^* is $d_1 \le d_2$. This implies that the nonlocal diffusion of the pioneer species accelerates the mild wave propagation. It seems that our results obtained for lattice system (1.5) are the same as in [19]. That is to say: the lattice distribution and diffusion of pioneer-climax species also accelerates the mild wave propagation. On the other hand, we see that time delay τ is a harmless delay.

This is the first time that the dynamics of the lattice pioneer-climax competition model with delay is studied, and we only considered one possible case about the traveling wave solution connecting one boundary equilibrium and the coexisting equilibrium. This model has complex dynamical properties, and is worth to be the target of further study.

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